TOPOLOGY OF THE NODAL AND CRITICAL POINT SETS
FOR EIGENFUNCTIONS OF ELLIPTIC OPERATORS

by

Jeffrey H. Albert
A.B. Columbia College
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ABSTRACT

The theorems announced here are the initial step of a program aimed at describing solutions of elliptic partial differential equations on a manifold by means of geometric properties, such as the layout of the zero set and the nature of the critical points.

Let $M$ be a compact, connected $C^\infty$ manifold without boundary. Let $C^\infty(M)$ denote the real $C^\infty$ functions on $M$. Let $L$ be a second-order, self-adjoint $C^\infty$ elliptic operator on $M$ and let $P = L + \rho$, where $\rho \in C^\infty(M)$. We say an eigenfunction $u \in C^\infty(M)$ of $L + \rho$ is generic if: (1) all the critical points of $u$ are non-degenerate, i.e., $u$ is a Morse function; and (2) the zeros are all regular points (implying that the zero set is a submanifold of codimension 1).

There are examples of operators, e.g., the Laplacian on the 2-sphere, having eigenfunctions which are not generic. However, if dimension $M = 2$, then for most $\rho \in C^\infty(M)$, $P = L + \rho$ has only simple eigenvalues and generic eigenfunctions. The statement "for most $\rho" means that the set of $\rho$ with these properties is a countable intersection of sets which are open and dense in $C^\infty(M)$ in the $C^s$ topology. (It is proved without any restriction on dimension $M$ that most operators $P = L + \rho$ have simple eigenvalues.)
3.

In addition to the above results, we have the following: if $M$ and $P$ are real-analytic, an eigenfunction of $P$ has only a finite number of critical values.

Thesis Supervisor: Victor W. Guillemin
Title: Associate Professor of Mathematics
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1. Introduction and Examples

(1.1) Outline (For definitions see (1.2).)

We are interested in investigating the question of what kinds of functions occur as solutions, and in particular, as eigenfunctions, of second-order elliptic partial differential operators. The aim is to characterize these functions by means of geometric properties, for example, by describing the nodal and critical point sets.

In the case of ordinary differential equations, results of this nature are well-known (the Sturm-Liouville theory). For examples: (1) the zeros are isolated; (2) the critical points are non-degenerate, so they are isolated; (3) the critical points occur away from the nodal set; (4) the n-th eigenfunction has exactly n zeros. [Ref. Ince].

We have aimed at deriving theorems analogous to these for partial differential equations. The situation is much more complicated, as expected. In (1.3), we discuss the properties exhibited by several eigenfunctions of the ordinary Laplacian on the 2-sphere. (1) to (4) are
all false. Except at critical points, the zero set is a 1-dimensional submanifold. (4) no longer makes sense, but can be replaced by: (4') there are exactly n connected components of the complement of the zero set of the n-th eigenfunction. It is known that there are always ≤ n components if the nodal set is sufficiently regular, but there are also examples of eigenfunctions yielding only two components, with arbitrarily high eigenvalue. [Ref. Courant-Hilbert]

The first of our theorems discusses the nature of the set of critical values of an eigenfunction. In the case of a real-analytic operator on a compact real-analytic manifold, the critical value set is finite. This is proved in section 2A and is followed in 2B by a discussion of the situation in the $C^\infty$ case. In section 2C, we show that if the dimension of $M$ is 2, the nodal critical points can be described very precisely. The diagram of a nodal set of an eigenfunction will always resemble the examples in (1.3).

The remaining sections (3 to 7) are devoted to the proof of one theorem describing a generic class of operators. The examples of (1.3) are peculiar because
of their symmetry. For "almost all" operators on a two-dimensional manifold, the eigenvalues are simple and the critical points of non-zero eigenfunctions are non-degenerate and occur away from the nodal lines. [See (3.2) for the precise statements.] I.e., (2) and (3) above are generically true and the nodal set consists of some number of 1-dimensional submanifolds, i.e. circles. A very natural next question (related to (4')) to ask is: Is the number of circles related to the eigenvalue and if so, how?

(1.2) **Terminology and Notation.**

Manifolds will be connected and without boundary, and at least $C^\infty$. If $M$ is a $C^\infty$ manifold, $C^\infty(M)$ denotes the real-valued $C^\infty$ functions on $M$. $C^\infty_0(M)$ denotes those with compact support. For $u \in C^\infty_0(M)$, the zero or nodal set of $u$ is \{p \in M : u(p) = 0\}. $p \in M$ is a critical point of $u$ if $du(p) = 0$; otherwise $p$ is a regular point. If $(x_1, ..., x_v)$ are local coordinates on $M$ about $p$, $du(p) = 0$ means $(\partial u/\partial x_i)(p) = 0$, $i=1, ..., v$. $a \in \mathbb{R}$ is a critical value of $u$ if there exists a critical point $p$ of $u$ such that $u(p) = a$. A critical point $p$
of \( u \) is non-degenerate or Morse if the matrix

\[
\left( \frac{\partial^2 u}{\partial x_i \partial x_j} (p) \right)
\]

is non-singular. This notion is independent of the choice of coordinates about \( p \). \( u \in C^\infty(M) \) is non-degenerate or Morse if all the critical points of \( u \) are non-degenerate. Non-degenerate critical points are isolated, hence there are a finite number of them on any compact manifold.

Let \( \alpha = (\alpha_1, \ldots, \alpha_v) \) be a multi-index, i.e. a \( v \)-tuple of non-negative integers. \( |\alpha| = \sum_1^v \alpha_j \) and

\[
D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_v)^{\alpha_v}.
\]

A point \( a \in M \) is a zero of order \( k \) of \( u \) if \( D^\alpha u(a) = 0 \) for all \( \alpha : |\alpha| < k \) but \( D^\alpha u(a) \neq 0 \) for some \( \alpha \) with \( |\alpha| = k \). A nodal critical point is a zero of order \( \geq 2 \).

