12.540 Principles of the Global Positioning System Lecture 03

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Review

- In last lecture we looked at conventional methods of measuring coordinates
- Triangulation, trilateration, and leveling
- Astronomic measurements using external bodies
- Gravity field enters in these determinations

Gravitational potential

- In spherical coordinates: need to solve

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\lambda^2}=0$$

- This is Laplace’s equation in spherical coordinates

Solution to gravity potential

- The homogeneous form of this equation is a “classic” partial differential equation.
- In spherical coordinates solved by separation of variables, r=radius, λ=longitude and θ=co-latitude

$$V(r,\theta,\lambda) = R(r)g(\theta)h(\lambda)$$

Solution in spherical coordinates

- The radial dependence of form r^n or r^{-n} depending on whether inside or outside body. N is an integer
- Longitude dependence is $\sin(m\lambda)$ and $\cos(m\lambda)$ where m is an integer
- The colatitude dependence is more difficult to solve

Colatitude dependence

- Solution for colatitude function generates Legendre polynomials and associated functions.
- The polynomials occur when m=0 in λ dependence. $t=\cos(\theta)$

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

Legendre Functions

$$P_o(t) = 1$$

$$P_1(t) = t$$

$$P_2(t) = \frac{1}{2}(3t^2 - 1)$$

$$P_3(t) = \frac{1}{2}(5t^3 - 3t)$$

$$P_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3)$$

- Low order functions. Arbitrary n values are generated by recursive algorithms

Associated Legendre Functions

- The associated functions satisfy the following equation

$$P_{nm}(t) = (-1)^m (1-t^2)^{m/2} \frac{d^m}{dt^m} P_n(t)$$

- The formula for the polynomials, Rodriques' formula, can be substituted

Associated functions

$$P_{00}(t) = 1$$
$$P_{10}(t) = t$$
$$P_{11}(t) = -(1-t^2)^{1/2}$$
$$P_{20}(t) = \frac{1}{2}(3t^2-1)$$
$$P_{21}(t) = -3t(1-t^2)^{1/2}$$
$$P_{22}(t) = 3(1-t^2)$$

- Pnm(t): n is called degree; m is order
- m<=n. In some areas, m can be negative. In gravity formulations m=>0

http://mathworld.wolfram.com/LegendrePolynomial.html

Ortogonality conditions

- The Legendre polynomials and functions are orthogonal:

