12.540 Principles of the Global Positioning System
Lecture 16

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Propagation: Ionospheric delay

- Summary
  - Quick review/introduction to propagating waves
  - Effects of low density plasma
  - Additional effects
  - Treatment of ionospheric delay in GPS processing
  - Examples of some results

Microwave signal propagation

- Maxwell's Equations describe the propagation of electromagnetic waves (e.g. Jackson, Classical Electrodynamics, Wiley, pp. 848, 1975)

$$\nabla \bullet \mathbf{D} = 4\pi\rho \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \bullet \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

Maxwell's equations

- In Maxwell's equations:
  - **E** = Electric field; ρ=charge density; **J**=current density
  - **D** = Electric displacement **D**=**E**+4π**P** where **P** is electric polarization from dipole moments of molecules.
  - Assuming induced polarization is parallel to E then we obtain **D**=ε**E**, where ε is the dielectric constant of the medium
  - **B**=magnetic flux density (magnetic induction)
  - **H**=magnetic field;**B**=μ**H**; μ is the magnetic permeability

Maxwell's equations

- General solution to equations is difficult because a propagating field induces currents in conducting materials which effect the propagating field.
- Simplest solutions are for non-conducting media with constant permeability and susceptibility and absence of sources.

Maxwell's equations in infinite medium

- With the before mentioned assumptions Maxwell's equations become:

$$\nabla \bullet \mathbf{E} = 0 \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \bullet \mathbf{B} = 0 \quad \nabla \times \mathbf{B} - \frac{\mu\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = 0$$

- Each cartesian component of **E** and **B** satisfy the wave equation

Wave equation

- Denoting one component by u we have:

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad v = \frac{c}{\sqrt{\mu\varepsilon}}$$

- The solution to the wave equation is:

$$u = e^{i\mathbf{k}.\mathbf{x} - i\omega t} \qquad |\mathbf{k}| = \frac{\omega}{v} = \sqrt{\mu\varepsilon}\frac{\omega}{c} \quad \text{wave vector}$$

$$\mathbf{E} = \mathbf{E_0} e^{i\mathbf{k}.\mathbf{x} - i\omega t} \qquad \mathbf{B} = \sqrt{\mu\varepsilon}\frac{\mathbf{k} \times \mathbf{E}}{|\mathbf{k}|}$$

Simplified propagation in ionosphere

- For low density plasma, we have free electrons that do not interact with each other.
- The equation of motion of one electron in the presence of a harmonic electric field is given by:

$$m\left[\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}\right] = -e\mathbf{E}(\mathbf{x},t)$$

- Where m and e are mass and charge of electron and γ is a damping force. Magnetic forces are neglected.

Simplified model of ionosphere

- The dipole moment contributed by one electron is **p**=-e**x**
- If the electrons can be considered free (ω_0=0) then the dielectric constant becomes (with f_0 as fraction of free electrons):

$$\varepsilon(\omega) = \varepsilon_0 + i\frac{4\pi N f_0 e^2}{m\omega(\gamma_0 - i\omega)}$$

High frequency limit (GPS case)

- When the EM wave has a high frequency, the dielectric constant can be written as for NZ electrons per unit volume:

$$e(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi NZe^2}{m} \Rightarrow \text{plasma frequency}$$

- For the ionosphere, NZ=10^4-10^6 electrons/cm^3 and ω_p is 6-60 of MHz
- The wave-number is

$$k = \sqrt{\omega^2 - \omega_p^2}\,/c$$

Effects of magnetic field

- The original equations of motion of the electron neglected the magnetic field. We can include it by modifying the F=Ma equation to:

$$m\ddot{\mathbf{x}} - \frac{e}{c}\mathbf{B_0} \times \dot{\mathbf{x}} = -e\mathbf{E}e^{-i\omega t} \quad \text{for } \mathbf{B_0} \text{ transverse to propagation}$$

$$x = \frac{e}{m\omega(\omega \mp \omega_B)}\mathbf{E} \quad \text{for } \mathbf{E} = (\mathbf{e_1} \pm i\mathbf{e_2})E$$

$$\omega_B = \frac{e|B_0|}{mc} \quad \text{precession frequency}$$

Effects of magnetic field

- For relatively high frequencies; the previous equations are valid for the component of the magnetic field parallel to the magnetic field
- Notice that left and right circular polarizations propagate differently: birefringent
- Basis for Faraday rotation of plane polarized waves