A \( C^\infty \) differential operator on \( M \) is a linear map \( P : C^\infty(M) \to C^\infty(M) \) such that on any open coordinate nbd \( U \subseteq M \) with coordinates \( (x_1, \ldots, x_v) \), \( P|_{C^\infty_0(U)} \) can be represented \( P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \) where \( \alpha = (\alpha_1, \ldots, \alpha_v) \) is a multi-index and \( a_\alpha \in C^\infty(M) \). \( m \) is the order of \( P \).
If \( M \) is a real-analytic manifold, \( P \) is an **analytic operator** if the \( a_\alpha \) are real-analytic.

Let \( \int f(x) \, dx \), or just \( \int f \), denote the integral of \( f \) over \( M \). Let \( \langle f, g \rangle = \int fg \) for \( f, g \in C^\infty(M) \).

\( P \) is **self-adjoint** if \( \langle Pf, g \rangle = \langle f, Pg \rangle \) for all \( f, g \).

The **symbol** of \( P \) is a real-valued function \( \sigma_p \) on the cotangent bundle defined as follows. Let \((x_1, \ldots, x_\nu)\) be coordinates on \( U \subset M \). If \( \xi = \sum_{i=1}^{\nu} \xi_i dx_i(p) \) is a cotangent vector at \( p \), \( \sigma_p(p, \xi) = \sum_{|\alpha| = m} a_\alpha(p) \xi_\alpha \).

\( P \) is **elliptic** at \( p \) if \( \sigma_p(p, \xi) \neq 0 \) for all non-zero cotangent vectors \( \xi \) at \( p \).

Let \( M \) be a compact \( C^\infty \) manifold. We will be considering second-order, self-adjoint \( C^\infty \) elliptic differential operators \( P \) on \( M \). An **eigenvalue** of \( P \) is a number \( \lambda \) (real, since \( P \) is self-adjoint) such that \( \ker (P+\lambda) \neq \{0\} \). \( \ker (P+\lambda) \) is the **eigenspace** corresponding to \( \lambda \) and \( u \in \ker (P+\lambda) \) is an **eigenfunction** of \( P \) with eigenvalue \( \lambda \). [Note
that \( \lambda \) is really the negative of an eigenvalue according to the usual definition.] \( P \) has a countable set of isolated eigenvalues having \( +\infty \) as its only limit point. The **multiplicity** of \( \lambda \) is the dimension of \( \ker (P+\lambda) \); it is finite. \( \lambda \) is a **simple** eigenvalue if its multiplicity is 1; otherwise it is **multiple**. We can arrange the eigenvalues of \( P \) in increasing order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots + \infty \), where an eigenvalue of multiplicity \( h \) is repeated \( h \) times in the listing. Thus we speak of the "\( n \)-th eigenvalue" or the "first \( n \) eigenvalues" of \( P \). The "\( n \)-th eigenfunction" is defined inductively to mean any eigenfunction which is in the \( n \)-th eigenspace and is orthogonal to the first \( n-1 \) eigenfunctions. It is not uniquely defined.

(1.3) **Examples**

On the sphere \( S^2 \subset \mathbb{R}^3 \) of radius \( r \), we use local (spherical) coordinates \( y_1 \) and \( y_2 \). In terms of rectangular coordinates \( (x_1, x_2, x_3) \) on \( \mathbb{R}^3 \), we have: 
\[
\begin{align*}
x_1 &= r \sin y_1 \sin y_2; \quad x_2 = r \sin y_1 \cos y_2; \\
x_3 &= r \cos y_1. \quad (y_1, y_2) \text{ are valid coordinates}
\end{align*}
\]
everywhere except at the north and south poles. The Laplacian on $S^2$ in these coordinates (in the metric induced from $\mathbb{R}^3$) is

$$\Delta = \frac{1}{r^2 \sin^2 y_1} \left\{ \sin y_1 \frac{\partial}{\partial y_1} \left( \sin y_1 \frac{\partial}{\partial y_1} \right) + \frac{\partial^2}{\partial y_2^2} \right\}$$

The eigenvalues of $\Delta$ have the form

$$\lambda = \frac{n(n-1)}{r^2}, \quad n = 1, 2, \ldots$$

and a basis for the eigenfunctions in the $n$-th eigenspace is given by

$$Y_{nm}^+(y_1, y_2) = p_{n-1}^m(\cos y_1)\sin my_2 \quad 0 < m \leq n-1$$

$$Y_{nm}^-(y_1, y_2) = p_{n-1}^m(\cos y_1)\cos my_2 \quad 0 \leq m \leq n-1$$

where $p_n^m(x) = K_{nm}(1 - x^2)^{m/2}(d/dx)^{n+m}(1 - x^2)^n$ are the Legendre polynomials ($K_{nm}$ normalizes $p_n^m$ so that $p_n^m(0) = 1$).

At the end of this section, there are diagrams of the nodal and critical point sets of some eigenfunctions in the second and fourth eigenspaces. These are
13.

\[ Y_{20}^- = \cos y_1 \quad Y_{42}^- = \sin^2 y_1 \cos y_1 \cos 2y_2 \]

\[ Y_{40}^- = (5 \cos^3 y_1 - 3 \cos y_1)/2 \quad Y_{43}^- = \sin^3 y_1 \cos 3y_2 \]

\[ Y_{41}^- = \sin y_1 (5 \cos^2 y_1 - 1) \cos y_2/4 \]

For \( Y_{n0}^- \), \( n = 2, 4 \), the zero set consists of \( n-1 \) distinct circles separating the sphere into \( n \) regions.

For \( Y_{20}^- \), there are exactly two non-degenerate critical points, a maximum and a minimum. For \( Y_{40}^- \), the absolute maximum and minimum are isolated, but there are degenerate local maxima and minima occurring in latitudinal circles lying between the components of the zero set.

For \( Y_{41}^- \), \( Y_{42}^- \) and \( Y_{43}^- \), the nodal sets consist of intersecting circles, with a different pattern in each case. At each intersection is a critical point. For \( Y_{41}^- \) and \( Y_{42}^- \), they are non-degenerate and two circles of the nodal set intersect at each place. \( Y_{43}^- \) has
two nodal critical points, at the north and south poles. They are degenerate; in fact, they are zeros of order 3. There are three longitudinal circles of the nodal lines intersecting at each nodal critical point.

