On the Slope Filtration of ϕ-modules over the Robba Ring

by

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Abstract

Given a p-adic representation of the Galois group of a local field, we show that its Galois cohomology can be computed using the associated étale (φ, Γ)-module over the Robba ring; this is a variant of a result of Herr. We then establish analogues, for not necessarily étale (φ, Γ)-modules over the Robba ring, of the Euler-Poincaré characteristic formula and Tate local duality for p-adic representations. These results are expected to intervene in the duality theory for Selmer groups associated to de Rham representations.

We introduce the notion of families of ϕ-modules which arises naturally from both rigid cohomology and p-adic Hodge theory. We then prove the local constancy of generic HN-polygons of families of overconvergent ϕ-modules and the semicontinuity of HN-polygons of families of ϕ-modules over reduced affinoid algebras. These results are prospective for a slope theory of families of (overconvergent) ϕ-modules.

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Contents

Chapter 1

Cohomology and Duality for (φ, Γ)-modules over the Robba Ring

Introduction

Two of the basic results in the theory of Galois cohomology over a local field are the Euler-Poincaré characteristic formula and Tate's local duality theorem. In this paper, we generalize these results to a larger category than the category of p-adic representations, namely the category of (φ, Γ)-modules over the Robba ring. We expect that these results will be relevant to the deformation theory of Galois representations, via a study of duality properties of Selmer groups associated to de Rham representations. This would extend the work of Bloch-Kato for ordinary representations [8]. See [30] for more details.

In the remainder of this introduction, we formulate more precise statements of our results (but we skip some definitions found in the body of the text). Let p be a prime number, and fix a finite extension K of $\mathbb{Q}_p$. Write $G_K = \mathrm{Gal}(\overline{K}/K)$ and $\Gamma = \Gamma_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$. By a "$p$-adic representation" let us mean a finite dimensional $\mathbb{Q}_p$-vector space V equipped with a continuous linear action of $G_K = \mathrm{Gal}(\overline{K}/K)$. Fontaine [18] constructed a functor $\mathbf{D}$ associating to each p-adic representation V an *étale* (φ, Γ)*-module* $\mathbf{D}(V)$ over a certain two-dimensional local field $\mathcal{E}_K$, and established an equivalence of categories between p-adic representations and étale (φ, Γ)-modules. For the moment, all we will say about $\mathbf{D}(V)$ is that $\mathcal{E}_K$ is constructed to carry a Frobenius operator φ and an action of Γ commuting with each other, and $\mathbf{D}(V)$ is a finite dimensional $\mathcal{E}_K$-vector space carrying semilinear actions of

φ and Γ. (The condition of étaleness is a certain extra restriction on the φ-action.)

Fontaine's equivalence suggests that one can compute Galois cohomology of a p-adic representation from the ostensibly simpler object $\mathbf{D}(V)$; this was worked out by Herr [20], who constructed an explicit complex from $\mathbf{D}(V)$ computing the Galois cohomology of V and the cup product. In case Γ is procyclic and topologically generated by γ, the complex is particularly easy to describe: it is the complex

$$0 \to \mathbf{D}(V) \stackrel{d_1}{\to} \mathbf{D}(V) \oplus \mathbf{D}(V) \stackrel{d_2}{\to} \mathbf{D}(V) \to 0$$

with $d_1(x) = ((\gamma-1)x, (\varphi-1)x)$ and $d_2((x,y)) = (\varphi-1)x-(\gamma-1)y$. Herr also showed [21] that one can easily recover the Euler-Poincaré characteristic formula and Tate local duality from this description.

More recently, Berger [2] (building on work of Cherbonnier and Colmez [10]) constructed a functor $\mathbf{D}^\dagger_{\mathrm{rig}}$ giving an equivalence between the category of p-adic representations and the category of étale (φ,Γ)-modules over a different ring, the *Robba ring* $\mathcal{R}_K$. Berger's original justification for introducing $\mathbf{D}^\dagger_{\mathrm{rig}}$ was to show that for V a de Rham representation, $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ can be used to construct a p-adic differential equation of the sort addressed by Crew's conjecture; this led Berger to prove that the p-adic monodromy conjecture of p-adic differential equations implies Fontaine's conjecture that de Rham representations are potentially semistable.

Subsequently, Colmez [13] observed that non-étale (φ,Γ)-modules over $\mathcal{R}_K$ play a role in the study of Galois representations, even though they do not themselves correspond to representations. Colmez specifically considered the class of two-dimensional representations which are *trianguline* (this idea goes back to Mazur), that is, their associated étale (φ,Γ)-modules over $\mathcal{R}_K$ admit filtrations by not necessarily étale (φ,Γ)-submodules with successive quotients free of rank 1. (The supply of such representations is plentiful: for instance, a result of Kisin [27] implies that many of the Galois representations associated to overconvergent p-adic modular forms are trianguline.) Colmez classified these representations in the dimension 2 case and showed that they fit naturally into the p-adic local Langlands correspondence of $\mathrm{GL}_2(\mathbb{Q}_p)$ initiated by Breuil.

In so doing, Colmez introduced an analogue of Herr's complex for an arbitrary (φ,Γ)-module over $\mathcal{R}_K$. Although he does not explicitly assert that this complex computes Galois cohomology, we infer that he had the following theorem in mind; we include its proof, an easy reduction to Herr's theorem, to fill a gap in the literature.

Theorem 1.0.1. *Let V be a p-adic representation of G_K. Then there are isomorphisms*

$$H^i(\boldsymbol{D}^\dagger_{\mathrm{rig}}(V)) \cong H^i(G_K, V) \qquad (i = 0, 1, 2)$$

which are functorial in V and compatible with cup products.

At this point, one may reasonably expect that the Euler-Poincaré characteristic formula and Tate local duality should extend to (φ, Γ)-modules over the Robba ring, using $\mathcal{R}(\omega) = \mathbf{D}^\dagger_{\mathrm{rig}}(\mathbb{Q}_p(1))$ as the dualizing object. The main goal of this article is to prove these results.

Theorem 1.0.2. *Let D be a (φ, Γ)-module over the Robba ring $\mathcal{R}_K$.*

(a) We have $H^i(D)$ are all finite dimensional $\mathbb{Q}_p$-vector spaces and

$$\chi(D) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Q}_p} H^i(D) = -[K : \mathbb{Q}_p] \operatorname{rank} D.$$

(b) For $i = 0, 1, 2$, the composition

$$H^i(D) \times H^{2-i}(D^\vee(\omega)) \to H^2((D \otimes D^\vee)(\omega)) \to H^2(\omega),$$

in which the first map is the cup product, is a perfect pairing into $H^2(\omega) \cong \mathbb{Q}_p$.

Our method of proof is to reduce to the known case of an étale (φ, Γ)-module, where by Theorem 1.0.1 we can invoke the standard form of the theorem. In doing so, we construct a bigger category, the category of generalized (φ, Γ)-modules, which allows us to consider the cohomology of the quotient of two (φ, Γ)-modules. Moreover, in case $K = \mathbb{Q}_p$ and $p > 2$, we provide an explicit calculation of H^2 of rank 1 (φ, Γ)-modules as a complement to Colmez's calculation on H^0 and H^1 in [13].

The author should mention that in a different direction, Seunghwan Chang has obtained some interesting results concerning extensions of (φ, Γ)-modules in his thesis [9].

1.1 Preliminaries

1.1.1 p-adic Hodge theory and (φ, Γ)-modules

This section is a brief summary of some basic constructions of p-adic Hodge theory and (φ, Γ)-modules. The results recalled here can be found in [19], [16], [32], [18], [10], and [2].

Let p be a prime number, and fix a finite extension K of $\mathbb{Q}_p$. Write $G_K = \mathrm{Gal}(\overline{K}/K)$. A *p-adic representation* V is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of G_K. The dimension of this representation is defined as the dimension of V as a $\mathbb{Q}_p$-vector space and is usually denoted by d.

Let k be the residue field of K, $W(k)$ be the ring of Witt vectors with coefficients in k, and $K_0 = W(k)[1/p]$ be the maximal unramified subfield of K. Let μ_{p^n} denote the group of p^n-th roots of unity. For every n, we choose a generator $\varepsilon^{(n)}$ of μ_{p^n}, with the requirement that $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. That makes $\varepsilon = \varprojlim_n \varepsilon^{(n)}$ a generator of $\varprojlim_n \mu_{p^n} \simeq \mathbb{Z}_p(1)$. We set $K_n = K(\mu_{p^n})$ and $K_\infty = \bigcup_{n=1}^{\infty} K_n$. The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ is defined by $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for all $n \in \mathbb{N}$ and $g \in G_K$. The kernel of χ is $H_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}/K_\infty)$, and χ identifies $\Gamma = \Gamma_K = G_K/H_K$ with an open subgroup of $\mathbb{Z}_p^\times$.

Let $\mathbf{C}_p$ denote the p-adic complex numbers, i.e. the completion of $\overline{\mathbb{Q}_p}$ for the p-adic topology, and set

$$\widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, ...) \mid (x^{(i+1)})^p = x^{(i)}\}.$$

A ring structure on $\widetilde{\mathbf{E}}$ is given by the following formulas: If $x = (x^{(i)})$ and $y = (y^{(i)})$, then their sum $x + y$ and product xy are given by

$$(x+y)^{(i)} = \lim_{j \to \infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \text{ and } (xy)^{(i)} = x^{(i)} y^{(i)}.$$

If $x = (x^{(i)})$, we define $v_{\widetilde{\mathbf{E}}}(x) = v_p(x^{(0)})$. This is a valuation on $\widetilde{\mathbf{E}}$ and the corresponding topology coincides with the projective limit topology; as a consequence, $\widetilde{\mathbf{E}}$ is a complete valuation ring with respect to $v_{\widetilde{\mathbf{E}}}$. Furthermore, the induced $G_{\mathbb{Q}_p}$-action on $\widetilde{\mathbf{E}}$ preserves this valuation. Let $\widetilde{\mathbf{E}}^+$ be the ring of integers of this valuation; i.e. $\widetilde{\mathbf{E}}^+$ is the set of $x \in \widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. From the construction of ε, we can naturally view it as an element of $\widetilde{\mathbf{E}}^+$. Set $\mathbf{E}_{K_0} = k((\varepsilon - 1))$, $\mathbf{E}$ the separable closure of $\mathbf{E}_{K_0}$ in $\widetilde{\mathbf{E}}$ and $\mathbf{E}_K = \mathbf{E}^{H_K}$. If K_0' denotes the maximal unramified extension of K_0 in K_∞ and k' is its residue field, then the discrete valuation ring $\mathbf{E}_K^+$ is just $k'[[\overline{\pi}_K]]$ where $\overline{\pi}_K$ is a uniformizer ([19], [32]).

Let $\widetilde{\mathbf{A}}^+$ (resp. $\widetilde{\mathbf{A}}$) be the ring $W(\widetilde{\mathbf{E}}^+)$ (resp. $W(\widetilde{\mathbf{E}})$) of Witt vectors with coefficients in $\widetilde{\mathbf{E}}^+$ (resp. $\widetilde{\mathbf{E}}$), and $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p]$ (resp. $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$). Set $\pi = [\varepsilon] - 1$, and $q = \varphi(\pi)/\pi$. Since $\widetilde{\mathbf{E}}^+$ is perfect, we have

$$\widetilde{\mathbf{A}}^+ = \{x = \textstyle\sum_{k=0}^{\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbf{E}}^+\},$$

where $[x_k]$ is the Teichmüller lift of x_k in $\widetilde{\mathbf{A}}^+$. This gives a bijection $\widetilde{\mathbf{A}}^+ \to (\widetilde{\mathbf{E}}^+)^{\mathbf{N}}$ which sends x to $(x_0, x_1, \ldots)$. Let $\widetilde{\mathbf{A}}^+$ be endowed with the topology induced from the product topology of the right hand side. Another way to get this topology is to define $([\overline{\pi}]^k, p^n)$ as a basis of neighborhoods of 0. The topology of $\widetilde{\mathbf{A}}$ is defined in the same way. The absolute Frobenius φ of $\widetilde{\mathbf{E}}$ lifts by functoriality of Witt vectors to the Frobenius operator φ of $\widetilde{\mathbf{A}}$ which commutes with the Galois action. It is easy to see that

$$\varphi(\textstyle\sum_{k=0}^{\infty} p^k[x_k]) = \sum_{k=0}^{\infty} p^k[x_k^p]$$

and therefore φ is an isomorphism. Now let $\mathbf{A}_{K_0}$ be the completion of $\mathcal{O}_{K_0}[\pi, \pi^{-1}]$ in $\widetilde{\mathbf{A}}$ for the topology given above. It is also the completion of $\mathcal{O}_{K_0}[[\pi]][\pi^{-1}]$ for the p-adic topology. This is a Cohen ring with residue field $\mathbf{E}_{K_0}$. Let $\mathbf{B}$ be the completion for the p-adic topology of the maximal unramified extension of $\mathbf{B}_{K_0} = \mathbf{A}_{K_0}[1/p]$. We then define $\mathbf{A} = \widetilde{\mathbf{A}} \cap \mathbf{B}$ and $\mathbf{A}^+ = \widetilde{\mathbf{A}}^+ \cap \mathbf{B}$. Note that these rings are endowed with the induced $G_{\mathbb{Q}_p}$ and Frobenius actions from $\widetilde{\mathbf{B}}$. For S any one of these rings, define $S_K = S^{H_K}$. Therefore $\mathbf{B}_K = \mathbf{A}_K[1/p]$ and $\mathbf{B}_K^+ = \mathbf{A}_K^+[1/p]$. When $K = K_0$, this definition of $\mathbf{A}_{K_0}$ coincides with the one given above.

We define $\mathbf{D}(V) = (\mathbf{B} \otimes V)^{H_K}$. It is a d-dimensional $\mathbf{B}_K$-vector space with Frobenius φ and Γ-action. Similarly, if T is a lattice of V, we define $\mathbf{D}(T) = (\mathbf{A} \otimes T)^{H_K}$, which is a free $\mathbf{A}_K$-module of rank d. We say a (φ, Γ)-module D over $\mathbf{B}_K$ is *étale* if there is a free $\mathbf{A}_K$-submodule T of D, which is stable under φ and Γ actions, such that $T \otimes_{\mathbf{A}_K} \mathbf{B}_K = D$. Then $\mathbf{D}(V)$ is an étale (φ, Γ)-module since $D(T)$ is such an $\mathbf{A}_K$-lattice. The following result is due to Fontaine [18].

Theorem 1.1.1. *The functor $V \mapsto \boldsymbol{D}(V)$ is an equivalence from the category of p-adic representations of G_K to the category of étale (φ, Γ)-modules over $\boldsymbol{B}_K$; the inverse functor is $D \mapsto (\boldsymbol{B} \otimes D)^{\varphi=1}$.*

We define the ring of overconvergent elements as follows:

$$\widetilde{\mathbf{B}}^{\dagger,r} = \{x = \sum_{k>-\infty}^{+\infty} p^k[x_k] \in \widetilde{\mathbf{B}}, \lim_{k\to+\infty} v_{\widetilde{\mathbf{E}}}(x_k) + kpr/(p-1) = +\infty\}$$

and $\widetilde{\mathbf{B}}^{\dagger} = \cup_{r\geqslant 0}\widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}^{\dagger,r} = (\widetilde{\mathbf{B}}^{\dagger,r}) \cap \mathbf{B}$, $\mathbf{B}^{\dagger} = \cup_{r\geqslant 0}\mathbf{B}^{\dagger,r}$. Note that $\varphi : \widetilde{\mathbf{B}}^{\dagger,r} \to \widetilde{\mathbf{B}}^{\dagger,pr}$ is an isomorphism. Let $\widetilde{\mathbf{A}}^{\dagger,r}$ be the set of elements of $\widetilde{\mathbf{B}}^{\dagger,r} \cap \widetilde{\mathbf{A}}$ such that $v_{\widetilde{\mathbf{E}}}(x_k) + kpr/(p-1) \geq 0$ for every k and similarly $\widetilde{\mathbf{A}}^{\dagger} = \cup_{r\geqslant 0}\widetilde{\mathbf{A}}^{\dagger,r}$, $\mathbf{A}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r} \cap \mathbf{A}$, $\mathbf{A}^{\dagger} = \cup_{r\geqslant 0}\mathbf{A}^{\dagger,r}$.

Define $\mathbf{D}^{\dagger,r}(V) = (\mathbf{B}^{\dagger,r} \otimes V)^{H_K}$ and $\mathbf{D}^{\dagger}(V) = \cup_{r\geqslant 0}\mathbf{D}^{\dagger,r}(V) = (\mathbf{B}^{\dagger} \otimes V)^{H_K}$. Similarly, if T is a lattice of V, we define $\mathbf{D}^{\dagger}(T) = (\mathbf{A}^{\dagger} \otimes T)^{H_K}$. We say a (φ, Γ)-module D over $\mathbf{B}_K^{\dagger}$ is *étale* if there is a free $\mathbf{A}_K^{\dagger}$-submodule T of D, which is stable under φ and Γ actions, such that $T \otimes_{\mathbf{A}_K^{\dagger}} \mathbf{B}_K^{\dagger} = D$. In [10], Cherbonnier and Colmez proved the following result.

Theorem 1.1.2. *There exists an $r(V)$ such that $\boldsymbol{D}(V) = \boldsymbol{B}_K \otimes_{\boldsymbol{B}_K^{\dagger,r}} \boldsymbol{D}^{\dagger,r}(V)$ if $r \geqslant r(V)$. Equivalently, $\boldsymbol{D}^{\dagger}(V)$ is a d-dimensional étale (φ,Γ)-module over $\boldsymbol{B}_K^{\dagger}$. Therefore, $V \mapsto \boldsymbol{D}^{\dagger}(V)$ is an equivalence from the category of p-adic representations of G_K to the category of étale (φ,Γ)-modules over $\boldsymbol{B}_K^{\dagger}$.*

We can take π_K to be an element of $\mathbf{A}_K^{\dagger}$ whose image modulo p is $\overline{\pi}_K$. Let e_K denote the ramification index of $K_\infty/(K_0)_\infty$. Then for $r \gg 0$, one can show that $\mathbf{B}_K^{\dagger,r}$ is given by

$$\mathbf{B}_K^{\dagger,r} = \{f(\pi_K) = \sum_{k=-\infty}^{+\infty} a_k \pi_K^k, \text{ where } a_k \in K_0' \text{ and } f(T) \text{ is convergent and bounded on } p^{-1/e_K r} \leq |T| < 1\}.$$

We see that the sup norms on closed annuli give a family of norms on $\mathbf{B}_K^{\dagger,r}$. Its Fréchet completion with respect to these norms is

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r} = \{f(\pi_K) = \sum_{k=-\infty}^{+\infty} a_k \pi_K^k, \text{ where } a_k \in K_0' \text{ and } f(T) \text{ is convergent on } p^{-1/e_K r} \leq |T| < 1\}.$$

Then the union $\mathbf{B}_{\mathrm{rig},K}^{\dagger} = \cup_{r\geq 0}\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ can be identified with the *Robba ring* $\mathcal{R}_K$ from the theory of p-adic differential equations, which is the set of holomorphic functions on the boundary of the open unit disk, by mapping π_K to T. And $\mathbf{B}_K^{\dagger} = \mathcal{E}_K^{\dagger}$ is the subring of $\mathcal{R}_K$ consisting of bounded functions. The p-adic completion of $\mathcal{E}_K^{\dagger}$ is $\mathcal{E}_K = \mathbf{B}_K$. When $K = K_0$, we can choose $\overline{\pi}_K = \overline{\pi} = \varepsilon - 1$ and $\pi_K = \pi = [\varepsilon] - 1$.

For any K, there exists an $r(K)$ such that for any $r \geq r(K)$ and $n \geq n(r) = (\log(r/(p-1))/\log p) + 1$, we have an injective morphism ι_n from $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ to $K_n[[t]]$ which satisfies

$\iota_n = \iota_{n+1} \circ \varphi$ (see [2, Chapter 2] for the construction). For example, when $K = K_0$, ι_n is defined by $\iota_n(\pi) = \varepsilon^{(n)} e^{t/p^n} - 1$.

Define the operator $\nabla = \log(\gamma)/\log(\chi(\gamma))$ which gives an action of $\mathrm{Lie}(\Gamma_K)$ on $\mathcal{R}_K$. Let $t = \log([\varepsilon])$ and set the differential operator $\partial = \nabla/t$ which satisfies:

$$\partial \circ \varphi = p\varphi \circ \partial \text{ and } \partial \circ \gamma = \chi(\gamma)\gamma \circ \partial.$$

In case $K = \mathbb{Q}_p$, we choose $\bar{\pi}_K = \bar{\pi} = \varepsilon - 1$ and $\pi_K = \pi = [\varepsilon] - 1$. Then we have $\nabla(f(\pi)) = \log(1+\pi)(1+\pi)df/d\pi$ and $\partial = (1+\pi)df/d\pi$.

Define $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{D}^{\dagger,r}(V) \otimes_{\mathbf{B}^{\dagger,r}_K} \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = \cup_{r \geqslant 0} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{D}^{\dagger}(V) \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ which is an étale (φ, Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Here we say a (φ,Γ)-module D over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is *étale* if D has a $\mathbf{B}^{\dagger}_K$-submodule D', which is an étale (φ,Γ)-module over $\mathbf{B}^{\dagger}_K$ under the restricted φ and Γ actions, such that $D = D' \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$. The following theorem is due to Kedlaya ([25]).

Theorem 1.1.3. *The correspondence $D \mapsto \boldsymbol{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\boldsymbol{B}^{\dagger}_K} D$ is an equivalence between the category of étale (φ,Γ)-modules over $\boldsymbol{B}^{\dagger}_K$ and the category of étale (φ,Γ)-modules over $\boldsymbol{B}^{\dagger}_{\mathrm{rig},K}$. As a consequence, $V \mapsto \boldsymbol{D}^{\dagger}_{\mathrm{rig}}(V)$ is an equivalence of categories from the category of p-adic representations of G_K to the category of étale (φ,Γ)-modules over $\boldsymbol{B}^{\dagger}_{\mathrm{rig},K}$.*

Suppose D is an arbitrary (φ,Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ of rank d. By a result of Berger [3, Theorem I.3.3], for r large enough, there is a unique $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ submodule D_r of D such that:

(1) D_r is a free $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-module of rank d, stable under Γ action, and $D_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{B}^{\dagger}_{\mathrm{rig},K} = D$;

(2) We can find a $\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}$-basis of $D_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}$ from elements of $\varphi(D_r)$.

