Introduction to Seismology: Lecture Notes
22 April 2005

SEISMOLOGY: NORMAL MODES (FREE OSCILLATIONS)

DECOMPOSING SEISMOLOGY INTO SUBDISCIPLINES

Seismology can be decomposed into three representative subdisciplines: body waves, surface waves, and normal modes of free oscillation. Technically, these domains form a continuum, each pertaining to particular frequency bands, spatial scales, etc. In all cases, these representations satisfy the wave equation, but each is subject to different boundary conditions and simplifying assumptions. Each is therefore relevant to particular types of subsurface investigation. Below is a table summarizing the salient characteristics of the three.

Seismic Domains	Type		Application	Data	Boundary Conditions
Body Waves	P-SV	SH	High frequency	travel times; waveforms	unbounded
Surface Waves	Rayleigh	Love	Lithosphere	dispersion; group c(w) & phase u(w) velocities	interfaces
Normal Modes	Spheroidal Modes	Toroidal Modes	Global	power spectra	spherical earth

As the table suggests, the normal modes provide a framework for representing global seismic waves. Typically, these modes of free oscillation are of extremely low frequency and are therefore difficult to observe in seismograms. Only the most energetic earthquakes are capable of generating free oscillations that are readily apparent on most seismograms, and then only if the seismograms extend over several days.

NORMAL MODES

To understand normal modes, which describe the modes of free oscillation of a sphere, it's instructive to consider the 1D analog of a vibrating string fixed at both ends as shown in panel Figure 1b. This is useful because the 3D case (Figure 1c), similar to the 1D case, requires that

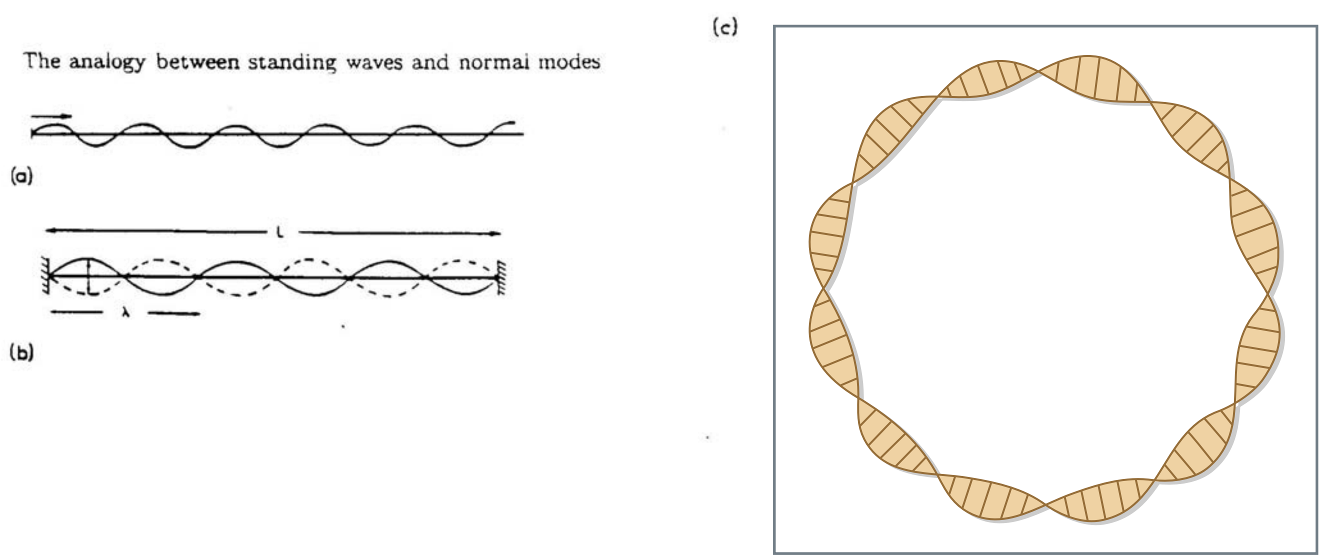

Figure by MIT OCW.