To summarize, we have exhibited critical points on the nodal lines, degenerate critical points both on and off the nodal lines, and (non-nodal) critical points which are not isolated. And each eigenspace exhibits a huge range of phenomena.
Diagrams of the Nodal and Critical Sets
for Some Eigenfunctions of the Laplacian on the 2-sphere

In all diagrams except the one at the left, only a planar projection of the front hemisphere is drawn.

\[ Y_{20} = \cos y_1 \]

\[ Y_{40} = 5 \cos^3 y_1 - 3 \cos y_1 \]
Critical points

\[ Y_{42} = \sin^2 y_1 \cos y_1 \cos 2y_2 \]

Nodal critical points

\[ Y_{43} = \sin^2 y_1 \cos 3y_2 \]

Nodal critical point

\[ Y_{41} = \frac{1}{4} \sin y_1 (5\cos^2 y_1 - 1) \cos y_2 \]
2. The Critical Value Set of an Eigenfunction

This section is concerned with the problem of whether or not an eigenfunction of an elliptic operator on a compact manifold always has a finite critical value set. First we prove that if the manifold and operator are analytic, the critical value set is finite. Notice that the example of the Laplacian on the sphere is analytic. The result follows from the fact that any analytic function on a compact manifold has a finite critical value set. In the $C^\infty$ case, the situation is different. There are plenty of $C^\infty$ functions having infinite critical value sets. In fact, we show that locally there are examples in which eigenfunctions of elliptic operators have infinite critical value sets. However, these local examples cannot be extended to global counter-examples. So the general question of whether or not there is a finite critical value theorem for the $C^\infty$ case is still open.

The section concludes with a discussion of what the nodal critical points look like if the dimension of $M$ is 2. They are isolated (hence 0 is an
isolated critical value) and near any nodal critical point the nodal set consists of a finite number of rays emanating from the critical point.

A. The Analytic Case

In this section, "analytic" will always mean "real-analytic". Let $M$ be an analytic manifold. $c^\omega(M)$ will denote the analytic functions on $M$.

(2.1) Definitions.

(1) $\Omega \subseteq M$ is an analytic variety if for all $p \in \Omega$, there is a nbd $V$ of $p$ and analytic functions $f_1, \ldots, f_m$ on $V$ such that

$\Omega \subseteq V = \{x \in V : f_1(x) = \ldots = f_m(x) = 0\}$.

(2) By an analytic arc connecting $p$ to $q$ through $\Omega$, we mean a continuous function $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$, $\gamma(1) = q$, $\gamma(t) \in \Omega$ for all $t \in [0, 1]$ and $\gamma$ is analytic on $(0, 1)$.

(3) A set $\Omega \subseteq M$ is locally analytically arcwise connected if for all $p \in \Omega$, there is a nbd $V$ of $p$ in $M$ such that $q \in \Omega \cap V$ implies there is an
analytic arc connecting \( p \) to \( q \) through \( \Omega \).

(2.2) **Theorem.** An analytic variety is locally analytically arcwise connected.

**Pf:** For \( M \) an open subset of \( \mathbb{R}^v \), there is a proof of this theorem in the paper by Whitney and Bruhat. Since this is a local theorem, it translates to a manifold easily. Let \( p \in \Omega \) and choose a nbd \( V \) of \( p \) such that \( V \) is the domain for a coordinate chart \( \phi : V \rightarrow \mathbb{R}^v \) and \( \Omega \cap V = \{ x \in V : f_1(x) = \ldots = f_m(x) = 0 \}, f_i \in C^\infty(V) \). Then \( \phi(\Omega \cap V) \) is an analytic variety in \( \phi(V) \subset \mathbb{R}^v \). There exists a nbd \( W \) of \( \phi(p) \) such that any point in \( W \) can be connected to \( \phi(p) \) through \( \phi(\Omega \cap V) \) via an analytic arc. Pulling back to \( M \), any point in \( \phi^{-1}(W) \) can be connected to \( p \) through \( \Omega \cap V \) via an analytic arc.

(2.3) **Proposition.** If \( f \in C^\infty(M) \),
\[ \Omega = \{ p \in M : df(p) = 0 \} \] is an analytic variety in \( M \).

**Pf:** On any coordinate nbd \( U \), with coordinates \( (x_1, \ldots, x_v) \), \( \Omega \cap U = \{ p \in U : \frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_v}(p) = 0 \} \).
(2.4) **THEOREM.** If $M$ is compact, $f \in C^0(M)$ has a finite number of critical values.

**Pf:** Let $a \in \mathbb{R}$ be a critical value of $f$. Let $p \in M$ be a critical point such that $f(p) = a$. Since the set $\Omega$ of critical points is an analytic variety, it is locally analytically arcwise connected, so there is a nbd $N_p$ of $p$ such that, for all $q \in \Omega \cap N_p$, there is an analytic arc $\gamma : [0, 1] \to N_p$ with $\gamma(0) = p$, $\gamma(1) = q$, $\gamma(t) \in \Omega$ for all $t$. This implies

$$\frac{d}{dt}(fo\gamma)(t_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t_0)) \frac{d(x_1 \circ \gamma)}{dt}(t_0) = 0$$

for $t_0 \in (0, 1)$, since $\gamma(t_0) \in \Omega$. Hence $fo\gamma$ is constant on $(0, 1)$ and, by continuity, also on $[0, 1]$ and thus $f(q) = f(\gamma(1)) = f(\gamma(0)) = f(p) = a$.

Therefore for each critical point $p \in f^{-1}(a)$, there is a nbd $N_p$ on which all critical points have critical value $a$. For each regular point $p \in f^{-1}(a)$, there is a nbd $N_p$ containing no critical points of $f$ at all.
Let $N = U\{N_p : p \in f^{-1}(a)\}$. $N$ is a nbd of $f^{-1}(a)$ containing no critical points with critical value $\neq a$. Also since $M$ is compact, $f$ is an open map, so $f(N)$ is a nbd of $a$ containing no critical values of $f$ different from $a$. Therefore $a$ is isolated. Since $f(M)$ is also compact, the set of critical values is finite.

(2.5) **Theorem.** Let $Pu = f$. If $f \in C^\infty(M)$ and $P$ is an analytic elliptic differential operator, then $u \in C^\infty(M)$.

