Today’s topics:
- Relationship between stress and strain
- Equation of motion
- Wave Equation

Review from Last Lecture
The stress tensor is symmetric and linearly related to traction T:

(1) $\sigma = \sigma_{ij} = \sigma_{ji} = T_i^{(j)}$
(2) $T_i = \sigma_{ijn_j}$

For all $T_i$, there are 3 corresponding stress elements on the i-surface working along all three axes:

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, p. 40, Blackwell Publishing.
The strain tensor $\varepsilon$ is also symmetric and can be written as the gradient of the displacement:

\[
\varepsilon = \varepsilon_{ij} = \varepsilon_i = \frac{1}{2} (\delta u_i/\delta u_j + \delta u_j/\delta u_i)
\]

The trace of the strain tensor is the relative change in volume, or the **cubic dilatation**:

\[
\text{Tr}(\varepsilon) = \delta u_i/\delta x_j = \nabla \cdot u = \Delta V/V
\]

**Relating Stress and Strain**

To relate stress and strain, we are going to assume a medium with perfect linear elasticity.

Aside: If we had a viscous medium, we could relate the stress and strain rate using viscosity $\mu$:

\[
\sigma \sim \mu \dot{\varepsilon},
\]

where the strain rate $\dot{\varepsilon}$ is given by

\[
\dot{\varepsilon} = \delta e/\delta t
\]

In one dimension, stress and strain can be related using **Young’s Modulus** $E$ is a proportionality constant:

\[
\sigma = E \varepsilon,
\]

where Young’s Modulus is equal to the ratio between $\sigma_x$ and $E_x$:

\[
E = \sigma_x/e_x
\]

e.g. Take a rubber band and apply a stress in the $\sigma_x$-direction to get $e_x$:

From $\sigma_x$ and $e_x$, $E$ can be found.

Note, however, that $e_y \neq 0$ even though $\sigma_y=0$; therefore, this simple scalar relationship does not hold in 2-D or 3-D. Instead, we need to be able to relate elements of the stress tensor to (combinations of) elements of the strain tensor. Because the stress tensor and the strain tensor are both 2nd-rank, we will need a 4th-rank tensor to do so, which gives us the generalized **Hooke’s Law**,

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl},
\]

also called the constitutive relationship between stress and strain, where $C_{ijkl}$ is known as the **stiffness tensor** or the **elasticity tensor**. Note that the generalized Hooke’s Law assumes perfect linear elasticity, so there are no effects from attenuation, etc.
More on $C_{ijkl}$:
This 4$^{th}$-rank tensor makes things complicated because it has 81 elements, so we must use symmetry to simplify $C_{ijkl}$. Because of symmetry in $\sigma$,
\begin{equation}
C_{ijkl} = C_{jikl}
\end{equation}
Because of symmetry in $E$,
\begin{equation}
C_{ijkl} = C_{ijlk} = C_{jilk}
\end{equation}
Therefore, we have reduced $C_{ijkl}$ to 36 independent elements. Another way to look at it is that there are 6 independent elements of $\sigma$ as well as 6 independent elements of $E$, giving 36 independent elements to $C_{ijkl}$. Furthermore, one can also demonstrate using the idea of strain energy that
\begin{equation}
C_{ijkl} = C_{klij},
\end{equation}
further reducing the tensor to 21 independent elements.

Unfortunately, displacement can be observed in at most 3 directions, meaning that only 3 observations are available to determine 21 unknowns. We need to make further assumptions to reduce the number of unknowns. Let’s assume that we are working in an isotropic medium, i.e. the medium has the same physical properties in all directions.

To reiterate, our assumptions about the medium are now:
- perfect linear elasticity
- isotropy

Isotropy allows $C_{ijkl}$ to be expressed in only two independent terms, giving
\begin{equation}
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\end{equation}
where $\lambda$ and $\mu$ are Lamé parameters (named for Gabriel Lamé, a 19$^{th}$ century French mathematician) and $\delta_{ij}$, $\delta_{kl}$, etc. are Kronecker symbols such that
\begin{equation}
\delta_{ij} = 0 \text{ when } i \neq j
\end{equation}
\begin{equation}
\delta_{ij} = 1 \text{ when } i = j
\end{equation}
As a consequence of the linear behavior of stress and strain, $\lambda$ and $\mu$ are not dependent on strain. $\mu$ refers to the rigidity, or shear modulus, of the medium. $\mu$ measures the resistance against shear, i.e. an easily sheared material has a smaller $\mu$ than one that is difficult to shear. For a fluid, $\mu = 0$.