If D_r is defined, then for any $r' \geq r$ we have $D_{r'} = D_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathbf{B}^{\dagger,r'}_{\mathrm{rig},K}$. We set

$$\mathbf{D}^{+,n}_{\mathrm{dif}}(D) = D_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\iota_n} K_n[[t]]$$

and call it the *localization at* $\varepsilon^{(n)} - 1$ of D. It is easy to see that $\mathbf{D}^{+,n}_{\mathrm{dif}}(D)$ is well defined, i.e. it does not depend on the choice of r. Let i_n denote the natural inclusion map from $K_n[[t]]$ to $K_{n+1}[[t]]$, and define the *connecting map* φ_n as

$$\varphi \otimes i_n : \mathbf{D}^{+,n}_{\mathrm{dif}}(D) \longrightarrow \mathbf{D}^{+,n+1}_{\mathrm{dif}}(D).$$

It is clear that

$$\varphi_n \otimes 1 : \mathbf{D}^{+,n}_{\mathrm{dif}}(D) \otimes_{K_n[[t]]} K_{n+1}[[t]] \longrightarrow \mathbf{D}^{+,n+1}_{\mathrm{dif}}(D)$$

is an isomorphism.

1.1.2 Slope theory of φ-modules

This section is a short collection of some basic facts concerning the slope theory of φ-modules over $\mathcal{R}_K$ (resp. $\mathcal{E}^\dagger_K$) which we will use later. For a complete treatment of this topic, see [25].

A *φ-module* M over $\mathcal{R}_K$ (resp. $\mathcal{E}^\dagger_K$) is a finitely generated free $\mathcal{R}_K$-module (resp. $\mathcal{E}^\dagger_K$-module) with a Frobenius action φ that satisfies $\varphi^*M \cong M$. We can view M as a left module over the twisted polynomial ring $\mathcal{R}_K\{X\}$. For a positive integer a, define the *a-pushforward* functor $[a]_*$ from φ-modules to φ^a-modules to be the restriction along the inclusion $\mathcal{R}_K\{X^a\} \to \mathcal{R}_K\{X\}$. Define the *$a$-pullback* functor $[a]^*$ from φ^a-modules to φ-modules to be the extension of scalars functor $M \to M \otimes_{\mathcal{R}_K\{X^a\}} \mathcal{R}_K\{X\}$. If $\operatorname{rank} M = n$, then $\wedge^n M$ has rank 1 over $\mathcal{R}_K$. Let v be a generator, then $\varphi(v) = \lambda v$ for some $\lambda \in \mathcal{R}_K^\times = (\mathcal{E}^\dagger_K)^\times$. Define the *degree* of M by setting $\deg(M) = w(\lambda)$, here w is the p-adic valuation of $\mathcal{E}_K$. Note that this does not depend on the choice of v. If M is nonzero, define the *slope* of M by setting $\mu(M) = \deg(M)/\operatorname{rank}(M)$. The following formal properties are easily verified:

(1) If $0 \to M_1 \to M \to M_2 \to 0$ is exact, then $\deg(M) = \deg(M_1) + \deg(M_2)$;

(2) We have $\mu(M_1 \otimes M_2) = \mu(M_1) + \mu(M_2)$;

(3) We have $\deg(M^\vee) = -\deg(M)$ and $\mu(M^\vee) = -\mu(M)$.

We say M is *étale* if M has a φ-stable $\mathcal{O}_{\mathcal{E}^\dagger_K}$-submodule M' such that $\varphi^*M' \cong M'$ and $M' \otimes_{\mathcal{R}^{int}_K} \mathcal{R}_K = M$. More generally, suppose $\mu(M) = s = c/d$, where c, d are coprime integers with $d > 0$. We say M is *pure* if for some φ-module N of rank 1 and degree $-c$, $([d]_*M) \otimes N$ is étale. We say a (φ, Γ)-module over $\mathcal{R}_K$ (resp. $\mathcal{E}^\dagger_K$) is *pure* if the underlying φ-module structure is pure. In the étale case, this definition is consistent with the one given in the last section. We have the following facts:

(1) A φ-module is pure of slope 0 if and only if it is étale;

(2) The dual of a pure φ-module of slope s is itself pure of slope $-s$;

(3) If M_1, M_2 are pure of slopes s_1, s_2, then $M_1 \otimes M_2$ is pure of slope $s_1 + s_2$.

We say M is *semistable* if for every nontrivial φ-submodule N, we have $\mu(N) \geq \mu(M)$. A difficult result is that M is semistable if and only if it is pure [25, Theorem 2.1.8]. As a consequence of this result, we have the following slope filtrations theorem.

Theorem 1.1.4. *(Kedlaya) Every φ-module M over $\mathcal{R}_K$ admits a unique filtration $0 = M_0 \subset M_1 \subset ... \subset M_l = M$ by saturated φ-submodules whose successive quotients are pure with $\mu(M_1/M_0) < ... < \mu(M_{l-1}/M_l)$. As a consequence, if M is a (φ, Γ)-module, then these M_i's are all (φ, Γ)-submodules.*

1.2 Cohomology of (φ, Γ)-modules

1.2.1 Construction of cohomology

Suppose D is a (φ, Γ)-module over $\mathcal{E}_K^\dagger$, $\mathcal{E}_K$, or $\mathcal{R}_K$. Let Δ_K be a torsion subgroup of Γ_K. Since Γ_K is an open subgroup of $\mathbb{Z}_p^\times$, Δ_K is a finite group of order dividing $p-1$ (or 2 if $p = 2$). Let p_Δ be the idempotent operator defined by $p_\Delta = (1/|\Delta_K|)\sum_{\delta \in \Delta_K} \delta$. Then p_Δ is the projection from D to $D' = D^{\Delta_K}$. In case Γ_K/Δ_K is procyclic, we set the following complex, where γ denotes a topological generator of Γ_K:

$$C^\bullet_{\varphi,\gamma}(D) : 0 \longrightarrow D' \xrightarrow{d_1} D' \oplus D' \xrightarrow{d_2} D' \longrightarrow 0,$$

with $d_1(x) = ((\gamma - 1)x, (\varphi - 1)x)$ and $d_2(x) = ((\varphi - 1)x - (\gamma - 1)y)$. Let $H^\bullet(D)$ denote cohomology groups of this complex. We need to check $H^\bullet(D)$ is well defined, i.e. it does not depend on the choice of Δ_K. In the following, we assume D is over $\mathcal{E}_K^\dagger$. The argument also works for (φ, Γ)-modules over $\mathcal{E}_K$ and $\mathcal{R}_K$.

First, it is obvious that $H^0(D) = D^{\Gamma=1, \varphi=1}$. For H^1, we claim $H^1(D)$ classifies all the extensions of $\mathcal{E}_K^\dagger$ by D in the category of (φ, Γ)-modules over $\mathcal{E}_K^\dagger$. In fact, if D is a (φ, Γ)-module over $\mathcal{E}_K^\dagger$ and D_1 is a such extension, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & D_1 & \longrightarrow & \mathcal{E}_K^\dagger & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_\triangle} & & \downarrow{\scriptstyle p_\triangle} & & \downarrow{\scriptstyle p_\triangle} & & \\ 0 & \longrightarrow & D' & \longrightarrow & (D_1)' & \longrightarrow & (\mathcal{E}_K^\dagger)' & \longrightarrow & 0. \end{array}$$

Since $|\triangle_K|$ divides $p-1$ (or 2 if p=2), all the characters of $\triangle_K$ take values in $\mathbb{Q}_p \subset \mathcal{E}_K^\dagger$. Then by standard representation theory, we have the eigenspace decomposition $D = \oplus_\chi D_\chi$ for any D. Here χ ranges over all the characters of $\triangle_K$, and D_χ is the χ-eigenspace. Any nonzero element x of $(\mathcal{E}_K^\dagger)_\chi$ (e.g. $\sum_{\delta\in\triangle_K}\chi(\delta^{-1})\delta(\varepsilon)$) gives an isomorphism $D' \cong D_\chi$ by mapping a to xa. Therefore we have $D' \otimes_{(\mathcal{E}_K^\dagger)'} \mathcal{E}_K^\dagger \cong D$, where the isomorphism respects φ and Γ_K-actions. So the extensions of $\mathcal{E}_K^\dagger$ by D as (φ, Γ_K)-modules over $\mathcal{E}_K^\dagger$ are in one-to-one correspondence with the extensions of $(\mathcal{E}_K^\dagger)'$ by $(D_1)'$ as $(\varphi, \Gamma_K/\triangle_K)$-modules over $(\mathcal{E}_K^\dagger)'$. The latter objects are clearly classified by $H^1(C^\bullet_{\varphi,\gamma}(D))$.

For H^2, suppose $\triangle'_K \supset \triangle_K$ is another torsion subgroup of Γ and $m = [\triangle'_K : \triangle_K]$. Then γ^m is a topological generator of $\Gamma/\triangle'_K$ and $p_{\triangle/\triangle'} = 1/m\sum_{i=0}^{m-1}\gamma^i$ is a projection from $D^{\triangle_K}$ to $D^{\triangle'_K}$. Obviously $p_{\triangle/\triangle'}$ reduces to a projection

$$D^{\triangle_K}/(\gamma-1) \xrightarrow{p_{\triangle/\triangle'}} D^{\triangle'_K}/(\gamma^m-1).$$

Similarly as above, we have $D^{\triangle_K} = D^{\triangle'_K} \oplus (\oplus_{\chi\neq 1} D_\chi^{\triangle_K})$ where χ ranges over all the nontrivial characters of $\triangle'_K/\triangle_K$. Note that $\gamma - 1$ acts bijectively on any $D_\chi^{\triangle_K}$ with $\chi \neq 1$, so the natural map

$$D^{\triangle'_K}/(\gamma^m-1) \xrightarrow{i} D^{\triangle_K}/(\gamma-1)$$

is surjective. We conclude that both i and $p_{\triangle/\triangle'}$ are isomorphisms. So there are canonical isomorphisms between $H^2(D)$'s respecting different choices of torsion subgroups.

Finally, we define cup products as follows:

$$H^0(M)\times H^0(N) \to H^0(M\otimes N) \qquad (x,y)\mapsto x\otimes y$$

$$H^0(M)\times H^1(N) \to H^1(M\otimes N) \qquad (x,(\bar{y},\bar{z}))\mapsto (\overline{x\otimes y},\overline{x\otimes z})$$

$$H^0(M)\times H^2(N) \to H^2(M\otimes N) \qquad (x,\bar{y})\mapsto \overline{x\otimes y}$$

$$H^1(M)\times H^1(N) \to H^2(M\otimes N) \qquad ((\bar{x},\bar{y}),(\bar{z},\bar{t}))\mapsto \overline{y\otimes\gamma(z) - x\otimes\varphi(t)}$$

1.2.2 Shapiro's lemma

If L is a finite extension of K, and D is a (φ,Γ)-module over $\mathcal{E}_L$ (resp. $\mathcal{E}_L^\dagger$, $\mathcal{R}_L$). Consider $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D = \{f : \Gamma_K \to D | f(hg) = hf(g) \text{ for } h \in \Gamma_L\}$, the induced Γ_K-representation of D as a Γ_L-representation. We can endow $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ with an $\mathcal{E}_K$ (resp. $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$) module structure and a Frobenius action φ by defining $(af)(g) = g(a)f(g)$ and $(\varphi(f))(g) = \varphi(f(g))$ for any element $f : \Gamma_K \to D$ of $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ and $g \in \Gamma_K$. In this way $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} D$ is now a (φ,Γ)-module over $\mathcal{E}_K$ (resp. $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$). We call it the *induced* (φ,Γ)*-module of* D from L to K, and denote it by $\mathrm{Ind}_L^K D$. Note that $\mathrm{rank}\,\mathrm{Ind}_L^K D = [L:K]\,\mathrm{rank}\,D$.

One can prove that the above definition of induced (φ,Γ)-modules is compatible with the definition of induced representations of Galois representations.

Proposition 1.2.1. *Suppose V is a p-adic representation of G_L, then $\boldsymbol{D}(\mathrm{Ind}_{G_L}^{G_K} V) = \mathrm{Ind}_L^K \boldsymbol{D}(V)$ (resp. $\boldsymbol{D}^\dagger$, $\boldsymbol{D}^\dagger_{\mathrm{rig}}$).*

Proof. For the functor $\mathbf{D}$, we define a map P from $\mathbf{D}(\mathrm{Ind}_{G_L}^{G_K} V) = ((\mathrm{Ind}_{G_L}^{G_K} V) \otimes_{\mathbb{Q}_p} \mathbf{B})^{H_K}$ to $\mathrm{Ind}_L^K \mathbf{D}(V)$ as follows: for $\sum f_i \otimes b_i \in ((\mathrm{Ind}_{G_L}^{G_K} V) \otimes_{\mathbb{Q}_p} \mathbf{B})^{H_K}$ and $\bar{g} \in \Gamma_K$, we put $P(\sum f_i \otimes b_i)(\bar{g}) = \sum f_i(g) \otimes g b_i$, where g is any lift of $\bar{g}$ in G_K. To see that P is well defined, we first need to show that it doesn't depend on the choice of g. In fact, for any $h \in H_K$ we have

$$\sum f_i(gh) \otimes ghb_i = \sum (hf_i)(g) \otimes g(hb_i) = \sum f_i(g) \otimes gb_i,$$

where the last equality is concluded from the fact that $\sum f_i \otimes b_i = \sum hf_i \otimes hb_i$, since $\sum f_i \otimes b_i$ is H_K-invariant. Then for $h \in H_L$, we have

$$h(\sum f_i(g) \otimes gb_i) = \sum hf_i(g) \otimes hgb_i = \sum f_i(hg) \otimes hgb_i = \sum f_i(g) \otimes gb_i,$$

since hg is also a lift of g. This implies that $P(\sum f_i \otimes b_i)(\bar{g})$ lies in $\mathbf{D}(V)$. For $\bar{h} \in \Gamma_L$, we have

$$P(\sum f_i \otimes b_i)(\bar{h}\bar{g}) = \sum f_i(hg) \otimes hgb_i = h(\sum f_i(g) \otimes gb_i) = \bar{h}(P(\sum f_i \otimes b_i)(\bar{g})).$$

It follows that $P(\sum f_i \otimes b_i)$ really lies in $\mathrm{Ind}_L^K \mathbf{D}(V)$. It is obvious that P is injective and commutes with φ. Now we check that P is a morphism of (φ,Γ)-modules. For $a \in \mathcal{E}_K$, we have

$$\begin{aligned} P(a(\sum f_i \otimes b_i))(\bar{g}) &= P(\sum f_i \otimes ab_i)(\bar{g}) = \sum f_i(g) \otimes g(a)g(b_i) \\ &= g(a)(\sum f_i(g) \otimes gb_i) = (a(P(\sum f_i \otimes b_i)))(\bar{g}). \end{aligned}$$

So P is a morphism of $\mathcal{E}_K$-modules. For $\bar{h} \in \Gamma_K$, we have

$$\begin{aligned} P(\bar{h}(\sum f_i \otimes b_i))(\bar{g}) = P(\sum hf_i \otimes hb_i))(\bar{g}) &= \sum (hf_i)(g) \otimes ghb_i = \\ \sum f_i(gh) \otimes ghb_i &= (\bar{h}P(\sum f_i \otimes b_i))(\bar{g}), \end{aligned}$$

hence P is Γ_K-equivariant. Now note that

$$\dim_{\mathcal{E}_K}(\operatorname{Ind}_L^K \mathbf{D}(V)) = [L:K] \dim_{\mathcal{E}_L} \mathbf{D}(V) = [L:K] \dim V = \dim_{\mathcal{E}_K} \mathbf{D}(\operatorname{Ind}_{G_L}^{G_K} V),$$

so P is an isomorphism. Since $\operatorname{Ind}_L^K \mathbf{D}^\dagger(V)$ is an étale (φ, Γ)-module over $\mathcal{E}_K^\dagger$ contained in $\operatorname{Ind}_L^K \mathbf{D}(V) = \mathbf{D}(\operatorname{Ind}_{G_L}^{G_K} V)$ and of maximal dimension, we conclude that $\operatorname{Ind}_L^K \mathbf{D}^\dagger(V) = \mathbf{D}^\dagger(\operatorname{Ind}_{G_L}^{G_K} V)$. Finally, we get

$$\operatorname{Ind}_L^K \mathbf{D}^\dagger_{\mathrm{rig}}(V) = \operatorname{Ind}_L^K \mathbf{D}^\dagger(V) \otimes \mathcal{R}_K = \mathbf{D}^\dagger(\operatorname{Ind}_{G_L}^{G_K} V) \otimes \mathcal{R}_K = \mathbf{D}^\dagger_{\mathrm{rig}}(\operatorname{Ind}_{G_L}^{G_K} V).$$

□

Theorem 1.2.2. *(Shapiro's Lemma for (φ, Γ)-modules) Suppose D is a (φ, Γ)-module over $\mathcal{E}_L$, $\mathcal{E}_L^\dagger$ or $\mathcal{R}_L$. Then there are isomorphisms*

$$H^i(D) \cong H^i(\operatorname{Ind}_L^K D) \qquad (i = 0, 1, 2)$$

which are functorial in D and compatible with cup products.

Proof. We first prove the theorem in the case that both of Γ_K and Γ_L are procyclic. Suppose $[\Gamma_K : \Gamma_L] = m$. Choose a topological generator γ_K of Γ_K, then $\gamma_L = \gamma_K^m$ is a topological generator of Γ_L. Define $Q : D \to \operatorname{Ind}_L^K D$ as follows: for any $x \in D$, $(Q(x))(e) = x$ and $(Q(x))(\gamma_K^i) = 0$ for $1 \leq i \leq m-1$. Then Q is a well defined φ, Γ_L-equivariant injective morphism of $\mathcal{R}_K$-modules. We claim that Q induces a φ-equivariant isomorphism from $D/(\gamma_L - 1)$ to $(\operatorname{Ind}_L^K D)/(\gamma_K - 1)$. Suppose $x \in D$, and $Q(x) = (\gamma_K - 1)f$ for some

$f \in \mathrm{Ind}_L^K D$. Then we have

$$\begin{aligned} x &= Q(x)(e) = (\gamma_K - 1)f(e) = f(\gamma_K) - f(e) \\ 0 &= Q(x)(\gamma_K^i) = (\gamma_K - 1)f(\gamma_K^i) = f(\gamma_K^{i+1}) - f(\gamma_K^i) \qquad 1 \le i \le m-1. \end{aligned}$$

Summing these equalities, we get $x = \sum_{i=0}^{m-1}(f(\gamma_K^{i+1}) - f(\gamma_K^i)) = f(\gamma_K^m) - f(e) = (\gamma_L - 1)f(e)$ since $\gamma_K^m = \gamma_L$. On the other hand, for any $f \in \mathrm{Ind}_L^K D$, suppose $f(\gamma_K^i) = x_i$ for $0 \le i \le m-1$. Then $f = \sum_{i=1}^m \gamma_K^i Q((\gamma_L)^{-1}x_{m-i})$ since for $0 \le j \le m-1$, we have

$$(\textstyle\sum_{i=1}^m \gamma_K^i Q((\gamma_L)^{-1}x_{m-i}))(\gamma_K^j) = \sum_{i=1}^m Q((\gamma_L)^{-1}x_{m-i})(\gamma_K^{i+j}) = Q((\gamma_L)^{-1}x_j)(\gamma_K^m) = x_j.$$

So both of f and $Q(x)$, where $x = \gamma_L^{-1}(\sum_{i=1}^m x_{m-i})$, have the same image in $(\mathrm{Ind}_L^K D)/(\gamma_K - 1)$.

For any $g \in \Gamma_K$, define the morphism Q^g by $Q^g(x) = g(Q(x))$ for any $x \in D$. Set $\widetilde{Q} = \sum_{i=0}^{m-1} Q^{\gamma_K^i}$ which is also φ, Γ_L-equivariant and injective since $(\widetilde{Q}(x))(e) = x$. We claim that $\widetilde{Q}$ induces an φ-equivariant isomorphism from D^{Γ_L} to $(\mathrm{Ind}_L^K D)^{\Gamma_K}$. The injectivity is obvious. Conversely, suppose $f : \Gamma_K \to D$ is an element of $(\mathrm{Ind}_L^K D)^{\Gamma_K}$ with $f(e) = x$. Then $f(g) = (gf)(e) = f(e) = x$ for any $g \in \Gamma_K$ since f is Γ_K-invariant. On the other hand, for $g \in \Gamma_L$, we have $f(g) = gf(e) = gx$. These imply that x is Γ_L-invariant. Therefore $\widetilde{Q}(x) = f$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} C^\bullet_{\varphi,\gamma'}(D) : 0 & \longrightarrow & D & \longrightarrow & D \oplus D & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{Q}} & & \downarrow{\scriptstyle Q\oplus\widetilde{Q}} & & \downarrow{\scriptstyle Q} & & \\ C^\bullet_{\varphi,\gamma}(\mathrm{Ind}_L^K D) : 0 & \longrightarrow & \mathrm{Ind}_L^K D & \longrightarrow & \mathrm{Ind}_L^K D \oplus \mathrm{Ind}_L^K D & \longrightarrow & \mathrm{Ind}_L^K D & \longrightarrow & 0. \end{array}$$

This induces morphisms α^i from $H^i(D)$ to $H^i(\mathrm{Ind}_L^K D)$. We will prove that they are isomorphisms.

For H^0, $\widetilde{Q}$ induces a φ-equivariant isomorphism from D^{Γ_L} to $(\mathrm{Ind}_L^K D)^{\Gamma_K}$. Taking φ-invariants, we conclude that α^0 is an isomorphism. For H^2, Q induces a φ-equivariant isomorphism from $D/(\gamma_L - 1)$ to $(\mathrm{Ind}_L^K D)/(\gamma_K - 1)$, so α^2 is also an isomorphism.