For a proof in $\mathbb{R}^n$, see Bers and Schechter. But this is entirely a local theorem, since analyticity (of functions and operators) is a local notion.

In particular, an eigenfunction of $P$ is analytic and has a finite number of critical values.

**B. The $C^\infty$ case**

(2.6) **Lemma.** Let $F(y) = \begin{cases} e^{-1/y^2} \left[p(1/y)\sin \left(\frac{1}{y}\right) + q(1/y)\cos \left(\frac{1}{y}\right)\right] & \text{for } y \neq 0 \\ 0 & \text{for } y = 0 \end{cases}$
where \( p \) and \( q \) are polynomials. Then \( F : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and differentiable and its derivative has the same form as \( F \).

**Corollary.** \( F \) is \( C^\infty \) and vanishes together with all its derivatives at \( y = 0 \). Hence \( F \) is not analytic.

**Pf. of lemma:** \( F \) is clearly continuous and differentiable for \( y \neq 0 \). Also we have:

\[
0 \leq \lim_{y \to 0^+} |p(1/y)e^{-1/y^2} \sin (1/y)|.
\]

\[
= \lim_{t \to +\infty} |p(t)e^{-t^2} \sin t|
\]

\[
< \lim_{t \to +\infty} |p(t)e^{-t}| \quad \text{since } |\sin t| \leq 1 \text{ and } t > 1 \Rightarrow e^{-t^2} < e^{-t}
\]

\[
= 0 \quad \text{since } "e^t \text{ dominates all polynomials}"
\]

Hence \( \lim_{y \to 0^+} p(1/y)e^{-1/y^2} \sin 1/y = 0 \). Similarly

\[
\lim_{y \to 0^+} q(1/y)e^{-1/y^2} \cos 1/y = 0 \quad \text{and} \quad \lim_{y \to 0^+} F(y) = 0.
\]
Also \( \lim_{y \to 0^-} F(y) = 0 \). Therefore, \( F \) is continuous at 0.

\[
F'(0) = \lim_{y \to 0} \frac{F(y) - F(0)}{y - 0} = \lim_{y \to 0} \frac{F(y)}{y} = 0
\]

since \( y \to \frac{F(y)}{y} \) has the same form as \( F(y) \), hence is continuous at 0. Thus \( F \) is differentiable at 0 and \( F'(0) = 0 \). For \( y \neq 0 \),

\[
F'(y) = e^{-1/y^2} \left[ p_1(1/y) \sin (1/y) + q_1(1/y) \cos (1/y) \right]
\]

where

\[
p_1(t) = -t^2 p'(t) + 2t^3 p(t) + t^2 q(t)
\]

\[
q_1(t) = -t^2 p(t) - t^2 q'(t) + 2t^3 q(t)
\]

so \( F' \) has the same form as \( F \).

(2.7) Lemma. Let \( F : \mathbb{R} \to \mathbb{R} \) be given by

\[
F(y) = \begin{cases} 
  e^{-1/y^2} \sin (1/y) & \text{if } y \neq 0 \\
  0 & \text{if } y = 0
\end{cases}
\]
Then \( F \in C^\infty(\mathbb{R}) \) and there is an infinite sequence of critical points of \( F \) with distinct critical values which converge to the critical point zero.

Proof: That \( F \in C^\infty(\mathbb{R}) \) and zero is a critical point with critical value zero follow from the corollary to lemma (2.6). Since \( \phi(y) = 1/y \) is a diffeomorphism of \( \mathbb{R}^+ \) onto itself, \( y \) is a critical point of \( F \) iff \( 1/y \) is a critical point of \( G = F \circ \phi \); the corresponding critical values of \( F \) and \( G \) are the same. Thus it suffices to show that \( G(t) = e^{-t^2} \sin t \) has an infinite sequence of critical points converging to \(+\infty\) and having distinct critical values. We have

\[
G'(t) = e^{-t^2} \left( -2t \sin t + \cos t \right)
\]

Therefore \( G \) has a critical point \( t \) whenever

\[
\cot t - 2t = 0.
\]

Since \( \lim_{t \to n\pi^+} (\cot t - 2t) = +\infty \) and \( \lim_{t \to n\pi^-} (\cot t - 2t) = -\infty \) for all integers \( n \), there is at least one zero of \( G' \), i.e. critical point of \( G \), in every interval \((n\pi, (n+1)\pi)\). Let \( t \) be such a
critical point. Then \( \cot t = 2t \), implying \( \sin t = \pm \frac{1}{\sqrt{1 + 4t^2}} \). If \( n \) is even, then sine is positive, so \( \sin t = \frac{1}{\sqrt{1 + 4t^2}} \) and

\[ G(t) = \frac{1}{e^{t^2}} \sqrt{1 + 4t^2} \]

is the critical value for the critical point \( C \). But the function \( t + \frac{1}{e^{t^2}} \sqrt{1 + 4t^2} \) is strictly decreasing on \( \mathbb{R}^+ \), so these critical values are distinct. Therefore there is an infinite sequence \( \{t_n\} \) of critical point of \( G \), one in each interval \( (n\pi, (n+1)\pi) \) for \( n \) even, having distinct critical values and converging to \( +\infty \). Hence the sequence \( \{y_n\}, y_n = \frac{1}{t_n} \) satisfies the conclusions of the theorem for \( F \).

(2.8) Lemma: Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
 f(x_1, x_2) = \begin{cases} 
 1 + x_1^2 + e^{-\frac{1}{x_2^2}} \sin \left( \frac{1}{x_2} \right) & \text{if } x_2 \neq 0 \\
 1 + x_1^2 & \text{if } x_2 = 0
\end{cases}
\]

Then \( f \in C^\infty(\mathbb{R}^2) \) and \( f \) has an infinite sequence of critical points which converge to the critical point zero and have distinct critical values.
Clearly \( f \in C^\infty(\mathbb{R}^2) \). We have \( \partial f/\partial x_1 = 2x_1 \), 
\( \partial f/\partial x_2 = F'(x_2) \), where \( F \) is the function of lemma 2.7. The critical points of \( f \) must have the form \((0, x_2)\), where \( x_2 \) is a critical point of \( F \). Also \( f(0, x_2) = F(x_2) \) so the critical values of \( f \) are exactly the same as those of \( F \). By lemma 2.7, we can construct an infinite sequence of critical points of \( f \) lying on the \( x_2 \)-axis, converging to zero, with distinct critical values converging to 1.