standing waves 'wrap around' and meet at a null point. The string obeys the 1D wave equation with fixed-end BCs, the general expression and solution to which are:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\therefore u(x,t) = Ae^{i\omega(t+x/c)} + Be^{i\omega(t-x/c)} + Ce^{-i\omega(t+x/c)} + De^{-i\omega(t-x/c)} \tag{1}$$

The BCs require that $u(0,t) = u(L,t) = 0$, which implies that A = -B and C = -D. Hence:

$$2i\left(Ae^{i\omega t} - Ce^{-i\omega t}\right)\sin\left(\frac{\omega L}{c}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\omega L}{c}\right) = 0 \tag{2}$$

$$\Leftrightarrow \frac{\omega L}{c} = (n+1)\pi \quad ,n = 0,1,2,\ldots.\infty$$

So there are infinitely many discrete frequencies, ω_n, that satisfy (1), and these are called eigenfrequencies.

Figure 2 depicts several modes or eigenfrequencies that satisfy (2). n=0 corresponds to the fundamental mode and all n≥1 correspond to higher modes (overtones).

> Aside: We have already seen ω-k plots for surface and body waves and have learned how to interpret and manipulate them. Normal modes are also frequently graphically depicted using ω-l plots, where ω has the normal meaning and where l is the characteristic length or *angular order*. But note that $l=2\pi R/\lambda$, and recall that $k=2\pi/\lambda$; so the angular order is like a wave number!

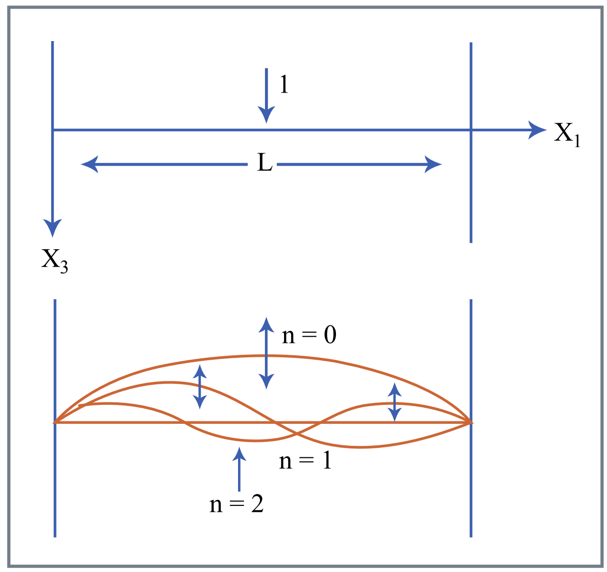

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Figure 2

Normal Modes in the Fourier Domain

The normal modes, just like the harmonic solutions we saw for the 1D case in (1) and (2), can be thought of as basis functions spanning the set of all possible waves we expect to encounter in a spherical body. Therefore, we can employ weighted *mode summation* to reconstruct or represent waveforms occurring in a spherical body. Heuristically, u can be represented by

$$u(x,t) = \sum A_n \text{modes}(\omega_n) \tag{3}$$

and, more precisely

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{i\omega_n t} + B_n e^{-i\omega_n t} \tag{4a}$$

Stein and Wysession report (4a) in the equivalent form

$$u(x,t) = \sum_{n=0}^{\infty} \underbrace{A_n}_{weight} \underbrace{U_n(x,\omega_n)\cos(\omega_n t)}_{basis\ function} \tag{4b}$$

In seismology, it is customary to express (4a) by

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n e^{i\omega_n t} \tag{4c}$$

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Aside: Earthquake magnitudes are initially estimated based upon the energy observed in the first-arriving body waves; magnitudes are then sometimes adjusted based upon the energy observed in the later-arriving surface waves; a final revision is also sometimes made once the energy in the long-wavelength normal modes is determined. For example, the Sumatra earthquake of 26 December 2004 was initially ascribed M = 8.0 when the body waves arrived. It was then adjusted to 9.0 when the surface waves were analyzed, and it was finally assessed a 9.3 when the normal modes were analyzed.