$\lambda$ does not have a specific physical characteristic to make it intuitively understandable. It is defined with respect to the shear modulus using the bulk modulus, $K$, such that
\begin{equation}
K = \lambda + 2\mu/3
\end{equation}
The bulk modulus measures a material’s incompressibility, a relative change in volume (cubic dilatation) $\Delta$ due to change in pressure $P$, such that
$K = -\delta P/\delta \Delta$
A material that is hard to compress or has a smaller relative volume will give a higher bulk modulus than a material that is easy to compress or has a larger relative volume.

A note on dimensions:
Because strain is dimensionless, $C_{ijkl}$ must have units of stress. Thus, $\lambda$ and $\mu$ are given in Pascal.
Using equation (13), Young’s modulus can be written in terms of Lamé’s parameters for an isotropic medium:

\[
E = \frac{\sigma_{xx}}{E_{xx}} = (3\lambda + 2\mu)\mu / (\lambda + \mu)
\]

Additionally, we often see Poisson’s ratio ν, where

\[
\nu = \lambda / 2(\lambda + \mu)
\]

Poisson’s ratio is often used to characterize the elastic properties of a medium, for example, for a fluid with μ=0, ν=0.5. As the rigidity of a material increases to infinity, Poisson’s ratio approaches 0.

A Poisson’s medium is an isotropic material with Lamé parameters such that λ = μ, giving ν=0.25. This value of Poisson’s ratio is reasonable for many crustal and mantle rocks, so it is often assumed in calculations. In the inner core, ν = 0.4, suggesting that the inner core is more “ mushy” and sponge-like than the mantle, but still maintains some rigidity.

When considering P- and S-waves in the crust and mantle, we can assume a Poisson’s medium in order to relate them, such that

\[
V_p \approx \sqrt{3} V_s
\]

Now, expanding \(\sigma_{ij}\) for an isotropic medium with perfect linear elasticity, we get

\[
\sigma_{ij} = C_{ijkl}e_{kl} = \begin{bmatrix}
\lambda\Delta + 2\mu e_{11} & 2\mu e_{12} & 2\mu e_{13} \\
2\mu e_{12} & \lambda\Delta + 2\mu e_{22} & 2\mu e_{23} \\
2\mu e_{13} & 2\mu e_{23} & \lambda\Delta + 2\mu e_{33}
\end{bmatrix}
\]

Notice that the off-diagonals are pure shear stresses, and they are only dependent on \(\mu\). The diagonals refer to the normal stress and depend on both \(\mu\), \(\lambda\), and \(\Delta\) (i.e., change in volume).

Aside:
Generic anisotropy takes us back to 21 unknowns in \(C_{ijkl}\), so we have to make assumptions about the symmetry of the medium. For instance, in an olivine-rich medium, such as the mantle, the olivine crystals tend to align themselves in a constant direction. Thus, seismic waves will propagate faster along the crystal alignment than in other directions, creating anisotropy and necessitating 5 independent elements in \(C_{ijkl}\) (hexagonal symmetry; transverse isotropy).

**Equation of Motion**

Let us revisit the stress tetrahedron. We have a traction \(T\) that can be broken up into 3 components, \((T_1, T_2, T_3)\). We also know, using Newton’s 2nd Law of Motion and a force balance on the tetrahedron, that

\[
\sum F = T_i\delta S - (\sigma_{i1}n_1\delta S + \sigma_{i2}n_2\delta S + \sigma_{i3}n_3\delta S) + f_d\delta V = ma = \rho \left( \frac{\delta^2 u}{\delta t^2} \right) dV
\]

where \(f_d\delta V\) represents the body forces on the tetrahedron. Ignoring the body forces and assuming \(a=0\), this gives us

\[
T_i = \sigma_{ijn}i,
\]