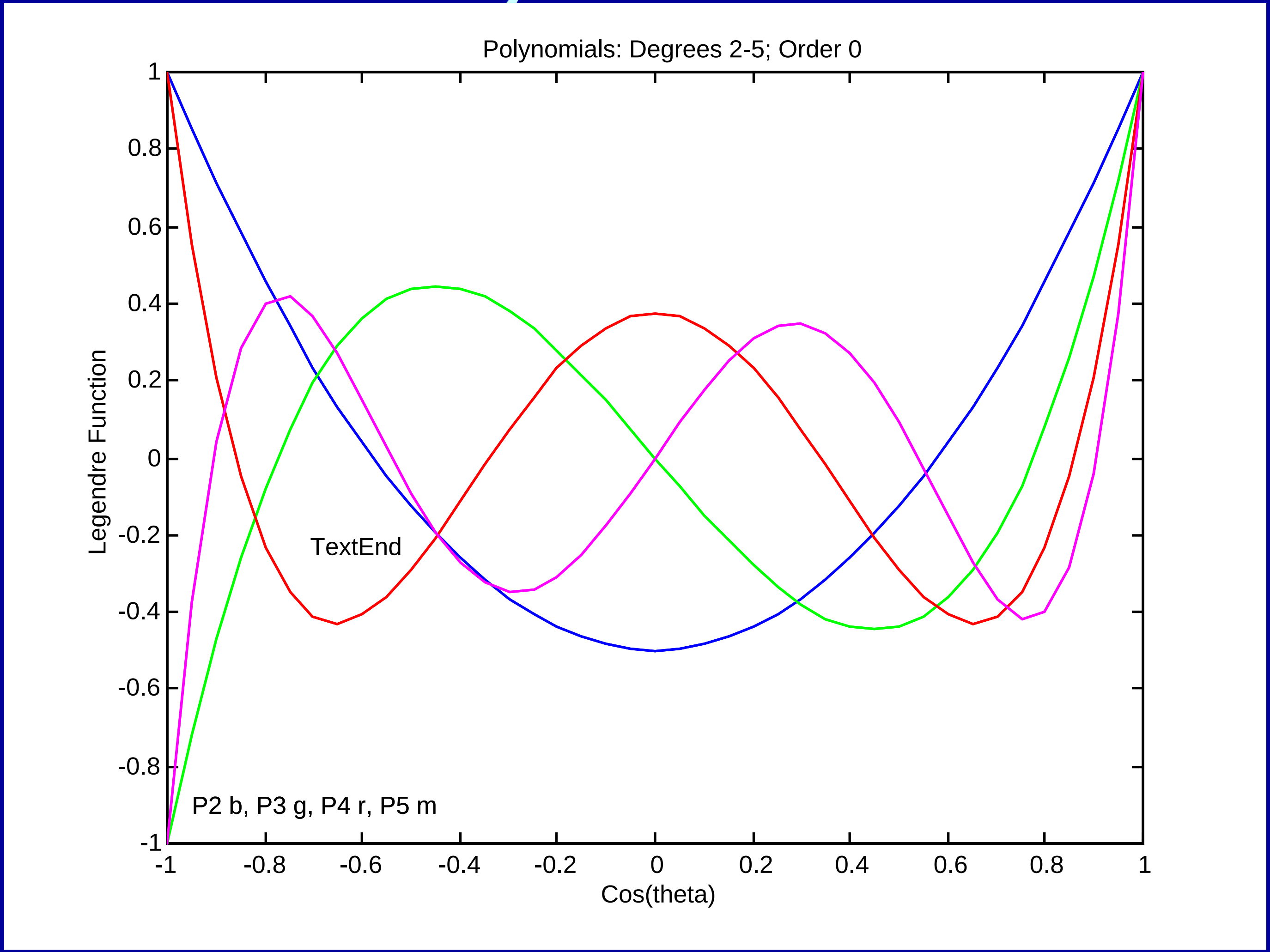
$$\int_{-1}^{1} P_{n'}(t)P_n(t)dt = \frac{2}{2n+1}\delta_{n'n}$$

$$\int_{-1}^{1} P_{n'm}(t)P_{nm}(t)dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n'n}$$