Equations 4a-c indicate that the *power spectrum* of $u(x,t)$ could be used to reveal the normal modes. Those discrete frequencies at which the Earth freely oscillates when it is excited by an earthquake will be evident in its normal mode power spectrum (see Figure 3).

Figure 3 clearly indicates that the Earth possesses more than one normal mode. Thinking of this as a forward modeling problem, we might begin to attempt to predict the Earth's normal modes by assuming the Earth is homogeneous. In that case, there would only be one peak apparent in the power spectrum of a seismogram. We could then refine our Earth model, adding layers at various depths and these would produce additional spectral peaks. Ultimately, we might apply PREM or some other Earth reference model to establish a reference power spectrum against which we could compare real data.

Synthetic Seismograms

Recall that we have already established a method for creating synthetic seismograms for body waves, using the following expression:

$$u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(k_x, k_y, \omega, z) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} dk_x dk_y d\omega\,, \tag{5}$$

This triple infinite integral is computationally prohibitive to calculate, and a number of simplifications have been introduced to circumvent these difficulties. Such techniques include the so-called WKBJ method, the Reflectivity method, and the Cagniard-de Hoop method, none of which will be developed here. The point is that simplifications are imperative. For plane waves, two useful simplifications are:

1. Integrate over a finite frequency band, $\omega_0 - \delta\omega < \omega < \omega_0 + \delta\omega$. This is useful if the phase of interest can be isolated within a practical frequency band.
2. Integrate over a finite 'wave number band', $k_0 - \delta k < k < k_0 + \delta k$. This is possible because arrivals at a particular station are only incident over a finite sub-range of all possible directions. The wave vector indicates the direction in which a plane wave travels, so we can reduce the range of integration to something finite, since only particular directions can physically arrive at a station for a given event.

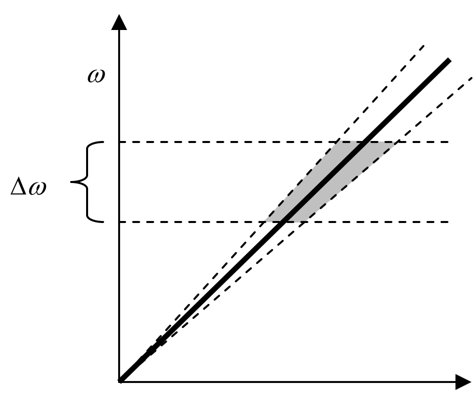

Figure 4

The same simplifications are applicable for normal modes. Whereas the preceding simplifications are applicable for body waves in the ω-k domain, with normal modes we work in the in the ω-l domain (recalling that l is the angular order – or basically a normal mode wave number). Figure 4 provides an explanatory cartoon of this in the ω-l domain. The frequency band of interest is $\Delta\omega$, which of course would correspond to waves having only particular frequency content. The waves might only arrive at a particular station in certain directions, allowing us to limit the range of integration in l as well. The shaded area therefore depicts the actual range over which modes must be summed to approximate waves having desired properties. Note that $c = \omega/k$ (body/surface waves) and $c = \omega/l$ (normal modes), so straight lines in the ω-l domain still correspond to constant wavespeeds.

And, just as in the ω-k domain, we can identify 'phases' in the ω-l domain (see Figure 5), which correspond to the average wavespeeds of rays interacting with various features at depth.

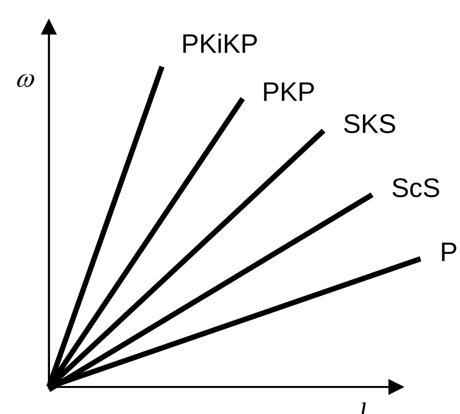

Figure 5

Normal Mode Nomenclature

The wave equation, subject to spherical boundary conditions, gives rise to the so-called *spherical harmonics*:

$$\ddot{u} = c^2 \nabla^2 u \xrightarrow[\text{BCs}]{\text{spherical}} \text{spherical harmonics}.$$

