Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.061 Introduction to Power Systems

Problem Set 2 Solutions   February 16, 2003

Problem 1: 1. The switch is *opened* at time $t = 0$, and the current source has a constant current $I_0 = 10A$. The *particular* solution is $V_p = 0$. Noting thta the homogeneous equation is:

$$\frac{L}{R}\frac{dv_0}{dt} + v_0 = 0$$

which is solved by:

$$v_0 = V_H e^{-\frac{L}{R}t}$$

The initial condition is simply:

$$V_H = RI_0 = 10A \times 100\Omega = 1,000V$$

So that the total solution is

$$v_0 = 1000e^{-1000t}$$

2. The source is changed to be $i_s = 10cos\omega t$ with $\omega = 2\pi \times 60Hz$. The *particular* solution is the sinusoidal steady state solution to

$$i_L + \frac{v_0}{R} = I_s = i_L + \frac{L}{R}\frac{di_L}{dt} = \text{Re}\left\{|I_s|\cos\omega t\right\}$$

That solution is simply:

$$i_L = \text{Re}\left\{\underline{I}_L e^{j\omega t}\right\}$$

where

$$\underline{I}_L = \frac{I_s}{1 + j\omega\frac{L}{R}}$$

So then the *particular* solution for inductor current is

$$i_{Lp} = \frac{I_s}{\sqrt{1 + (\frac{\omega L}{R})^2}}\cos\omega t - \phi$$

where the phase angle is $\phi = \tan^{-1}\frac{\omega L}{R}$ The whole solution is

$$v_o = V_H e^{-\frac{R}{L}t} + I_s\left(\cos\omega t - \cos(\omega t - phi)\right)$$

Since the same initial condition as in the first part of the problem holds, the full solution is:

$$v_o = RI_s\left(\cos\phi e^{-\frac{R}{L}t} + \cos\omega t - \cos(\omega t - \phi)\right)$$

3. **For 6.979** Here is a script which calculates and plots the results derived here and which does the simulation too:

```
\% PS2 Problem 1
R=100;
L=.1;
t = 0:.000001:.01;
V1 = 1000 .* exp(-(R/L) .* t);
I10 = 0;
[ts, I1] = ode23('isubs', t, I10);
Vs1 = R .* (10 - I1);
figure(1)
plot(t, V1, ts, Vs1);
om = 2*pi*60;
phi = atan(om*L/R);
t2 = 0:.000001:.05;
Is = 10;
I1p = (Is/sqrt(1+(om*L/R)^2)) .* cos(om .* t2 - phi);
V2 = R*Is*cos(phi) .* exp(-(R/L) .* t2) + R*Is.* cos(om .* t2) - R .* I1p;
[tss, I12] = ode45('isubs2', t2, I10);
V2s = R .* (Is .* cos(om .* tss) - I12);
figure(2)
plot(t2, V2, tss, V2s);
```

Below are the two subroutines called by ode23:

```
function didt = isubs(t, i)
R = 100;
L=.1;
Is = 10;
didt = R*(Is-i)/L;
------------------------------
function didt = isubs2(t, i)
R = 100;
L=.1;
om = 2*pi*60;
Is = 10*cos(om*t);
didt = R*(Is-i)/L;
------------------------------
```

Problem 2: 'Buck converter'

1. The *average* output voltage is simply

$$< v_o >= dV_s$$

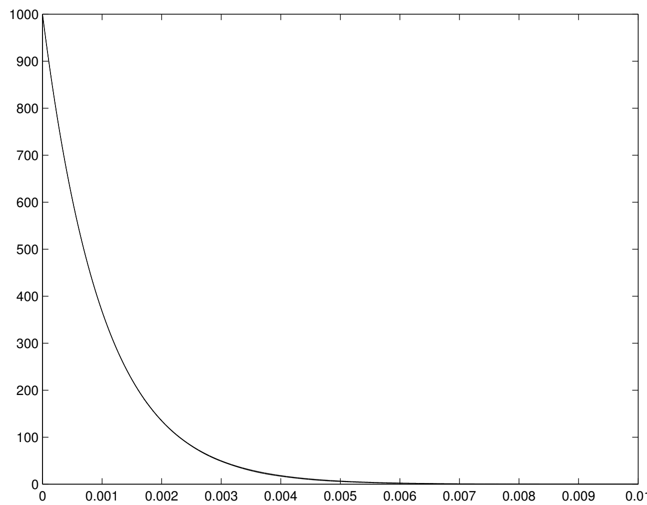

Figure 1: Solution to Problem 1, part 1

2. We know the maximum current (in steady state) is

$$i_m = \frac{V_s}{R}\frac{1-e^{-\frac{R}{L}t_{on}}}{1-e^{-\frac{R}{L}T}}$$

where $t_{on} = dT$ and of course the minimum current is:

