16.323 | Handout #5
Prof. J. P. How | April 13, 2006
Due: April 27, 2006

16.323 Homework Assignment #5

1. For the system

$$\dot{x} = -\frac{1}{\tau}x + u + w$$
$$y = x + v$$

with $w(t) \sim N(0,q)$, $v(t) \sim N(0,r)$, and

$$J = E\left\{\lim_{t_f \to \infty} \frac{1}{2t_f}\int_{t_0}^{t_f}(ax^2 + bu^2)dt\right\}$$

show that the optimal LQG controller is

$$\dot{x}_c = \left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} + \frac{a}{b}} - \sqrt{\frac{1}{\tau^2} + \frac{q}{r}}\right)x_c + \left(-\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} + \frac{q}{r}}\right)y$$
$$u = -\left(-\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} + \frac{a}{b}}\right)x_c$$

Use these results to chart the location of the compensator poles and zeros as a function of a/b and q/r (do this numerically, for $\tau = 1$).

2. Stochastic Optimal Output Feedback Regulator. Consider the simple benchmark system shown in the figure. Use the equations of motion in terms of the two states given (i.e. x_1 and x_2).

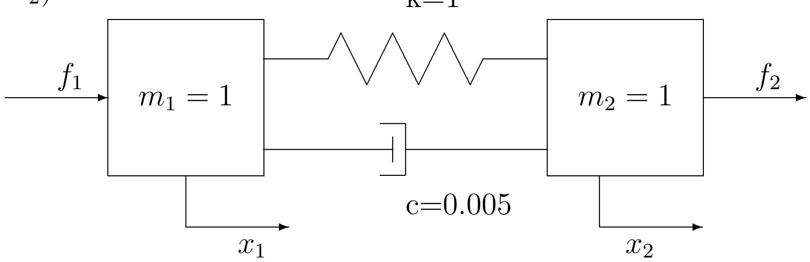

Figure 1: Benchmark system for problem 2

For the system with state $x = [x_1, x_2, \dot{x}_1, \dot{x}_2]$, we could consider a variety of control topologies using the following actuator and sensor locations:

Topology	Z	Y	W	U	Perf	Feedback
(1)	x_1	x_1	f_1	f_1	C	C
(4)	x_1	x_2	f_1	f_2	C	C

(a) **LQG design** For each of the given topologies, compute the optimal steady-state continuous feedback control using LQR ($R_{\rm zz} = 1$ and $R_{\rm uu} = 5e^{-2}$) and LQE using ($V_{\rm ww} = 1$ and $V_{\rm yy} = 2.5e^{-2}$).

(b) As best as possible, provide a "classical interpretation" of these two LQG controllers. Recall the difficulty of designing classical controllers for a system with a flexible mode (especially the non-collocated cases), and comment on the ease of designing high bandwidth controllers using this approach.

(c) **Frequency response** For each topology compare the open and closed-loop transfer functions from the disturbance w to the performance variable z. Comment of the apparent effectiveness of the controllers in improving the low frequency response and the response at the modal frequency.

(d) **Transient Response** For three systems (one OL and two CL) compute and plot the response to an initial condition

$$x_0 = [1\,, -0.5\,, 0.5\,, -1]^T\,,\ \hat{x}_0 = [0\,, 0\,, 0\,, 0]^T$$

For each case, be sure to look at both the displacement states ($x_1(t)$ and $x_2(t)$), the estimates of the displacement states ($\hat{x}_1(t)$ and $\hat{x}_2(t)$), and the performance output variable. Only plot the response to 10 sec.

(e) Compute/tabulate the mean square regulation error and mean square control input for each case.

3. **Estimator of a non-minimum phase system** Consider the system dynamics

$$\begin{aligned} \dot{x}_1 &= w \\ \dot{x}_2 &= -x_1 - 2x_2 + w \end{aligned} \tag{1}$$

with continuous measurements $y(t) = x_2(t) + v(t)$. The measurement and process noise are both independent white noise processes (spectral densities R_v and R_w, respectively).

(a) Find the transfer function $G_{\rm yw}(s)$ from $w(s)$ to $y(s)$ and identify the poles and zeroes. Recall that right-half plane zeroes are called non-minimum phase.

(b) Use this transfer function to sketch the symmetric root locus for the time-invariant estimator poles versus R_w/R_v. Clearly demonstrate that one of the estimator poles tends to $s = -1$ as $R_v/R_w \to 0$, even though there is no zero there (only a reflected one). Confirm that this implies that part of the estimation error will never attenuate faster than e^{-t} even with noise free measurements.

(c) Use the solution of the steady state Riccati equation to show that as $R_v/R_w \to 0$, then

$$P \to \begin{bmatrix} 2R_w & -\sqrt{R_w R_v} \\ \cdot & \sqrt{R_w R_v} \end{bmatrix}$$

Use this result to confirm that x_1 cannot be estimated perfectly, even with $R_v \to 0$.

(d) Comment on the impact of the nonminimum-phase zero on our ability to do good estimation for this system.