For example, the gravitational potential can be expressed in spherical harmonics by:

$$U_{\text{grav}} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{1}{r}\right)^{l} \underbrace{\left\{A_n^m \cos m\varphi + B_n^m \sin m\varphi\right\}}_{\text{Longitude-dependent}} \underbrace{P_l^m(\cos\theta)}_{\substack{\text{Legendre Polynomial} \\ \text{Latitude-dependent}}} . \tag{6}$$

Equation (6) displays a 2*l*+1 *degeneracy*. That is, for each *l* there exist 2*l*+1 modes (solutions). For example, for *l* = 0 there is only one mode; for *l* = 1, there are three modes corresponding to A_1^0, A_1^1, and B_1^1. *n* indicates the number of nodes along the radius of the Earth (also called the *overtone number*), and *l* is the angular order, which indicates the number of nodal planes on the surface (see Figures 6 and 7).

Spheroidal modes (~P-SV; changes in volume) are denoted by ${}_nS_l$ and are sensitive to compressional and shear wavespeed as well as density. Toroidal modes (~SH; rotation or shear; no change in volume) are denoted by ${}_nT_l$ and are sensitive only to shear wavespeed. There are more spheroidal than toroidal modes.

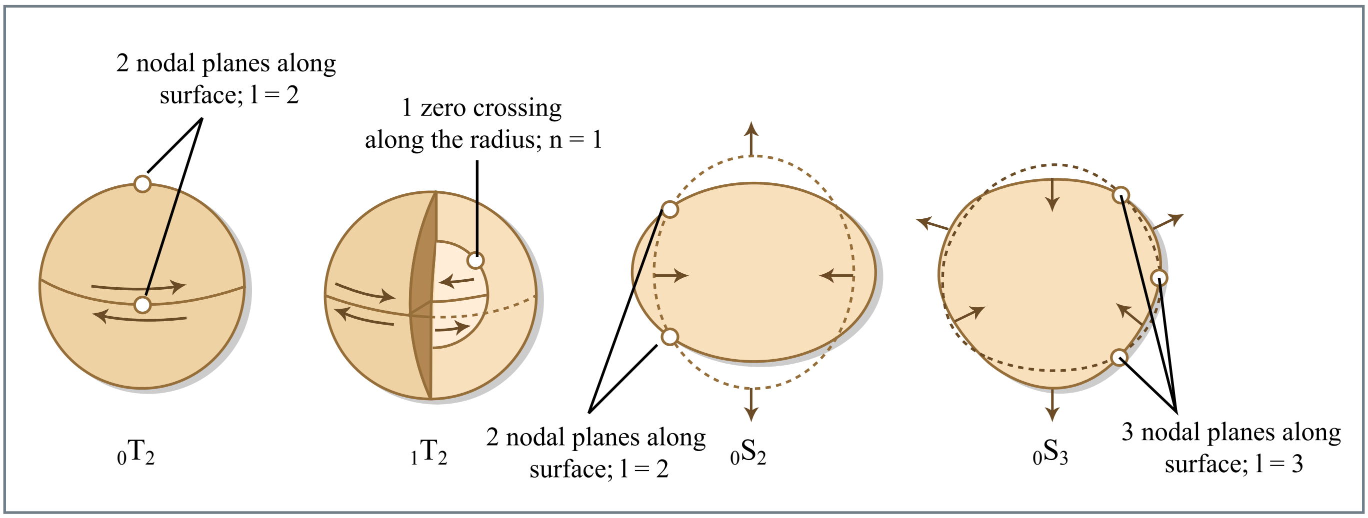

Figure by MIT OCW.