which is true for pure equilibrium.
Now, consider an accelerating seismic wave. The equation of motion will be
\[ (T_i - \sigma_{ij}n_j)\delta S + f_i dV = \rho \left( \delta^2 u_i / \delta t^2 \right) dV \]
If the traction cancel, i.e., \( T_i - \sigma_{ij}n_j \), equation (22) would give
\[ f_i = \rho \left( \delta^2 u_i / \delta t^2 \right), \] that is acceleration would only be due to the body forces; but if there is a net change in stress, \( (T_i - \sigma_{ij}n_j) \) can be considered as the non-lithostatic (deviatoric) stress, \( \sigma_{ij}' \). Therefore,
\[ \sigma_{ij}'u_j \delta S + f_i dV = \rho \left( \delta^2 u_i / \delta t^2 \right) dV \]
From now on, \( \sigma_{ij} \) will be used to refer to the deviatoric stress, \( \sigma_{ij}' \).

Now, to develop equation (25) further we want to get rid of either \( \delta S \) or \( dV \). We will use Gauss’ Divergence Theorem to transform a surface integral into a volume integral. Gauss’ theorem uses flux to relate volume to surface area.

Consider a field \( \mathbf{a} \) with a flux through a surface with area \( \delta S \):

\[ \delta S = \delta \mathbf{S} \]

The total flux of the field in and out of the surface is given by
\[ \int \mathbf{a} \cdot d\mathbf{S} = \int \mathbf{f} \cdot d\mathbf{S} \]
which is related to the amount of field generated or absorbed in the volume within \( dS \). In other words
\[ \int \mathbf{f} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{a} \ dV, \]
where \( \nabla \cdot \mathbf{a} \) is the source (or sink) of the field. This is Gauss’ Divergence Theorem. If \( \nabla \cdot \mathbf{a} = 0 \), the field is source/sink-free.

So, taking equation (25) and integrating both sides, we get
\[ \int \sigma_{ij}'u_j \delta S + \int f_i dV = \int \rho \left( \delta^2 u_i / \delta t^2 \right) dV \]
Now we apply Gauss’ Theorem and combine terms to get
\[ \int (\delta \sigma_{ij} / \delta x_j + f_i) dV = \int \rho \left( \delta^2 u_i / \delta t^2 \right) dV, \]
which leads to
\[ \rho \left( \delta^2 u_i / \delta t^2 \right) = \delta \sigma_{ij} / \delta x_j + f_i = \sigma_{ij,j} + f_i \] (Stein and Wysession not.) = \( \delta \sigma_{ij} + f_i \) (van der Hilst not.)

In vector form, we get the equation of motion,
\[ \rho \ddot{\mathbf{u}} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma} \]
Equation (30 or 31) is known as either Navier’s or Cauchy’s Equation of Motion, as they independently worked on developing it in the 19th century.
Aside:
If the medium is stationary, \( f_i = \delta_j \sigma_{ij} \), where \( f_i \) describes the body forces such as gravity and electromagnetic forces. Later, we'll use \( f \) as a seismic source, e.g. a point source such that
\[
(32) \quad f_i = A \delta(x-x_0) \delta(t-t_0) \delta n_i,
\]
where \( A \) is the source magnitude, \( x_0 \) is the source position, \( t_0 \) is the origin time, and \( n_i \) is the direction. Solving equations (30) for \( f_i \) as in (32) gives a Green's Function \( \sim G(x,x_0; t,t_0) \), and the displacement field to a more complex source can then be obtained by summing (integrating) over different Green's functions. We will get back to this later.

