Anomalous Scattering

When the incident radiation has sufficient energy to promote an electronic transition in atoms contained in the crystal, we observe anomalous scattering in non-centrosymmetric crystals.

E.g. Selenium

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Anomalous Scattering

When the incident radiation has sufficient energy to promote an electronic transition in atoms contained in the crystal, we observe anomalous scattering in non-centrosymmetric crystals.

For instance, in Selenium, $\lambda = 0.9795 \, \text{Å}$.

E.g. Selenium

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\( \Delta E \) is a function of the periodic table.

\( \Delta E \) is near 8keV for most heavy and some light elements, so anomalous signal can be measured on a home X-ray source with CuKa radiation \((8\text{keV } \lambda=1.54\text{Å})\).

At a synchrotron, the energy of the incident radiation can be tuned to match \( \Delta E \) (accurately).

Importantly, \( \Delta E_s \) for C,N,O are out of the X-ray range.

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Anomalous Scattering

Anomalous scattering causes small but measurable differences in intensity between the reflections \( hkl \) and \( -h-k-l \) not normally present. That means for non-centrosymmetric crystals in the presence of atoms heavier than sulfur, Friedel’s law is not strictly true.

Under normal conditions, electron distributions within atoms are centrosymmetric:

![Diagram of electron distributions](Image)

Courtesy of Michael R. Sawaya. Used with permission.
The centrosymmetry in the scattering atoms is reflected in the centrosymmetry in the pattern of scattered X-ray intensities.

The positions of the reflections $hkl$ and $-h-k-l$ on the reciprocal lattice are related by a center of symmetry through the reciprocal lattice origin $(0,0,0)$.

Pairs of reflections $hkl$ and $-h-k-l$ are called Friedel pairs.

They share the same intensity by Friedel’s law.

\[
I_{(hkl)} = I_{(-h-k-l)}
\]
and

\[
\phi_{(hkl)} = -\phi_{(-h-k-l)}
\]
On an Argand diagram, $F_{(hkl)}$ and $F_{(-h-k-l)}$ appear to be reflected across the real axis.

$F_{h,k,l}$
imaginary

$|F_{h,k,l}|$

$h,k,l \phi_{hkl}$

real

$I_{(hkl)} = I_{(-h-k-l)}$
and
$\phi_{(hkl)} = -\phi_{(-h-k-l)}$

Courtesy of Michael R. Sawaya. Used with permission.
On an Argand diagram, $F_{(hkl)}$ and $F_{(-h-k-l)}$ appear to be reflected across the real axis.

$I_{(hkl)} = I_{(-h-k-l)}$

and

$\phi_{(hkl)} = -\phi_{(-h-k-l)}$

True for any crystal in the absence of anomalous scattering. Normally, $I_{hkl}$ and $I_{-h-k-l}$ are averaged together.
Anomalous Scattering

Anomalous scattering causes small but measurable differences in intensity between the reflections hkl and -h-k-l not normally present. That means for non-centrosymmetric crystals in the presence of atoms heavier than sulfur, Friedel’s law is not strictly true.

Under normal conditions, electron distributions within atoms are centrosymmetric:

Under conditions of anomalous scattering, electrons are perturbed from their centrosymmetric distributions; electrons are jumping between orbitals.

The breakdown of centrosymmetry in the scattering atoms is reflected in a loss of centrosymmetry in the pattern of scattered X-ray intensities.

Thus \( I_{(hkl)} = I_{(-h-k-l)} \) is no longer true.

The differences between \( I_{hkl} \) and \( I_{-h-k-l} \) are small; typically between 1-3%.

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Anomalous Scattering

In the presence of anomalous scattering, $F_{(hkl)}$ and $F_{(-h-k-l)}$ obey neither of the two relationships of Friedel’s Law and we are able to measure the differences between $F_{(hkl)}$ and $F_{(-h-k-l)}$.

$\mathbf{I}_{(hkl)} \neq \mathbf{I}_{(-h-k-l)}$

and

$\phi_{(hkl)} \neq -\phi_{(-h-k-l)}$

How to calculate F for an anomalous scatterer?