For H^1, we use the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{\Gamma_L}/(\varphi-1) & \longrightarrow & H^1(D) & \longrightarrow & (D/(\gamma_L-1))^{\varphi=1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{Q}} & & \downarrow{\scriptstyle \alpha^1} & & \downarrow{\scriptstyle Q} & & \\ 0 & \longrightarrow & (\mathrm{Ind}_L^K D)^{\Gamma_K}/(\varphi-1) & \longrightarrow & H^1(\mathrm{Ind}_L^K D) & \longrightarrow & (\mathrm{Ind}_L^K D/(\gamma_K-1))^{\varphi=1} & \longrightarrow & 0. \end{array}$$

We have proved that $\widetilde{Q}$ and Q are isomorphisms. So α^1 is an isomorphism by the Five Lemma.

For the general case, let Δ_K and Δ_L be the torsion subgroups of Γ_K and Γ_L respectively. Then Γ_L/Δ_L is a subgroup of Γ_K/Δ_K. Let γ_K be a topological generator of Γ_K/Δ_K. Suppose $[\Gamma_K/\Delta_K : \Gamma_L/\Delta_L] = m$, then $\gamma_L = \gamma_K^m$ is a topological generator of Γ_L/Δ_L. Set $Q' : D' \to (\mathrm{Ind}_L^K D)'$ as follows: for any $x \in D'$, $(Q'(x))(e) = x$ and $(Q'(x))(y) = 0$ for any other representative of Γ_K/Γ_L. We define $\widetilde{Q}' = \sum_{i=0}^{m-1} \gamma^i Q'$. Replacing Q by Q', and $\widetilde{Q}$ by $\widetilde{Q}'$ in the above argument, we are done. □

1.2.3 Comparison theorems

For a $\mathbb{Z}_p$-representation V (of finite length or not), define $H^\bullet(\mathbf{D}(V))$ using the same complex as in the last section. The groups $H^\bullet(\mathbf{D}(V))$ are also well defined by the same argument (Note: for $p = 2$, there is only one choice of Δ_K, so it is well defined automatically. However, the description of H^1 in terms of extensions does not apply to $\mathbb{Z}_2$, because the projection p_Δ is not integral.). The following theorem was first proved by Herr ([20]) in case Γ_K is procyclic. Our result is a small improvement of his result since Γ is always procyclic for $p \neq 2$.

Theorem 1.2.3. *Let V be a $\mathbb{Z}_p$-representation of G_K. Then there are isomorphisms*

$$H^i(\mathbf{D}(V)) \cong H^i(G_K, V) \qquad (i = 0, 1, 2)$$

which are functorial in V and compatible with cup products. The same conclusion therefore also holds for $\mathbb{Q}_p$-representations.

Proof. For V of finite length, we adapt the proof given by [12, Theorem 5.2.2] to the case, where Γ is not necessarily procyclic. Let H'_K denote the preimage of Δ_K in G_K. Replacing H_K by H'_K and $\mathbf{D}(V)$ by $(\mathbf{D}(V))'$ in their proof then it works for general Γ.

For general V, note that the inverse system $\{H^i(\mathbf{D}(V/p^nV)) \cong H^i(G_K, V/p^nV)\}$ satisfies the Mittag-Leffler condition, so we can conclude the result by taking the inverse limit of $\{H^i(\mathbf{D}(V/p^nV)) \cong H^i(G_K, V/p^nV)\}$. □

In the remainder of this section, V is a $\mathbb{Q}_p$-representation.

Lemma 1.2.4. *The morphism* $\gamma - 1 : ((\boldsymbol{D}^\dagger(V))')^{\psi=0} \longrightarrow ((\boldsymbol{D}^\dagger(V))')^{\psi=0}$ *has a continuous inverse.*

Proof. Note that $\chi(\Gamma_{K_1}) \subset 1 + p\mathbb{Z}_p$ is procyclic. We can choose a topological generator γ' of Γ_{K_1} such that $\gamma' = \gamma^m$ in Γ/Δ_K for some $m \in \mathbb{N}$. Consider the commutative diagram:

$$\begin{array}{ccc} \mathbf{D}^\dagger(V)^{\psi=0} & \xrightarrow{\gamma'-1} & \mathbf{D}^\dagger(V)^{\psi=0} \\ \downarrow{\scriptstyle p_{\Delta_K}} & & \downarrow{\scriptstyle p_{\Delta_K}} \\ ((\mathbf{D}^\dagger(V))')^{\psi=0} & \xrightarrow{\gamma^m-1} & ((\mathbf{D}^\dagger(V))')^{\psi=0}. \end{array}$$

Since $\mathbf{D}^\dagger(V)^{\psi=0} \xrightarrow{\gamma'-1} \mathbf{D}^\dagger(V)^{\psi=0}$ has a continuous inverse by [10, Proposition 2.6.1], and p_{Δ_K} is an idempotent operator, we get that $((\mathbf{D}^\dagger(V))')^{\psi=0} \xrightarrow{\gamma^m-1} ((\mathbf{D}^\dagger(V))')^{\psi=0}$ has a continuous inverse. Then $(\gamma-1)^{-1} = (\gamma^m-1)^{-1}(1+\gamma+\ldots+\gamma^{m-1})$ is also continuous. □

Lemma 1.2.5. *Let* $C^\bullet_{\psi,\gamma}(\boldsymbol{D}^\dagger(V))$ *be the complex*

$$0 \longrightarrow (\boldsymbol{D}^\dagger(V))' \xrightarrow{d_1} (\boldsymbol{D}^\dagger(V))' \oplus (\boldsymbol{D}^\dagger(V))' \xrightarrow{d_2} (\boldsymbol{D}^\dagger(V))' \longrightarrow 0$$

with $d_1(x) = ((\gamma-1)x, (\psi-1)x)$ *and* $d_2((x,y)) = ((\psi-1)x - (\gamma-1)y)$. *Then we have a commutative diagram of complexes*

$$\begin{array}{lccccccc} C^\bullet_{\psi,\gamma}(\boldsymbol{D}^\dagger(V)) : 0 \longrightarrow & (\boldsymbol{D}^\dagger(V))' & \longrightarrow & (\boldsymbol{D}^\dagger(V))' \oplus (\boldsymbol{D}^\dagger(V))' & \longrightarrow & (\boldsymbol{D}^\dagger(V))' & \longrightarrow 0 \\ & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle -\psi\oplus id} & & \downarrow{\scriptstyle -\psi} & \\ C^\bullet_{\psi,\gamma}(\boldsymbol{D}^\dagger(V)) : 0 \longrightarrow & (\boldsymbol{D}^\dagger(V))' & \longrightarrow & (\boldsymbol{D}^\dagger(V))' \oplus (\boldsymbol{D}^\dagger(V))' & \longrightarrow & (\boldsymbol{D}^\dagger(V))' & \longrightarrow 0 \end{array}$$

which induces an isomorphism on cohomology.

Proof. Since ψ is surjective, the cokernel complex is 0. The kernel complex is

$$0 \longrightarrow 0 \longrightarrow ((\mathbf{D}^\dagger(V))')^{\psi=0} \xrightarrow{\gamma-1} ((\mathbf{D}^\dagger(V))')^{\psi=0} \longrightarrow 0.$$

which has trivial cohomology by Lemma 1.2.2. □

Lemma 1.2.6. *Let T be a G_K-stable $\mathbb{Z}_p$-lattice of V. Then the natural morphism $\boldsymbol{D}^\dagger(T)/(\psi-1) \to \boldsymbol{D}(T)/(\psi-1)$ is an isomorphism.*

Proof. We can view $\mathbf{D}(T)/(\psi-1)$ (resp. $\mathbf{D}^\dagger(T)$) as an étale φ-module over $\mathbf{A}_{\mathbb{Q}_p}$ (resp. $\mathbf{A}^\dagger_{\mathbb{Q}_p}$). For $x \in \mathbf{A}_{\mathbb{Q}_p}$ and $n \in \mathrm{N}$, define $w_n(x) \in \mathrm{N}$ to be the smallest integer k such that $x \in \pi^{-k}\mathbf{A}_{\mathbb{Q}_p} + p^{n+1}\mathbf{A}_{\mathbb{Q}_p}$. Short computations show that $w_n(x+y) \leq \sup\{w_n(x), w_n(y)\}$, $w_n(xy) \leq w_n(x) + w_n(y)$ and $w_n(\varphi(x)) \leq pw_n(x)$. By [11, Proposition III 2.1], for any interger $m > 1$, $x \in \mathbf{A}^{\dagger,m}_{\mathbb{Q}_p}$ if and only if $w_n(x) - n(p-1)(p^{m-1}-1) \leq 0$ for every n, and moreover approaches $-\infty$ as $n \to \infty$. For a vector or matrix X with entries in $\mathbf{A}_{\mathbb{Q}_p}$, define $w_n(X)$ as the maximal w_n among the entries. Pick a basis $\{e_1, e_2, ..., e_d\}$ of $\mathbf{D}^\dagger(T)$ over $\mathbf{A}^\dagger_{\mathbb{Q}_p}$. For any $x \in \mathbf{D}(T)$, define $w_n(x) = w_n(X)$ if $x = X(e_1, e_2, ..., e_d)^t$. Let $A \in \mathrm{GL}(\mathbf{A}^\dagger_{\mathbb{Q}_p})$ defined by $\varphi(e_1, e_2, ..., e_d)^t = A(e_1, e_2, ..., e_d)^t$.

Suppose $x = \psi(y) - y$, for $x = X(e_1, e_2, ..., e_d)^t \in \mathbf{D}^\dagger(T)$ and $y = Y(e_1, e_2, ..., e_d)^t \in \mathbf{D}(T)$. Then from [11, Lemma I.6.4] we have

$$w_n(y) \;\leq\; \max\{w_n(x), \frac{p}{p-1}(w_n(A^{-1})+1)\}.$$

Now suppose all the entries of X and A^{-1} lie in $\mathbf{A}^{\dagger,m}_{\mathbb{Q}_p}$ for some m. It follows that all the entries of Y are in $\mathbf{A}^{\dagger,m+1}_{\mathbb{Q}_p}$, hence $y \in \mathbf{D}^\dagger(T)$. This proves the injectivity of $\mathbf{D}^\dagger(T)/(\psi-1) \to \mathbf{D}(T)/(\psi-1)$.

Since $\mathbf{D}(T)/(\psi-1)$ is a finite $\mathbb{Z}_p$-module ([20, Proposition 3.6]), so too is $\mathbf{D}^\dagger(T)/(\psi-1)$. Note that

$$\mathbf{D}^\dagger(T)/(p) = \mathbf{D}^\dagger(T/(p)) = \mathbf{D}(T/(p)) = \mathbf{D}(T)/(p)$$

since $\mathbf{D}^\dagger$ and $\mathbf{D}$ are identical at torsion level. Therefore

$$(\mathbf{D}^\dagger(T)/(\psi-1))/(p) = (\mathbf{D}^\dagger(T)/(p))/(\psi-1) = (\mathbf{D}(T)/(p))/(\psi-1) = (\mathbf{D}(T)/(\psi-1))/(p).$$

This implies $\mathbf{D}^\dagger(T)/(\psi-1) \to \mathbf{D}(T)/(\psi-1)$ is surjective by Nakayama's Lemma. Hence $\mathbf{D}^\dagger(T)/(\psi-1) \to \mathbf{D}(T)/(\psi-1)$ is an isomorphism. □

Proposition 1.2.7. *Let V be a p-adic representation of G_K. Then the natural morphisms*

$$H^i(\boldsymbol{D}^\dagger(V)) \xrightarrow{\alpha_i} H^i(\boldsymbol{D}^\dagger_{\mathrm{rig}}(V)), \quad H^i(\boldsymbol{D}^\dagger(V)) \xrightarrow{\beta_i} H^i(\boldsymbol{D}(V)) \qquad i = 0,1,2$$

are all isomorphisms which are functorial in V and compatible with cup products.

Proof. It is clear that these morphisms are functorial in V and compatible with cup products. To prove they are isomorphisms, first note that $H^1(\mathbf{D}^\dagger(V))$ (resp. $H^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V)$, $H^1(\mathbf{D}(V))$) classifies all the extensions of $\mathcal{E}^\dagger_K$ (resp. $\mathcal{R}_K$, $\mathcal{E}_K$) by $\mathbf{D}^\dagger(V)$ (resp. $\mathbf{D}^\dagger_{\mathrm{rig}}(V), \mathbf{D}(V)$) in the category of étale (φ, Γ)-modules over $\mathcal{E}^\dagger_K$ (resp. $\mathcal{R}_K$, $\mathcal{E}_K$). Since these categories are all equivalent to the category of p-adic representations by Theorems 1.1.1, 1.1.2 and 1.1.3, we conclude that both α_1, β_1 are isomorphisms.

From [25, Proposition 1.5.4], the natural maps $\mathbf{D}^\dagger(V)^{\varphi=1} \to \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\varphi=1}$ and $\mathbf{D}^\dagger(V)/(\varphi - 1) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V)/(\varphi-1)$ are bijective. Taking Δ_K-invariants of the first map, we have that $((\mathbf{D}^\dagger(V))')^{\varphi=1} \to ((\mathbf{D}^\dagger_{\mathrm{rig}}(V))')^{\varphi=1}$ is an isomorphism. As in Lemma 1.2.4, by the following commutative diagram

$$\begin{array}{ccc} \mathbf{D}^\dagger(V)/(\varphi-1) & \longrightarrow & \mathbf{D}^\dagger_{\mathrm{rig}}(V)/(\varphi-1) \\ \big\downarrow{\scriptstyle p_{\Delta_K}} & & \big\downarrow{\scriptstyle p_{\Delta_K}} \\ (\mathbf{D}^\dagger(V))'/(\varphi-1) & \longrightarrow & (\mathbf{D}^\dagger_{\mathrm{rig}}(V))'/(\varphi-1) \end{array}$$

and the fact that p_{Δ_K} is an idempotent operator, we get $(\mathbf{D}^\dagger(V))'/(\varphi-1) \to (\mathbf{D}^\dagger_{\mathrm{rig}}(V))'/(\varphi-1)$ is also an isomorphism. Therefore α_0 and α_2 are isomorphisms.

Since $H^0(\mathbf{D}^\dagger(V)) = V^{\Gamma_K} = H^0(\mathbf{D}(V))$, we conclude that β_0 is an isomorphism. By Lemmas 1.2.5 and 1.2.6 we have

$$H^2(\mathbf{D}^\dagger(V)) = (\mathbf{D}^\dagger(V))'/(\psi-1, \gamma-1) = (\mathbf{D}(V))'/(\psi-1,\gamma-1) = H^2(\mathbf{D}(V)).$$

Hence β_2 is an isomorphism. □

As a consequence of this proposition, there are canonical isomorphisms

$$H^i(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \xrightarrow{\beta_i\alpha_i^{-1}} H^i(\mathbf{D}(V)) \qquad i = 0,1,2.$$

Composing them with isomorphisms in Theorem 1.2.3, we get the following theorem.

Theorem 1.2.8. *Let V be a p-adic representation of G_K. Then there are isomorphisms*

$$H^i(\boldsymbol{D}^\dagger(V)) \cong H^i(G_K, V) \qquad (i = 0, 1, 2)$$

$$H^i(\boldsymbol{D}^\dagger_{\mathrm{rig}}(V)) \cong H^i(G_K, V) \qquad (i = 0, 1, 2)$$

which are functorial in V and compatible with cup products.

Corollary 1.2.9. *The Euler-Poincaré characteristic formula and Tate local duality hold for all étale (φ, Γ)-modules over the Robba ring.*

Proof. From the above theorem, we have that $H^2(\mathbf{D}^\dagger_{\mathrm{rig}}(\mathbb{Q}_p(1)))$ is canonically isomorphic to $H^2(\mathbb{Q}_p(1))$, and then the Euler-Poincaré characteristic formula and Tate local duality for étale (φ, Γ)-modules follow from the usual Euler-Poincaré characteristic formula and Tate local duality for Galois cohomology. □

1.2.4 Cohomology of rank 1 (φ, Γ)-modules

In this section, we provide an explicit computation of H^2 of rank 1 (φ, Γ)-modules over the Robba ring in case $K = \mathbb{Q}_p$ and $p > 2$ as a complement to Colmez's results on H^0 and H^1. Although we don't need this for the main theorems, it is useful for some purposes (see [6, Lemma 2.3.11]). In this section, all (φ, Γ)-modules are over the Robba ring and $K = \mathbb{Q}_p$. Moreover, to be consistent with Colmez's set up, we fix L a finite extension of $\mathbb{Q}_p$ as the coefficient field. This means we consider (φ, Γ)-modules over $\mathcal{R}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$, where φ and Γ act on L trivially, $\gamma(T) = (1+T)^{\chi(\gamma)} - 1$ and $\varphi(T) = (1+T)^p - 1$. Following Colmez's notation, we use $\mathcal{R}_L$ to denote $\mathcal{R}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$ in this section only. Note that this is different from our usual definition of $\mathcal{R}_L$.

If δ is a continuous character from $\mathbb{Q}_p^\times$ to $L^\times$, we can associate a rank 1 (φ, Γ)-module $\mathcal{R}(\delta)$ to δ. Namely, there is a basis v of $\mathcal{R}(\delta)$ such that

$$\varphi(xv) = \delta(p)\varphi(x)v \text{ and } \gamma(xv) = \delta(\chi(\gamma))\gamma(x)v$$

for any $x \in \mathcal{R}_L$. Here χ is the cyclotomic character. It is obvious that such v is unique up to a nonzero scalar of L. In the sequel, for $a \in \mathcal{R}_L$, we use a to denote the element av of $\mathcal{R}(\delta)$. Conversely, Colmez ([13, Proposition 4.2, Remark 4.3]) proved that if D is a (φ, Γ)-module of rank 1, then there is a unique character δ such that D is isomorphic to $\mathcal{R}(\delta)$.

For simplicity, let $H^i(\delta)$ denote $H^i(\mathcal{R}(\delta))$. In the following, the character x is the identity character induced by the inclusion of $\mathbb{Q}_p$ into L and $|x|$ is the character mapping x to $p^{-v_p(x)}$. We use ω to denote $x|x|$; then $\mathcal{R}(\omega) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(L(1))$, as described in the introduction.

In [13], Colmez computed H^0 and H^1 for all the (φ, Γ)-modules of rank 1 when $p > 2$. More precisely, he proved the following result ([13, Proposition 3.1, Theorem 3.9]).

Proposition 1.2.10. *If $p > 2$, then the following are true.*

(1) For $i \in \mathbb{N}$, $H^0(x^{-i}) = L \cdot t^i$. If $\delta \neq x^{-i}$ for $i \in \mathbb{N}$, then $H^0(\delta) = 0$.

(2) For $i \in \mathbb{N}$, $H^1(x^{-i})$ is a 2-dimensional L-vector space generated by $(0, \overline{t^i})$ and $(\overline{t^i}, 0)$, and $H^1(\omega x^i)$ is also 2-dimensional. If $\delta \neq x^{-i}$ or ωx^i for $i \in \mathbb{N}$, then $H^1(\delta)$ is 1-dimensional.

In fact, we can follow Colmez's method to compute H^2 easily. For $f = \sum_{k \in \mathbb{Z}} a_k T^k \in \mathcal{R}_L$, we define the residue of the differential form $\omega = f dT$ by the formula $\mathrm{res}(\omega) = a_{-1}$. Then ω is closed if and only if $\mathrm{res}(\omega) = 0$. Recall that $\partial = (1+T)\frac{d}{dT}$, then $\ker(\partial) = L$ and $df = \partial f \frac{dT}{1+T}$. We define $\mathrm{Res}(f) = \mathrm{res}(f\frac{dT}{1+T})$, then f is in the image of ∂ if and only if $\mathrm{Res}(f) = 0$.

Recall that we have the following formulas:

$$\partial \circ \varphi = p\varphi \circ \partial \text{ and } \partial \circ \gamma = \chi(\gamma)\gamma \circ \partial.$$

If $\mathrm{Res}(f) = 0$, then there exists a $g \in \mathcal{R}_L$ such that $\partial(g) = f$. Therefore we have $\partial(\frac{1}{p}\varphi(g)) = \varphi(f)$ and $\partial(\frac{1}{\chi(\gamma)}\gamma(g)) = \gamma(f)$. Hence $\mathrm{Res}(\varphi(f)) = \mathrm{Res}(\gamma(f)) = 0$. In general, if $\mathrm{Res}(f) = a \in L$, then $\mathrm{Res}(f - a(\frac{1+T}{T})) = 0$. So $\mathrm{Res}(\gamma(f)) = a\mathrm{Res}(\gamma(\frac{1+T}{T}))$ and $\mathrm{Res}(\varphi(f)) = a\mathrm{Res}(\varphi(\frac{1+T}{T}))$. Note that $\log\frac{\gamma(T)}{T}$ and $\log\frac{\varphi(T)}{T^p}$ are defined in $\mathcal{R}_L$. Hence

$$0 = \mathrm{res}(d\log\frac{\gamma(T)}{T}) = \mathrm{res}(\frac{d\gamma(T)}{\gamma(T)} - \frac{dT}{T}) = \mathrm{res}(\frac{\chi(\gamma)(1+T)^{\chi(\gamma)-1}dT}{\gamma(T)}) - 1 = \mathrm{Res}(\chi(\gamma)\gamma(\frac{1+T}{T})) - 1,$$
$$0 = \mathrm{res}(d\log\frac{\varphi(T)}{T^p}) = \mathrm{res}(\frac{d\varphi(T)}{\varphi(T)} - \frac{pdT}{T}) = \mathrm{res}(\frac{p(1+T)^{p-1}dT}{\varphi(T)}) - p = \mathrm{Res}(p\varphi(\frac{1+T}{T})) - p,$$

therefore $\mathrm{Res}(\gamma(\frac{1+T}{T})) = 1/\chi(\gamma)$ and $\mathrm{Res}(\varphi(\frac{1+T}{T})) = 1$. So we get

$$\mathrm{Res}(\gamma(f)) = 1/\chi(\gamma)\mathrm{Res}(f) \text{ and } \mathrm{Res}(\varphi(f)) = \mathrm{Res}(f).$$

For any $x \in \mathcal{R}_L$, by the formulas $\partial \circ \varphi = p\varphi \circ \partial$ and $\partial \circ \gamma = \chi(\gamma)\gamma \circ \partial$, we have

$$\partial((x^{-1}\delta)(p)\varphi(x)) = \delta(p)\varphi(\partial(x))$$
$$\partial((x^{-1}\delta)(\chi(\gamma))\gamma(x)) = \delta(\chi(\gamma))\gamma(\partial(x)).$$

So ∂ induces an L-linear morphism, which commutes with φ and Γ, from $\mathcal{R}(x^{-1}\delta)$ to $\mathcal{R}(\delta)$ by mapping x to ∂x. Then ∂ induces an L-linear morphism from $H^i(x^{-1}\delta)$ to $H^i(\delta)$.