### Wave Equation

We will look at the homogeneous equation of motion, i.e. \( f = 0 \). Combining equation (31) with Hooke's Law, equation (9), we get
\[
(33) \quad \rho (\delta^2 u / \delta t^2) = \delta_j \{ C_{ijkl} e_{kl} \}
\]
Since we are still in an isotropic medium, \( \delta_j \{ C_{ijkl} e_{kl} \} \) becomes \( \delta_j \{ \lambda(...) + \mu(...) + \ldots \} \), giving gradients in \( \lambda \) and \( \mu \) which need to be dealt with. Obviously, we can simplify these expressions by assuming a homogeneous medium so that \( \lambda \) and \( \mu \) are not changing with position \( x \), thus,
\[
(34) \quad \delta \lambda / \delta x = 0; \quad \delta \mu / \delta x = 0
\]
Therefore, in an isotropic, homogeneous, perfect linear elastic medium,
\[
(35) \quad \delta_j \{ C_{ijkl} e_{kl} \} = C_{ijkl} \delta e_{kl} / \delta x_j
\]

Aside:
Realistically, the earth is not homogeneous. To get around this, we can divide the subsurface into piecewise homogeneous regions, e.g. layers:

We can also look at ray theory to show that the spatial derivative of the parameters of the medium are related to \( 1 / \omega \), where \( \omega \) is the angular frequency:
\[
(36) \quad \delta \lambda / \delta x (\text{or} \delta \mu / \delta x) \sim 1 / \omega \quad \text{so that for infinite frequency} \quad \delta \lambda / \delta x (\text{or} \delta \mu / \delta x) \sim 0
\]
From equations (33) and (35), we can show that the 2nd derivative in time is balanced by the 2nd derivative in space, so that it begins to look like a wave equation:

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \mu \nabla^2 u_i \]

Dividing both sides by \( \rho \), we obtain the equation of motion under assumptions of a homogeneous, isotropic, linear perfect elastic medium, with no body forces, and undergoing infinitesimal strain:

\[ \frac{\partial^2 u_i}{\partial t^2} = \left[ \frac{(\lambda + 2\mu)}{\rho} \right] \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \left( \frac{\mu}{\rho} \right) \nabla^2 u_i \]

or

\[ \ddot{u} = \left[ \frac{(\lambda + 2\mu)}{\rho} \right] \nabla (\nabla \cdot u) + \left( \frac{\mu}{\rho} \right) \nabla^2 u \]

Note that the general form of the wave equation is \( \frac{\partial^2 \circ}{\partial t^2} = c^2 \nabla^2 \circ \) with \( c \) the propagation speed of disturbance \( \circ \), so that we can see that the terms \( (\mu/\rho) \) and \( (\lambda + 2\mu)/\rho \) are related to wave speed.

Some useful vector identities:
- The divergence of a curl is zero:
  \[ \nabla \cdot (\nabla \times u) = 0 \]
- The curl of a gradient is zero:
  \[ \nabla \times (\nabla \Phi) = 0 \]
- The Laplacian of a vector \( u \) is given by
  \[ \nabla^2 u = \nabla (\nabla \cdot u) - \nabla \times (\nabla \times u) \]
  (NB \( \nabla (\nabla \cdot u) = \nabla^2 u \) only if \( u \) is a conservative, rotation free field).

Combining equations (38) and (41), we get

\[ \ddot{u} = \left[ \frac{(\lambda + 2\mu)}{\rho} \right] \nabla (\nabla \cdot u) - \left( \frac{\mu}{\rho} \right) \nabla \times (\nabla \times u) \]

Some remarks on equation (42):
Remember that in an isotropic medium such that
\[ \sigma_{ij} = \lambda \delta_{ij} \Delta + 2\mu \varepsilon_{ij}, \]
The diagonals of \( \sigma_{ij} \) have terms in both \( \Delta \) and \( \mu \). The off-diagonals are only in terms of \( \mu \). The same thing occurs with \( \mu (\rho) [\nabla \times (\nabla \times u)] \).

Also, \( \nabla \cdot u = \Delta = e_{kk} \), which shows that in a general elastic medium an accelerating system has a volume change (e.g. P-wave) component as well as a rotation (curl) component with no volume change.

Equation (43) can be separated into a pure volume change component \( (\lambda \delta_{ij} \Delta) \) and a pure shear component \( (2\mu \varepsilon_{ij}) \). However, \( \lambda \) depends on both \( \kappa \) and \( \lambda \); that is, a pure volume change cannot occur without a change in shear because volume change is nonzero only when \( i=j \), and when \( i=j \), there still exists a shear component in equation (43). Thus, there must always be slip somewhere to account for a volume change in the system.

Notes: Lori Eich (Feb. 14, 2005)