Figure 6

There are a number of modes that have been given special names. One such mode is ${}_0S_0$ (see Figure 7, bottom left), which is called the *breathing mode* because the entire spherical volume periodically expands and contracts. Another is ${}_0S_2$ (see Figure 7, top left), which is called the *football mode* because the extrema of this free oscillation are shaped like an American football (also because a European football displays this oscillation when it is kicked). Two modes that do not exist naturally are the ${}_0S_1$ and ${}_0T_1$ modes. The ${}_0S_1$ spheroidal mode cannot exist because it requires the displacement of the center of gravity, which cannot happen. The ${}_0T_1$ toroidal mode cannot exist because it requires the entire sphere to 'twist' back and forth, which contradicts the conservation of angular momentum for a rotating sphere.

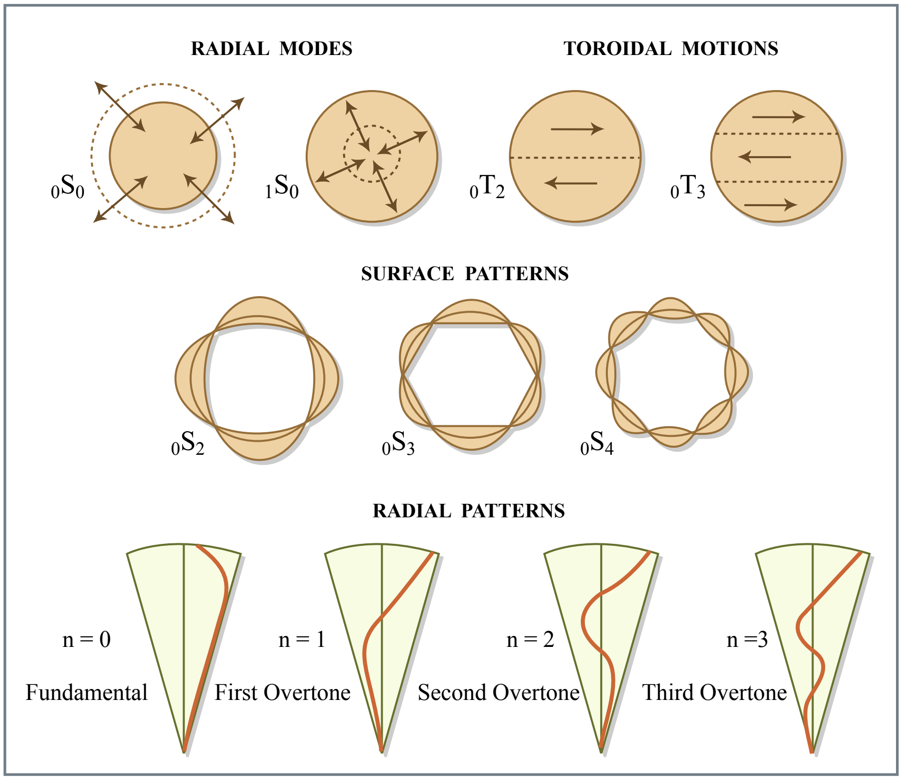

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Figure 7

3D examples of several of spherical harmonics are provided in Figure 9. Each spherical harmonic or normal mode of the Earth can be treated as a basis function. Any wave on Earth can be represented as a weighted sum of normal modes or spherical harmonics, as heuristically expressed in (4a). Therefore, if we take the normal mode power spectrum of a seismogram (as in Figure 3) we shall see spectral peaks corresponding to the frequency of these modes. But, because of the $2l+1$ degeneracy described above, there are multiple modes of free oscillation for each l. Each of these modes should oscillate at the same frequency. However, because of the Earth's rotation, the $2l+1$ modes will not be observed to be oscillating at precisely the same frequency. Hence, while we should observe a sharp spike in the power spectrum for each l (this is called a *singlet*), we often observed a broadened or 'smeared' spike around the expected frequency of the normal mode (this is called a *multiplet*). These ideas are illustrated in Figure 8.

In fact, a multiplet can sometimes become so spread out that the power spectrum for a particular mode appears multimodal. This is aptly termed *normal mode splitting*. It is currently hypothesized that normal mode splitting is the effect of anisotropy, not rotational frequency modulation.