Proposition 1.2.11. *In case $p > 2$, if $v_p(\delta(p)) < 0$, then $H^2(\delta) = 0$.*

Proof. For any $\bar{f} \in H^2(\delta)$, from [13, Corollary 1.3], there is a $b \in \mathcal{R}_L$ such that $c = f - (\delta(p)\varphi - 1)b$ lies in $(\mathcal{E}_L^\dagger)^{\psi=0}$. Since $\bar{c} = \bar{f}$ in $H^2(\delta)$, we can just assume f is in $(\mathcal{E}_L^\dagger)^{\psi=0}$. Then there exists an $a \in \mathcal{E}_L^\dagger$ such that $f = (\delta(\chi(\gamma))\gamma - 1)a$ by Lemma 1.2.2. Therefore $\bar{f} = 0$ in $H^2(\delta)$. □

Proposition 1.2.12. *If $p > 2$, then the following are true.*

(1) If $\delta \neq x$, then $\partial : H^2(\delta x^{-1}) \to H^2(\delta)$ is injective. If $\delta \neq \omega$, then $\partial : H^2(\delta x^{-1}) \to H^2(\delta)$ is surjective. Therefore $\partial : H^2(\delta x^{-1}) \to H^2(\delta)$ is an isomorphism if $\delta \neq \omega, x$.

(2) $H^2(\omega)$ is a 1-dimensional L-vector space, generated by $\overline{1/T}$.

(3) $H^2(x^k) = 0$ for any $k \in \mathbb{Z}$. Combining with (1), we conclude that ∂ is always injective.

Proof. For (1), first suppose $\delta \neq x$. If $\partial(\bar{f}) = 0$ for some $\bar{f} \in H^2(x^{-1}\delta)$, this means that there exist a, $b \in \mathcal{R}_L$ such that $\partial(f) = (\delta(\chi(\gamma))\gamma - 1)a - (\delta(p)\varphi - 1)b$. Now since $\mathrm{Res}(\partial(f)) = 0$, we have $\mathrm{Res}((\delta(\chi(\gamma))\gamma - 1)a) = \mathrm{Res}((\delta(p)\varphi - 1)b)$. Therefore

$$(\delta(\chi(\gamma))\chi(\gamma)^{-1} - 1)\mathrm{Res}(a) = (\delta(p) - 1)\mathrm{Res}(b).$$

If $\delta(p) - 1 = 0$, then $v_p(\delta x^{-1}(p)) < 0$. Therefore $H^2(\delta x^{-1}) = 0$ by Proposition 1.2.11. If $\delta(p) - 1$ is not zero, let $c = (\delta(p) - 1)^{-1}\mathrm{Res}(a)\frac{1+T}{T}$, $a' = a - (\delta(p)\varphi - 1)c$ and $b' = b - (\delta(\chi(\gamma))\gamma - 1)c$. Then we have $\mathrm{Res}(a') = \mathrm{Res}(b') = 0$ and $\partial(f) = (\delta(\chi(\gamma))\gamma - 1)a' - (\delta(p)\varphi - 1)b'$.

So we can assume that $\mathrm{Res}(a) = \mathrm{Res}(b) = 0$. Now suppose $\partial(\tilde{a}) = a$ and $\partial(\tilde{b}) = b$. Let $\tilde{f} = f - ((\delta(\chi(\gamma))\chi(\gamma)^{-1}\gamma - 1)\tilde{a} - (\delta(p)p^{-1}\varphi - 1)\tilde{b})$, then $\partial(\tilde{f}) = 0$. This implies $\tilde{f} \in L$. Since $\delta \neq x$, we have either $\delta(\chi(\gamma))\chi(\gamma)^{-1} - 1 \neq 0$ or $\delta(p)p^{-1} - 1 \neq 0$. If $\delta(\chi(\gamma))\chi(\gamma)^{-1} - 1 \neq 0$, let $\tilde{a}' = \tilde{a} + (\delta(\chi(\gamma))\chi(\gamma)^{-1} - 1)^{-1}\tilde{f}$, then $f = (\delta(\chi(\gamma))\chi(\gamma)^{-1}\gamma - 1)\tilde{a}' - (\delta(p)p^{-1}\varphi - 1)\tilde{b}$. So $\bar{f} = 0$ in $H^2(\delta x^{-1})$. If $\delta(p)p^{-1} - 1$ is not zero, let $\tilde{b}' = \tilde{b} - (\delta(p)p^{-1} - 1)^{-1}\tilde{f}$, then $f = (\delta(\chi(\gamma))\chi(\gamma)^{-1}\gamma - 1)\tilde{a} - (\delta(p)p^{-1}\varphi - 1)\tilde{b}'$. So $\bar{f}$ is also zero.

If $\delta \neq \omega$, then either $\delta(\chi(\gamma))\chi(\gamma)^{-1} - 1$ or $\delta(p) - 1$ is not zero. Hence for any $\bar{f} \in H^2(\delta)$, we can choose a, b such that $\mathrm{Res}(f - (\delta(\chi(\gamma))\gamma - 1)a - (\delta(p)\varphi - 1)b) = 0$. Then there exists an f' such that $\partial(f') = f - (\delta(\chi(\gamma))\gamma - 1)a - (\delta(p)\varphi - 1)b$. So $\partial(\bar{f'}) = \bar{f}$. This proves the surjectivity.

For (2), we have that both $\omega(\chi(\gamma))\chi(\gamma)^{-1} - 1$ and $\omega(p) - 1$ are zero. So we can define a map Res : $H^2(\omega) \to L$ by $\mathrm{Res}(\bar{f}) = \mathrm{Res}(f)$. We claim that it is an isomorphism. If $\mathrm{Res}(\bar{f}) = 0$, then $\bar{f}$ is in the image of $\partial : H^2(|x|) \to H^2(\omega)$. But $H^2(|x|) = 0$ by Proposition 1.2.11, so $\bar{f} = 0$. Therefore Res is injective. Note that $\mathrm{Res}(\overline{1/T}) = 1$, so Res is also surjective.

For (3), if $k < 0$, then $H^2(x^k) = 0$ by proposition 1.2.11. If $k \in \mathbb{N}$, then $\partial : H^2(x^{k-1}) \to H^2(x^k), ..., \partial : H^2(1) \to H^2(x)$ and $\partial : H^2(x^{-1}) \to H^2(1)$ are all surjective by (1). So $H^2(x^k) = 0$ since $H^2(x^{-1}) = 0$. □

Corollary 1.2.13. *Suppose p is not equal to 2. If $\delta = \omega x^k$ for $k \in \mathbb{N}$, then $H^2(\delta)$ is a 1-dimensional L-vector space generated by $\overline{\partial^k(1/T)}$. Otherwise, $H^2(\delta) = 0$.*

Proof. This is an easy consequence of Propositions 1.2.11 and 1.2.12. In fact, for any δ, we can find a $k_0 \in \mathbb{N}$ such that $v_p(\delta x^{-k_0}(p)) < 0$. Then $H^2(\delta x^{-k_0}) = 0$. If ω does not appear in the sequence δx^{-k_0},..., δx^{-1}, δ, then $H^2(\delta) = 0$ by Proposition 1.2.12(1). If ω appears, then $\delta = \omega x^k$ for some $k \in \mathbb{Z}$. If $k < 0$, then $H^2(\delta) = 0$ since $v_p(\omega x^k(p)) < 0$. For $k \in \mathbb{N}$, by Proposition 1.2.12(2), $H^2(\omega)$ is generated by $\overline{1/T}$. Repeatedly applying (1) of Proposition 1.2.12, we get that $H^2(\omega x^k)$ is generated by $\overline{\partial^k(1/T)}$. □

Corollary 1.2.14. *If $p > 2$, then the Euler-Poincaré characteristic formula holds for all rank 1 (φ, Γ)-modules .*

1.3 Generalized (φ, Γ)-modules

In the rest of this paper, all (φ, Γ)-modules are over the Robba ring. For simplicity, we only consider the usual Robba ring without an additional coefficient field. However, it is easy to see that the same argument works to prove the results in the general case.

1.3.1 Generalized (φ, Γ)-modules

In this section we will investigate generalized (φ, Γ)-modules. Define a *generalized (φ, Γ)-module over $\mathcal{R}_K$* as a finitely presented $\mathcal{R}_K$-module D with commuting φ, Γ-actions such that $\varphi^* D \to D$ is an isomorphism. Since $\mathcal{R}_K$ is a Bezout domain ([15, Proposition 4.6]), it is a coherent ring (i.e. the kernel of any map between finitely presented modules is again finitely presented), so the generalized (φ, Γ)-modules form an abelian category. Define a *torsion* (ϕ, Γ)*-module* as a generalized (φ, Γ)-module which is $\mathcal{R}_K$-torsion. We say a torsion (φ, Γ)-module S is a *pure* t^k*-torsion* (φ, Γ)*-module* if it is a free $\mathcal{R}_K/t^k$-module. For a generalized (φ, Γ)-module D, its torsion part S is a torsion (φ, Γ)-module and D/S is a (φ, Γ)-module. We define the rank of D as the rank of D/S.

Proposition 1.3.1. *If* $K = \mathbb{Q}_p$*, then a torsion* (φ, Γ)*-module* S *is a successive extensions of pure* t*-torsion* (φ, Γ)*-modules.*

Proof. From [2, Proposition 4.12(5)], we can find a set of elements $\{e_1,...,e_d\}$ of S and principal ideals $(r_1), (r_2), ..., (r_d)$ of $\mathcal{R}_K$ such that $S = \oplus_{i=1}^d \mathcal{R}e_i$, $\mathrm{Ann}(e_i) = (r_i)$, and $(r_1) \subset (r_2) \subset ... \subset (r_d)$. Furthermore, these ideals $(r_1), (r_2), ..., (r_d)$ are unique. Therefore they are Γ-invariant. Since $\mathcal{R}_K$ is a free $\mathcal{R}_K$-module via φ, we have $\mathrm{Ann}(1 \otimes e_i) = \mathrm{Ann}(e_i) = (r_i)$ in φ^*S for every i. Hence $\mathrm{Ann}(\varphi(e_i)) = (r_i)$ because $\varphi^*S \to S$ is an isomorphism. This implies $(\varphi(r_i)) \subset \mathrm{Ann}(\varphi(e_i)) = (r_i)$.

We claim that if a principal ideal I of $\mathcal{R}_{\mathbb{Q}_p}$ is stable under φ and Γ, then it is (t^k) for some k. In fact, from the proof of [4, Lemma I.3.2], since I is stable under φ and Γ it is generated by $\prod_{n=1}^{\infty}(\varphi^{n-1}(q)/p)^{j_n}$ for a decreasing sequence $\{j_n\}_n$. Therefore these j_n's are eventually constant, let $k \in \mathbb{N}$ denote this constant. This implies $I = (t^k)$. So we conclude that $(r_i) = (t^{k_i})$ for every i, and $\{k_i\}_i$ is decreasing. Then we can construct a filtration $0 = t^{k_1}S \subset t^{k_1-1}S \subset ... \subset S$ of S such that each quotient is a pure t-torsion (φ, Γ)-module, hence the result.

□

In case $K = \mathbb{Q}_p$, for any pure t^k-torsion (φ, Γ)-module S, let $d = \mathrm{rank}_{\mathcal{R}/t^k} S$, and choose a basis $\{e_1,...,e_d\}$ of S. Let A be the matrix of φ in this basis. Since $\varphi^*S \cong S$, there is another matrix B such that $AB = BA = I_d$. Furthermore, since Γ is topologically finite generated, we can choose an r_0 large enough such that A, B and the elements of Γ have all entries lie in $\mathbf{B}^{\dagger, r_0}_{\mathrm{rig}, \mathbb{Q}_p}/(t^k)$ For $r \geq r_0$, set S_r be the $\mathbf{B}^{\dagger, r}_{\mathrm{rig}, \mathbb{Q}_p}/(t^k)$-submodule of S spanned by $\{e_1,...,e_d\}$. Then Γ acts on S_r and $\varphi : S_r \to S_{pr}$ induces an isomorphism

$$1 \otimes \varphi : \mathbf{B}^{\dagger, pr}_{\mathrm{rig}, \mathbb{Q}_p}/(t^k) \otimes_{\mathbf{B}^{\dagger, r}_{\mathrm{rig}, \mathbb{Q}_p}/(t^k)} S_r \cong S_{pr}.$$

Here we view $\mathbf{B}^{\dagger,pr}_{\mathrm{rig},\mathbb{Q}_p}/(t^k)$ as a $\mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k)$ algebra via φ.

Lemma 1.3.2. *For $r \geq p-1$, we have the following.*

(1) The natural maps $\boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k) \to \boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k))$ for $n \geq n(r)$ induce an isomorphism

$$\boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k) \cong \textstyle\prod_{n\geq n(r)}^{\infty} \boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k)).$$

(2) If $n \geq n(r)$, the localization $\pi \mapsto \varepsilon^{(n)}e^{t/p^n} - 1$ induces a Γ-equivariant isomorphism from $\boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k))$ to $\mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k)$.

(3) For $r' \geq r$, $\boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k)) \to \boldsymbol{B}^{\dagger,r'}_{\mathrm{rig},\mathbb{Q}_p}/(\varphi^n(q^k))$ is the identity map via the isomorphism of (2).

(4) The morphism $\varphi : \boldsymbol{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k) \to \boldsymbol{B}^{\dagger,pr}_{\mathrm{rig},\mathbb{Q}_p}(t^k)$ can be described via the isomorphism of (1) as follows: $\varphi((x_n)_{n\geq n(r)}) = ((y_n)_{n\geq n(r)+1})$ where $y_{n+1} = x_n$ for $n \geq n(r)$.

Proof. See [13, Lemma 3.15]. □

Using (2) of Lemma 1.3.2, for $n \geq n(r)$, we set $S^n = S_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}/(t^k)} \mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k)$. Then S^n is a free $\mathbb{Q}_p(\varepsilon^{(n)})/(t^k)$-module of rank d with Γ-action. The injective map $\varphi : S^n \to S^{n+1}$ induces an isomorphism

$$1 \otimes \varphi : \mathbb{Q}_p(\varepsilon^{(n+1)})[t]/(t^k) \otimes_{\mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k)} S^n \cong S^{n+1}.$$

It allows us to regard S^n as a submodule of S^{n+1}.

Theorem 1.3.3. *With notations as above, the following are true.*

(1) The natural maps $S_r \to S^n$ for $n \geq n(r)$ induce $S_r \cong \prod_{n\geq n(r)}^{\infty} S^n$ as $(\mathbb{Q}_p(\varepsilon^{(n(r))})[t]/(t^k))[\Gamma]$-modules.

(2) For $r' \geq r$, under the isomorphism of (1), the natural map $S_r \to S_{r'}$ is $((x_n)_{n\geq n(r)}) \mapsto ((x_n)_{n\geq n(r')})$.

(3) Under the isomorphism of (1), $\varphi : S_r \to S_{pr}$ is $(x_n)_{n\geq n(r)} \mapsto ((y_n)_{n\geq n(r)+1})$, where $y_{n+1} = x_n$ for $n \geq n(r)$.

Proof. For (1), we have

$$\begin{aligned}
S_r &= S_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}} \mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/(t^k) \\
&= S_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}} \prod_{n\geq n(r)}^{\infty} \mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/((\varphi^n(q))^k) \qquad \text{(by (1) of Lemma 3.2)} \\
&= \prod_{n\geq n(r)}^{\infty} S_r \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}} \mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}/((\varphi^n(q))^k) \\
&= \prod_{n\geq n(r)}^{\infty} S_r \otimes_{\mathbb{Q}_p(\varepsilon^{(n)})[t]} \mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k) \qquad \text{(by (2) of Lemma 3.2)} \\
&= \prod_{n\geq n(r)}^{\infty} S^n.
\end{aligned}$$

Then (2) and (3) follow from (3) and (4) of Lemma 1.3.2 respectively. □

The natural examples of torsion (φ, Γ)-modules are *quotient* (φ, Γ)*-modules* which are of the forms D/E, here $E \subset D$ are two (φ, Γ)-modules of the same rank. For sufficiently large r, we have $E_r \subset D_r$. For $n \geq n(r)$, localizing at $\varepsilon^{(n)} - 1$, we get $D^{+,n}_{\mathrm{dif}}(E) \subset D^{+,n}_{\mathrm{dif}}(D)$. It is natural to view the quotient $D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$ as the localization at $\varepsilon^{(n)} - 1$ of the quotient (φ, Γ)-module D/E. The connecting map φ_n of D and E induces a connecting map

$$\varphi_n : D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E) \to D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$$

of D/E such that

$$\varphi_n \otimes 1 : D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E) \otimes_{\mathbb{Q}_p(\varepsilon^{(n)})[t]} \mathbb{Q}_p(\varepsilon^{(n+1)})[t]/(t^k) \to D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$$

is an isomorphism. Note that if we apply the proof of Theorem 1.3.2 to D/E by replacing S_r by D_r/E_r and S^n by $D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$, then we get the following formulas which might be useful for other purposes.

Proposition 1.3.4. *Suppose $K = \mathbb{Q}_p$. With notation as above, the following are true.*

(1) The localization maps $D_r/E_r \to D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$ induce an isomorphism

$$D_r/E_r \cong \textstyle\prod_{n\geq n(r)}^{\infty} D^{+,n}_{\mathrm{dif}}(D)/D^{+,n}_{\mathrm{dif}}(E)$$

as $(\mathbb{Q}_p(\varepsilon^{(n(r))})[t]/(t^k))[\Gamma]$ *modules.*

(2) *For* $r' \geq r$, *under the isomorphism of (1), the natural map* $D_r/E_r \to D_{r'}/E_{r'}$ *is* $((x_n)_{n\geq n(r)}) \mapsto ((x_n)_{n\geq n(r')})$.

(3) *Via the isomorphism of (1),* $\varphi : D_r/E_r \to D_{pr}/E_{pr}$ *is* $((x_n)_{n\geq n(r)}) \mapsto ((y_n)_{n\geq n(r)+1})$, *where* $y_{n+1} = x_n$ *for* $n \geq n(r)$.

1.3.2 Cohomology of Generalized (φ, Γ)-modules

We also use Herr's complex to define the cohomology of generalized (φ,Γ)-modules. If L is a finite extension of K, and D is a generalized (φ,Γ)-module over $\mathcal{R}_L$, then we define the *induced* (φ,Γ)-*module of* D from L to K in the same way as for (φ,Γ)-modules as in the beginning of section 1.2.2 and also denote it by $\mathrm{Ind}_L^K D$.

Theorem 1.3.5. *(Shapiro's Lemma for generalized* (φ,Γ)*-modules) Suppose* D *is a* (φ,Γ)*-module over* $\mathcal{R}_L$*. Then there are isomorphisms*

$$H^i(D) \cong H^i(\mathrm{Ind}_L^K D) \qquad (i = 0, 1, 2)$$

which are functorial in D *and compatible with cup products.*

Proof. The proof is the same as the proof of Theorem 1.2.2. □

Suppose $\eta : \mathbb{Z}_p^* \to \mathcal{O}_L$ is a character of finite order with conductor $p^{N(\eta)}$ ($N(\eta) = 0$ if $\eta = 1$; otherwise it is the smallest integer n such that η is trivial on $1 + p^n\mathbb{Z}_p$). We define the Gauss sum $G(\eta)$ associated to η by $G(\eta) = 1$ if $\eta = 1$, otherwise

$$G(\eta) = \sum_{x \in (\mathbb{Z}/p^{N(\eta)}\mathbb{Z})^*} \eta(x)\mu_{p^{N(\eta)}}^x \in L_{N(\eta)}^\times.$$

Lemma 1.3.6. *Let* k *be a positive integer.*

(1) *If* $\eta : \mathbb{Z}_p^* \to \mathcal{O}_L$ *is of finite order and* $0 \leq i \leq k-1$, *then* $g(G(\eta)t^i) = (\eta^{-1}\chi^i)(g) \cdot (G(\eta)t^i)$ *for every* $g \in \Gamma$.

(2) *For every* $n \in \mathbb{N}$, *we have* $\mathbb{Q}_p(\varepsilon^{(n)})[t]/t^k = \oplus_{\eta, N(\eta)\leq n} \oplus_{0\leq i\leq k-1} \mathbb{Q}_p \cdot G(\eta)t^i$.

Proof. See [13, Prop 3.13]. □

Theorem 1.3.7. *Suppose S is a torsion (φ, Γ)-module. Then we have the following.*

(1) $\dim_{\mathbb{Q}_p} H^0(S) = \dim_{\mathbb{Q}_p} H^1(S) < \infty$;

(2) $\varphi - 1$ *is surjective on* S, *and therefore* $H^2(S) = 0$.

Proof. By Theorem 1.3.5, we first reduce the theorem to the case $K = \mathbb{Q}_p$. Suppose we are given a short exact sequence $0 \to S' \to S \to S'' \to 0$ of torsion (φ, Γ)-modules and the theorem holds for S' and S''. Then (2) also holds for S by Five Lemma. From the long exact sequence of cohomology we get

$$0 \to H^0(S') \to H^0(S) \to H^0(S'') \to H^1(S') \to H^1(S) \to H^1(S'') \to 0.$$

Then we see that $\dim_{\mathbb{Q}_p} H^0(S) = \dim_{\mathbb{Q}_p} H^1(S) < \infty$ since $\dim_{\mathbb{Q}_p} H^0(S_i) = \dim_{\mathbb{Q}_p} H^1(S_i) < \infty$ for $i = 1, 2$.

