ANISOTROPY

So far we have assumed isotropy i.e. wavespeeds do not depend on the direction of wave propagation. We solved the wave equation assuming plane waves:

\[
\eta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]

\[
\sin i = \sin j
\]

\[
\alpha / \beta \rightarrow i / j
\]

\[
V_p = \alpha
\]

\[
V_{sH} = V_{SV} = \beta
\]

Anisotropy however, cannot be ignored as it is the focus of increasing research in seismology.

In an isotropic medium wavefronts are concentric circles with radius dependent on the velocity (Vp or Vs) as shown in Fig. 2a. The raypaths are perp. to wavefronts and the slowness vector is perp. to the wavefront. The energy goes with group velocity and the group velocity is perp. to wavefronts.

In anisotropic media the wavefronts are distorted (Fig. 2b). The raypaths are not perp. to the wavefronts, therefore the direction of the group velocity is not the same as the direction of the phase velocity.

In a homogeneous media there are still three solutions but now they are called quasi P (q P), quasi SH (q SH) and quasi Sv (q Sv).
Now we look at some basic theory to get insights into the problem and concepts. A full treatment is beyond the scope of this class.

We have used the Generalized Hooke’s Law:

\[
\tau_{ij} = c_{ijkl} \varepsilon_{kl}
\]

\[
\tau_{ij} = \tau_{ji} \Rightarrow c_{ijkl} = c_{jikl}
\]

\[
\varepsilon_{kl} = \varepsilon_{lk} \Rightarrow c_{ijkl} = c_{jikl}
\]

And from thermodynamics

\[
c_{ijkl} = c_{klij}
\]

These relationships reduce the number of independent elements from 81 to 21 elements.

In the isotropic case:

\[
c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{jk})
\]

Where only the two independent parameters are the Lamé constants, \(\lambda\) and \(\mu\).

\[
\begin{align*}
    c_{1111} &= c_{2222} = c_{3333} = \lambda + 2\mu \\
    c_{1122} &= c_{1133} = c_{2233} = \lambda \\
    c_{1212} &= c_{1313} = c_{2323} = \mu \\
    \text{Else} &= 0
\end{align*}
\]

An anisotropic medium is a more complex system and uses symmetries such as:

- Orthorhombic- (e.g. olivine) with 9 elements
- Hexagonal- with 5 elements
- Cubic- (e.g. MgO) with 3 elements

Anisotropy is not due to individual minerals but the whole medium or lattice. Lattice preferred orientation (LPO) is the deformation of olivine by plate motion.

Hexagonal symmetry is very useful for seismology where there is rotational symmetry around a symmetry axis (not necessarily vertical but often is).

![Figure 3](image)

For example a layered system with a vertical symmetry axis which is the most useful for Earth (Fig. 3). In the transverse direction (i.e. direction perpendicular to the symmetry axis) you have isotropy. This is also known as Vertical Transverse Isotropy (VTI).
The advantages are:
1. q SH and q Sv can still be treated separately
2. the velocities vary only with incidence angle and not with azimuth

If the symmetry axis is horizontal you have azimuthal anisotropy or Horizontal Transverse Isotropy (HTI).

For mathematical convenience we introduce a new notation convention:
\( c_{ijkl} \rightarrow C_{IJ} \)

\[
\begin{array}{c}
11 \rightarrow 1 \\
22 \rightarrow 2 \\
33 \rightarrow 3 \\
23 \rightarrow 4 \\
13 \rightarrow 5 \\
12 \rightarrow 6 \\
\end{array}
\]
e.g. \( c_{1122} = C_{12} \)

So for isotropic media:
\[
C_{IJ} = \begin{pmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix}
\]

The Love convention is used in global seismology (principle axis along \( x_3 \)):
\[
\begin{align*}
C_{11} &= A \\
C_{33} &= C \\
c_{1133} &= c_{13} = F \\
c_{2323} &= c_{1313} = C_{55} = L \\
c_{1212} &= C_{66} = N \\
c_{1122} &= C_{12} = C_{21} = A - 2N
\end{align*}
\]

\[
C_{IJ} = \begin{pmatrix}
A & A - 2N & F & 0 & 0 & 0 \\
A - 2N & A & F & 0 & 0 & 0 \\
F & F & C & 0 & 0 & 0 \\
0 & 0 & 0 & L & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 \\
0 & 0 & 0 & 0 & 0 & L
\end{pmatrix}
\]
The exploration industry convention is similar but used Thomsen parameters $\varepsilon, \gamma, \delta$

Looking at wave propagation:

$$\tau_{ij} = c_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2} \left( \partial_i u_j + \partial_j u_i \right)$$

$$\tau_{ij} = c_{ijkl} \partial_j u_k$$

The Equation of Motion for a homogeneous medium:

$$\rho \ddot{u}_i = (\nabla \cdot \sigma)_i = \partial_j \sigma_{ij} = \partial_j (c_{ijkl} \partial_l u_k) = c_{ijkl} \partial_j \partial_l u_k$$

Where $i$ is the only free index.