$$i_\ell = i_m e^{-\frac{R}{L}t_{off}}$$

and of course $t_{off} = (1-d)T$ This little script here calculates and plots the ripple for the full range of duty cycle and prints the value for a duty cycle of 1/2.:

```
R = 10;
L = .02;
T = 1e-4;
dc = .5;
ton = T*dc;
toff = T*(1-dc);
Vs = 100;
Vm =  Vs *(1-exp(-(R/L)*ton))/(1-exp(-(R/L)*T));
Vl = Vm * exp(-(R/L)*toff);
Vr = Vm-Vl;
fprintf('50 percent duty cycle\n');
fprintf('Max Voltage = %g\n',Vm);
fprintf('Min Voltage = %g\n',Vl);
fprintf('Ripple Voltage = %g\n',Vr);
d = 0:.01:1;
t_on = T .* d;
t_off = T .* (1-d);
Vm =  Vs .*(1-exp(-(R/L) .*t_on)) ./(1-exp(-(R/L)*T));
```

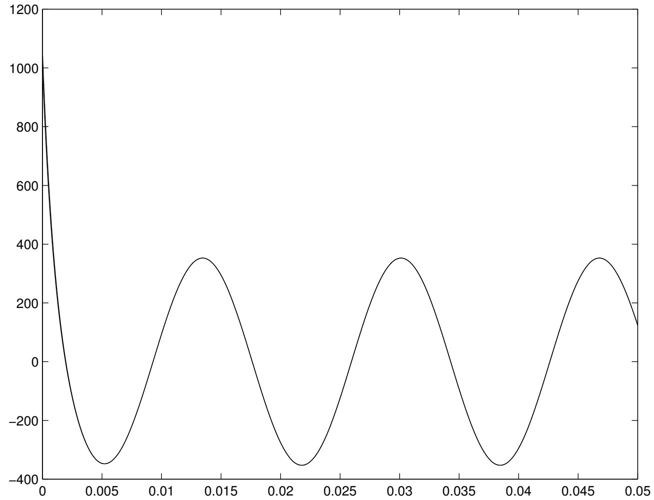

Figure 2: Solution to Problem 1, part 2

```
Vl = Vm .* exp(-(R/L) .*t_off);
Vr = Vm-Vl;
figure(3)
plot(d, Vr)
title('Problem Set 2, Problem 2')
ylabel('Ripple Voltage');
xlabel('Duty Cycle')

-----------------------
Here is the output for 50\%:
>> p2
50 percent duty cycle
Max Voltage = 50.625
Min Voltage = 49.375
Ripple Voltage = 1.24993
```

3. Simulation: Here is a script which does the simulation. Note that this repeats a few cycles at the end to get a better idea of steady state operation. These may be compared with the numbers obtained above

```
---------------------------
% Problem set 2, PRoblem 2 simulation
% buck converter
global Vs L R
Vs=100;
L=.02;
R=10;
il=[];
t = [];
T = 1e-4;
```