**Remark.** If \( U \) is any nbd of 0, \( f|U \in C^\infty(u) \) and the conclusion of the above lemma is valid by extracting from the previous sequence a subsequence lying in \( U \). Thus \( f|U \) has an infinite sequence of distinct critical values converging to 0.

**Proposition.** There is a nbd \( U \) of 0 in \( \mathbb{R}^2 \) and an elliptic operator \( L \) on \( U \) such that \( Lf = f \), \( f \) as in lemma 2.8.

**Pf.** Let \( \Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 \) be the ordinary Laplacian on \( \mathbb{R}^2 \). \( \Delta f(x_1, x_2) = 2 + F''(x_2) \), so \( \Delta f(0, 0) = 2 \).
Let $\phi = \Delta f$ and choose a mbd $U$ of $0$ on which $\phi \neq 0$. Define $L = (f/\phi)\Delta$. Then $L$ is a $C^\infty$ elliptic operator on $U$ since it is a multiple of the Laplacian by a non-zero $C^\infty$ function. Also $Lf = (f/\phi)\Delta f = f$ since $\phi = \Delta f$.

Remarks. We have just constructed a local example of a function which is an eigenfunction of a $C^\infty$ elliptic differential operator and has a non-isolated critical value. To see that this cannot be extended to a global example, notice that $L$ fails to be elliptic at any point where the eigenfunction $f$ vanishes, for $f/\phi$ then vanishes. But an elliptic operator which annihilates constants, such as $L$ above, has zero as an eigenvalue and an eigenfunction in a different eigenspace must be orthogonal to constants and hence must vanish somewhere.

C. The Nodal Critical Points

(2.10) Notation.

(1) Suppose $U \subset M$ is the domain for a coordinate chart $\phi : U \to \mathbb{R}^v$ with coordinates $\phi(x) = (x_1, \ldots, x_v)$. 
Let $a \in U$ and $f \in C^\infty(M)$. We will write the Taylor series of $f$ about $a$ as

$$f(x) = \sum D^\alpha f(a)[x - a]^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_v)$ is a multi-index; it is understood that we are writing the Taylor series of $f \circ \phi^{-1}$ about $\phi(a)$ and evaluating it at $\phi(x)$.

(2) Let $(\cdot, \cdot)$ be a metric on the cotangent bundle of $M$. This defines a norm in every cotangent space. Given $f \in C^\infty(M)$, $|df(p)|$ will mean $(df(p), df(p))^{1/2}_p$. Define

$$f_{(1)}(p) = |f(p)| + |df(p)|$$

Then:

(i) $f_{(1)}(p) \geq 0$ for all $p$ and $f_{(1)}(p) = 0$ if and only if $p$ is a nodal critical point of $f$.

(ii) $(f + g)_{(1)}(p) \leq f_{(1)}(p) + g_{(1)}(p)$

(iii) $|f|_{(1)} = \sup \{f_{(1)}(p) : p \in M\}$ defines a norm on $C^\infty(M)$
(iv) given coordinates \((x_1, \ldots, x_v)\) on an open set \(U \subset M\), \(|df|\) is equivalent to \(\sum_{i=1}^{v} |\partial f / \partial x_i|\) on \(U\).

(3) Given coordinates \((x_1, \ldots, x_v)\) on an open set \(U\), let \(\Delta = \sum_{i=1}^{v} \partial^2 \partial x_i^2\) be the ordinary Laplacian in these coordinates.

(4) \(u \in C^\infty(M)\) has a zero of order \(k\) at \(a \in M\) (see defn., p. 9) iff there is a nbd \(U\) of \(a\) and a constant \(c > 0\) such that
\(|u(x)| < c|x-a|^k\), all \(x \in U\). Facts
(i) If \(P\) is a differential operator of order \(m\) and \(u\) has a zero of order \(k\) at \(a\), then \(Pu\) has a zero of order \(k-m\) at \(a\),
(ii) If \(f \in C^\infty(M)\) has a zero of order \(j\) and \(u\) has a zero of order \(k\) at \(a\), then \(fu\) has a zero of order \(k+j\) at \(a\), (iii) a non-zero polynomial in \(x-a\), homogeneous of degree \(k\), has a zero of order \(k\) at \(a\).

(2.11) Remark. Let \(P\) be a second order \(C^\infty\) elliptic differential operator on \(M\). For \(a \in M\), there exist coordinates \((x_1, \ldots, x_v)\) on a nbd \(U\) of \(a\) such that
\[ \sigma_p(a, \xi) = \sum_{i=1}^{\nu} \xi_i^2 \] in these coordinates.

(2.12) **Theorem.** Let \( P \) be a second-order \( C^\infty \) elliptic differential operator on \( M \) and suppose \( Pu = 0 \). Given \( a \in M \), there exist coordinates \((x_1, \ldots, x_\nu)\) on a nbd \( U \) of \( a \) such that if \( u \) has a zero of order \( k \) at \( a \), the \( k \)-th Taylor polynomial
\[ u_k(x) = \sum_{|\alpha|=k} D^\alpha u(a)[x - a]^\alpha \] of \( u \) about \( a \) satisfies \( \Delta u_k = 0 \).

**Pf.** If \( k = 0 \) or 1, this is true for any coordinates, for a second order operator annihilates linear functions. Assume \( k > 2 \). Using (2.11), choose coordinates \((x_1, \ldots, x_\nu)\) on a nbd \( U \) of \( a \) so that
\[ \sigma_p(a, \xi) = \sum_{i=1}^{\nu} \xi_i^2 \] i.e. the principal part of \( P \) at \( a \) is \( \Delta \). Then
\[ P = \sum_{i,j=1}^{\nu} g_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + T \] where \( T \) is a first order operator
\[ = \sum_{i,j=1}^{\nu} \left[ g_{ij}(a) - g_{ij}(a) \right] \frac{\partial^2}{\partial x_i \partial x_j} + T \]
\[ = \Delta + \sum_{i,j=1}^{\nu} \left[ g_{ij}(a) \right] \frac{\partial^2}{\partial x_i \partial x_j} + T \]
Now let $v = u - u_k$ be the remainder of the $k$-th Taylor polynomial. $v$ has a zero of order $\geq k+1$ at $a$.