The displacement:

$$u = g e^{i\omega(t - \mathbf{x})}$$

Where:

$g$=$polarization$ vector (in the direction of particle motion)

$s$=$slowness$ vector or $1/phase$ velocity (direction is $\frac{1}{c} k$)

$x$=$position$ vector

$s$ is perpendicular to the wavefront since $s \cdot x$ is unchanged for $dx$ perpendicular to $s$.

If we substitute the displacement into the equation of motion:

$$\rho g_i = g_i c_{ijkl} s_j s_l$$

or

$$\left(c_{ijkl} s_j s_l - \rho \delta_{ik} \right) g_k = 0$$

Where $i$ is the only free parameter.

Now, we introduce the density- normalized elastic tensor:

$$\frac{c_{ijkl}}{\rho} = \Gamma_{ijkl}$$

And re-write the Equation of Motion as:

$$\left(\Gamma_{ijkl} \hat{s}_j \hat{s}_l - c^2 \delta_{ik} \right) g_k = 0$$

The Christoffel Matrix:

$$M_{ik} = \Gamma_{ijkl} \hat{s}_j \hat{s}_l$$
The Christoffel Equation:

\[
(M_{ik} - c^2 \delta_{ik}) g_k = 0
\]

Where \( c \) = phase velocity and \( \hat{s} = \frac{s}{|s|} = \frac{1}{c} \).

The Christoffel Equation is simply an eigenvalue equation. For each direction of \( s \) there are three solutions that correspond to the three eigenvalues, \( c_1, c_2, c_3 \), and corresponding eigenvectors, \( g = (g_1, g_2, g_3) \), of the Christoffel matrix.

Now we will look at 2 specific cases.

**CASE 1**

A plane wave in a VTI medium where propagation is in the direction of the symmetry axis. The \( x_3 \) direction is the axis of symmetry and propagation direction.

The displacement is:

\[
u_i = A_i \sin \left\{ \omega \left( \frac{x_3}{c} - t \right) \right\}
\]

Christoffel Matrix:

\[
M_{ik} = \frac{1}{\rho} \begin{pmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & C \end{pmatrix}
\]

Christoffel Equation:

\[
\begin{bmatrix}
\frac{L}{\rho} - c^2 \\
0 & \frac{L}{\rho} - c^2 \\
0 & \frac{C}{\rho} - c^2
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
ge_3
\end{bmatrix} = 0
\]

The three solutions are:

\[
c_1 = \sqrt{\frac{L}{\rho}},
\quad
\frac{L}{\rho},
\quad
\frac{C}{\rho}
\]

\[
g_1 = (1,0,0)
\]

\[
g_2 = (0,1,0)
\]

\[
g_3 = (0,0,1)
\]
Therefore the velocity in the \( x_3 \) direction is different from \( x_1 \) and \( x_2 \) (which are the same).

\[
\text{(In the isotropic case } c_1 = \sqrt{\frac{\mu}{\rho}}, c_2 = \sqrt{\frac{\mu}{\rho}}, c_3 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \text{)}
\]

The solutions are degenerate eigenvalues and there is no shear wave splitting.

Therefore remember that if you observe no shear wave splitting the cause is either:
  1. Propagation along symmetric axis
  2. No anisotropy
You cannot tell the difference between the two.

**CASE 2**

Propagation in direction \( x_1 \) and axis of symmetry in \( x_3 \) direction.

\[
u_i = B_i \left\{ \omega \left( \frac{x_1}{c} - t \right) \right\}
\]

Christoffel Matrix:

\[
M_{ij} = \frac{1}{\rho} \begin{pmatrix} A & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & L \end{pmatrix}
\]

Christoffel Equation:

\[
\begin{bmatrix}
\frac{A}{\rho} - c^2 & 0 & 0 \\
0 & \frac{N}{\rho} - c^2 & 0 \\
0 & 0 & \frac{L}{\rho} - c^2
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
g_3
\end{bmatrix} = 0
\]

The solutions are:

\[
c_1 = \sqrt{\frac{A}{\rho}}, c_2 = \sqrt{\frac{N}{\rho}}, c_3 = \sqrt{\frac{L}{\rho}}
\]

\[
g_1 = (1,0,0)
g_2 = (0,1,0)
g_3 = (0,0,1)
\]
The wavespeeds in all three directions are different therefore **there is shear-wave splitting** (quasi P, SH, and SV) where three polarization directions will propagate at three different speeds.

**quasi P** \[ \alpha' = \frac{A}{\sqrt{\rho}} \quad (\alpha = \frac{\sqrt{\lambda + 2\mu}}{\rho}, \lambda_{1111} = \lambda + 2\mu = C_{11}A) \]

The quasi P wave is the fastest wave, polarized parallel to the direction of propagation \((x_1)\).

**First quasi S** \[ \beta' = \frac{N}{\sqrt{\rho}} \quad (C_{66} = c_{1212} = N = \mu, \beta = \sqrt{\frac{\mu}{\rho}}) \]

If \(N>L\) the first quasi S wave is associated with the \(x_2\) direction.

**Second quasi S** \[ \beta'' = \frac{L}{\sqrt{\rho}} \quad (C_{44} = c_{2323} = \lambda = L, c_3 = \sqrt{\frac{\lambda}{\rho}}) \]

The second quasi S is the slowest wave and polarized parallel to the \(x_3\) direction.