16.323 Lecture 8

Constrained Optimal Control - 2

Minimum Time Problems

- Can repeat the analysis in Lecture 7 for minimum time and energy problems using the PMP
  - Issue is that the process of a developing a solution by analytic construction is laborious and very hard to extend to anything nonlinear and/or linear with more than 2 states
- Need to revisit the problem statement and develop a new approach.
- **Goal:** develop the control input sequence

$$M_i^- \leq u_i(t) \leq M_i^+$$

that drives the system (nonlinear, but linear control inputs)

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}$$

from an arbitrary state $\mathbf{x}_0$ to the origin to minimize maneuver time

$$\min J = \int_{t_0}^{t_f} dt$$

- **Solution:** form the Hamiltonian

$$\begin{aligned} H &= 1 + \mathbf{p}^T(t)\{A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}\} \\ &= 1 + \mathbf{p}^T(t)\{A(\mathbf{x}, t) + \left[\, \mathbf{b}_1(\mathbf{x}, t) \;\; \mathbf{b}_2(\mathbf{x}, t) \;\; \cdots \;\; \mathbf{b}_m(\mathbf{x}, t) \,\right] \mathbf{u}\} \\ &= 1 + \mathbf{p}^T(t)A(\mathbf{x}, t) + \sum_{i=1}^{m} \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i(t) \end{aligned}$$

- Now use the PMP: select $u_i(t)$ to minimize H, which gives

$$u_i(t) = \begin{cases} M_i^+ & \texttt{if} \;\; \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t) < 0 \\ M_i^- & \texttt{if} \;\; \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t) > 0 \end{cases}$$

which gives us the expected **Bang-Bang** control
- Then solve for the costate

$$\dot{\mathbf{p}} = -H_{\mathbf{x}}^T = -\left(\frac{\partial A}{\partial \mathbf{x}} + \frac{\partial B}{\partial \mathbf{x}} u\right)^T \mathbf{p}$$

  - Could be very complicated for a nonlinear system.

- Note: shown how to pick $u(t)$ given that $\mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x},t) \neq 0$
  - Not obvious what to do if $\mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x},t) = 0$ for some finite time interval.
  - In this case the coefficient of $u_i(t)$ is zero, and PMP provides no information on how to pick the control inputs.
  - Will analyze this **singular condition** in more detail later.

- To develop further insights, restrict the system model further to LTI, so that

$$A(\mathbf{x},t) \rightarrow A\mathbf{x} \qquad B(\mathbf{x},t) \rightarrow B$$

  - Assume that $[A, B]$ controllable
  - Set $M_i^+ = -M_i^- = u_{m_i}$

- Just showed that if a solution exists, it is Bang-Bang
  - **Existence:** if $\mathbb{R}(\lambda_i(A)) \leq 0$, then an optimal control exists that transfers any initial state $\mathbf{x}_0$ to the origin.
    - ◇ Must eliminate unstable plants from this statement because the control is bounded.
  - **Uniqueness:** If an extremal control exists (i.e. solves the necessary condition and satisfies the boundary conditions), then it is unique.
    - ◇ Satisfaction of the PMP is both necessary and sufficient for time-optimal control of a LTI system.

- If the eigenvalues of A are all real, and a unique optimal control exists, **then each control input can switch at most $n-1$ times.**
  - Still need to find the costates to determine the switching times – but much easier in the linear case.

Min Fuel Problems

- **Goal:** develop the control input sequence

$$M_i^- \leq u_i(t) \leq M_i^+$$

that drives the system

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}$$

from an arbitrary state $\mathbf{x}_0$ to the origin in a fixed time t_f and optimizes the cost

$$\min J = \int_{t_0}^{t_f} \sum_{i=1}^{m} c_i |u_i(t)| dt$$

- **Solution:** form the Hamiltonian

$$\begin{aligned} H &= \sum_{i=1}^{m} c_i |u_i(t)| + \mathbf{p}^T(t)\{A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}\} \\ &= \sum_{i=1}^{m} c_i |u_i(t)| + \mathbf{p}^T(t)A(\mathbf{x}, t) + \sum_{i=1}^{m} \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i(t) \\ &= \sum_{i=1}^{m} \left[c_i |u_i(t)| + \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i(t)\right] + \mathbf{p}^T(t)A(\mathbf{x}, t) \end{aligned}$$

- Use the PMP, which requires that we select $u_i(t)$ to ensure that for all admissible $u_i(t)$

$$\sum_{i=1}^{m} \left[c_i |u_i^\star(t)| + \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i^\star(t)\right] \leq \sum_{i=1}^{m} \left[c_i |u_i(t)| + \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i(t)\right]$$

- If the components of $\mathbf{u}$ are independent, then can just look at

$$c_i |u_i^\star(t)| + \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i^\star(t) \leq c_i |u_i(t)| + \mathbf{p}^T(t)\mathbf{b}_i(\mathbf{x}, t)u_i(t)$$