We have

$$0 = Pu = \Delta u_k + \Delta v + \sum_{i,j=1}^{\nu} \left[ e_{ij} - e_{ij}(a) \right] \frac{\partial^2 u}{\partial x_i \partial x_j} + Tu$$

$$= \Delta u_k + f$$

Claim $f$ has a zero of order $\geq k-1$ at $a$. This will imply $\Delta u_k = 0$ for otherwise $\Delta u_k$ must have a zero of order $k-2$ at $a$, hence so must $f$. To show the claim,

(1) $v$ has a zero of order $\geq k+1$ and $\Delta$ is of order 2, so $\Delta v$ has a zero of order $\geq k-1$.

(2) $u$ has a zero of order $k$ and $T$ has order 1, so $Tu$ has a zero of order $k-1$.

(3) $\partial^2 u / \partial x_i \partial x_j$ has a zero of order $k-2$, but $e_{ij} - e_{ij}(a)$ vanishes at $a$ (zero of order $\geq 1$), so

$$\sum_{i,j=1}^{\nu} \left[ e_{ij} - e_{ij}(a) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}$$

has a zero of order $\geq k-1$. 
(2.13) Assume \( \dim M = 2 \).

**Proposition.** Let \((x_1, x_2)\) be coordinates on a nbd \(U\) of \(a \in M\). Suppose that \(u\) has a zero of order \(k\) at \(a\) and \(\Delta u_k = 0\). Then there is a nbd \(V \subset U\) and a constant \(K > 0\) such that

\[
|du(x)| \geq K|x-a|^{k-1}
\]

**Pf.** Since \( \dim M = 2 \), view \( \mathbb{R}^2 \) as \( \mathbb{C} \); this allows us to use complex variables techniques \((z = x_1 + x_2\sqrt{-1})\).

Since \( \Delta u_k = 0 \), \(u_k = \text{Re } f\), for some complex analytic function \(f\). Requiring \(f(a) = 0\), which we may do, implies that \(f\) is homogeneous of degree \(k\) in \(z-a\).

Hence \(f(z) = \zeta(z-a)^k\) for some \(\zeta \in \mathbb{C}\), so

\[
u_k(z) = \frac{1}{2i}[\zeta(z-a)^k + \bar{\zeta}(\bar{z}-\bar{a})^k]
\]

Now note that \(|du_k(x)| = 2|\frac{\partial u_k}{\partial z}(z)|\) where

\[
\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2}).
\]

But \(\frac{\partial u_k}{\partial z} = \frac{k}{2}\zeta(z-a)^{k-1}\).
so \[ |d_{u,k}(x)| = k|\zeta||x-a|^{k-1} \]

\[ |d_u(x)| \geq |d_{u,k}(x)| - |d_v(x)| \]

\[ \geq k|\zeta||x-a|^{k-1} - c|x-a|^k \] since \( v \) has a zero of order \( k+1 \)

\[ \geq \frac{1}{2}k|\zeta||x-a|^{k-1} \]

provided \( |x-a| < k|\zeta|/2c \) (this defines \( V \))

**Remark.** If \( P\mu = 0 \) (or equivalently, \( (P+\lambda)u = 0 \)), (2.12) and (2.13) imply that if \( a \in M \) is a zero of order \( k \) of \( u \), then there is a nbd \( V \) of \( a \) on which \( |u(x)| \leq K|x-a|^k \) and \( |du(x)| \geq K'|x-a|^{k-1} \) for constants \( K, K' > 0 \).

**Corollary.** Let \( P\mu = 0 \) with \( P \) as above.

(i) Nodal critical points of \( u \) are isolated

(ii) The nodal set \( u^{-1}(0) \) has measure zero

(iii) Zero is an isolated critical value

**Pf:** (i) is immediate for the proposition implies that \( a \) is the only critical point in \( V \). For (ii), the
set $N$ of regular nodal points has measure zero in $M$ since there exists a countable covering of $N$ by coordinate nbds $U_i$ in which $N \cap U_i$ is one-dimensional, hence has measure zero. Since the nodal critical points are isolated and $M$ is compact, there are only a finite number of them. Hence $u^{-1}(0)$ has measure zero. (iii): For each $p \in u^{-1}(0)$, if $p$ is a critical point, there is a nbd $N_p$ containing no critical points other than $p$ by (i) and if $p$ is a regular point, there is a nbd $N_p$ containing no critical points at all. Let $N = \bigcup \{N_p : p \in u^{-1}(0)\}$. Then $u(N)$ is a nbd of 0 containing no critical values other than zero.

Remarks.

(1) Notice that in the example in section B above, the critical value is 1.

(2) These results also enter into the proof of the theorem on generic eigenfunctions (see section 7).

(3) We actually know more about the structure of the nodal set of $u$, that in a nbd of a nodal critical
point \( a \), the nodal set consists of \( 2k \) rays (curves) emanating from \( a \) (where \( k \) is the order of the zero). For this is true of \( f(z) = \text{Re } z^k \) and there exists a nbd \( W \) of \( a \) and a homeomorphism from \( W \) onto an open set in \( \mathbb{R}^2 \) such that \( a \) gets mapped onto 0 and the nodal set of \( u \) gets mapped onto the nodal set of \( f \). This follows from theorem I in Kuo.
3. Generic Elliptic Operators

(3.1) The $C^s$ norms. Let $M$ be a compact $C^\infty$ manifold of dimension $v$. Fix open sets $U_1, \ldots, U_r$ covering $M$, with coordinates $(x_1^{(i)}, \ldots, x_v^{(i)})$ on $U_i$ and let 
\{\psi_i\} be a partition of unity subordinate to \{U_i\}. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_v)$, set

$$D^\alpha_{(i)} = \left(\partial / \partial x_1^{(i)}\right)^{\alpha_1} \ldots \left(\partial / \partial x_v^{(i)}\right)^{\alpha_v}.$$ 
In addition to using $D^\alpha_{(i)}$ for pointwise differentiations, we will consider it as a map $C^s_0(U_i) \rightarrow C^\infty(M)$.