So conditions (1) and (2) are preserved by extensions. By Proposition 1.3.1 we only need to treat the case in which S is a pure t^k-torsion (φ, Γ)-module. We claim that the map $\varphi - 1 : S_r \to S_{pr}$ is surjective for any $r \geq p - 1$. This will prove (2), since S is the union of S_r's. In fact, for any $((y_n)_{n \geq n(r)+1}) \in S_{pr}$, if we let $x_n = -\sum_{i=n(r)}^{n} y_i$ for $n \geq n(r)$, where we put $y_{n(r)} = 0$, then we have $(\varphi - 1)((x_n)_{n \geq n(r)}) = ((y_n)_{n \geq n(r)+1})$ by (2) and (3) of Theorem 1.3.3.

For (1), we set $S'_r = S_r^{\triangle_K}$ and $(S^n)' = (S^n)^{\triangle_K}$. Then we have that $S'_r \cong \prod_{n \geq n(r)}^{\infty} (S^n)'$. By Theorem 1.3.3(2), if $a = ((a_n)_{n \geq n(r)}) \in S'_r$, then $a = 0$ if and only if $a_n = 0$ for almost all n. For any $a \in H^0(S)$, suppose a is represented by $((a_n)_{n \geq n(r)}) \in S'_r$; then $(\varphi - 1)(a) = 0$ implies a_n becomes constant for n large enough. Therefore we have

$$H^0(S) = \varinjlim_{n \to \infty} ((S_n)')^{\Gamma/\triangle_K} = \varinjlim_{n \to \infty} (S_n)^{\Gamma}.$$

Suppose $(a, b) \in Z^1(S)$. By (2) which we have proved, there exists a $c \in S$ such that $(\varphi - 1)c = b$. Then (a, b) is homogeneous to $(a - (\gamma_K - 1)c, 0)$, so we can assume that $b = 0$. Suppose a is represented by $((a_n)_{n \geq n(r)}) \in S'_r$ for some r. Then $(\varphi - 1)a = 0$ implies a_n becomes constant for n is large enough, say for $n \geq n_0$. Also $(a, 0)$ is a coboundary if and only if $a_n \in (\gamma_K - 1)(S^n)'$ for some $n \geq n_0$. Then we have

$$H^1(S) = \varinjlim_{n \to \infty} (S^n)'/(\gamma_K - 1).$$

Since $(S^n)'$ is a finite dimensional $\mathbb{Q}_p$-vector space, we have that $\dim_L(S^n)^\Gamma = \dim_L(S^n)'/(\gamma_K-1)$. Since $(S^n)^\Gamma \to (S^{n+1})^\Gamma$ is injective by Lemma 1.3.3(2), in order to prove (1), we need only to verify two things: (*a*) $(S^n)'/(\gamma-1) \to (S^{n+1})'/(\gamma-1)$ is injective, and (*b*) $\dim_L(S^n)^\Gamma$ has an upper bound independent of n.

From Lemma 1.3.6(2), $\mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k)$ is a direct summand of $\mathbb{Q}_p(\varepsilon^{(n+1)})[t]/(t^k)$ as Γ-modules. Hence S^n is a direct summand of S^{n+1} as Γ-modules, then $(S^n)'$ is also a direct summand of $(S^{n+1})'$ as Γ-modules and this proves (*a*).

For $s \in \mathbb{N}$, using Lemma 1.3.6(2) for $S^{n+s} = \mathbb{Q}_p(\varepsilon^{(n+s)})[t]/(t^k) \otimes_{\mathbb{Q}_p(\varepsilon^{(n)})[t]/(t^k)} S^n$, we see

$$\dim_{\mathbb{Q}_p}(S^{n+s})^\Gamma \leq \sum_{N(\eta)\leq n+s} \sum_{0\leq i\leq k-1} \dim_{\mathbb{Q}_p}(S^n(\eta^{-1}\chi^i))^\Gamma).$$

But the right hand side is no more than $\dim_{\mathbb{Q}_p} S^n$ because these characters $\eta^{-1}\chi^i$ are distinct. Hence $\dim_{\mathbb{Q}_p}(S^{n+s})^\Gamma \leq \dim_{\mathbb{Q}_p} S^n$ for any s, and this proves (*b*). □

Corollary 1.3.8. *For any torsion (φ,Γ)-module S, we have $\chi(S)=0$.*

1.4 Main Theorems

1.4.1 Euler-Poincaré characteristic formula

The main goal of this section is to prove the Euler-Poincaré characteristic formula.

Lemma 1.4.1. *For any (φ,Γ)-module D and $0 \leq i \leq 2$, $\dim_{\mathbb{Q}_p} H^i(D)$ is finite if and only if $\dim_{\mathbb{Q}_p} H^i(D(x))$ is finite. Furthermore, if all of $\dim_{\mathbb{Q}_p} H^i(D)$ are finite, then $\chi(D) = \chi(D(x))$.*

Proof. We identify $D(x)$ with tD, then apply Theorem 1.3.7 and Corollary 1.3.8 to $D/D(x)$. □

Lemma 1.4.2. *We can find a (φ,Γ)-module E of rank d such that*

(1) E is pure and $\mu(E) = 1/d$;

(2) E is a successive extensions of $\mathcal{R}(x^i)$'s, where i is either 0 or 1.

Proof. We proceed by induction on d. For $d=1$, take $E=\mathcal{R}(x)$. Now suppose $d>1$ and the lemma is true for $d-1$. Choose such an example E_0. By Lemma 1.4.1, we have

$\chi(\mathcal{R}(x)) = \chi(\mathcal{R})$. Since E_0 is a successive extensions of $\mathcal{R}(x^i)$'s where i is either 0 or 1, we have

$$\dim_{\mathbb{Q}_p} H^1(E_0) \geq -\chi(E_0) = (\operatorname{rank} E_0)(-\chi(\mathcal{R})) = (d-1)[K:\mathbb{Q}_p] \geq 1,$$

where the last equality follows from Corollary 1.2.9. Therefore we can find a nontrivial extension E of $\mathcal{R}_K$ by E_0. Then $\mu(E) = 1/d$. We claim that E is pure. In fact, suppose P is a submodule of E such that $\mu(P) < 1/d$. Since $\operatorname{rank} P \leq d$, we get $\deg P \leq 0$, and hence $\mu(P) \leq 0$. Therefore, $P \cap E_0 = 0$, since E_0 is pure of positive slope. Therefore the composite map $P \to E \to \mathcal{R}_K$ is injective, we get that $\mu(P) \geq 0$ with equality if and only if it is an isomorphism [25, Corollary 1.4.10]. But this forces the extension to be trivial, which is a contradiction. Obviously E also satisfies (2), so we finish the induction step. □

Theorem 1.4.3. *(Euler-Poincaré characteristic formula) For any generalized (φ,Γ)-module D, we have*

(1) $\dim_{\mathbb{Q}_p} H^i(D) < \infty$ *for* $i = 0, 1, 2$

(2) $\chi(D) = -[K:\mathbb{Q}_p] \operatorname{rank} D$.

Proof. First by Theorem 3.6, we reduce to the case D is a (φ,Γ)-module. Then by Theorem 1.2.2, we can further reduce to the case where $K = \mathbb{Q}_p$. We first show that $\dim_{\mathbb{Q}_p} H^0(D) \leq d = \operatorname{rank} D$ for any D. For r large enough, D_r is defined and we have $D_r \hookrightarrow D_{\mathrm{dif}}^{+,n}(D)[1/t]$ for $n \geq n(r)$. We claim that $\dim_{\mathbb{Q}_p}(D_{\mathrm{dif}}^{+,n}(D)[1/t])^{\Gamma} \leq d$. Otherwise we can find $e_1, e_2, ..., e_{d+1} \in (D_{\mathrm{dif}}^{+,n}(D)[1/t])^{\Gamma}$ that are linearly independent over $\mathbb{Q}_p$. But $D_{\mathrm{dif}}^{+,n}(D)[1/t]$ is a d-dimensional vector space over $\mathbb{Q}_p(\varepsilon^{(n)})((t))$. So $e_1, e_2, ..., e_{d+1}$ are linearly dependent over $\mathbb{Q}_p(\varepsilon^{(n)})((t))$. Then there is a minimal k such that k of these vectors are linearly dependent over $\mathbb{Q}_p(\varepsilon^{(n)})((t))$. Assume $e_1, e_2, ..., e_k$ are k such vectors and $\sum_{i=1}^k a_i e_i = 0$. Obviously $a_1 \neq 0$ since k is minimal, so $e_1 + \sum_{i=2}^k (a_i/a_1) e_i = 0$. Using γ, we get $e_1 + \sum_{i=2}^k \gamma(a_i/a_1) e_i = 0$. By minimality of k, we must have $\gamma(a_i/a_1) = a_i/a_1$. But $(\mathbb{Q}_p(\varepsilon^{(n)})((t)))^{\Gamma} = \mathbb{Q}_p$, so $e_1, e_2, ..., e_k$ are linearly dependent over $\mathbb{Q}_p$. That is a contradiction. So we get $\dim_{\mathbb{Q}_p}(D_r)^{\Gamma} \leq d$ for any r; therefore $\dim_{\mathbb{Q}_p} H^0(D) \leq \dim_{\mathbb{Q}_p} D^{\Gamma} \leq d$.

We will prove Theorem 1.4.3 by induction on the rank of D. Assume for some $d \geq 1$ the theorem holds for all (φ,Γ)-modules which have rank less than d. Now suppose $\operatorname{rank} D = d$. Note that both of (1) and (2) are preserved under taking extensions. Thus by the slope filtration theorem we can further assume that D is pure. Suppose $\mu(D) = c/d$. Let E

be as in Lemma 1.4.2. Then $(\otimes_{i=1}^{s} E)(x^k)$ is pure of slope $k + s/d$. In particular, we can find a pure (φ, Γ)-module F which is a successive extensions of $\mathcal{R}(x^i)$'s and $\mu(F) = -c/d$. Consider the étale (φ, Γ)-module $D \otimes F$. By Corollary 1.2.9 we get

$$\chi(D \otimes F) = -\operatorname{rank}(D \otimes F) = -\operatorname{rank} F \operatorname{rank} D.$$

On the other hand, by the construction of F, $D \otimes F$ is a successive extensions of $D(x^i)$'s. So in particular there exists a $j \in \mathbb{Z}$ such that $D(x^j)$ is a saturated submodule of $D \otimes F$. Let G be the quotient, so we have the long exact sequence of cohomology:

$$\cdots \to H^0(G) \to H^1(D(x^j)) \to H^1(D \otimes F) \to \ H^1(G) \to H^2(D(x^j)) \to H^2(D \otimes F) \cdots.$$

Since $D \otimes F$ is étale, $\dim_{\mathbb{Q}_p} H^1(D \otimes F)$ is finite by Theorem 2.6. Hence $\dim_{\mathbb{Q}_p} H^1(D(x^j))$ is finite; then $\dim_{\mathbb{Q}_p} H^1(D(x^i))$ is finite for any $i \in \mathbb{Z}$. If $\dim_{\mathbb{Q}_p} H^2(D) = \infty$, by Lemma 1.4.1 we have that $\dim_{\mathbb{Q}_p} H^2(D(x^i)) = \infty$ for any i. This implies $\dim_{\mathbb{Q}_p} H^2(D \otimes F) = \infty$ from the above sequence. But this is a contradiction since $D \otimes F$ is étale. Therefore $\dim_{\mathbb{Q}_p} H^2(D(x^i))$ is finite for any i. By Lemma 1.4.1, we have that $\chi(D) = \chi(D(x^i))$. By the additivity of χ, we get

$$\chi(D \otimes F) = (\operatorname{rank} F)\chi(D);$$

hence

$$(\operatorname{rank} F)\chi(D) = -\operatorname{rank} F \operatorname{rank} D,$$

$$\chi(D) = -\operatorname{rank} D.$$

The induction step is finished. □

1.4.2 Tate local duality theorem

The main topic of this section is to prove the Tate local duality theorem: the cup product

$$H^i(D) \times H^{2-i}(D^\vee(\omega)) \to H^2(\omega) \cong \mathbb{Q}_p$$

is a perfect pairing for any (φ, Γ)-module D and $0 \le i \le 2$.

Lemma 1.4.4. *Suppose $0 \to D' \to D \to D'' \to 0$ is an exact sequence of (φ, Γ)-modules. If Tate local duality holds for any two of them, it also holds for the third one.*

Proof. First note that the pairing $H^i(D) \times H^{2-i}(D^\vee(\omega)) \to H^2(\omega) \cong \mathbb{Q}_p$ is perfect if and only if the induced map $H^{2-i}(D^\vee(\omega)) \to H^i(D)^\vee$ is an isomorphism. From the long exact sequence of cohomology, we get the following commutative diagram.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{2-i}(D'^\vee(\omega)) & \longrightarrow & H^{2-i}(D^\vee(\omega)) & \longrightarrow & H^{2-i}(D'^\vee(\omega)) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^i(D'')^\vee & \longrightarrow & H^i(D)^\vee & \longrightarrow & H^i(D')^\vee & \longrightarrow \cdots \end{array}$$

Then the lemma follows from the Five Lemma. □

Lemma 1.4.5. *Tate local duality is true for $\mathcal{R}(|x|)$.*

Proof. By the Euler-Poincaré formula, we get $\dim_{\mathbb{Q}_p} H^1(x|x|^{-1}) \geq -\chi(\mathcal{R}(x^{-1}|x|)) = [K : \mathbb{Q}_p]$. Hence there exists a nonsplit short exact sequence of (φ, Γ)-modules

$$0 \to \mathcal{R}(x) \longrightarrow D \longrightarrow \mathcal{R}(|x|) \longrightarrow 0.$$

Then $\deg(D) = \deg(\mathcal{R}(x)) + \deg(\mathcal{R}(|x|)) = 0$ and furthermore we see D is forced to be étale. In fact, suppose P is a submodule of D such that $\mu(P) < 0$, then P is necessary of rank 1 and hence $\mu(P) \leq -1$. Then $P \cap \mathcal{R}(x) = 0$, hence P maps injectively to $\mathcal{R}(|x|)$. So we have $\mu(P) \geq -1$. Therefore we conclude that $\mu(P) = -1$; but this forces P to map isomorphically to $\mathcal{R}(|x|)$, which is a contradiction. If $a \in H^0(x)$, then $\varphi(a) = a/p$. It implies $\varphi(at) = at$, yielding at is a constant, therefore $a = 0$. If $a \in H^0(|x|)$, then $\gamma(a) = a$ for any $\gamma \in \Gamma$, so a is a constant. But $\varphi(a) = pa$, hence $a = 0$. So $H^0(D) = 0$ by the long exact sequence of cohomology. Take the dual exact sequence

$$0 \longrightarrow \mathcal{R}(x) \longrightarrow D^\vee(\omega) \longrightarrow \mathcal{R}(|x|) \longrightarrow 0.$$

By usual Tate duality, $H^0(D)$ is dual to $H^2(D^\vee(\omega))$, so $H^2(D^\vee(\omega)) = 0$. Hence $H^2(|x|) = 0$, so $\dim_{\mathbb{Q}_p} H^1(|x|) = [K : \mathbb{Q}_p]$ by the Euler-Poincaré formula. The cup pairing gives a morphism of long exact sequences:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & H^1(x) & \longrightarrow & H^1(D^\vee(\omega)) & \longrightarrow & H^1(|x|) & \longrightarrow & H^2(x) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^2(x)^\vee & \longrightarrow & H^1(|x|)^\vee & \longrightarrow & H^1(D)^\vee & \longrightarrow & H^1(x)^\vee & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

in which $H^1(D^\vee(\omega)) \to H^1(D)^\vee$ is an isomorphism by usual Tate duality. Then diagram chasing shows that $H^1(x) \to H^1(|x|)^\vee$ is injective, so $H^1(x)$ has $\mathbb{Q}_p$-dimension $\leq [K:\mathbb{Q}_p]$. Then by the Euler-Poincaré formula, $\dim_{\mathbb{Q}_p} H^1(x) = [K:\mathbb{Q}_p]$ and $H^2(x) = 0$. Therefore $H^1(x) \to H^1(|x|)^\vee$ is an isomorphism. □

Remark 1.4.6. Note that we can use $\mathcal{R}(x^{-1})$ instead $\mathcal{R}(|x|)$ in the proof of Theorem 1.4.7 (see below). In case $K = \mathbb{Q}_p$ and $p > 2$, we can verify the Tate duality for $\mathcal{R}(x^{-1})$ by explicit calculations. Recall that Res : $H^2(\omega) \to \mathbb{Q}_p$ is an isomorphism. For $i = 0$, $H^0(x^{-1}) = \mathbb{Q}_p \cdot t$, $H^2(\omega x) = \mathbb{Q}_p \cdot \overline{(1+T)/T^2}$, the cup product of t and $\overline{(1+T)/T^2}$ is $\overline{t(1+T)/T^2}$, and $\mathrm{Res}(t(1+T)/T^2) = 1$. For $i = 1$, $H^1(x^{-1})$ has a basis$\{(\bar{t}, 0), (0, \bar{t})\}$. From [13, Proposition 3.8], $H^1(\omega)$ has a basis $\{(\bar{a}, \overline{1/T + 1/2}), (\overline{1/T}, \bar{b})\}$, where $a \in T\mathcal{R}^+$ and $b \in (\mathcal{E}^\dagger)^{\psi=0}$. Furthermore, $\partial : H^1(\omega) \to H^1(\omega x)$ is an isomorphism, therefore $\{(\overline{\partial a}, \overline{-(1+T)/T^2}), (\overline{-(1+T)/T^2}, \overline{\partial b})\}$ is a basis of $H^1(\omega x)$. A short computation shows that under the given basis, the matrix of cup product is $\left(\begin{smallmatrix} 1 & * \\ 0 & -1 \end{smallmatrix}\right)$. For $i = 2$, there is nothing to say since $H^2(-1) = H^0(\omega x) = 0$.

Theorem 1.4.7. *The Tate local duality is true for all* (φ, Γ)*-modules.*

Proof. By Lemma 1.4.4 and the slope filtration theorem, we need only to prove the theorem for pure (φ, Γ)-modules. Suppose D is a pure (φ, Γ)-module of rank d. By passing to $D^\vee(\omega)$, we can further assume $\mu(D) = s/d \geq 0$. We proceed by induction on $s = \deg(D)$.

If $s = 0$, D is étale, so the theorem follows from Corollary 1.2.6. Now suppose $s > 0$ and the theorem is true for any pure (φ, Γ)-module D which satisfies $0 \leq \deg(D) < s$. By the Euler-Poincaré formula, we have $\dim_{\mathbb{Q}_p} H^1(D(|x|^{-1})) \geq d \geq 1$. Hence we can find a nontrivial extension E of $\mathcal{R}(|x|)$ by D. Then $\deg(E) = s - 1$ and $\mu(E) = (s-1)/(d+1) < \mu(D)$. Suppose the slope filtration of E is $0 = E_0 \subset E_1 \subset \ldots \subset E_l = E$. Then $\mu(E_1) \leq \mu(E) < \mu(D)$. Note that $\deg(E_1) = \deg(E_1 \cap D) + \deg(E_1/(E_1 \cap D))$. Since D is pure of positive slope, $\deg(E_1 \cap D) > 0$ unless $E_1 \cap D = 0$. Since $E_1/(E_1 \cap D)$ is the image of $E_1 \to \mathcal{R}(|x|)$, $\deg(E_1/(E_1 \cap D)) \geq -1$. Consequently, $\deg(E_1) \geq 0$ unless $E_1 \cap D = 0$ and $E_1/(E_1 \cap D) \cong \mathcal{R}(|x|)$, but these imply that the extension splits, which it does not by construction. So we have $\mu(E_j/E_{j-1}) \geq 0$ for each j. Note that $\sum_{j=1}^{l} \deg(E_{j-1}/E_j) = \deg E = s - 1$. Thus for each j, we have $\deg(E_{j-1}/E_j) < s$. Hence E_{j-1}/E_j satisfies the theorem by induction. Therefore the theorem is true for E by Lemma 1.4.4. By Lemma 1.4.5 the theorem holds for $\mathcal{R}(|x|)$. Therefore the same is true for D by Lemma 4.5 again.

This finishes the induction step. □

Remark 1.4.8. Our approach to Tate local duality is similar to the way we established the Euler-Poincaré formula. However, in the case of Tate local duality, Euler-Poincaré formula has provided the existence of nontrivial extensions, so we don't need the reduction steps on torsion (φ, Γ)-modules that were used in the proof of Theorem 1.4.3.

Chapter 2

Slope Filtrations in Family

Introduction

The slope filtration theorem gives a partial analogue of the eigenspace decomposition of a linear transformation, for a Frobenius-semilinear endomorphism of a finite free module over the Robba ring (the ring of holomorphic functions on the boundary of the p-adic open unit disk). It was originally introduced in the context of rigid cohomology by Kedlaya as the key ingredient in his proof of the Crew's conjecture, a p-adic analogue of the l-adic local monodromy theorem. It also has important applications in p-adic Hodge theory via Berger's construction of the (φ, Γ)-modules associated to Galois representations. For instance, it allows Berger to prove Fontaine's conjecture that de Rham implies potentially semistable and to give an alternate proof of the Colmez-Fontaine theorem that weakly admissible implies admissible.

This chapter grew out of an attempt to generalize the slope theory to families of Frobenius modules over a nontrivial base, i.e. over the Robba ring with coefficients not in a p-adic field but in, e.g. an affinoid algebra. In fact, in both rigid cohomology and p-adic Hodge theory, one is led to study such families of Frobenius modules. The difference between these two cases is that in rigid cohomology, the Frobenius acts on the base as a lift of a p-power map, but in p-adic Hodge theory the Frobenius does not move the base at all. The families of Frobenius modules we consider in this chapter fits the set up of the latter case, which is the main motivation of our work.

In this chapter, we mainly concern the variation of HN-polygons (generic HN-polygons) of families of (overconvergent) Frobenius modules. We establish the local constancy of generic HN-polygons of families of overconvergent Frobenius modules and the semicontinuity of HN-polygons of families of Frobenius modules over reduced affinoid algebras. Besides, we proved the existence of HN filtration of Frobenius modules over Robba rings over spherically complete fields. To build up a slope theory in this setting, we still need to prove that the semistability is preserved under tensor product. We expect to verify this point in the near future.