  – As before, this boils down to a comparison of c_i and $\mathbf{p}^T(t)\mathbf{b}_i$

  – Resulting control law is:

$$u_i^\star(t) = \begin{cases} M_i^- & \texttt{if} \quad c_i < \mathbf{p}^T(t)\mathbf{b}_i \\ 0 & \texttt{if} \quad -c_i < \mathbf{p}^T(t)\mathbf{b}_i < c_i \\ M_i^+ & \texttt{if} \quad \mathbf{p}^T(t)\mathbf{b}_i < -c_i \end{cases}$$

Example: 8–1

- Consider $G(s) = 1/s^2 \Rightarrow$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\min J = \int_{t_0}^{t_f} c|u(t)|dt$$

  – Drive state to the origin with t_f fixed.

$$H = |u| + p_1 x_2 + p_2 u$$

- Consider case with $c = 1$, final control $u(t_f) = u_m \Rightarrow p_2(t_f) < -1$

$$p_2(t) = c_2 - c_1 t$$

- As before, integrate EOM forward from 0 to t_2 using $-u_m$, then from t_2 to t_1 using $u = 0$, and from t_1 to t_f using u_m
  – Apply terminal conditions and solve for c_1 and c_2

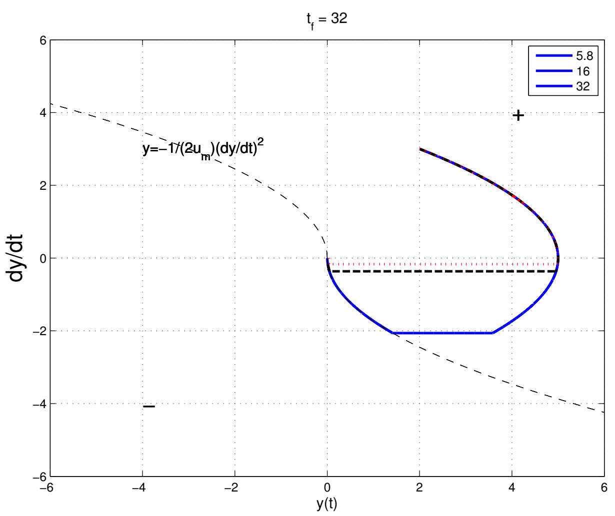

Figure 1: Min Fuel for varying final times

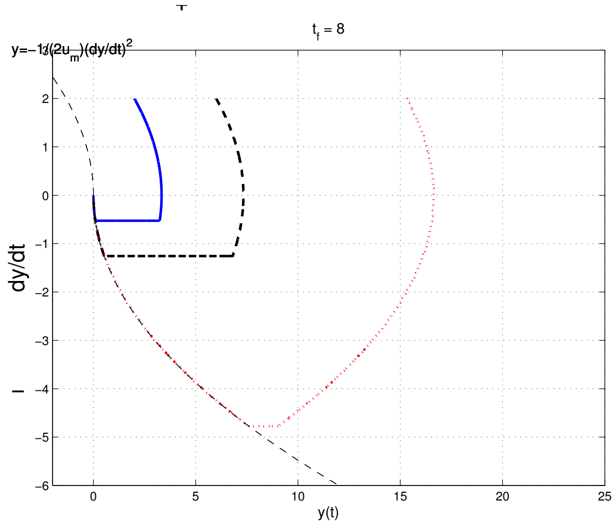

Figure 2: Min fuel for fixed final time, varying IC's

- First switch depends on IC, t_f, and $c \Rightarrow$ no clean closed-form solution
  – Larger c results in longer coast (deadband)
  – Larger t_f leads to longer coast.
  – For given t_f, limit to the IC from which we can reach the origin.

Minimum Energy Problem

- Goal: for a fixed final time and terminal constraints

$$\min J = \frac{1}{2}\int_0^{t_f} \mathbf{u}^T R\mathbf{u}\, dt \qquad R > 0$$

- Again use special dynamics:

$$\begin{aligned}\dot{\mathbf{x}} &= A(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u} \\ H &= \frac{1}{2}\mathbf{u}^T R\mathbf{u} + \mathbf{p}^T\{\ A(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}\}\end{aligned}$$

- Obviously with no constraints on $\mathbf{u}$, solve $H_{\mathbf{u}} = 0$, to get

$$\mathbf{u} = -R^{-1}B^T\mathbf{p}(t)$$

- But with bounded controls, must solve:

$$\mathbf{u}^\star(t) = \arg\min_{\mathbf{u}(t)\in\mathcal{U}} \left[\frac{1}{2}\mathbf{u}^T R\mathbf{u} + \mathbf{p}^T B(\mathbf{x},t)\mathbf{u}\right]$$

which is a constrained quadratic program in general

– However, for diagonal R, the effects of the controls are independent

$$\mathbf{u}^\star(t) = \arg\min_{\mathbf{u}(t)\in\mathcal{U}} \left[\sum_{i=1}^{m} \frac{1}{2}R_{ii}u_i^2 + \mathbf{p}^T\mathbf{b}_i u_i\right]$$