**Definition.** Let $f \in C^\infty(M)$.

$$|f| = \sup \{|f(p)| : p \in M\}$$

$$|f|_s = \sum_{i=1}^r \sum_{|\alpha| \leq s} |D^\alpha_{(i)}(\psi_i f)|$$

Note that $|f| \leq |f|_0 \leq r|f|$ so these norms are equivalent. Also $|f|_{(1)}$ of (2.10) is equivalent to $|f|_1$. 

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By the $C^S$ topology on $C^\infty(M)$, we mean the topology induced by the norm $|...|_S$. This is equivalent to the usual definition.

**Facts:**

(1) If $P$ is a differential operator of order $m$, then

$$|Pf|_S \leq (\text{const})|f|_{S+m}$$

(2) $|fg|_S \leq (\text{const})|f|_S|g|_S$

(3.2) **Main Theorem.**

Let $L$ be a second order, self-adjoint, $C^\infty$ elliptic differential operator on $M$. $L$ will be fixed from now on and we will study operators of the form $L + \rho$, where $\rho \in C^\infty(M)$ is called a forcing function.

**Definition.** Let $u$ be an eigenfunction of $L + \rho$, $u$ is **generic** if (1) all the critical points of $u$ are Morse and non-nodal or (2) $u$ is identically zero.

Let $A = \{\rho \in C^\infty(M) : \text{all eigenvalues of } L + \rho \text{ are simple}\}$. Let $B = \{\rho \in A : \text{all eigenfunctions of } L + \rho \text{ are generic}\}$. 
MAIN THEOREM: If dim M = 2, B is a countable intersection of sets which are open and dense in the $C^s$-topology on $C^\infty(M)$, provided $s \geq 2$.

Remark, It follows that B is dense in $C^\infty(M)$; in particular B is non-empty.

For $n = 1, 2, \ldots$, let

$A_n = \{ \rho \in C^\infty(M) : \text{the first } n \text{ eigenvalues of } L+\rho \text{ are simple} \}$

and let $A_0 = C^\infty(M)$. Then

$A \subset \ldots \subset A_n \subset \ldots \subset A_2 \subset A_1 \subset A_0 = C^\infty(M)$ and $A = \bigcap_{n=1}^{\infty} A_n$.

THEOREM I: $A_n$ is open in the $C^s$ topology on $C^\infty(M)$, for all $n$.

THEOREM II: $A_n$ is dense in $A_{n-1}$ in the $C^s$ topology, for all $n$.

Theorem II implies $A_n$ is dense in $C^\infty(M)$ for all $n$. Thus theorems I and II imply that A is a countable intersection of open, dense sets in $C^\infty(M)$. Theorems I and II are proved in section 5.
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Let \( B_n = \{ \rho \in A_n : u \in \bigcup_{j=1}^{n} \ker (L+\rho+\lambda_j) \text{ implies } u \text{ is generic} \} \) for \( n = 1, 2, \ldots \) and let \( B_0 = C^\infty(M) \).

**THEOREM III:** \( B_n \) is open in the \( C^s \) topology for all \( n \), provided \( s \geq [\nu/2] + 1 \).

**Note:** \([x]\) means the greatest integer \( \leq x \).

**THEOREM IV:** Assume \( \dim M = 2 \). \( B_n \) is dense in \( A_n \cap B_{n-1} \) in the \( C^s \) topology for all \( n \).

From theorems II and IV, we have

\[
B_n \subset A_n \cap B_{n-1} \subset A_{n-1} \cap B_{n-2} \subset \cdots \subset A_1 \cap B_0 = A_1 \subset C^\infty(M)
\]

dense dense dense dense dense

implying \( B_n \) is dense in \( C^\infty(M) \). Since \( B = \bigcap_{n=1}^{\infty} B_n \), we have the main theorem. Theorem III is proved in section 6 and theorem IV is proved in section 7. Note that theorem IV is the only place \( \dim M = 2 \) is mentioned; see also the introduction to section 7.
Intuitive Discussion: Theorem I just says that the property of having the first n eigenvalues of \( L + \rho \) simple is stable under small perturbations of \( \rho \). Theorem III says that the property of having generic eigenfunctions is stable under small enough perturbations (which must be at least small enough to keep the first n eigenvalues simple). Both of the density theorems use these openness results to show that you can change the n-th eigenvalue and/or eigenspace appropriately by a small perturbation while not really changing the previous n-1. In theorem II, if the n-th eigenvalue has multiplicity \( h > 1 \) but the previous n-1 eigenvalues are simple, we may break up the n-th eigenvalue into \( h \) simple eigenvalues by a small perturbation, thus producing \( n + h - 1 \) simple eigenvalues. Similarly, in theorem IV, we show that starting with the first n eigenvalues simple and the eigenfunctions in the first n-1 eigenspaces generic, it is possible to preserve these properties while making the n-th eigenfunction generic. So we perform a perturbation in \( A_n \cap B_{n-1} \) to get into \( B_n \).
Having simple eigenvalues is extremely important for these perturbation arguments, for if the n-th eigenvalue is simple, it is only necessary to produce one non-zero generic eigenfunction in the n-th eigenspace. All others will then be generic since they will be constant multiples of the first one.
4. Results from Elliptic Operator Theory and Perturbation Theory

(4.1) The $H^s$ norms:

For $f \in C^\infty(M)$, $||f|| = \int f^2$. Let

$$||f||_s = \sum_{i=r}^{r} \sum_{|\alpha| \leq s} ||D^\alpha(\psi f)||$$

The completion of $C^\infty(M)$ in the norm $||\ldots||_s$ is the Sobolev space $H^s(M)$. Note that $||f||_0 = ||f||$ and $H^0(M) = L^2(M)$.