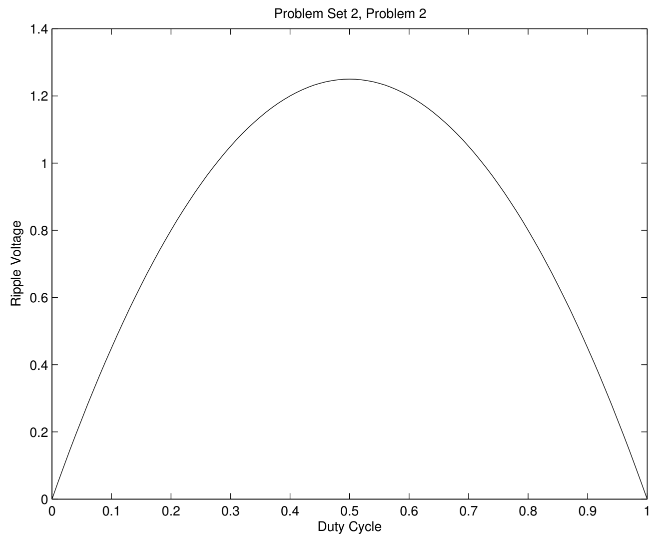

Figure 3: Buck Converter Ripple vs. duty cycle

```
d = .5;
ton = T*d;
toff = T*(1-d);
S0=0;
for n = 0:100
    [tt, S] = ode23('bon', [n*T n*T+ton], S0);
    t = [t' tt']';
    il = [il S'];
    S0 = S(length(tt));
    [tt, S] = ode23('boff', [n*T+toff (n+1)*T], S0);
    t = [t' tt']';
    il = [il S'];
    S0 = S(length(tt));
end
vo = R .* il;
figure
plot(t, vo)
title('Buck Converter Simulation')
ylabel('Volts');
xlabel('Time, sec');
% now just to get the last few cycles
tf =[];
ilf = [];
for n = 100:102
    [tt, S] = ode23('bon', [n*T n*T+ton], S0);
    tf = [tf' tt']';
    ilf = [ilf S'];
    S0 = S(length(tt));
```

```
    [tt, S] = ode23('boff', [n*T+toff (n+1)*T], S0);
    tf = [tf' tt']';
    ilf = [ilf S'];
    S0 = S(length(tt));
end
vof = R*ilf;
figure
plot(tf, vof)
title('Buck Converter Simulation')
ylabel('Volts');
xlabel('Time, sec');
-------------------------------------
function DS = bon(t, il)
global Vs L R
DS = (Vs - R*il)/L;
--------------------------------------
function DS = boff(t, il)
global Vs L R
DS = (- R*il)/L;
-------------------------------------
```

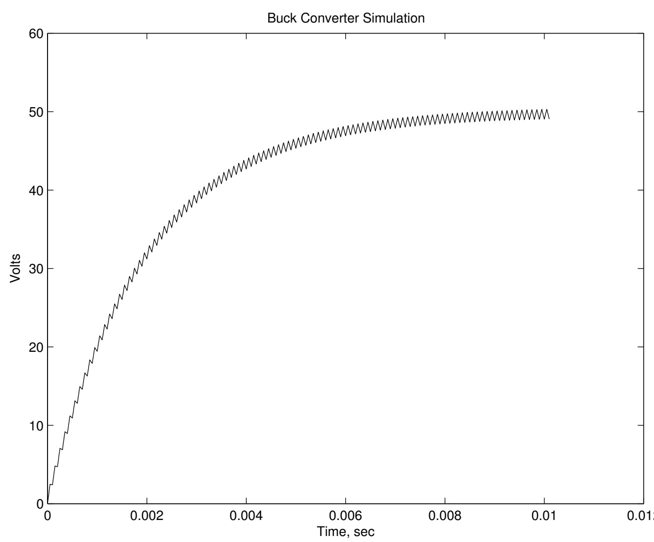

Figure 4: Buck Converter Voltage Buildup

Problem 3: This problem has two loops and two energy storage elements. The two loop equations are:

$$
\begin{aligned}
L\frac{di_L}{dt} + Ri_L - RC\frac{dv_c}{dt} &= V_s \\
2RC\frac{dv_c}{dt} + v_c - Ri_L &= 0
\end{aligned}
$$

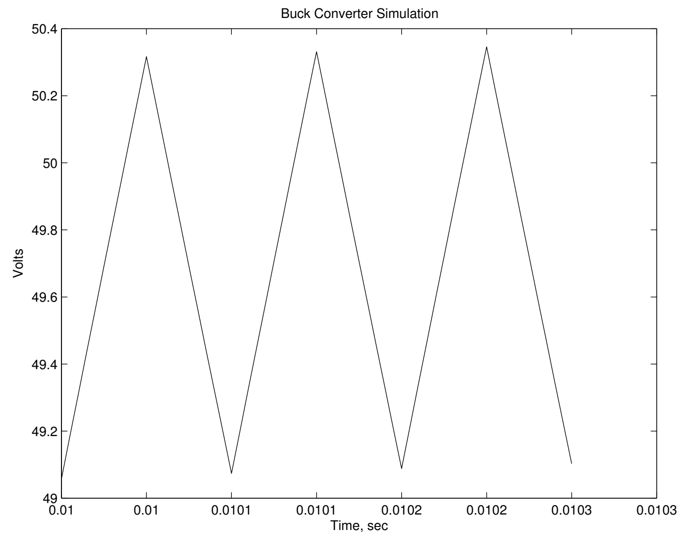

Figure 5: Buck Converter in Steady State

Now by multiplying the second of these by 1/2 and adding it to the first, we get a separated set of state equations which we can solve directly and use in a simulation:

$$\begin{aligned} V_s &= L\frac{di_L}{dt} + \frac{R}{2}i_L + \frac{1}{2}v_c \\ 0 &= 2RC\frac{dv_c}{dt} + v_c - Ri_L \end{aligned}$$

The characteristic equation for this is the determinant of:

$$\begin{bmatrix} SL + \frac{R}{2} & \frac{1}{2} \\ -R & 2RCS + 1 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

which is

$$\left(SL + \frac{R}{2}\right)(2RCS + 1) + \frac{R}{2} = 0$$

or

$$S^2 + S\left(\frac{1}{2RC} + \frac{R}{2L}\right) + \frac{1}{2LC} = 0$$

which is solved by

$$S = -\alpha \pm j\omega$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\left(\frac{1}{2RC} + \frac{R}{2L}\right) \\ \omega &= \sqrt{\frac{1}{2LC} - \alpha^2} \end{aligned}$$

It is clear that the particular solution to this problem is $v_P = 0$ (note that the excitation is DC and the output is in series with a capacitor). So the whole solution must be:

$$v_o = Ae^{(-\alpha+j\omega)t} + Be^{(-alpha-j\omega)t}$$

Noting the inductor in series with the source, we can see that the voltage at $t = 0+ = 0$, or $A + B = 0$.