2.1 Preliminaries

This section contains some basic definitions and facts from the slope theory of ϕ-modules over the Robba ring. The novel feature of our treatment is that we allow the coefficient field of Robba ring to be non-discrete. We prove some new results in this case, particularly the existence of HN filtrations for ϕ-modules over Robba rings with spherically complete coefficient fields. Moreover, we introduce the notion of Robba rings over a Banach algebra, which arises from both rigid cohomology [24] and p-adic Hodge theory [6]. Some of our presentations follow [25, Chapter 1] closely.

2.1.1 The Robba ring and ϕ-modules

Throughout this paper, K is a field of characteristic 0 and complete for a non-archimedean valuation v_K which is not necessarily discrete. Let $\mathcal{O}_K$ denote the valuation subring of K. Let $\mathfrak{m}_K$ denote the maximal ideal of $\mathcal{O}_K$. We make the assumption that the residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ is of characteristic p. The valuation v_K is normalized so that $v_K(p) = 1$ and the corresponding norm $|\cdot|$ is defined as $|\cdot| = p^{-v_K(\cdot)}$. We extend v_K and $|\cdot|$ uniquely to a valuation and a norm of $\widehat{\overline{K}}$ which are still denoted by v_K and $|\cdot|$. Let $\mathcal{A}$ be a commutative Banach algebra over K, and let $\mathcal{M}(\mathcal{A})$ be the spectrum of $\mathcal{A}$ in the sense of Berkovich [7].

Definition 2.1.1. For any subinterval $I \subset (0, \infty]$, define

$$\mathcal{R}_K^I = \{f = \sum_{k=-\infty}^{+\infty} a_k T^k, \text{ where } a_k \in K \text{ and } f(T) \text{ is convergent on } v_K(T) \in I\}.$$

Equivalently, $\mathcal{R}_K^I$ is the ring of rigid analytic functions on the annulus $v_K(T) \in I$ over K. For any $s \in I$, we define a valuation $w_{s,k}$ on $\mathcal{R}_K^I$ as $w_{s,k}(f) = v_K(a_k) + ks$. We set the valuation w_s on $\mathcal{R}_K^I$ as $w_s(f) = \inf_{k\in\mathbb{Z}}\{w_{s,k}\}$. The corresponding norm is $|f|_s = p^{-w_s(f)}$. Note that $|\cdot|_s$ is multiplicative for any $s \in I$. Geometrically, $|f|_s$ is the maximal value of f

on the circle $|x| = p^{-s}$. It is clear that $\mathcal{R}_K^I$ is Fréchet complete with respect to w_s for $s \in I$. For $I = (0, r]$, we also use $\mathcal{R}_K^r$ instead of $\mathcal{R}_K^{(0,r]}$. Let $\mathcal{R}_K$ be the union of $\mathcal{R}_K^r$ for all $r > 0$. The ring $\mathcal{R}_K$ is called the *Robba ring* over K.

Definition 2.1.2. Let $\mathcal{R}_K^{\text{int}}$ be the subring of $\mathcal{R}_K$ consisting of series with coefficients in $\mathcal{O}_K$. Let $\mathcal{R}_K^{\text{bd}}$ be the subring of $\mathcal{R}_K$ consisting of series with bounded coefficients. We equip $\mathcal{R}_K^{\text{bd}}$ with a valuation w by setting $w(f) = \inf_{k\in\mathbb{Z}}\{v_K(a_k)\}$, and the corresponding norm is $|f| = \sup_{k\in\mathbb{Z}} |a_k|$. In case v_K is discrete, $\mathcal{R}_K^{\text{int}}$ is a henselian ring with residue field $k((T))$, and $\mathcal{R}_K^{\text{bd}}$ is the fraction field of $\mathcal{R}_K^{\text{int}}$ [22].

Remark 2.1.3. In case v_K is discrete, by Lazard's work [28] we know that the units in $\mathcal{R}_K$ are precisely the nonzero elements of $\mathcal{R}_K^{\text{bd}}$. If v_K is not discrete, $\mathcal{R}_K^{\text{bd}}$ is no longer a field, let alone the fraction field of $\mathcal{R}_K^{\text{int}}$. But we still have that the units of $\mathcal{R}_K$ are contained in $\mathcal{R}_K^{\text{bd}}$.

Similarly, we can define the Robba ring over the Banach algebra $\mathcal{A}$.

Definition 2.1.4. For any subinterval $I \subset (0, \infty]$, we set

$$\mathcal{R}_{\mathcal{A}}^I = \{f = \sum_{k=-\infty}^{+\infty} a_k T^k, \text{ where } a_k \in \mathcal{A} \text{ and } f(T) \text{ is convergent on } v_K(T) \in I\}.$$

For any $s \in I$, we define a valuation $w_{s,k}$ on $\mathcal{R}_{\mathcal{A}}^I$ as $w_{s,k}(f) = v_{\mathcal{A}}(a_k) + ks$. We set the valuation w_s on $\mathcal{R}_{\mathcal{A}}^I$ as $w_s(f) = \inf_{k\in\mathbb{Z}}\{v_{\mathcal{A}}(a_k) + ks\}$. The corresponding norm is $|f|_s = p^{-w_s(f)}$. It is clear that $\mathcal{R}_{\mathcal{A}}^I$ is Fréchet complete with respect to w_s for $s \in I$. For general $\mathcal{A}$, $|\cdot|_s$ is no longer multiplicative; instead we have that $w_s(fg) \geq w_s(f) + w_s(g)$, $|fg|_s \leq |f|_s|g|_s$ for any $f, g \in \mathcal{R}_{\mathcal{A}}^I$. Similarly, for $I = (0, r]$, we use $\mathcal{R}_{\mathcal{A}}^r$ instead of $\mathcal{R}_{\mathcal{A}}^{(0,r]}$. We let $\mathcal{R}_{\mathcal{A}}$ be the union of $\mathcal{R}_{\mathcal{A}}^r$ for all $r > 0$. It is called the *Robba ring* over $\mathcal{A}$.

Definition 2.1.5. Let $\mathcal{R}_{\mathcal{A}}^{\text{int}}$ be the subring of $\mathcal{R}_{\mathcal{A}}$ consisting of series with coefficients having norm less than or equal to 1. Let $\mathcal{R}_{\mathcal{A}}^{\text{bd}}$ be the subring of $\mathcal{R}_{\mathcal{A}}$ consisting of series with bounded coefficients. For $f \in \mathcal{R}_{\mathcal{A}}^{\text{bd}}$, define $w(f) = \inf_{k\in\mathbb{Z}}\{v_{\mathcal{A}}(a_k)\}$ and $|f| = \sup_{k\in\mathbb{Z}}\{|a_k|\}$. Then $\mathcal{R}_{\mathcal{A}}^{\text{int}} = \{f \in \mathcal{R}_{\mathcal{A}}^{\text{bd}}, w(f) \geq 0\}$.

Let $\mathcal{R}_{\mathcal{A}}^{\text{int},r}$ denote the intersection of $\mathcal{R}_{\mathcal{A}}^{\text{int}}$ and $\mathcal{R}_{\mathcal{A}}^r$.

Definition 2.1.6. Fix an integer $q > 1$. A *relative (q-power) Frobenius lift* on the Robba ring $\mathcal{R}_K$ (resp. $\mathcal{R}_{\mathcal{A}}$) is a homomorphism $\phi : \mathcal{R}_K \to \mathcal{R}_K$ (resp. $\phi : \mathcal{R}_{\mathcal{A}} \to \mathcal{R}_{\mathcal{A}}$) of the form

$$\sum_{k=-\infty}^{+\infty} a_k T^k \mapsto \sum_{k=-\infty}^{+\infty} \phi_K(a_k) S^k$$

(resp. $\sum_{k=-\infty}^{+\infty} a_k T^k \mapsto \sum_{k=-\infty}^{+\infty} \phi_{\mathcal{A}}(a_k) S^k$), where ϕ_K (resp. $\phi_{\mathcal{A}}$) is an isometric endomorphism of K (resp. $\mathcal{A}$) and $S \in \mathcal{R}_K^{\text{int}}$ (resp. $S \in \mathcal{R}_{\mathcal{A}}^{\text{int}}$) is such that $w(S - T^q) > 0$. If k has characteristic $p > 0$ and q is a power of p, we define an *absolute (q-power) Frobenius lift* as a relative Frobenius lift for which ϕ_K or $\phi_{\mathcal{A}}$ is a q-power Frobenius lift.

In the rest of the paper, we always equip K and $\mathcal{A}$ with isometries ϕ_K and $\phi_{\mathcal{A}}$ respectively. They then become a ϕ-field and a ϕ-ring respectively in the sense of following definition.

Definition 2.1.7. Define a *ϕ-ring/field* to be a ring/field R equipped with an endomorphism ϕ; we say R is *inversive* if ϕ is bijective. Define a *(strict) ϕ-module* over a ϕ-ring R to be a finite free R-module M equipped with an isomorphism $\phi^* M \to M$, which we also think of as a semilinear ϕ-action on M; the semilinearity means that for $r \in R$ and $m \in M$, $\phi(rm) = \phi(r)\phi(m)$. Note that the category of ϕ-modules admits tensor products, symmetric and exterior powers, and duals.

From now on, if R is a ϕ-ring/field, then we always use ϕ_R to denote the endomorphism ϕ from the definition. For a ϕ-subring of a ϕ-ring R, we mean a subring of R stable under ϕ with the restricted ϕ-action.

Definition 2.1.8. We can view a ϕ-module M over a ϕ-ring R as a left module over the twisted polynomial ring $R\{X\}$. For a positive integer a, define the *a-pushforward* functor $[a]_*$ from ϕ-modules to ϕ^a-modules to be the restriction along the inclusion $R\{X^a\} \hookrightarrow R\{X\}$. Define the *a-pullback* functor $[a]^*$ from ϕ^a-modules to ϕ-modules to be the extension of scalars functor $M \to M \otimes_{R\{X^a\}} R\{X\}$.

In the rest of the paper, a ϕ-field is always complete with a valuation such that the ϕ-action is an isometric endomorphism. For a ϕ-subring, we mean a subring with the induced ϕ-action and valuation. We always equip the Robba ring $\mathcal{R}_K$ (resp. $\mathcal{R}_{\mathcal{A}}$) with a Frobenius lift ϕ, and our main objects will be ϕ-modules over $\mathcal{R}_K$ (resp. $\mathcal{R}_{\mathcal{A}}$). Note that by the definition of Frobenius lift, we have that ϕ maps $\mathcal{R}_K^{\text{bd}}$ or $\mathcal{R}_{\mathcal{A}}^{\text{bd}}$ to itself. Therefore we can also talk about ϕ-modules over $\mathcal{R}_K^{\text{bd}}$ (resp. $\mathcal{R}_{\mathcal{A}}^{\text{bd}}$). They are called *overconvergent ϕ-modules.*

2.1.2 Slope theory of ϕ-modules

Definition 2.1.9. Let M be a ϕ-module over $\mathcal{R}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$) of rank n. Then its top exterior power $\bigwedge^n M$ has rank 1 over $\mathcal{R}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$). Let v be a generator of $\bigwedge^n M$, and suppose $\phi(v) = \lambda v$ for some $\lambda \in \mathcal{R}_K^{\mathrm{bd}}$. Define the *degree* of M by setting $\deg(M) = w(\lambda)$. Note that this is independent of the choice of v because ϕ is an isometry on $\mathcal{R}_K^{\mathrm{bd}}$. If M is nonzero, define the *slope* of M by setting $\mu(M) = \deg(M)/\operatorname{rank}(M)$.

For $c \in \mathbb{Z}$, we set $\mathcal{R}_K(c)$ as the rank 1 ϕ-module over $\mathcal{R}_K$ with a generator v such that $\phi(v) = p^c v$. For a ϕ-module M over $\mathcal{R}_K$, we set $M(c)$ as $M \otimes \mathcal{R}_K(c)$. The following formal properties are easily verified; see [25] for the proof.

(1) If $0 \to M_1 \to M \to M_2 \to 0$ is exact, then $\deg(M) = \deg(M_1) + \deg(M_2)$;

(2) We have $\mu(M_1 \otimes M_2) = \mu(M_1) + \mu(M_2)$;

(3) We have $\deg(M^\vee) = -\deg(M)$ and $\mu(M^\vee) = -\mu(M)$;

(4) We have $\mu(M(c)) = \mu(M) + c$;

(5) If M is a ϕ-module, then $\mu([a]_* M) = a\mu(M)$;

(6) If M is a ϕ^a-module, then $\mu([a]^* M) = a^{-1}\mu(M)$.

Definition 2.1.10. We say a ϕ-module M over $\mathcal{R}_K$ is *(module-)semistable* if for any nonzero ϕ-submodule N, we have $\mu(N) \geq \mu(M)$. We say M is *(module-)stable* if for any proper nonzero ϕ-submodule N, we have $\mu(N) > \mu(M)$. Note that both properties are preserved under *twisting* (tensoring with a rank 1 module).

Lemma 2.1.11. *For $a \in \mathcal{R}_K$, if there exists a $\lambda \in \mathcal{R}_K^\times$ such that $w(\lambda) \leq 0$ and $\phi(a) = \lambda a$, then we have $a \in \mathcal{R}_K^\times$.*

Proof. Recall that a is a unit in $\mathcal{R}_K$ if and only if there exists an $r > 0$ such that a has no roots in the annulus $0 < v_K(T) < r$. By [25, Proposition 1.2.6] (although this is under the hypothesis that v_K is discrete, the proof works for the general case), we get that $a \in \mathcal{R}_K^{\mathrm{bd}}$. Note that there exists an $r_0 > 0$ such that for any $r \in (0, r_0)$, ϕ induces a finite étale map of degree q from the annulus $0 < v_K(T) < r/q$ to the annulus $0 < v_K(T) < r$ and $v_K(\phi(x)) = q v_K(x)$ for $0 < v_K(x) < r/q$. Now if a is not a unit in $\mathcal{R}_K$, then it has roots $c_1, c_2, \cdots$ in the open unit disk such that $\lim v_K(c_i) = 0$. Moreover, by the theory of

Newton polygons, we have that $\sum v_K(c_i) < \infty$ since $a \in \mathcal{R}_K^{\mathrm{bd}}$. Now we choose an $r < r_0$ such that λ has no roots in the annulus $0 < |T| < r$. Pick a root c of a in this annulus. For any c' such that $\phi(c') = \phi_K(c)$, we have that it is a root of $\phi(a)$, hence a root of a by $\phi(a) = \lambda a$. Now we can find q different roots c' in the annulus $0 < v_K(x) < r/q$ such that $\phi(c') = \phi_K(c)$; each of them has valuation $v_K(c)/q$. By iterating the above process, we get q^2 roots of a having valuations $v_K(c)/q^2$, and so on. But $\sum_{i\in\mathbb{N}} q^i(v_K(c)/q^i) = \infty$, this yields a contradiction. □

Remark 2.1.12. In the rest of the paper, we fix such a r_0 as in the above proof.

Proposition 2.1.13. *Any rank 1 ϕ-module over $\mathcal{R}_K$ is stable.*

Proof. By twisting, we only need to prove this for the trivial ϕ-module $M \cong \mathcal{R}_K$. Suppose that N is a nonzero ϕ-submodule of M; we write $N = \mathcal{R}_K a$ for some $a \in \mathcal{R}_K$. Then we have that $\lambda = \phi(a)/a \in \mathcal{R}_K^\times$, and $\mu(N) = w(\lambda)$ by definition. If $w(\lambda) = \mu(N) \leq \mu(M) = 0$, then $\phi(a) = \lambda a$ implies that $a \in \mathcal{R}_K^\times$ by the above lemma. Hence $N = M$. In other words, $\mu(N) > \mu(M)$ unless $N = M$, as desired. □

Corollary 2.1.14. *Suppose that $N \subset M$ are two ϕ-modules over $\mathcal{R}_K$ of the same rank; then $\mu(N) \geq \mu(M)$, with equality if and only if $N = M$.*

Proof. Suppose that $\operatorname{rank} M = n$. Apply the above lemma to the inclusion $\bigwedge^n N \subseteq \bigwedge^n M$. □

Definition 2.1.15. Let M be a ϕ-module over $\mathcal{R}_K$. A *semistable filtration* of M is a filtration $0 = M_0 \subset M_1 \cdots \subset M_l = M$ of M by saturated ϕ-submodules, such that each successive quotient M_i/M_{i-1} is a semistable ϕ-module of some slope s_i. A *Harder-Narasimhan (HN) filtration* of M is a semistable filtration such that $s_1 < \cdots < s_l$. These s_i are called the *slopes* of M.

Definition 2.1.16. Let M be a ϕ-module over $\mathcal{R}_K$. Define the *slope multiset* of a semistable filtration of M as the multiset in which each slope of a successive quotient occurs with multiplicity equal to the rank of that quotient. Define the *slope polygon* of this filtration as the Newton-polygon of the slope multiset. For the definition of Newton polygon of a multiset, we refer to [23, Definition 3.5.1].

Proposition 2.1.17. *In case $\mathcal{R}_K$ is a Bézout domain, every ϕ-module over $\mathcal{R}_K$ has a unique maximal ϕ-submodule of minimal slope, which is semistable.*

Proof. Let M be a ϕ-module over $\mathcal{R}_K$. First note that by the property for Bézout domains, the saturation of a ϕ-submodule of M and the sum of two ϕ-submodules of M are still ϕ-submodules of M. So we can talk about the saturation of a ϕ-submodule and the sum of two ϕ-submodules.

We will prove this proposition by induction on the rank of the ϕ-module. The rank 1 case is trivial. Now suppose the theorem is true for ϕ-modules over $\mathcal{R}_K$ of rank $\leq d-1$ for some $d \geq 2$. Let M be a rank d ϕ-module of slope s. If M is semistable, then we are done. Otherwise, let P be a ϕ-submodule of slope less than s and of maximal rank. By passing to the saturation, we suppose that P is saturated. If $\operatorname{rank} P = d$, then we have $\mu(M) = \mu(P) < s$. That is a contradiction. So P is a ϕ-module of rank $\leq d-1$. By the inductive hypothesis, P has a unique maximal ϕ-submodule P_1 of minimal slope. We claim that P_1 is also the unique maximal ϕ-submodule of M of minimal slope. In fact, suppose that Q is a ϕ-submodule of M such that $\mu(Q) \leq \mu(P_1) < s$. If $Q \nsubseteq P$, consider the following exact sequence

$$0 \longrightarrow P \cap Q \longrightarrow P \oplus Q \longrightarrow P + Q \longrightarrow 0.$$

Since $\mu(P \cap Q) \geq \mu(P_1) \geq \mu(Q)$, we conclude that $\mu(P+Q) \leq \max\{\mu(P), \mu(Q)\} < s$. But $\operatorname{rank}(Q+P) > \operatorname{rank} P$, which contradicts the definition of P. We conclude that $Q \subset P$. Therefore we must have $\mu(Q) = \mu(P_1)$ and $Q \subseteq P_1$ by the definition of P_1. The induction step is finished. □

Theorem 2.1.18. *In case $\mathcal{R}_K$ is a Bézout domain, every ϕ-module over $\mathcal{R}_K$ admits a unique HN filtration.*

Proof. This is a formal consequence of the above proposition. In fact, by the definition of HN filtration, for any $i \geq 1$, M_i can be characterized as the unique maximal ϕ-submodule of M/M_{i-1} of minimal slope. □

Corollary 2.1.19. *For K spherically complete, every ϕ-module over $\mathcal{R}_K$ admits a unique HN filtration.*

Proof. By [28], $\mathcal{R}_K$ is a Bézout domain if K is spherically complete. □

Definition 2.1.20. Let M be a ϕ-module over $\mathcal{R}_K$. If K is spherically complete, by the above corollary, M admits a unique HN filtration. The slope polygon of the HN filtration of M is called the *HN-polygon* of M. For general K, taking the spherical completion K' of K, we define the *HN-polygon* of M as the HN-polygon of $M \otimes_{\mathcal{R}_K} \mathcal{R}_{K'}$.

Proposition 2.1.21. *The HN-polygon lies above the slope polygon of any semistable filtration and has the same endpoint.*

Proof. This is a formal consequence of the definition of HN filtration. We refer to [23, Proposition 3.5.4] for a proof. □

In fact, when v_K is discrete, we can say more about semistable ϕ-modules over $\mathcal{R}_K$.

Definition 2.1.22. Let M be a ϕ-module over $\mathcal{R}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$). We say M is *étale* if M has a ϕ-stable $\mathcal{R}_K^{\mathrm{int}}$-submodule M' such that $\phi^* M' \cong M'$ and $M' \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K = M$ (resp. $M' \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}} = M$). More generally, suppose $\mu(M) = s = c/d$, where c, d are coprime integers with $d > 0$. We say M is *pure* if for some ϕ-module N of rank 1 and degree $-c$, $([d]_* M) \otimes N$ is étale.

It is not difficult to prove that a pure ϕ-module is always semistable [25, Theorem 1.6.10] (although the proof is under the hypothesis that v_K is discrete, it works for the general case). But the converse is much more difficult. We know this for discrete v_K thanks to the following theorem of Kedlaya [25].

Theorem 2.1.23. *(Slope filtration theorem) In case v_K is discrete, every semistable ϕ-module over $\mathcal{R}_K$ is pure. As a consequence, every ϕ-module M over $\mathcal{R}_K$ admits a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated ϕ-submodules whose successive quotients are pure with $\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1})$.*

Remark 2.1.24. It would be interesting to generalize the slope filtration theorem to spherically complete K, i.e. to prove semistable implies pure in this case. See Remark 2.3.10 for one of the motivations.

2.2 Generic and special slope filtrations

In this section, we suppose that K is of discrete valuation and π_K is a uniformizer.

Definition 2.2.1. Let $\mathcal{E}_K$ be the completion of $\mathcal{R}_K^{\mathrm{bd}}$ with respect to w; the induced valuation on $\mathcal{E}_K$ is still denoted by w. Then $\mathcal{E}_K$ is just the set of Laurent series $f = \sum_{k=-\infty}^{+\infty} a_k T^k$ with $a_k \in K$, satisfying the condition that the $|a_i|$'s are bounded, and that $|a_k| \to 0$ as $k \to -\infty$. Then w can be written as $w(f) = \min_{i\in\mathbb{Z}}\{v_K(a_i)\}$. The ϕ-action on $\mathcal{R}_K^{\mathrm{bd}}$ induces a ϕ-action on $\mathcal{E}_K$. In this way, $\mathcal{E}_K$ is equipped with a ϕ-ring structure.