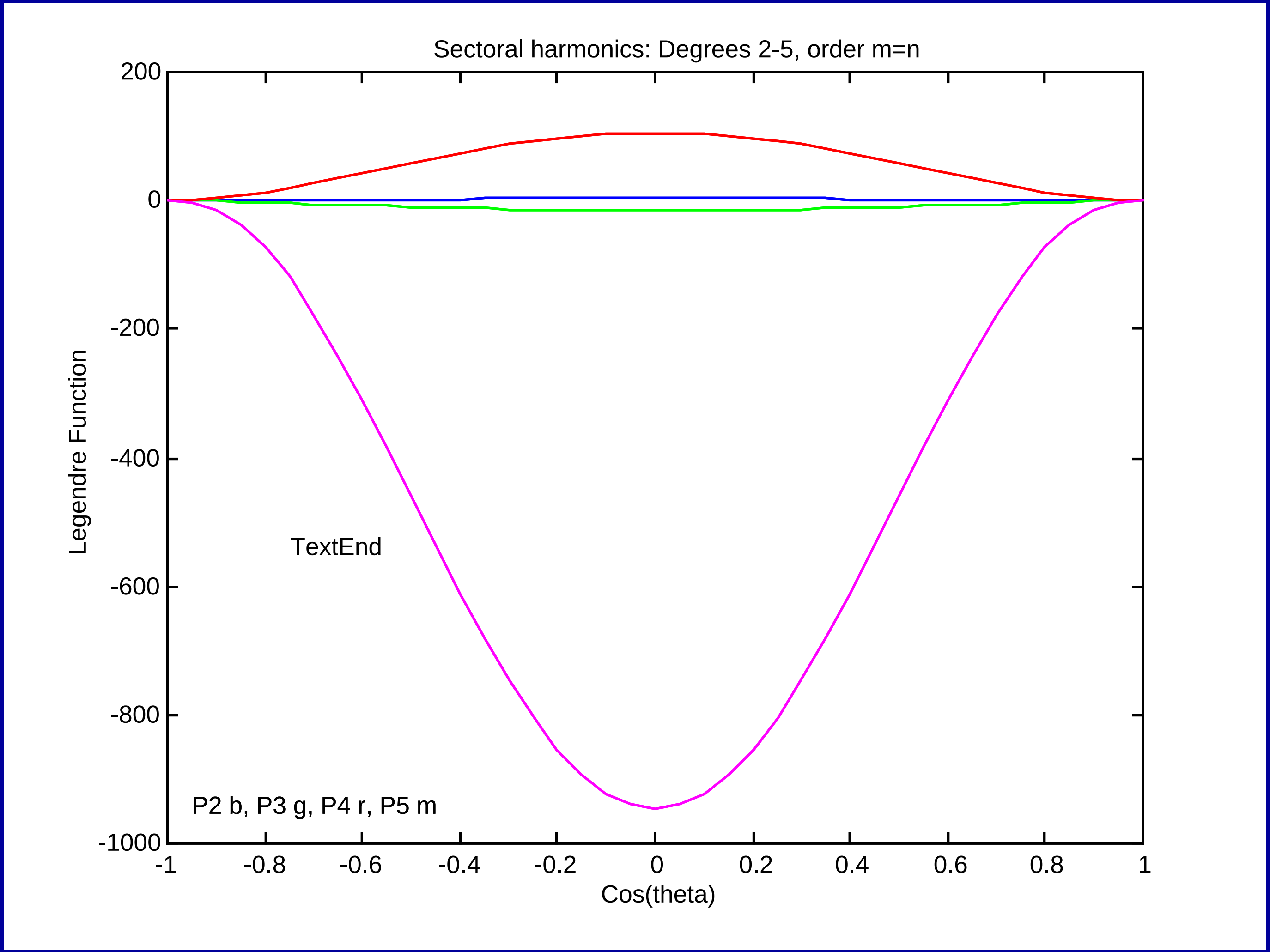
Examples from Matlab

- Matlab/Harmonics.m is a small matlab program to plots the associated functions and polynomials
- Uses Matlab function: Legendre