The *time derivative* of output voltage at $t = 0+$ is:

$$\frac{dv_o}{dt} = \frac{1}{2}\frac{R}{L}V_s = 2j\omega A$$

so that whole solution is simply:

$$v_o = \frac{V_s}{2}\frac{R}{L}\frac{1}{\omega}e^{-\alpha t}\sin\omega t$$

Here is a script which simulates this and which plots both the simulated and analytical solutions. Note they plot right on top of each other:

```
\% Problem Set 2, Problem 3
global Vs R L C
Vs = 100;
R = 10;
L = .001;
C = 1e-5;
alf = 1/(4*R*C)+R/(4*L);
om = sqrt(1/(2*L*C)-alf^2);
Vm = .5*(Vs/om)*(R/L);
t=0:5e-6:2e-3;
Vo = Vm .* exp(-alf .* t) .* sin(om .* t);
[ts X] = ode23('rlcsim', t, [0 0]');
i_L = X(:,1);
v_c = X(:,2);
V_o = .5*R .* i_L - .5 .* v_c;
figure
plot(t, Vo, ts, V_o)
title('Problem Set 2, Problem 3')
ylabel('Volts')
xlabel('Seconds');
------------------------------------------
function xdot = rlcsim(t, X);
global Vs R L C
il = X(1);
vc = X(2);
didt = (Vs - .5*R*il - .5*vc)/L;
dvdt = (R*il-vc)/(2*R*C);
xdot = [didt dvdt]';
-------------------------------------------
```

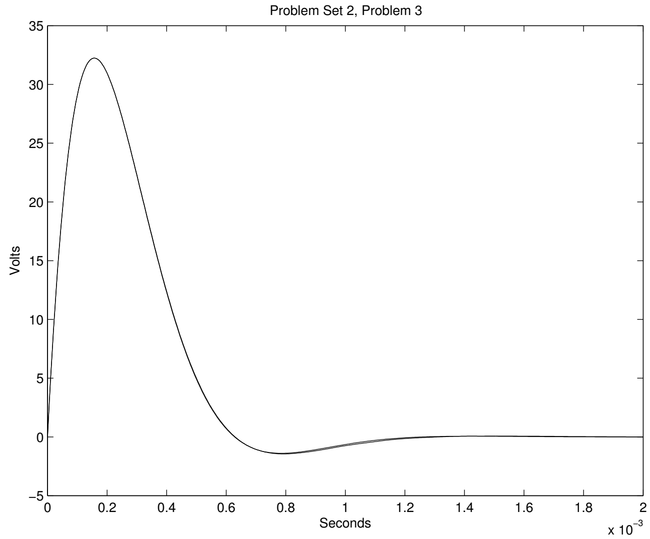

Figure 6: Solution for Problem 3: Analytic and Simulated

Problem 4: for 6.979 This one is much easier than it looks!

Note that there are two periods of time. First, when the switch is ON (for 10 mS) the diode is back-biased and inductor current is simply:

$$i_L = \frac{V}{L} t$$

With a drive voltage of 100 volts and an inductance of 100 mHy, the rate of change of current in the inductance is $\frac{V}{L} = 1000 A/sec$. After 10 mS the inductor current is 10 A.

As the switch is opened current circulates through the capacitance and diode. At first it looks like a simple harmonic oscillator with $i_L = I_0 \cos \omega t$. The frequency of oscillation is $\omega = \frac{1}{\sqrt{LC}} = 1000\text{rad/sec}$. Now, note that this can keep up only for 1/4 of a cycle, or until the current tries to reverse. The diode becomes forward biased and the voltage on the capacitance becomes steady. That voltage is:

$$V_c = -\frac{1}{C} \int_0^{\frac{\pi}{2\omega}} I_0 \cos \omega t dt = -\frac{I_0}{\omega C} = -I_0 \sqrt{\frac{L}{C}}$$

Since the current I_0 is 10 A and $\sqrt{\frac{L}{C}} = 100\Omega$, the final output voltage is 1000 V.