The correction to the atomic scattering factor is derived from classical physics and is based on an analogy of the atom to a forced oscillator under resonance conditions.
Examples of forced oscillation:

A tuning fork vibrating when exposed to periodic force of a sound wave.

The housing of a motor vibrating due to periodic impulses from an irregularity in the shaft.

A bridge swaying under the influence of marching soldiers or gusts of wind.

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Courtesy of Michael R. Sawaya. Used with permission.
An atom can be viewed as a dipole oscillator where the electron oscillates around the nucleus.

The oscillator is characterized by

- Mass = m
- Position = x, y
- Natural circular frequency = $\nu_B$

Characteristic of the atom. Bohr frequency from Bohr’s representation of the atom.

Courtesy of Michael R. Sawaya. Used with permission.
An incident photon’s electric field can exert a force on the $e^{-}$, affecting its oscillation frequency.

What happens when the external force matches the natural frequency of the oscillator (a.k.a. resonance condition)?

Courtesy of Michael R. Sawaya. Used with permission.
If the external force is a periodic gust of wind and the oscillator is the Tacoma Narrows bridge, then disaster occurs.

Photograph removed due to copyright reasons.

Tacoma Narrows bridge, 1940
In the case of an atom, resonance \((n = n_B)\) leads to electronic transition analogous to the condition \(h \nu = \Delta E\) discussed earlier.

The amplitude of the oscillator (electron) is given by classical physics:

\[
A = \frac{e}{mc^2} \frac{\nu^2 E_o}{\nu_B^2 - \nu^2 + i k \nu}
\]

- \(m\) = mass of oscillator
- \(e\) = charge of the oscillator
- \(c\) = speed of light
- \(E_o\) = max value of electric vector of incident photon
- \(\nu\) = frequency of external force (photon)
- \(\nu_B\) = natural resonance frequency of oscillator (e\(^-\))

Courtesy of Michael R. Sawaya. Used with permission.
Knowing the amplitude of the e\textsuperscript{-} leads to a definition of the scattering factor, \( f \).

The amplitude of the scattered radiation is defined by the oscillating electron. The oscillating electron is the source of the scattered electromagnetic wave which will have the same frequency and amplitude as the e\textsuperscript{-}.

Keep in mind, the frequency and amplitude of the e\textsuperscript{-} is itself strongly affected by the frequency and amplitude of the incident photon as indicated on the previous slide.

\[
f = \frac{\text{Amplitude of scattered radiation from the forced e-}}{\text{Amplitude of scattered radiation by a free e-}}
\]

Courtesy of Michael R. Sawaya. Used with permission.
The scattering factor is a complex number, with value dependent on $\nu$.

$$f = f_o + \Delta f' + i\Delta f''$$

$\nu =$ frequency incident photon
$\nu_B =$ Bohr frequency of oscillator (e$^-$)
(corresponding to electronic transition)

$$\Delta f' = g \left( \frac{\nu_B}{\nu} \right)^2 \log_e \left( \left( \frac{\nu}{\nu_B} \right)^2 - 1 \right)$$

$$\Delta f'' = g \frac{\pi\nu_B^2}{\nu^2}$$

Normal scattering factor
REAL

Correlation factor
REAL

Correction factor
IMAGINARY

Courtesy of Michael R. Sawaya. Used with permission.
Correction factors are largest near \( \nu = \nu_B \).

\[
f = f_0 + \Delta f' + i\Delta f''
\]

\[
\Delta f' = g \left( \frac{\nu_B}{\nu} \right)^2 \log_e \left( \frac{\nu}{\nu_B} \right)^2 - 1
\]

\[
\Delta f'' = g \frac{\pi \nu_B^2}{\nu^2} \quad \text{when } \nu > \nu_B
\]

\[
\nu = \text{frequency of external force (incident photon)}
\]

\[
\nu_B = \text{natural frequency of oscillator (e^-)}
\]

The REAL COMPONENT becomes negative near \( \nu = \nu_B \).