Figure 8

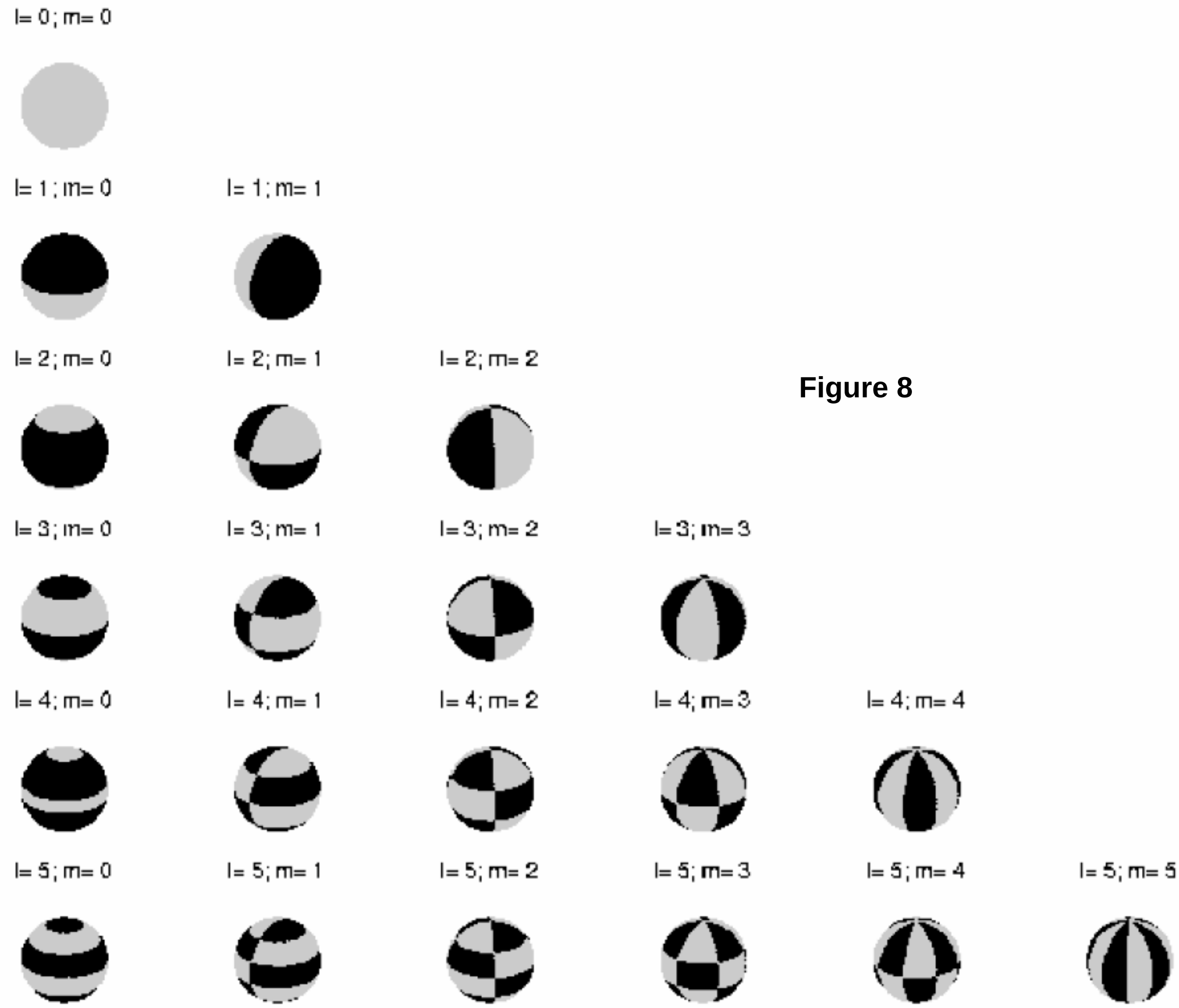

Figure 2.14: Some spherical harmonics.

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