– In the unconstrained case, each $u_i(t)$ can easily be determined by minimizing

$$\frac{1}{2}R_{ii}u_i^2 + \mathbf{p}^T\mathbf{b}_i u_i \qquad \rightarrow \qquad \tilde{u}_i = R_{ii}^{-1}\mathbf{p}^T\mathbf{b}_i$$

- The resulting controller inputs are $u_i(t) = \texttt{sat}(\tilde{u}_i(t))$

$$u_i(t) = \begin{cases} M_i^- & \texttt{if} \qquad\quad \tilde{u}_i < M_i^- \\ \tilde{u}_i & \texttt{if}\ \ M_i^- < \tilde{u}_i < M_i^+ \\ M_i^+ & \texttt{if}\ \ M_i^+ < \tilde{u}_i \end{cases}$$

Min Fuel

```
%
% Min fuel for double integrator
% 16.323 Spring 2006
% Jonathan How
%
c=1;
t=[0:.01:t_f];
alp=(1/2/u_m) % switching line
T_2=roots([-u_m/2 yd0 y0] + conv([-u_m yd0],[-2 t_f+yd0/u_m])-alp*conv([-u_m yd0],[-u_m yd0]));%
t_2=min(T_2);
yd2=-u_m*t_2+yd0;yd1=yd2;
t_1=t_f+yd1/u_m;
c_1=2/(t_1-t_2);c_2=c_1*t_1-1;

G=ss([0 1;0 0],[0 1]',eye(2),zeros(2,1));
arc1=[0:.001:t_2]'; arc2=[t_2:.001:t_1]';arc3=[t_1:.001:t_f]'; %
[Y1,T1,X1]=lsim(G,-u_m*ones(length(arc1),1),arc1,[y0 yd0]'); %
[Y2,T2,X2]=lsim(G,0*ones(length(arc2),1),arc2,Y1(end,:)'); %
[Y3,T3,X3]=lsim(G,u_m*ones(length(arc3),1),arc3,Y2(end,:)'); %
plot(Y1(:,1),Y1(:,2),zzz,'Linewidth',2); hold on%
plot(Y2(:,1),Y2(:,2),zzz,'Linewidth',2); plot(Y3(:,1),Y3(:,2),zzz,'Linewidth',2);%
ylabel('dy/dt','Fontsize',18); xlabel('y(t)','Fontsize',12);%
text(-4,3,'y=-1/(2u_m)(dy/dt)^2','Fontsize',12)%
text(4,4,'+','Fontsize',18);text(-4,-4,'-','Fontsize',18);grid on;hold off
title(['t_f = ',mat2str(t_f)],'Fontsize',12)%

hold on;% plot the switching curves
kk=[0:.1:8]'; plot(-alp*kk.^2,kk,'k--');plot(alp*kk.^2,-kk,'k--');
hold off;axis([-4 4 -4 4]/4*6);

figure(2);%
p2=c_2-c_1*t;%
plot(t,p2,'Linewidth',4);%
hold on; plot([0 t_f],[c c],'k--','Linewidth',2);hold off; %
hold on; plot([0 t_f],-[c c],'k--','Linewidth',2);hold off; %
hold on; plot([t_1 t_1],[-2 2],'k:','Linewidth',3);hold off; %
text(t_1+.1,1.5,'t_1','Fontsize',12)%
hold on; plot([t_2 t_2],[-2 2],'k:','Linewidth',3);hold off; %
text(t_2+.1,-1.5,'t_2','Fontsize',12)%
title(['c = ',mat2str(c),' u_m = ',mat2str(u_m)],'Fontsize',12);%
ylabel('p_2(t)','Fontsize',12); xlabel('t','Fontsize',12);%
text(1,c+.1,'c','Fontsize',12);text(1,-c+.1,'-c','Fontsize',12)%
axis([0 t_f -3 3]);grid on; %

return

figure(1);clf
y0=2;yd0=3;t_f=5.8;u_m=1.5;zzz='-';minu;
figure(1);hold on
y0=2;yd0=3;t_f=16;u_m=1.5;zzz='k--';minu;
figure(1);hold on
y0=2;yd0=3;t_f=32;u_m=1.5;zzz='r:';minu;
figure(1);
legend('5.8','16','32')
print -f1 -depsc uopt1.eps;jpdf('uopt1');


figure(1);clf
y0=2;yd0=2;t_f=8;u_m=1.5;zzz='-';minu
figure(1);hold on
y0=6;yd0=2;t_f=8;u_m=1.5;zzz='k--';minu
figure(1);hold on
y0=15.3;yd0=2;t_f=8;u_m=1.5;zzz='r:';minu
figure(1);axis([-2 25 -6 3])
print -f1 -depsc uopt2.eps;jpdf('uopt2');
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