16.323 | Handout #4
Prof. J. P. How | March 23, 2006
Due: April 13, 2006

16.323 Homework Assignment #4

1. Find the curve $x^\star(t)$ that minimizes the functional $J(x) = \int_0^{t_f} \sqrt{1+\dot{x}^2}\, dt$ with $x(0) = 5$ and the end points lie on the circle $x^2(t) + (t-5)^2 - 4 = 0$. Draw a plot of the solution and provide a geometric interpretation of the result.

2. The derivation on pages 6–1 – 6–2 was done for the case of free or fixed $x(t_f)$. Repeat the derivation for the more general boundary condition $m(x(t_f), t_f) = 0$ that we originally considered on 5–17.

   (a) State the necessary and boundary conditions for this case (equivalent to what is on page 6–2)

   (b) Use this result to derive an optimal controller for the double integrator system (see 6–4) starting at $y(0) = 10$, $\dot{y}(0) = 0$ with the objective of minimizing
   $$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_0^{t_f} bu^2(t)dt \qquad b > 0$$
   subject to the terminal constraints that $\dot{y}(t_f) = 0$ and
   $$y^2(t_f) + (t_f - 5)^2 - 4 = 0$$
   Note: if the controller is too difficult to compute, just clearly what conditions need to be solved.

3. Complete the proof on page 6–18 for the Kalman Frequency domain equality - removing all of the "hand-waving" done in class.

4. Show that if in the Hamiltonian H, a and g are independent of time t, then H is a constant, and that if t_f is free, this constant is zero.

5. Read the posted article by Betts "Survey of numerical methods for. trajectory optimization," AIAA J. of Guidance,. Control and Dynamics, 21:193-207, 1998, and write a 1 page summary of his suggestions/conclusions.

6. Solve the min time-fuel problem ($b = 1$) for the double integrator system for the four initial conditions $x_0 \in \{0.5,\ 5\}$, $\dot{x}_0 \in \{0.5,\ 5\}$ using $u_m \leq 1$

   - For each case, use the calculated final time from the min time-fuel problem in a second control calculation that solves the finite-time LQR problem.
   $$J_{LQR} = \frac{1}{2}\mathbf{x}^T(t_f)P_{t_f}\mathbf{x} + \int_0^{t_f} [\mathbf{x}^T R_{xx}\mathbf{x} + \rho u^T u]dt$$
   with $P_{t_f} = 100I_2$, $R_{xx} = I_2$. Simulate the state response, and then tune your choice of ρ to ensure that $|u(t)| \leq u_m$ for this LQR controller.

- Plot the state response on the phase plot along with the fuel-time optimal result, and compare the paths followed and the amount of fuel used.

  Note: This problem will require that you solve the time-varying LQR problem, which means solving the DRE for $P(t)$ (This was done in the codes for Lecture 4, which should be in the updated notes). As noted in class, this is done backwards in time from t_f, but this can easily be switched to forwards in time by introducing $\tau = t_f - t$ and then converting the DRE from d/dt to $d/d\tau$. Furthermore, since P is $2x2$ and symmetric, there are really only three equations that need to be solved. You can use Matlab ODE solvers if you rewrite the matrix as a vector and write a function of the form:

$$\begin{bmatrix} \dot{p}_{11} \\ \dot{p}_{12} \\ \dot{p}_{22} \end{bmatrix} = ftn(P)$$

7. We discussed in class that LQR is a great **regulator** in that it quickly returns the system states to 0 while balancing the amount of control used. However, we are also interested in tracking a reference command, so that $y(t) = r(t)$ as $t \to \infty$.

   (a) Design a steady state LQR controller for the system using $R_{xx} = I_2$, $R_{uu} = 0.01$

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

   A naive way to implement a reference tracker is to modify the LQR controller from $u = -Kx$ to $u = r - Kx$:

$$\dot{x} = (A - BK)x + Br \ , \quad y = Cx$$

   Verify that this leads to particularly poor tracking of a step input!

   (b) An alternative strategy is to use $u = Nr - Kx$, where N is a constant. What is a good way to choose N to ensure zero steady state error for this closed-loop system? What are the consequences of this change in the step response of the closed-loop system?

   (c) A completely different approach to ensuring zero steady-state error is to use what is often called an LQ-servo. The approach is to add a new state to the system that integrates the tracking error: $\dot{x}_i = r - y = r - Cx$, giving:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} &= \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} \end{aligned}$$

   The LQR problem statement can now be modified (ignore r in the design of u) to place a high weighting on x_i to penalize the tracking error. Use this technique to design a new controller (keep R_{xx} and R_{uu} the same as part(a) and tune the weight on x_i to achieve a performance that is similar to part (b)). Compare the transient responses for the approaches in (b) and (c) - do you see any advantages to one approach over the other?