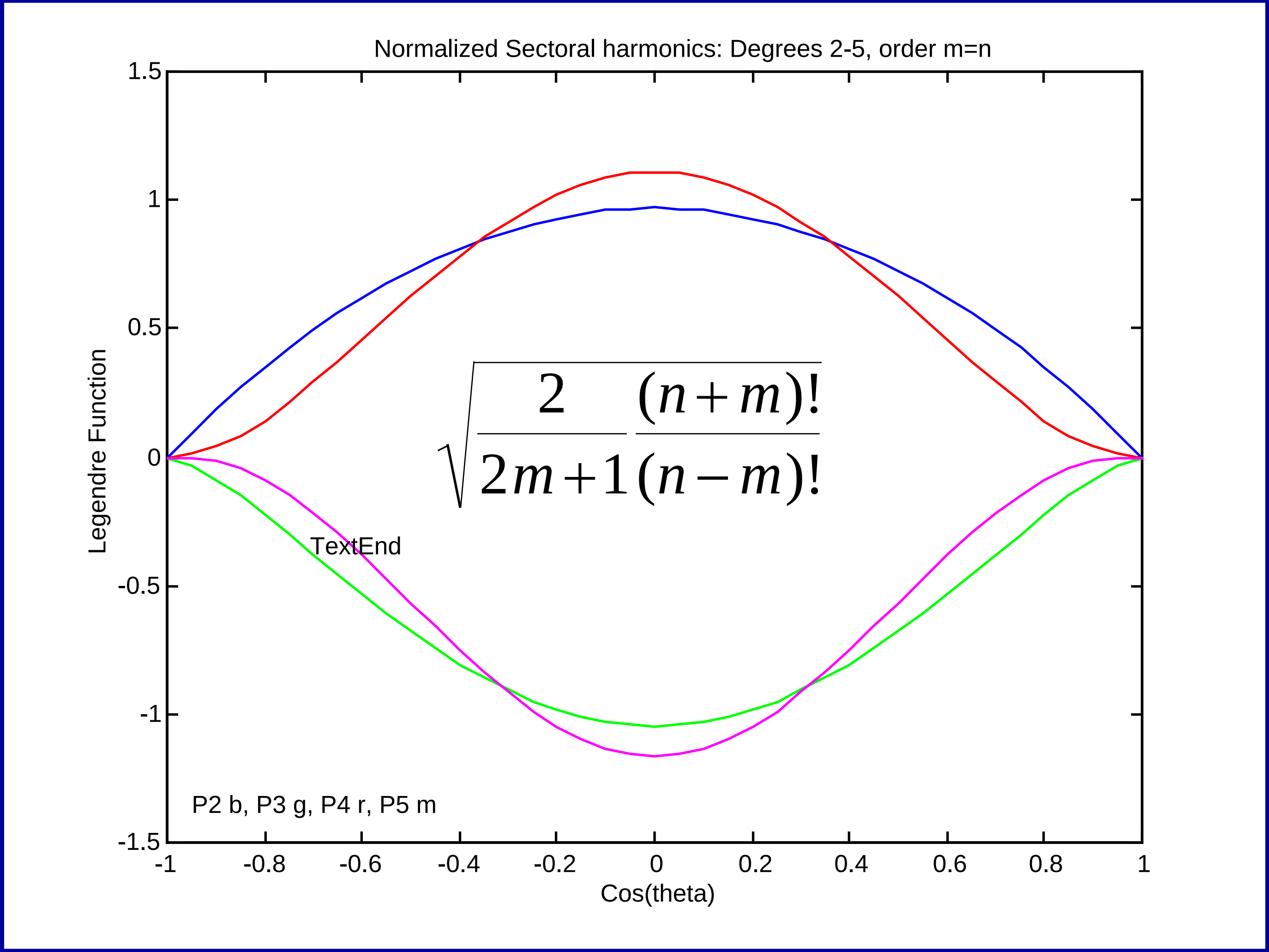
Polynomials



“Sectoral Harmonics” m=n



Normalized



$$\sqrt{\frac{2}{2m+1}\frac{(n+m)!}{(n-m)!}}$$

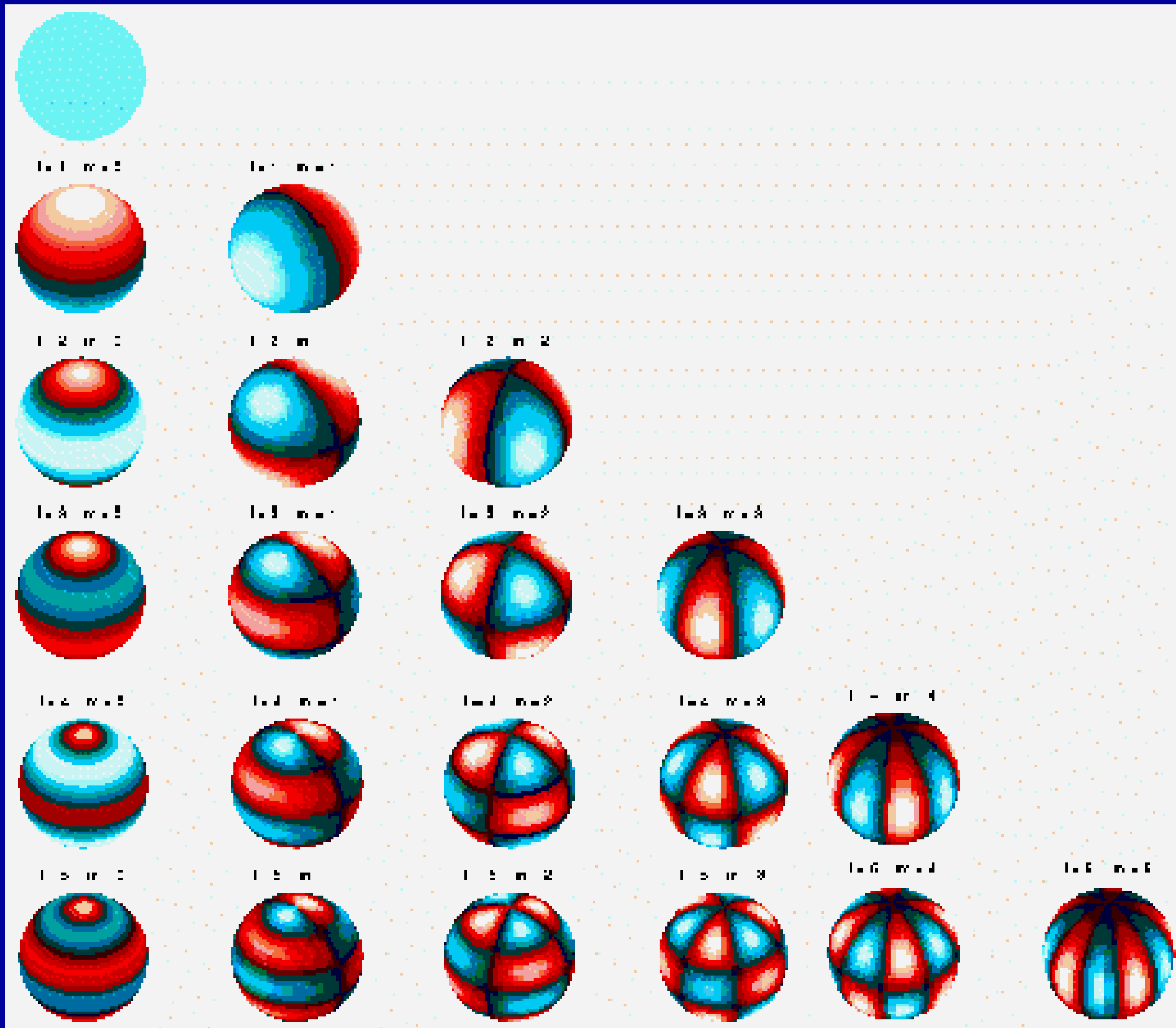
Surface harmonics

- To represent field on surface of sphere; surface harmonics are often used

$$Y_{nm}(\theta,\lambda)=\sqrt{\frac{2m+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_{nm}(\theta)e^{im\lambda}$$

- Be cautious of normalization. This is only one of many normalizations
- Complex notation simple way of writing cos(mλ) and sin(mλ)

Surface harmonics



Zonal ---- Terreserals ----------------------Sectorials

Gravitational potential

- The gravitational potential is given by:

$$V = \iiint \frac{G\rho}{r} dV$$

- Where ρ is density,
- G is Gravitational constant 6.6732×10^{-11} $m^3 kg^{-1} s^{-2}$ ($N\ m^2 kg^{-2}$)
- r is distance
- The gradient of the potential is the gravitational acceleration

Spherical Harmonic Expansion

- The Gravitational potential can be written as a series expansion

$$V = -\frac{GM}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\cos\theta)[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)]$$