Let $\tilde{\mathcal{R}}_K$, $\tilde{\mathcal{R}}_K^{\mathrm{bd}}$ be the extended Robba ring, and extended bounded Robba ring defined in [25].

Definition 2.2.2. Let $\tilde{\mathcal{E}}_K$ be the set of formal sums $f = \sum_{i\in\mathbb{Q}} a_i u^i$ with bounded coefficients satisfying the condition that for each $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geq c$ is well-ordered, and that $|a_i| \to 0$ as $i \to -\infty$. It is also the completion of $\tilde{\mathcal{R}}_K^{\mathrm{bd}}$; the induced valuation w on $\tilde{\mathcal{E}}_K$ is $w(f) = \min_{i\in\mathbb{Q}} v_K(a_i)$. We define the ϕ-action on $\tilde{\mathcal{E}}_K$ as $\phi(\sum_{i\in\mathbb{Q}} a_i u^i) = \sum_{i\in\mathbb{Q}} \phi(a_i) u^{qi}$.

Proposition 2.2.3. *There exists a ϕ-equivariant isometric embedding $\psi : \mathcal{E}_K \hookrightarrow \tilde{\mathcal{E}}_K$. Thus we can view $\mathcal{E}_K$ as a ϕ-subfield of $\tilde{\mathcal{E}}_K$ via ψ.*

Proof. By the proof of [25, Proposition 2.2.6], we have a ϕ-equivariant isometric embedding from $\mathcal{E}_K^{\mathrm{bd}}$ to $\tilde{\mathcal{E}}_K^{\mathrm{bd}}$. Taking the completion of this embedding, we get the ψ required. $\square$

We can develop the slope theory for ϕ-modules over $\mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$) as for ϕ-modules over $\mathcal{R}_K$. For P a ϕ-module over $\mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$) of rank n, let v be a generator of $\bigwedge^n P$, and suppose $\phi(v) = \lambda v$ for some $\lambda \in \mathcal{E}_K$. Define the *degree* of P as $\deg(P) = w(\lambda)$. This is independent of the choice of v because of the ϕ-equivariance of w. If P is nonzero, define the *slope* of P by setting $\mu(P) = \deg(P)/\operatorname{rank}(P)$. We say M is *stable* (resp. *semistable*) if for any nonzero proper ϕ-submodule N of M, we have $\mu(N) > \mu(M)$ (resp. $\mu(N) \geq \mu(M)$). We then define *HN filtration* and *HN-polygon* similarly. Since $\mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$) is a field, we have that every ϕ-module over $\mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$) admits a unique HN filtration.

For $\lambda \in \mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$), let $V_{\lambda,d}$ be the ϕ-module over $\mathcal{E}_K$ (resp. $\tilde{\mathcal{E}}_K$) with basis $e_1, \ldots, e_d$ such that $\phi(e_1) = e_2, \ldots, \phi(e_{d-2}) = e_{d-1}, \phi(e_{d-1}) = e_d, \phi(e_d) = \lambda e_1$ and any such basis is called a *standard* basis. Then $V_{\lambda,d}$ is a semistable ϕ-module with slope $w(\lambda)/d$. For a ϕ-module P over $\mathcal{E}_K$ or $\tilde{\mathcal{E}}_K$, a *Dieudonné-Manin decomposition* of P is a decomposition of ϕ-modules $P = \oplus P_i$ such that each P_i is of the form $V_{\lambda,d}$ for some λ, d.

Proposition 2.2.4. *If K is such that every ϕ-module over K admits a Dieudonné-Manin decomposition, then every ϕ-module P over $\tilde{\mathcal{E}}_K$ admits a Dieudonné-Manin decomposition.*

Proof. See [26, Theorem 14.6.3] for a proof (although this is for an absolute Frobenius lift, the proof works for the general case). □

Lemma 2.2.5. *Let L be a ϕ-field extension of K. Put $S = \tilde{\mathcal{E}}_L \otimes_{\mathcal{E}_K} \tilde{\mathcal{E}}_L$. Set $i_1, i_2 : \tilde{\mathcal{E}}_L \to S$ by $i_1(a) = a \otimes 1$ and $i_2(a) = 1 \otimes a$. Suppose that λ_1 and λ_2 are elements of $\tilde{\mathcal{E}}_L$ such that $w(\lambda_1) > w(\lambda_2)$. If $z \in S$ satisfies $i_2(\lambda_2^{-1})i_1(\lambda_1)\phi(z) = z$, then $z = 0$.*

Proof. Let l be the residue field of L, fix a basis $\overline{B}$ of l over k containing 1, and lift $\overline{B}$ to a subset B of $\mathcal{O}_L$ containing 1. Then every element $x \in \tilde{\mathcal{E}}_L$ can be written uniquely as a formal sum

$$\sum_{\alpha\in[0,1)\cap\mathbb{Q}} \sum_{b\in B} x_{\alpha,b} u^\alpha b \qquad (x_{\alpha,b} \in \mathcal{E}_K).$$

Given x presented as above, we set $f_{\alpha,b}(x) = x_{\alpha,b}$; one checks that $f_{\alpha,b}$ is an $\mathcal{E}_K$-linear map from $\tilde{\mathcal{E}}_L$ to $\mathcal{E}_K$ satisfying $w(f_{\alpha,b}(x)) \geq w(x)$. Then it is clear that $x = 0$ if and only if $f_{\alpha,b}(x) = 0$ for any $\alpha \in [0,1) \cap \mathbb{Q}$ and $b \in B$. We also use $f_{\alpha,b}$ to denote the map from S to $\tilde{\mathcal{E}}_L$ defined by setting $f_{\alpha,b}(x \otimes y) = f_{\alpha,b}(x)y$; then for any $s \in S$, $s = 0$ if and only if $f_{\alpha,b}(s) = 0$ for any $\alpha \in [0,1) \cap \mathbb{Q}$ and $b \in B$. Now from $z = i_2(\lambda_2^{-1})i_1(\lambda_1)\phi(z)$, we get

$$z = i_2(\phi^{n-1}(\lambda_2^{-1}) \cdots \lambda_2^{-1}) i_1(\phi^{n-1}(\lambda_1) \cdots \lambda_1)\phi^n(z)$$

for any $n \geq 1$. Write $z = \sum_{i=1}^m x_i \otimes y_i$ and suppose that $c = \min_{i=1}^m \{w(x_i)\}$. For any $\alpha \in [0,1) \cap \mathbb{Q}$, $b \in B$, by the above equality, we get that

$$f_{\alpha,b}(z) = \sum_{i=1}^m f_{\alpha,b}(\phi^{n-1}(\lambda_1) \cdots \lambda_1 x_i)\phi^{n-1}(\lambda_2^{-1}) \cdots \lambda_2^{-1} y_i.$$

Then we get that $w(f_{\alpha,b}(z)) \geq n(w(\lambda_1) - w(\lambda_2)) + 2c$ for any n. We conclude that $f_{\alpha,b}(z) = 0$ since $w(\lambda_1) > w(\lambda_2)$. Hence we have $z = 0$.

□

Proposition 2.2.6. *Let L be a ϕ-field containing K with discrete valuation such that every ϕ-module over L has a Dieudonné-Manin decomposition. Suppose P is a ϕ-module over*

$\mathcal{E}_K$. Then the HN filtration of P, tensored up to $\tilde{\mathcal{E}}_L$, gives the HN filtration of $P \otimes_{\mathcal{E}_K} \tilde{\mathcal{E}}_L$. In particular, P and $P \otimes_{\mathcal{E}_K} \tilde{\mathcal{E}}_L$ have the same HN-polygon.

Proof. Suppose that $P_L = P \otimes_{\mathcal{E}_K} \tilde{\mathcal{E}}_L$ has the HN filtration $0 = P_{L,0} \subset \cdots \subset P_{L,l} = P_L$ with $\mu(P_{L,i}/P_{L,i-1}) = s_i$. We will show that this filtration descends to a filtration $0 = P_0 \subset \cdots \subset P_l = P$ of P. Since $P_i \otimes_{\mathcal{E}_K} \tilde{\mathcal{E}}_L = P_{L,i}$ is semistable, we have that P_i is also semistable. Hence the descended filtration is the HN filtration of P and we are done.

To do this, we apply faithfully flat descent for modules. We will show that $P_{L,1} \otimes_{i_2} S \subset P_{L,i} \otimes_{i_1} S$ for $i = l, l-1, \ldots, 1$ inductively; the starting point $i = l$ is trivial. Now given that $P_{L,1} \otimes_{i_2} S \subset P_{L,i} \otimes_{i_1} S$ for some $i > 1$; we get a homomorphism $P_{L,1} \otimes_{i_2} S \to (P_{L,i}/P_{L,i-1}) \otimes_{i_1} S$. We will show that this homomorphism is forced to vanish. Indeed, since every ϕ-module over $\tilde{\mathcal{E}}_L$ admits a Dieudonné-Manin decomposition, we can suppose that $P_{L,1}$ is a direct sum of some V_{λ_m, d_m} and that $P_{L,i}/P_{L,i-1}$ is a direct sum of some V_{δ_n, c_n}; since $P_{L,1}$ and $P_{L,i}/P_{L,i-1}$ are semistable of slopes s_1 and s_i respectively, we have that $w(\lambda_m) = d_m s_1$ and $w(\delta_n) = c_n s_i$. We only need to show that any ϕ-homomorphism h from $V_{\lambda_m, d_m} \otimes_{i_2} S$ to $V_{\delta_n, c_n} \otimes_{i_1} S$ vanishes. Suppose that $e'_1, \ldots, e'_{d_m}$ and $e''_1, \ldots, e''_{c_n}$ are standard bases of V_{λ_m, d_m} and V_{δ_n, c_n} respectively; write $h(e'_1) = \sum_{j=1}^{c_n} a_j e''_j$. Then $\phi^{c_n d_m}(e'_1) = \phi^{(c_n-1)d_m}(\lambda_m) \cdots \lambda_m e'_1$ implies that $\phi^{c_n d_m}(a_1) i_1(\phi^{(d_m-1)c_n}(\mu_n) \cdots \mu_n) = i_2(\phi^{(c_n-1)d_m}(\lambda_m) \cdots \lambda_m) a_1$. Since we have that $w(\phi^{(d_m-1)c_n}(\mu_n) \cdots \mu_n) = d_m w(\mu_n) = c_n d_m s_i > c_n d_m s_1 = w(\phi^{(c_n-1)d_m}(\lambda_m) \cdots \lambda_m)$, we conclude that a_1 is forced to be zero by the above lemma. Similarly, we get that a_2, $\ldots$, a_{c_n} are all zero; then so is $h(e'_1)$. Hence $h(e'_s) = h(\phi^s(e'_1)) = \phi^s(h(e'_1)) = 0$ for any $1 \leq s \leq d_m$. Therefore $P_{L,1} \otimes_{i_2} S \subset P_{L,j-1} \otimes_{i_1} S$, the induction step is finished. Then by symmetry, we conclude that $P_{L,1} \otimes_{i_2} S = P_{L,1} \otimes_{i_1} S$. This shows that $P_{L,1}$ satisfies the condition for faithfully flat descent, so it descends to $\mathcal{E}_K$. Similarly, each $P_{L,i}$ descends to $\mathcal{E}_K$. □

Definition 2.2.7. Let M be a ϕ-module over $\mathcal{R}_K^{bd}$. Define the *generic slope filtration* of M as the HN filtration of $M \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{E}_K$. The slope polygon of the generic slope filtration is called the *generic HN-polygon*. Similarly, define the *special slope filtration* of M as the HN filtration of $M \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K$. The slope polygon of the special slope filtration is called the *special HN-polygon*.

Proposition 2.2.8. *Let M be a ϕ-module over $\mathcal{R}_K$. Then the special HN-polygon of M lies above the generic HN-polygon of M and has the same endpoint.*

Proof. We refer to [23, Proposition 5.5.1] for a proof (although this is for an absolute Frobenius lift, the proof works for the general case). □

Definition 2.2.9. For a ϕ-module M over $\mathcal{R}_K$, we say a ϕ-submodule N over $\mathcal{R}_K^{\mathrm{bd}}$ is a *model* of M if $N \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K = M$. We say it is a *good model* if the generic HN-polygon of N coincides with the special HN-polygon of N, i.e. the HN-polygon of M.

Proposition 2.2.10. *Every ϕ-module over $\mathcal{R}_K$ admits a good model.*

Proof. A pure ϕ-module over $\mathcal{R}_K$ has a unique good model. So by the slope filtration theorem, we only need to prove that if M is an extension of ϕ-modules M_1, M_2 such that the HN-polygon of M is the sum of HN-polygons of M_1, M_2 and the proposition holds for M_1, M_2, then it also holds for M. In fact, suppose that N_1, N_2 are good models of M_1, M_2, then by [23, Lemma 7.4.1] we can find a model N of M such that N is an extension of N_2 by N_1. Note that the generic HN-polygons are additive; we conclude that N is a good model of M. □

We pick a basis $e = \{e_1, e_2, \dots, e_{n_1+n_2}\}$ of N such that $\{e_1, e_2, \dots, e_{n_1}\}$ is a basis of N_1 and $\{e_{n_1+1}, e_{n_1+2}, \dots, e_{n_1+n_2}\}$ is a lift of a basis of N_2. Then the matrix F of ϕ under e is of the form

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix},$$

where F_{11} is the matrix of ϕ under $\{e_1, e_2, \dots, e_{n_1}\}$ and F_{22} is the matrix of ϕ under the image of $\{e_{n_1+1}, e_{n_2+2}, \dots, e_{n_1+n_2}\}$.

Lemma 2.2.11. *With notations as above, we can make $w(F_{12})$ arbitrarily large by choosing suitable e.*

Proof. If we change e to $\{e_1, e_2, \dots, e_{n_1}, \lambda e_{n_1+1}, \dots, \lambda e_{n_1+n_2}\}$ for $\lambda \in (\mathcal{R}_K^{bd})^\times$, then F_{12} changes to $\phi(\lambda)F_{12}$. □

By iterating the above procedure, we get the following

Corollary 2.2.12. *Let M be a ϕ-module over $\mathcal{R}_K$. Suppose that the HN filtration of M is*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M.$$

Then we can choose a basis $e = \{e_1, e_2, \ldots, e_n\}$ *of* M *such that the matrix of* ϕ *under* e *is of the form*

$$F = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1l} \\ 0 & F_{22} & \cdots & F_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{ll} \end{pmatrix},$$

where F *has entries in* $\mathcal{R}_K^{\mathrm{bd}}$ *and* F_{ii} *is the matrix of* ϕ *of* M_i/M_{i-1} *under some basis. Hence the* $\mathcal{R}_K^{\mathrm{bd}}$*-module* N *generated by* $\{e_1, e_2, \ldots, e_n\}$ *is a good model of* M*. Furthermore, we can make* $\min_{i<j}\{w(F_{ij})\}$ *arbitrarily large.*

2.3 Variation of HN-polygons

Throughout this section, K is of discrete valuation v_K with uniformizer π_K. Let A be a reduced affinoid algebra over K, and suppose that ϕ_K is extended to an isometric endomorphism ϕ_A of A such that for every prime ideal p of A, $\phi_A(\mathrm{p}) \subseteq \mathrm{p}$, and that ϕ_A induces an isometric endomorphism of A/p. We use $M(A)$ to denote the associated affinoid space of A as in rigid geometry, and let $k(x)$ denote the residue field of x. By *a family of* ϕ*-modules* (resp. *a family of overconvergent* ϕ*-modules*) over A, we mean a ϕ-module over $\mathcal{R}_A$ (resp. $\mathcal{R}_A^{\mathrm{bd}}$). Let M_A (resp. N_A) be a family of ϕ-modules (resp. a family of overconvergent ϕ-modules) over A. For any $x \in M(A)$, we set $M_x = M_A \otimes_{\mathcal{R}_A} \mathcal{R}_{k(x)}$ (resp. $N_x = N_A \otimes_{\mathcal{R}_A^{\mathrm{bd}}} \mathcal{R}_{k(x)}^{\mathrm{bd}}$); it is a ϕ-module over $\mathcal{R}_{k(x)}$ (resp. $\mathcal{R}_{k(x)}^{\mathrm{bd}}$). In section 3.1, we will prove that the generic HN-polygons of a family of overconvergent ϕ-modules are locally constant over $M(A)$. In section 3.2, we prove some results concerning the semicontinuity of HN-polygons of a family of ϕ-modules.

2.3.1 Local constancy of generic HN-polygons of families of overconvergent ϕ-modules

Definition 2.3.1. For an invertible square matrix F over $\mathcal{R}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$), we define the *HN-polygon* (resp. *generic HN-polygon and special HN-polygon*) of F as the HN-polygon (resp. generic HN-polygon and special HN-polygon) of the ϕ-module over $\mathcal{R}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$) defined by F.

Proposition 2.3.2. *Let* F *be an invertible* $n \times n$ *matrix over* $\mathcal{R}_K^{\mathrm{bd}}$*. Then there exists a* *constant C_F depending only on F satisfying the following property. For any ϕ-field $L \supset K$ which is complete for a discrete valuation extending v_K, F' a $n \times n$ matrix over $\mathcal{R}_L^{\mathrm{bd}}$, if $w(F'-F) > C_F$, then F' has the same generic HN-polygon as F.*

Proof. We choose a ϕ-field extension K' of K with discrete valuation such that every étale ϕ-module over K' is trivial. We then have the Dieudonné-Manin decomposition for any ϕ-modules over $\widetilde{\mathcal{E}}_{K'}$, and furthermore there is an invertible n by n matrix U over $\widetilde{\mathcal{E}}_{K'}$ such that $D = U^{-1}FU^{\phi}$ is diagonal. Then the generic slopes of F are equal to the valuations of the diagonal entries of D. For any L, F' given in the proposition, choose a ϕ-field extension L' of K' with discrete valuation such that L can be embedded into L'. We then view all of F', F, U, D as matrices over $\tilde{\mathcal{E}}_{L'}$. Now if $w(F'-F) > -w(D^{-1}) - w(U^{-1}) - w(U^{\phi})$, then we get $w(U^{-1}F'U^{\phi}D^{-1} - I_n) = w(U^{-1}F'U^{\phi}D^{-1} - U^{-1}FU^{\phi}D^{-1}) \geq w(U^{-1}) + w(F' - F) + w(U^{\phi}) + w(D^{-1}) > 0$. Then by [23, Lemma 5.2.6] (although it is for an absolute Frobenius lift, the proof works for the general case), the generic slopes of $U^{-1}F'U^{\phi}$, or equivalently the the generic slopes of F', are equal to the valuations of the diagonal entries of D, hence are the same as generic slopes of F. This shows that we can take C_F to be $-w(D^{-1}) - w(U^{-1}) - w(U^{\phi})$. □

Lemma 2.3.3. *For any $x \in M(A)$ and $\lambda > 0$, there exists an affinoid neighborhood $M(B)$ of x such that for any $f \in A$ vanishing at x, $|f(y)| \leq \lambda |f|$ for any $y \in M(B)$.*

Proof. We first prove the lemma for $A = K\langle x_1, \cdots, x_n\rangle$, the n-dimensional Tate algebra. Without loss of generality we suppose that x is the origin point $x_1 = \cdots = x_n = 0$. Choosing a rational number $\lambda' < \lambda$, the affinoid domain $\{(x_1, \ldots, x_n) | |x_1| \leq \lambda', \ldots, |x_n| \leq \lambda'\}$ satisfies the required property.

For general A, the reduction $\bar{A} = A^o/\pi_K A^o$ is a finite type scheme over k. For n big enough, we take a surjective k-algebra homomorphism $\overline{\alpha} : k[\overline{x_1}, \ldots, \overline{x_n}] \twoheadrightarrow \bar{A}$, where $k[\overline{x_1}, \ldots, \overline{x_n}]$ is the polynomial algebra of variables $\overline{x_1}, \ldots, \overline{x_n}$. We then lift $\overline{\alpha}$ to a K-affinoid algebra homomorphism $\alpha : K\langle x_1, \ldots, x_n\rangle \to A$ by mapping x_i to a lift of $\overline{\alpha}(\overline{x_i})$ in A^o. Then it follows from Nakayama's lemma that α maps $\mathcal{O}_K\langle x_1, \ldots, x_n\rangle$ onto A^o. We still use α to denote the induced K-affinoid homomorphism from $M(A)$ to $M(K\langle x_1, \ldots, x_n\rangle)$. By the case of $K\langle x_1, \ldots, x_n\rangle$, we can find an affinoid neighborhood $M(B)$ of $\alpha(x)$ satisfying the required property. Now for any nonzero $f \in A$ vanishing at x, choosing $c \in K$ such that $|c| = |f|$, we can find a $f' \in \mathcal{O}_K\langle x_1, \ldots, x_n\rangle$ mapping to f/c via α. Then $f'(\alpha(x)) = (f/c)(x) = 0$

implies that $|f'(y)| \leq \lambda$ for any $y \in M(B)$. Then for any $y \in \alpha^{-1}(M(B))$, we have $|f(y)|/c = |f'(\alpha(y))| \leq \lambda$. Hence $\alpha^{-1}(M(B))$ is an affinoid neighborhood of x satisfying the property we need. □

Theorem 2.3.4. *For a family of overconvergent ϕ-modules over A, the generic HN-polygons are locally constant over $M(A)$.*

Proof. Let N_A be a family of overconvergent ϕ-modules over A. For any $x \in M(A)$, by passing from A to $A_{k(x)} = A \otimes_K k(x)$, we can suppose that x is rational. Choose a basis e of N_A, and let F be the matrix of ϕ under e. For any $y \in M(A)$, e maps to e_y, a basis of N_y, and the matrix of ϕ under e_y is F_y, the image of F in $\mathcal{R}^{\mathrm{bd}}_{k(y)}$. Let C_{F_x} be the constant for F_x as in Proposition 2.3.2. Since F is over $\mathcal{R}^{\mathrm{bd}}_K$, by the above lemma there exists an affinoid neighborhood $M(B)$ of x such that $w(F_y - F_x) > C_{F_x}$ for any $y \in M(B)$. Hence the generic HN-polygon of F_y, or in other words, the generic HN-polygon of N_y, is the same as the generic HN-polygon of N_x. □

2.3.2 Semicontinuity of HN-polygons of families of ϕ-modules

For a family of ϕ-modules M_A over A, we say a a family of overconvergent ϕ-submodule N_A of M_A is a model of M_A if $N_A \otimes_{\mathcal{R}^{\mathrm{bd}}_A} \mathcal{R}_A = M_A$. We say it is a *good model* if N_x is a good model of M_x for any $x \in M(A)$.