Refractive indices

- Results so far have shown behavior of single frequency waves.
- For wave packet (ie., multiple frequencies), different frequencies will propagate a different velocities: Dispersive medium
- If the dispersion is small, then the packet maintains its shape by propagates with a velocity given by $d\omega/dk$ as opposed to individual frequencies that propagate with velocity ω/k

Group and Phase velocity

- The phase and group velocities are

$$v_p = c / \sqrt{\mu\varepsilon} \quad v_g = \frac{1}{\frac{d}{d\omega}\left(\sqrt{\mu\varepsilon(\omega)}\right)\frac{\omega}{c} + \sqrt{\mu\varepsilon(\omega)}/c}$$

- If ε is not dependent on ω, then $v_p=v_g$
- For the ionosphere, we have $\varepsilon<1$ and therefore vp>c. Approximately $v_p=c-\Delta v$ and $v_g=c+\Delta v$ and Δv depends of ω^2

Dual Frequency Ionospheric correction

- The frequency squared dependence of the phase and group velocities is the basis of the dual frequency ionospheric delay correction

$$R_1 = R_c + I_1 \qquad R_2 = R_c + I_1 (f_1 / f_2)^2$$
$$\phi_1 \lambda_1 = R_c - I_1 \qquad \phi_2 \lambda_2 = R_c - I_1 (f_1 / f_2)^2$$

- R_c is the ionospheric-corrected range and I_1 is ionospheric delay at the L1 frequency

Linear combinations

- From the previous equations, we have for range, two observations (R_1 and R_2) and two unknowns R_c and I_1

$$I_1 = (R_1 - R_2)/(1 - (f_1/f_2)^2)$$

$$R_c = \frac{(f_1/f_2)^2 R_1 - R_2}{(f_1/f_2)^2 - 1} \qquad (f_1/f_2)^2 \approx 1.647$$

- Notice that the closer the frequencies, the larger the factor is in the denominator of the R_c equation. For GPS frequencies, $R_c = 2.546R_1 - 1.546R_2$

Approximations

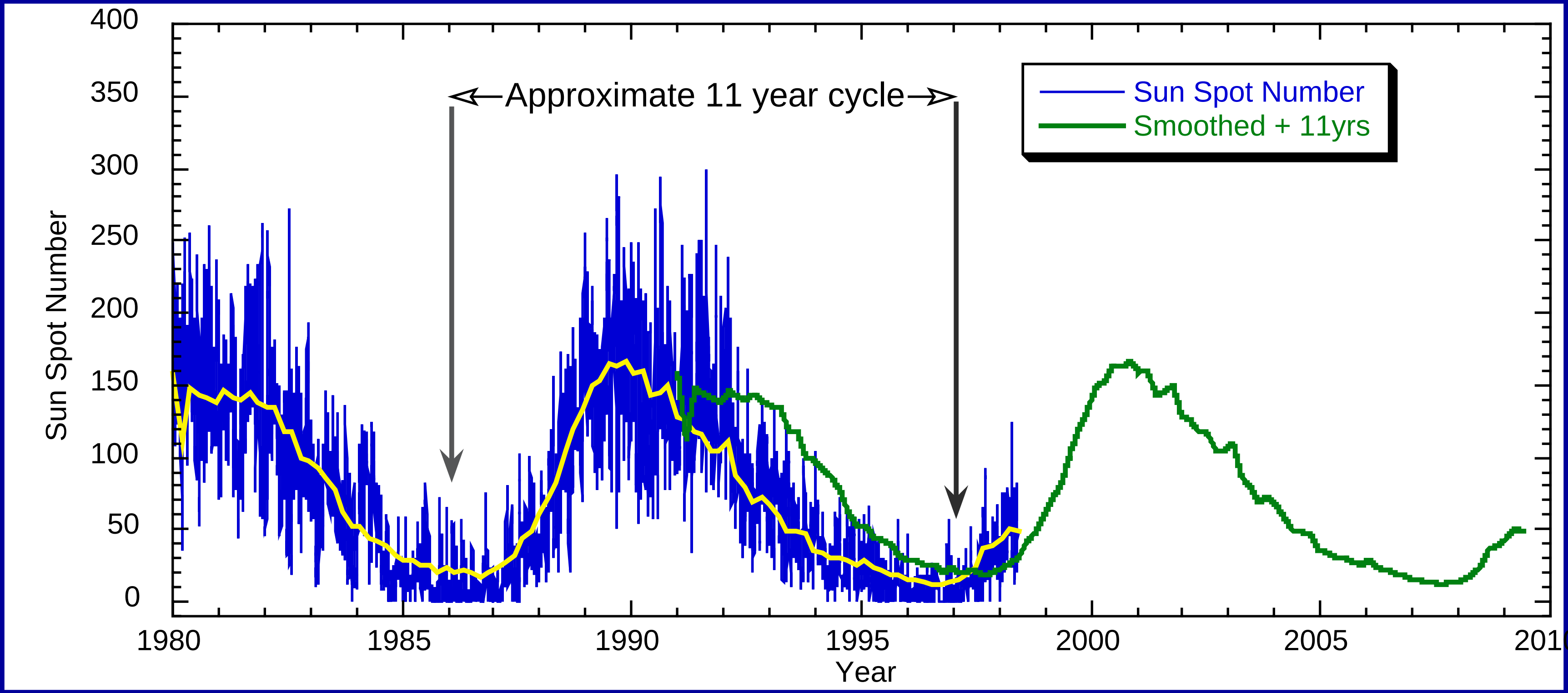
- If you derive the dual-frequency expressions there are lots of approximations that could effect results for different (lower) frequencies
  - Series expansions of square root of ε (f^4 dependence)
  - Neglect of magnetic field (f^3). Largest error for GPS could reach several centimeters in extreme cases.
  - Effects of difference paths traveled by f_1 and f_2. Depends on structure of plasma, probably f^4 dependence.