- Cnm and Snm are called Stokes coefficients

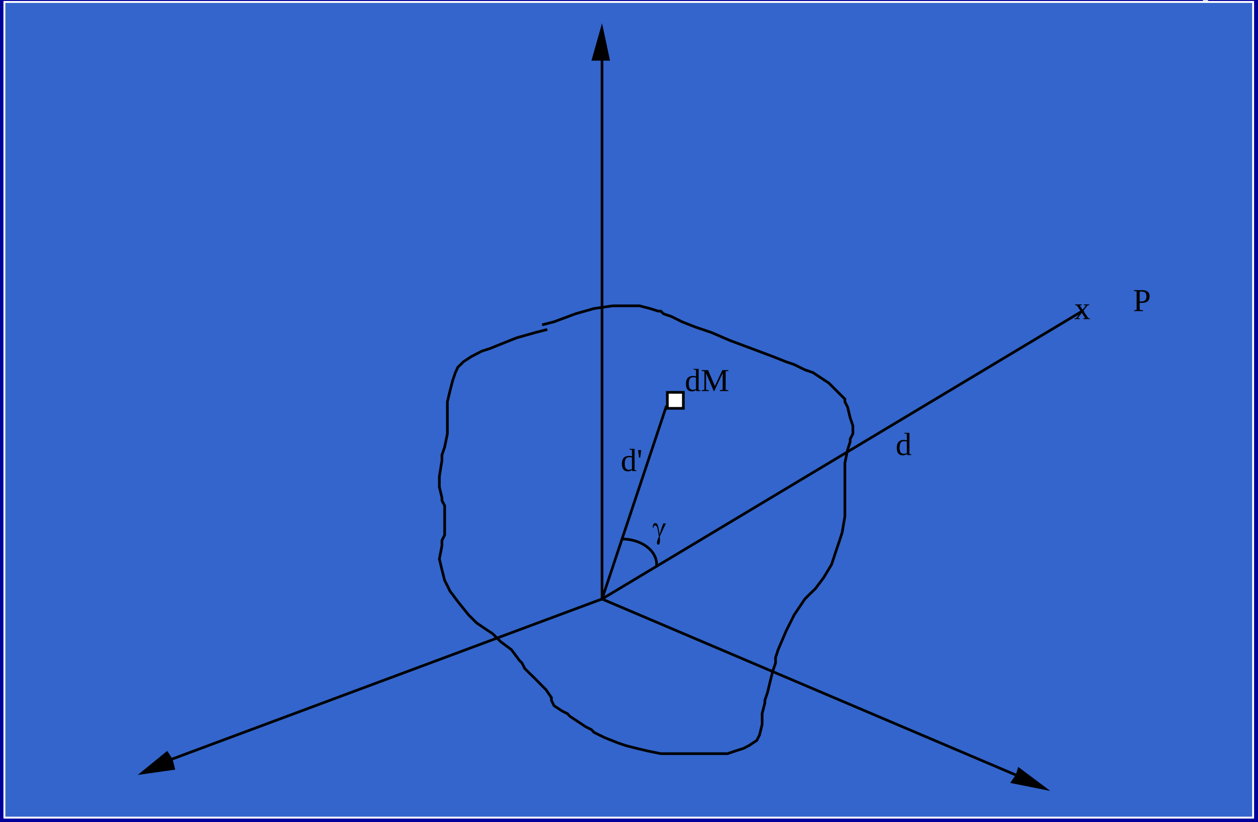
Stokes coefficients

- The Cnm and Snm for the Earth's potential field can be obtained in a variety of ways.
- One fundamental way is that 1/r expands as:

$$\frac{1}{r} = \sum_{n=0}^{\infty} \frac{d'^{n}}{d^{n+1}} P_n(\cos\gamma)$$

1/r expansion

- Pn(cosγ) can be expanded in associated functions as function of θ,λ



Spherical harmonics

- The Stokes coefficents can be written as volumn integrals
- C_{00} = 1 if mass is correct
- C_{10}, C_{11}, S_{11} = 0 if origin at center of mass
- C_{21} and S_{21} = 0 if Z-axis along maximum moment of inertia

Global coordinate systems

- If the gravity field is expanded in spherical harmonics then the coordinate system can be realized by adopting a frame in which certain Stokes coefficients are zero.
- What about before gravity field was well known?