The IMAGINARY COMPONENT becomes large and positive near \( \nu = \nu_B \).

\[
\Delta f'
\]

\[
\Delta f''
\]

\[
\nu = \nu_B
\]

After dampening correction

Courtesy of Michael R. Sawaya. Used with permission.
Physical interpretation of the real and imaginary correction factors of $f$.

$$f = f_o + \Delta f' + i\Delta f''$$

**real component, $\Delta f''$**

A small component of the scattered radiation is $180^\circ$ out of phase with the normally scattered radiation given by $f_o$.

Always diminishes $f_o$.

Absorption of X-rays

**imaginary component, $\Delta f''$**

A small component of the scattered radiation is $90^\circ$ out of phase with the normally scattered radiation given by $f_o$.

Courtesy of Michael R. Sawaya. Used with permission.
**90° phase shift analogy to a child on a swing**

**Forced Oscillator Analogy**

- **Maximum negative force**
  - Zero speed

- **Zero force**
  - Maximum +/- speed

- **Maximum positive force**
  - Zero speed

Swing speed is 90° out of phase with the applied force.

Courtesy of Michael R. Sawaya. Used with permission.
90° phase shift analogy to a child on a swing

Forced Oscillator Analogy

Maximum negative force
Zero speed

Maximum positive force
Zero speed

Zero force
Maximum +/- speed

force : speed
incident photon : re-emitted photon.

time->

Courtesy of Michael R. Sawaya.
Used with permission.
Argand diagram for $F_{H(hkl)}$ under conditions of anomalous scattering

$$F_H = [f_o + \Delta f'(\lambda) + i\Delta f''(\lambda)] \, e^{2\pi i(hx_H + ky_H + lz_H)}$$

scattering factor for H

<table>
<thead>
<tr>
<th>real</th>
<th>real</th>
<th>imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive number</td>
<td>180° out of phase</td>
<td>90° out of phase</td>
</tr>
</tbody>
</table>

Courtesy of Michael R. Sawaya. Used with permission.
Argand diagram for $F_H(-h-k-l)$ is constructed in a similar way as $F_H(hkl)$ except $\phi$ is negative.

\[
F_H(hkl) = [f_o + \Delta f'(\lambda) + i\Delta f''(\lambda)] e^{2\pi i(hx + ky + lz)}
\]

\[
F_H(-h-k-l) = [f_o + \Delta f'(\lambda) + i\Delta f''(\lambda)] e^{2\pi i(-hx -ky -lz)}
\]

Courtesy of Michael R. Sawaya. Used with permission.
Friedel’s Law is broken

$\phi_{H(hkl)} \neq \phi_{H(-h-k-l)}$

Courtesy of Michael R. Sawaya. Used with permission.
Friedel’s Law is broken

For only the contribution of one type of heavy atom:

\[ I_{H(hkl)} = I_{H(-h-k-l)} \]

This situation changes in the presence of other atoms. E.g. lighter atoms that do not show anomalous scattering.

\[ I_{H(hkl)} = I_{H(-h-k-l)} \]

\[ \phi_{H(hkl)} \neq \phi_{H(-h-k-l)} \]

Courtesy of Michael R. Sawaya. Used with permission.
Friedel’s Law is broken

There is an observable difference between $|F_{(-h-k-l)}|$ and $|F_{(hkl)}|$ called a Bijvoet or Friedel difference or an anomalous difference.

This difference is only small (typically 1-3% of the measured intensity) and it is not always trivial to determine it accurately.

\[ I_{(hkl)} \neq I_{(-h-k-l)} \]
\[ \phi_{(hkl)} \neq \phi_{(-h-k-l)} \]

Courtesy of Michael R. Sawaya. Used with permission.
Anomalous Signal

This weak signal can be used to solve the phase problem in several ways (anomalous Patterson, MAD), and also to determine the absolute configuration of a molecule.

\[ F_{(hkl)} \neq F_{(-h-k-l)} \]

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