Magnitudes

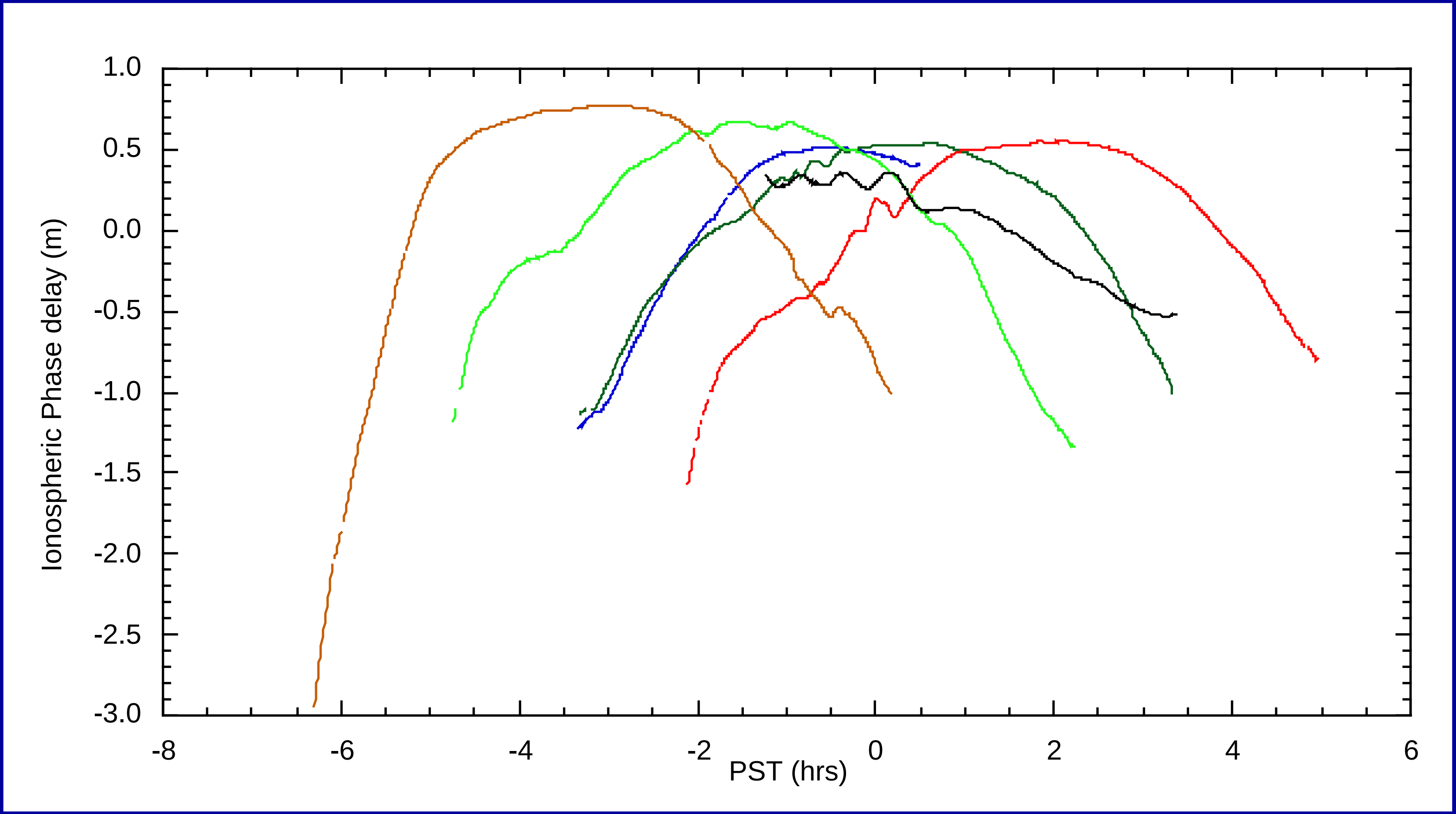
- The factors 2.546 and 1.546 which multiple the L1 and L2 range measurements, mean that the noise in the ionospheric free linear combination is large than for L1 and L2 separately.
- If the range noise at L1 and L2 is the same, then the R_c range noise is 3-times larger.
- For GPS receivers separated by small distances, the differential position estimates may be worse when dual frequency processing is done.
- As a rough rule of thumb; the ionospheric delay is 1-10 parts per million (ie. 1-10 mm over 1 km)