The following is a family version of [23, Lemma 6.1.1].

Lemma 2.3.5. *For $r < r_0/q$, let D be an invertible $n \times n$ matrix over $\mathcal{R}^{[r,r]}_A$, and put $h = -w_r(D) - w_r(D^{-1})$. Let F be an $n \times n$ matrix over $\mathcal{R}^{[r,r]}_A$ such that $w_r(FD^{-1} - I_n) \geq c + h/(q-1)$ for a positive number c. Then for any positive integer k satisfying $2(q-1)(k-1) \leq c$, there exists an invertible $n \times n$ matrix U over $\mathcal{R}^{[r,qr]}_A$ such that $U^{-1}FU^{\phi}D^{-1} - I_n$ has entries in $\pi^k\mathcal{R}^{\mathrm{int},r}_A$ and $w_r(U^{-1}FU^{\phi}D^{-1} - I_n) \geq c + h/(q-1)$.*

Proof. To prove the lemma, we first introduce some valuations on $\mathcal{R}_A$. For $i \in \mathbb{Z}$, $r > 0$, $f = \sum_{k=-\infty}^{+\infty} a_k T^k \in \mathcal{R}_A$, we set $v_i(f) = \min\{k : v_K(a_k) \leq n\}$ and $v_{i,r}(f) = rv_i(f) + i$. They were first introduced in [23, p. 458] and named as v_i^{naive}, $v_{i,r}^{\mathrm{naive}}$ respectively.

We define a sequence of invertible matrices U_0, U_1,... over $\mathcal{R}^{[r,qr]}_A$ and a sequence of matrices F_0, F_1,... over $\mathcal{R}^{[r,r]}_A$ as follows. Set $U_0 = I_n$. Given U_l, put $F_l = U_l^{-1}FU_l^{\phi}$. Suppose $F_lD^{-1} - I_n = \sum_{m=-\infty}^{\infty} V_mT^m$ where the V_m's are $n \times n$ matrices over A. Let $X_l =$

$\sum_{v_A(V_m)\leq k-1} V_m T^m$, and put $U_{l+1} = U_l(I_n + X_l)$. Set

$$c_l = \min_{i\leq k-1}\{v_{i,r}(F_l D^{-1} - I_n) - h/(q-1)\}.$$

We now prove by induction that $c_l \geq \frac{l+1}{2}c$, $w_r(F_l D^{-1} - I_n) \geq c + h/(q-1)$ and U_l is invertible over $\mathcal{R}_A^{r,qr}$ for any $l \geq 0$. This is obvious for $l = 0$. Suppose that the claim is true for $l \geq 0$. Then for any $s \in [r, qr]$, since $c_l \geq \frac{l+1}{2}c \geq (q-1)(k-1)$, we have

$$w_s(X_l) \geq (s/r)w_r(X_l) - (s/r-1)(k-1) \geq (s/r)(c_l + h/(q-1)) - (s/r-1)(k-1) > 0.$$

Hence U_{l+1} is also invertible over $\mathcal{R}_A^{[r,qr]}$. Furthermore, we have

$$\begin{aligned}
w_r(DX_l^\phi D^{-1}) &\geq w_r(D) + w_r(X_l^\phi) + w_r(D^{-1}) \\
&= w_{qr}(X_l) - h \\
&\geq q(c_l + h/(q-1)) - h - (q-1)(k-1) \\
&= qc_l + h/(q-1) - (q-1)(k-1) \\
&\geq c_l + \frac{1}{2}c + h/(q-1) + (\frac{1}{2}c - (q-1)(k-1)) \\
&\geq \frac{(l+2)}{2}c + h/(q-1)
\end{aligned}$$

since $c_l \geq c$. Note that

$$\begin{aligned}
F_{l+1}D^{-1} - I_n &= (I_n + X_l)^{-1}F_l D^{-1}(I_n + DX_l^\phi D^{-1}) - I_n \\
&= ((I_n + X_l)^{-1}F_l D^{-1} - I_n) + (I_n + X_l)^{-1}(F_l D^{-1})DX_l^\phi D^{-1}.
\end{aligned}$$

Since $w_r(F_l D^{-1}) \geq 0$ and $w_r((I_n+X_l)^{-1}) \geq 0$, we have $w_r((I_n+X_l)^{-1}(F_l D^{-1})DX_l^\phi D^{-1}) \geq \frac{(l+2)}{2}c + h/(q-1)$. Write

$$\begin{aligned}
(I_n + X_l)^{-1}F_l D^{-1} - I_n &= (I_n + X_l)^{-1}(F_l D^{-1} - I_n - X_l) \\
&= \sum_{j=0}^{\infty}(-X_l)^j(F_l D^{-1} - I_n - X_l).
\end{aligned}$$

For $j \geq 1$, we have

$$w_r((-X_l)^j(F_l D^{-1} - I_n - X_l)) \geq c + c_l + 2h/(q-1) > \tfrac{l+2}{2}c + h/(q-1).$$

By definition of X_l, we also have $v_i(F_lD^{-1} - I_n - X_l) = \infty$ for $i \leq k-1$ and $w_r(F_lD^{-1} - I_n - X_l) \geq c + h/(q-1)$. Putting these together, we get that

$$v_{i,r}(F_{l+1}D^{-1} - I_n) \geq \frac{l+2}{2}c + h/(q-1)$$

for any $i \leq k-1$, i.e. $c_{l+1} \geq \frac{l+2}{2}c$, and that $w_r(F_{l+1}D^{-1} - I_n) \geq c + h/(q-1)$. The induction step is finished.

Now since $w_s(X_l) \geq (s/r)(c_l + h/(q-1))$ for $s \in [r, qr]$, and $c_l \to \infty$ as $l \to \infty$, the sequence U_l converges to a limit U, which is an invertible $n \times n$ matrix over $\mathcal{R}_A^{[r,qr]}$ satisfying $w_r(U^{-1}FU^{\phi}D^{-1} - I_n) \geq c + h/(q-1)$. Furthermore, we have

$$v_{m,r}(U^{-1}FU^{\phi}D^{-1} - I_n) = \lim_{l\to\infty} v_{m,r}(U_l^{-1}FU_l^{\phi}D^{-1} - I_n) = \lim_{l\to\infty} v_{m,r}(F_{l+1}D^{-1} - I_n) = \infty,$$

for any $m \leq k-1$. Therefore $U^{-1}FU^{\phi}D^{-1} - I_n$ has entries in $\pi^k\mathcal{R}_A^{\mathrm{int},r}$. □

The following is an analogue of [23, Lemma 6.2.1].

Lemma 2.3.6. *In the above lemma, suppose that F and D are both invertible over $\mathcal{R}_A^r$, then U is invertible over $\mathcal{R}_A^{qr}$.*

Proof. Set $B = U^{-1}FU^{\phi}D^{-1}$. Then B is invertible over $\mathcal{R}_A^{\mathrm{int},r}$ since $B - I_n$ has entries in $\pi\mathcal{R}_A^{\mathrm{int},r}$ and $w_r(B - I_n) > 0$. In the equation

$$BD = U^{-1}FU^{\phi},$$

we have F, B and D are all invertible over $\mathcal{R}_A^{[r/q,r]}$. We also have that U^{ϕ} is invertible over $\mathcal{R}_A^{[r/q,r]}$ since U is invertible over $\mathcal{R}_A^{[r,qr]}$. So U is also invertible over $\mathcal{R}_A^{[r/q,r]}$. Therefore U is in fact invertible over $\mathcal{R}_A^{[r/q,qr]}$. Repeating this argument, we conclude that U is invertible over $\mathcal{R}_A^{[r/q^i,qr]}$ for all positive integers i, yielding the desired result. □

Proposition 2.3.7. *Let M_A be a family of ϕ-modules over A. Then for any $x \in M(A)$ and model N_x of M_x, there is an affinoid neighborhood $M(B)$ of x and a model $N_{B_{k(x)}}$ of $M_{B_{k(x)}} = M_A \otimes_{\mathcal{R}_A} \mathcal{R}_{B_{k(x)}}$ such that $(N_{B_{k(x)}})_{x'} = N_x$ for any $x' \in M(B_{k(x)})$ above x under the identification of $(M_{B_{k(x)}})_{x'}$ with M_x, and the generic HN-polygons are constant over $M(B)$.*

Proof. By passing to $A_{k(x)}$, we only need to treat the case that x is rational. We choose a basis e_x of N_x, and lift it to a basis e of M_A. Let F be the matrix of ϕ under e; then F_x is the matrix of ϕ under e_x; hence is over $\mathcal{R}_K^{\mathrm{bd}}$. Suppose that F and F_x are over $\mathcal{R}_A^r$ and $\mathcal{R}_K^r$ respectively for some $r > 0$. Put $h = -w_r(F_x) - w_r((F_x)^{-1})$. Let C_{F_x} be the constant for F_x as in Proposition 2.3.2. Pick a positive integer $k > C_{F_x} - w(F_x)$ and a positive number $C \geq 2(q-1)(k-1)$. We take the affinoid neighborhood $M(B)$ of x defined by $w_r(F-F_x) > C - w_r((F_x)^{-1}) + h/(q-1)$. So we have $w_r(F-F_x) > C - w_r((F_x)^{-1}) + h/(q-1)$ as matrices over $\mathcal{R}_B^r$. Hence $w_r(F(F_x)^{-1} - I_n) > C + h/(q-1)$. By Lemma 2.3.5, 2.3.6, there exists an invertible matrix U over $\mathcal{R}_B^{qr}$ such that $U^{-1}FU^{\phi}(F_x)^{-1} - I_n$ has entries in $\pi^k \mathcal{R}_B^{\mathrm{int},r}$. This implies that $w((U^{-1}F_yU^{\phi})_y - F_x) > C_{F_x}$ for any $y \in M(B)$. Then we have $w(U^{-1}F_yU^{\phi} - F_x) \geq k + w(F_x) > C_{F_x}$. So the generic HN-polygon of $U^{-1}F_yU^{\phi}$, or equivalently the generic HN-polygon of N_y, is the same as that of N_x. Furthermore, note that by the construction of U, we have $U_x = I_n$. Therefore the $\mathcal{R}_B^{\mathrm{bd}}$-module N_B generated by eU is a model satisfying all the desired properties. □

Now for any $y \in M(B_{k(x)})$, N_y is a model of M_y. Since the HN-polygon of M_y lies above the generic HN-polygon of N_y and has the same endpoint, if we choose N_x a good model of M_x, then we attain the following theorem.

Theorem 2.3.8. *(Semicontinuity of HN-polygons) Let M_A be a family of ϕ-modules over A. Then for any $x \in M(A)$, there is an affinoid neighborhood $M(B)$ of x such that for any $y \in M(B)$ the HN-polygon of M_y lies above the HN-polygon of M_x and has the same endpoint.*

Corollary 2.3.9. *(Local existence of good models) For M_A a family of ϕ-modules over A, if M_x is pure of slope s for some $x \in M(A)$, then there exists an affinoid neighborhood $M(B)$ of x such that M_y is pure of slope s for any $y \in M(B)$, and M_B has a good model.*

Proof. Using Proposition 2.3.7 with N_x the good model of M_x, we get an affinoid neighborhood $M(B)$ of x and a model N_B of M_B such that for any $M(B)$ the generic HN-polygon of N_y is the same as that N_x. Since the only convex polygon lying above the HN-polygon of M_x is the HN-polygon of M_x itself, we conclude that M_y is pure of slope s for any $y \in M(B)$, and N_B is a good model of M_B. □

Remark 2.3.10. By Definition 2.1.20, we can now talk about HN-polygons for ϕ-modules over Robba rings with non-discrete coefficient fields. So one can think consider the variation of HN-polygons over $\mathcal{M}(A)$, the Berkovich space associated to A, rather than only over classical points $M(A)$. Ideally, we should have the semicontinuity of HN-polygons in this case. However, to follow the strategy we used for classical points, there are two difficulties. Namely, we need to verify the existence of good models and that special HN-polygons lie above generic HN-polygons for ϕ-modules over Robba rings with non-discretely valued coefficient fields. These amount to a generalization of Kedlaya's slope filtration theorem and de Jong's reverse filtration theorem to spherically complete coefficient fields.

Lemma 2.3.11. *Let M_A be a family of ϕ-modules over A such that M_x is pure of slope s with s independent of $x \in M(A)$. Suppose that N_A is a good model of M_A. Then for any $x \in M(A)$, there exists an affinoid neighborhood $M(B)$ of x such that for every generic point y of $M(B)$, M_y is pure of slope s as ϕ-module over $\mathcal{R}_{\mathcal{H}(y)}$, and N_y is the good model of it.*

Proof. As in the proof of Proposition 2.3.7, we suppose that x is rational. Suppose the Frobenius matrix of N_A is F under a basis e. Then F_x is the Frobenius matrix of N_x under the basis e_x the image of e in N_x. Then we can find an affinoid neighborhood $M(B)$ of x such that $w(F_y - F_x) < C_{F_x}$ for any $y \in M(B)$ by Lemma 2.3.3. Then we have $w(F - F_x) < C_{F_x}$ as matrices over $\mathcal{R}_B^{\mathrm{bd}}$, hence as matrices over $\mathcal{R}_{\mathcal{H}(y)}^{\mathrm{bd}}$ for any generic point y of $M(B)$. Then we get that the generic HN-polygon of N_y is the same as the generic HN-polygon of N_x. Since M_x is pure of slope s, we conclude that M_y is also pure of slope s and that N_y is the good model of it. □

Corollary 2.3.12. *(Local uniqueness of good models) Let M_A be a family of ϕ-modules over $M(A)$ such that M_x is pure of the slope s with f independent of $x \in M(A)$. Suppose that N'_A and N''_A are two good models of M_A. Then for any $x \in M(A)$, there exists an affinoid neighborhood $M(B)$ of x such that N'_A and N''_A coincide on $M(B)$.*

Proof. Let U be a transformation matrix between N'_A and N''_A under some basis. By the above lemma, we can find an affinoid neighborhood $M(B)$ of x such that for any generic point y of $M(B)$, M_y is pure of slope s, and that both N'_y and N''_y are good models of it. Therefore $U_{\mathcal{H}(y)}$ are over $\mathcal{R}_{\mathcal{H}(y)}^{\mathrm{bd}}$. Since B can be embedded into the product of $\mathcal{H}(y)$'s for all the generic points y, we conclude that U_B is over $\mathcal{R}_B^{\mathrm{bd}}$. Hence we have $N'_B = N''_B$. □

Let M_A be a family of ϕ-modules over $M(A)$. Suppose that the HN-polygons of M_x are constant on $M(A)$. By a *global filtration* of M_A, we mean a filtration $0 = M_{0,A} \subset M_{1,A} \cdots \subset M_{l,A} = M_A$ of M_A by ϕ-submodules over $\mathcal{R}_A$ such that each $M_{i,A}$ is a direct summand of M_A as an $\mathcal{R}_A$-module, and that $0 = (M_{0,A})_x \subset (M_{1,A})_x \cdots \subset (M_{l,A})_x = M_x$ is the slope filtration of M_x for any $x \in M(B)$.

Corollary 2.3.13. *(Local uniqueness of global filtrations) Let M_A be a family of ϕ-modules over $M(A)$ such that the HN-polygons of M_x's are constant over $M(A)$. Suppose that $0 = M'_{0,A} \subset M'_{1,A} \cdots \subset M'_{l,A} = M_A$ and $0 = M''_{0,A} \subset M''_{1,A} \cdots \subset M''_{l,A} = M_A$ are two global filtrations of M_A. Then for any $x \in M(A)$, there exists an affinoid neighborhood $M(B)$ of x such that these two filtrations coincide on $M(B)$.*

Proof. Let U be a transformation matrix between these two filtrations. Suppose the slopes of M_x are $s_1, \cdots, s_l$. From Lemma 2.2.10, we can find an affinoid neighborhood $M(B)$ of x such that for any generic point y of $M(B)$, $(M'_{i,A})_y/(M'_{i-1,A})_y$ and $(M''_{i,A})_y/(M''_{i-1,A})_y$ are pure of slope s_i for any $1 \leq i \leq l$. Therefore $0 = (M'_{0,A})_y \subset (M'_{1,A})_y \cdots \subset (M'_{l,A})_y = M_y$ and $0 = (M''_{0,A})_y \subset (M''_{1,A})_y \cdots \subset (M''_{l,A})_y = M_y$ are both the slope filtration of M_y. We then have that $U_{\mathcal{H}(y)}$ is upper triangular. Since B can be embedded into the product of $\mathcal{H}(y)$'s for all the generic points y, we conclude that U_B itself is upper triangular. Hence these two filtrations coincide on $M(B)$. □

Conjecture 2.3.14. *(Local existence of global filtrations) Let M_A be a family of ϕ-modules over A such that the HN-polygons of M_x are constant over $M(A)$. Then for any $x \in M(A)$, there exists an affinoid neighborhood $M(B)$ of x such that M_B admits a global filtration.*

2.3.3 An example

In this section, we give an example to show that unlike the case of families of overconvergent ϕ-modules, there is no local constancy property for families of ϕ-modules. This example is actually a family of (ϕ, Γ)-modules. From now on, let K be a finite extension of $\mathbb{Q}_p$ with trivial ϕ, Γ-actions. The ϕ, Γ-actions on $\mathcal{R}_K$ are given by $\phi(T) = (1+T)^p - 1$, $\gamma(T) = (1+T)^{\chi(\gamma)} - 1$.

Lemma 2.3.15. *For $f \in \mathcal{R}_K$, if $\phi(f) = \lambda f$ for some $\lambda \in K$, then we must have $\lambda = p^i$ and $f = ct^i$ for some $i \in \mathbb{N}$ and $c \in K$.*

Proof. Write $f = \sum_{k=-\infty}^{+\infty} a_k T^k$. If f is not in $\mathcal{R}_K^+$, let k_0 be the maximal $k < 0$ such that $a_k \neq 0$. Note that $\phi(T^{-1}) = 1/((1+T)^p - 1) = T^{-p}(1 + pT^{-1} + \cdots + pT^{1-p})^{-1}$. Hence pk_0 is the maximal $k < 0$ such that $\phi(f)$ has nonzero coefficient of T^k. Since $\phi(f) = \lambda f$, we conclude that $pk_0 = k_0$; this yields a contradiction. Therefore we have $f \in \mathcal{R}_K^+$ and the lemma follows from [14, Lemma 1.1] □

Lemma 2.3.16. *Let M be a (ϕ, Γ)-module over $\mathcal{R}_K$. Suppose that M satisfies the short exact sequence*

$$0 \longrightarrow \mathcal{R}_K(1) \xrightarrow{\alpha} M \xrightarrow{\beta} \mathcal{R}_K(-1) \longrightarrow 0$$

of (ϕ, Γ)-modules. Then M is étale if and only if the exact sequence is non-split.

Proof. The 'only if' part is obvious. For the 'if' part, we first have $\deg(M) = (-1) + 1 = 0$. If M is not étale, then it has a rank 1 (ϕ, Γ)-submodule N such that $\deg(N) < 0$. We may suppose N is of the form $\mathcal{R}_K(\delta)$ by the classification of rank 1 (ϕ, Γ)-modules [14]. Since $\mathcal{R}_K(1)$ is a saturated ϕ-submodule of positive slope of M, we have that $N \cap \mathcal{R}_K(1) = 0$ by Corollary 1.14. This implies that N maps into a ϕ-submodule of $\mathcal{R}_K(-1)$. Now let v and v' be the canonical generators of N and $\mathcal{R}_K(-1)$ respectively. Suppose that $\beta(v) = fv$ for some $f \in \mathcal{R}_K$. Then we have

$$p^{-1}\phi(f)v' = \phi(fv') = \phi(\beta(v)) = \beta(\phi(v)) = \beta(\delta(p)v) = \delta(p)fv'.$$

Hence $\phi(f) = p\delta(p)f$. Then by the above lemma, we have that $p\delta(p)$ must be a nonnegative power of p. Since $v_K(\delta(p)) < 0$, we conclude that $\delta(p) = p^{-1}$. Therefore we have $\phi(f) = f$. This implies that f is a scalar. So β induces an isomorphism between N and $\mathcal{R}_K(-1)$. Hence the exact sequence is split. □

Example 2.3.17. Let A be the 1-dimensional Tate algebra $K\langle x\rangle$ with trivial ϕ-action. By [14, Theorem 3.9], we have $\dim_K H^1(\mathcal{R}_K(2)) = 1$. Now suppose that (a, b) represents a nonzero element of $H^1(\mathcal{R}_K(2))$. Let M_A be a family of rank 2 (φ, Γ)-modules over A such that the ϕ, γ-actions are given by the matrices $\left(\begin{smallmatrix} p & xa \\ 0 & p^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & pxb \\ 0 & 1 \end{smallmatrix}\right)$ respectively. Then it is clear that at each closed point $x = c \in \overline{K}$, M_x is an extension of $\mathcal{R}_{k(x)}(-1)$ by $\mathcal{R}_{k(x)}(1)$, which is represented by the cocycle (pca, pcb) in $H^1(\mathcal{R}_{k(x)}(1) \otimes \mathcal{R}_{k(x)}(-1)^\vee) = H^1(\mathcal{R}_{k(x)}(2))$. Since (a, b) is nonzero, (pca, pcb) vanishes if and only if $c = 0$, i.e. M_x is the trivial extension if and only $x = 0$. So from the above lemma, we conclude that M_x is étale if and only if x

is not the origin. Hence we do not have an affinoid neighborhood of the origin over which HN-polygons are constant.

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