Variations in ionosphere

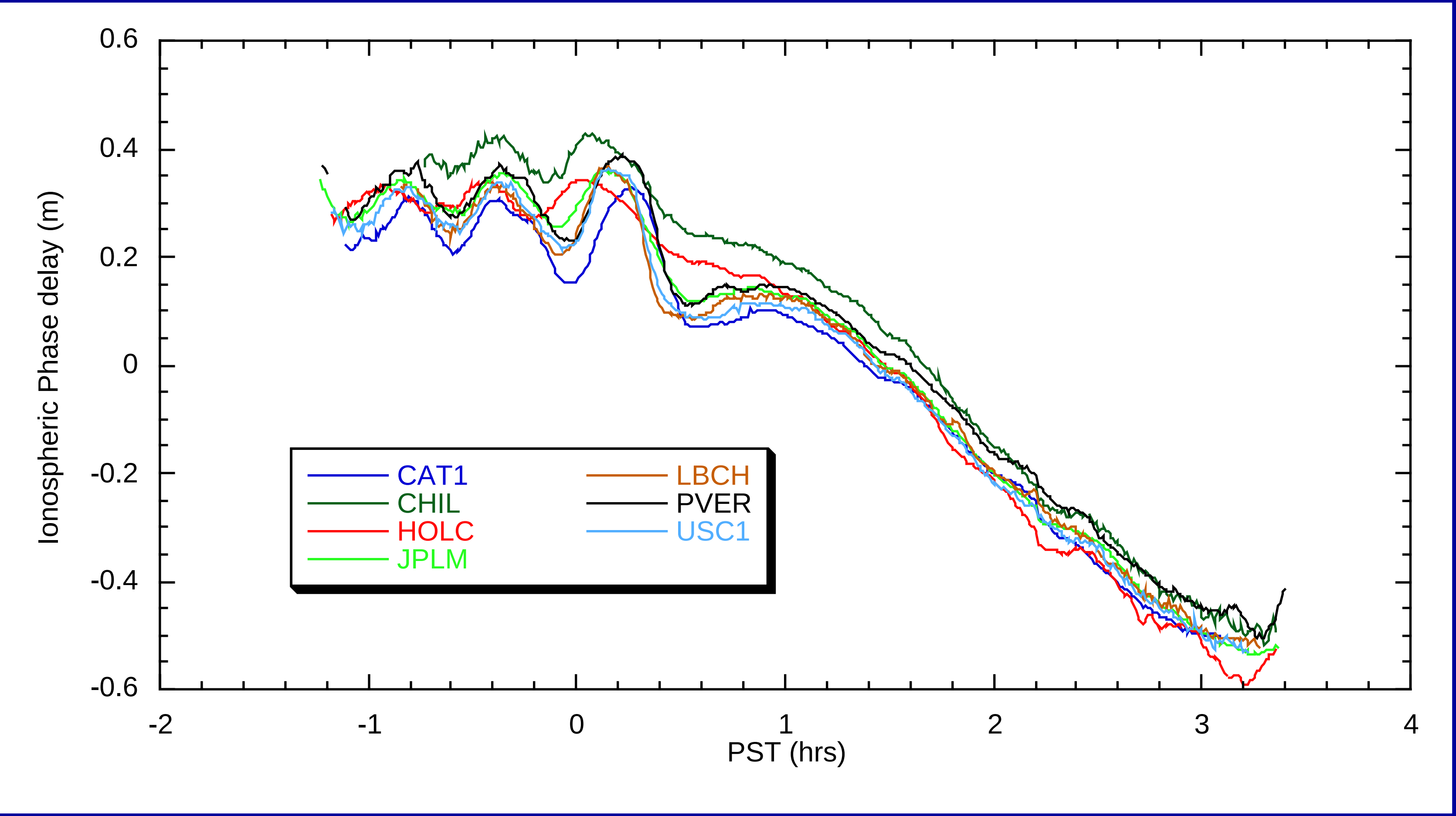
- 11-year Solar cycle



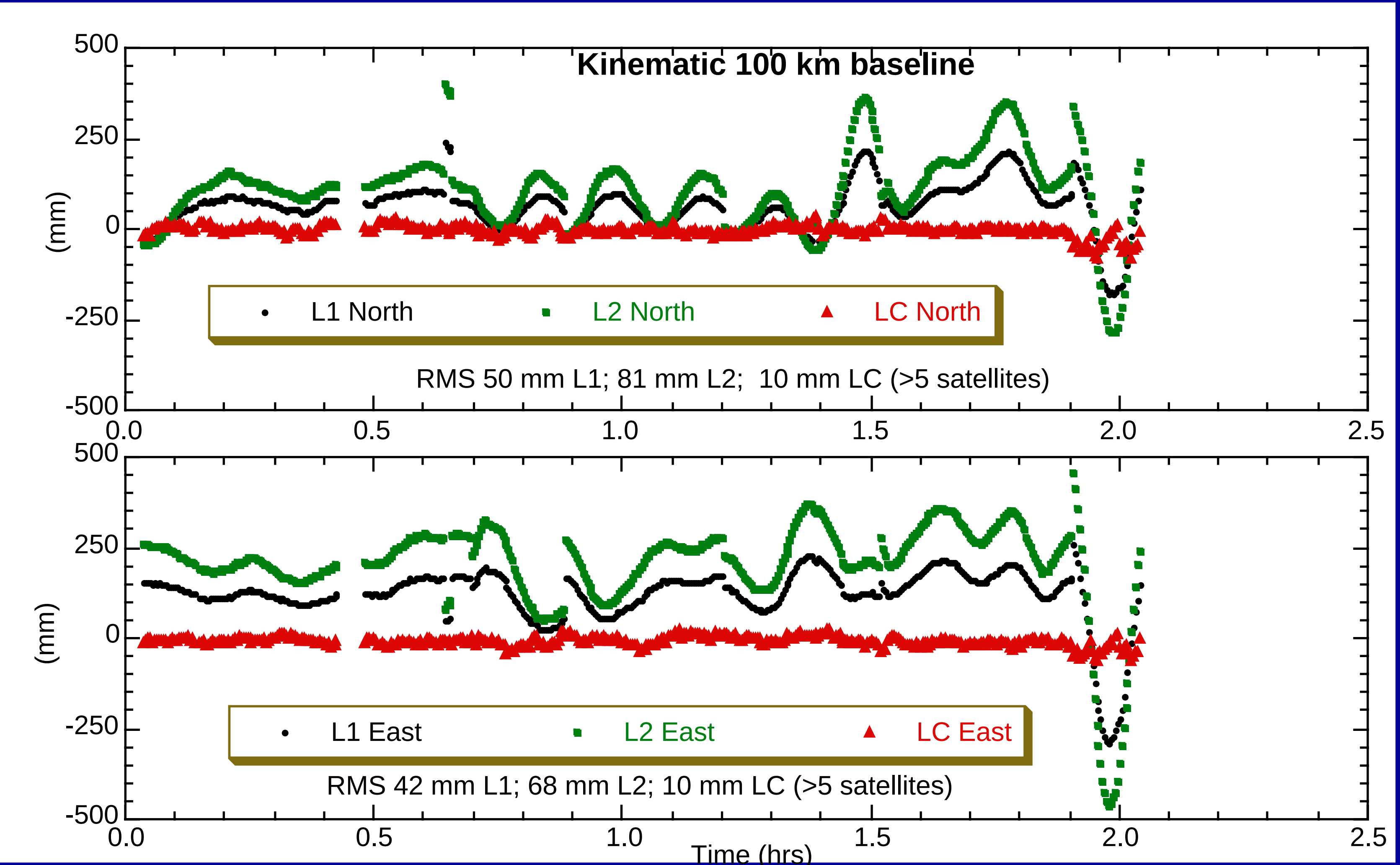
Example of JPL in California



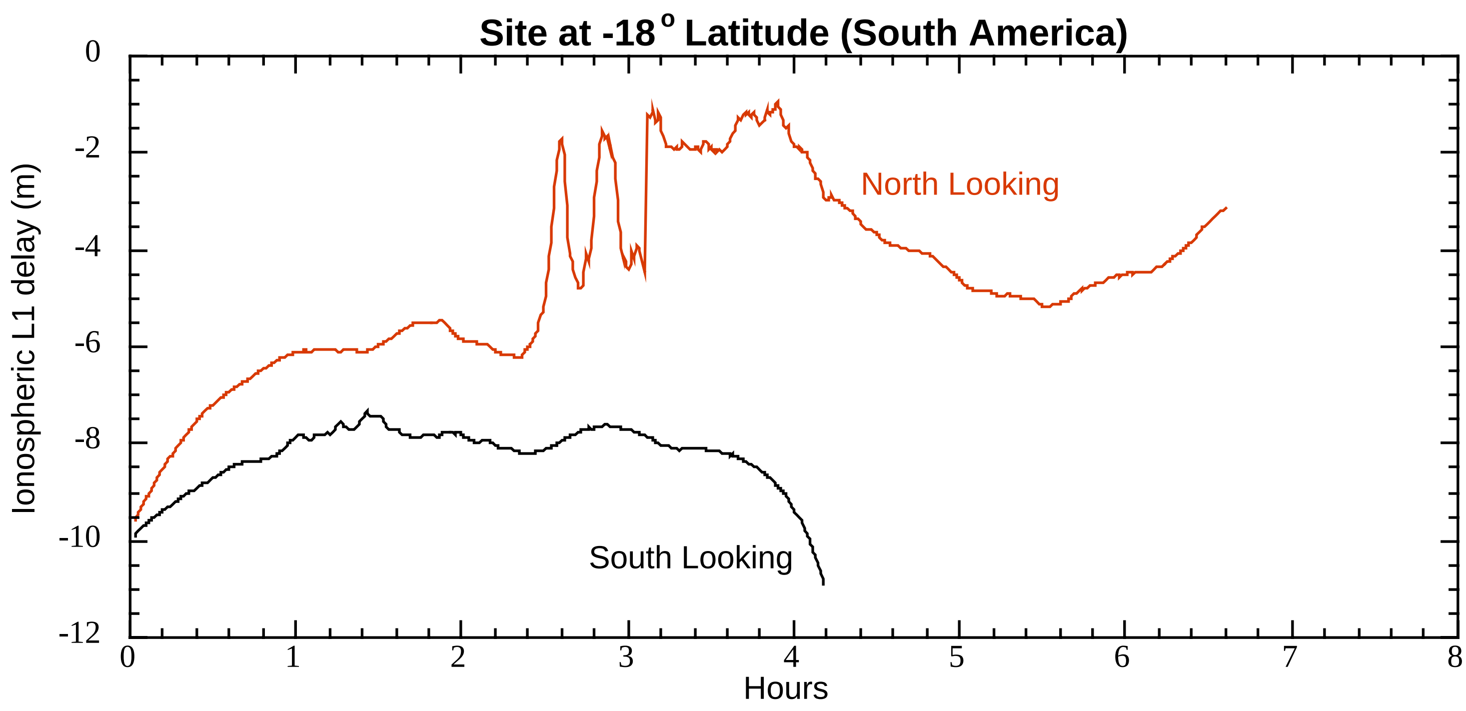
PRN03 seen across Southern California



Effects on position (New York)



Equatorial Electrojet (South America)



Summary

- Effects of ionospheric delay are large on GPS (10’s of meters in point positioning); 1-10ppm for differential positioning
- Largely eliminated with a dual frequency correction at the expense of additional noise (and multipath)
- Residual errors due to neglected terms are small but can reach a few centimeters when ionospheric delay is large.