**Lemma 1.** (Sobolev). For $f \in C^\infty(M)$,

$$|f|_s \leq (\text{const}) ||f||_t$$

provided $t \geq [\nu/2] + s + 1$

**Lemma 2.** $||fg||_s \leq (\text{const}) |f|_s ||g||_s$ for $f,g \in C^\infty(M)$.

**Remark.** The use of the partition of unity enables you to globalize the standard local results. Reference: Bers and Schechter.
(4.2) **The Minimax Principle.** Let $P$ be a second order, self-adjoint elliptic differential operator on $M$. For $u \in C^\infty(M)$, we define a quadratic form

$$A(u) = -\int u(Pu).$$

Let $\{v_1, \ldots, v_{n-1}\}$ be a linearly independent set of $C^\infty$ functions on $M$ and define

$$A_n(v_1, \ldots, v_{n-1}) = \min \left\{ A(u) : u \in C^\infty(M), \|u\| = 1, \langle u, v_i \rangle = 0, 1 \leq i \leq n-1 \right\}$$

Let the eigenvalues of $P$ be arranged in increasing order $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, and let $u_1, u_2, \ldots, u_n, \ldots$ be corresponding eigenfunctions with $\|u_n\| = 1$.

**Theorem.**

1. $\lambda_n = \max \{A_n(v_1, \ldots, v_{n-1}) : \{v_1, \ldots, v_{n-1}\} \text{ linearly independent in } C^\infty(M)\}$

2. $u = u_n, v_1 = u_1, \ldots, v_{n-1} = u_{n-1}$ are a set of extremals for the problem. (I.e. $\lambda_n = A_n(u_1, \ldots, u_{n-1}$) and $\lambda_n = -\int u_n Pu_n$, the latter being obvious.
(4.3) **Pseudo-Differential Inverse.** With $P$ as in (4.2), let $N = \ker P \subset C^\infty(M)$. Let $\pi_N : L_2(M) \rightarrow C^\infty(M)$ be the projection onto $N$.

**THEOREM.** There is a pseudo-differential operator $Q : C^\infty(M) \rightarrow C^\infty(M)$ of order $-2$ such that

$$QPu = u - \pi_N u$$

and

$$Q\pi_N u = 0 \quad \text{for } u \in C^\infty(M)$$

Furthermore, $Qf(x) = \int K(x,y)f(y)dy$, where the kernel $y \mapsto \kappa(x,y)$ is in $L_p(M)$ for fixed $x \in M$, where

$$p = \begin{cases} 
2 & \text{if } \nu = 2 \\
(\nu-1)/(\nu-2) & \text{if } \nu > 2
\end{cases}$$

A proof is given in appendix I.

**Regularity.** If $\Psi$ is a pseudo-differential operator of order $m$, then $||\Psi u||_s \leq (\text{const})||u||_{s+m}$ for $u \in C^\infty(M)$. 


(4.4) **Perturbation Theory**

**Defn.** Let \((E, ||...||)\) be a normed space. Let \(T > 0\) and let \(f\) be a function defined (at least) on the interval \((-T,T)\), with values in \(E\). We say \(f\) is **analytic in \(E\)** for \(|\tau| < T\) if there is a sequence \(\{f_i\}\) of elements of \(E\) such that \(\lim_{m \to \infty} \left| f(\tau) - \sum_{i=0}^{m} f_i \tau^i \right| = 0\) for \(|\tau| < T\). \(f\) is **absolutely analytic** for \(|\tau| < T\) if in addition, the real series \(\sum_{i=0}^{\infty} |f_i||\tau|^i\) converges if \(|\tau| < T\).

**Remarks.** (1) If \(f(\tau)\) is analytic for \(|\tau| < T\), \(f : (-T,T) \to E\) is a continuous function, hence \(\|f(\tau)\|\) is bounded on \(\{\tau : |\tau| < T'\}\) for all \(T' < T\).

(2) \(f(\tau)\) analytic for \(|\tau| \leq T\) means \(f(\tau)\) analytic for \(|\tau| < T'\) where \(T < T'\).

(3) If we write \(f(\tau) = \sum_{i=0}^{m-1} f_i \tau^i + \tau^m F_m(\tau)\), then \(F_m(\tau)\) is analytic for \(|\tau| < T\) if \(f\) is.

(4) \(f\) analytic in \(R\) means with respect to the usual norm. In this case, analytic and absolutely analytic are equivalent.
(4.5) **Perturbation Theorem.** Let \( \lambda \) be an eigenvalue of multiplicity \( h \) of \( L+\rho \). Suppose \( \rho(\tau) \) is absolutely analytic in \((C^\infty(M), \| \cdot \|_s)\) for all \( s \), on some nbd of \( \tau = 0 \), with \( \rho(0) = \rho \). Then: there exists a number \( T_0 > 0 \), \( h \) functions \( \lambda^{(1)}(\tau), \ldots, \lambda^{(h)}(\tau) \) analytic in \( \mathbb{R} \) for \( |\tau| < T_0 \) and \( h \) functions \( u^{(1)}(\tau), \ldots, u^{(h)}(\tau) \) analytic in \((C^\infty(M), \| \cdot \|_s)\) for \( |\tau| \leq T_0 \), such that:

(a) \( \lambda^{(i)}(0) = \lambda \), \( i = 1, \ldots, h \);

(b) \( u^{(i)}(\tau) \) is an eigenfunction of \( L+\rho(\tau) \), with eigenvalue \( \lambda^{(i)}(\tau) \) for \( |\tau| \leq T_0 \), \( i = 1, \ldots, h \);

(c) \( \{u^{(1)}(\tau), \ldots, u^{(h)}(\tau)\} \) is an orthonormal set in \( L^2(M) \) for each \( \tau : |\tau| \leq T_0 \);

(d) for every open interval \( I \subset \mathbb{R} \) such that \( \overline{I} \)
contains only the eigenvalue \( \lambda \) of \( L+\rho \), there is a
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T > 0 such that for |τ| < T there are exactly h eigenvalues (counting multiplicity)
λ(1)(τ),...,λ(h)(τ) of L+p(τ) in I.

This theorem follows from a minor variation of a theorem of Rellich (Theorem I, p. 57). However since neither the theorem nor Rellich's book seem to be well-known, we give a proof in Appendix II.

(4.6) Remarks. One can write down explicit expressions for the coefficients of these power series by equating coefficients in the equations

\[(L + p(τ) + \lambda^{(j)}(τ))u^{(j)}(τ) = 0 \quad (1)\]

However, rather than using the entire power series, we will generally only need the first two terms. We will write \(\lambda^{(j)}(τ) = \lambda + α_j τ + β_j(τ)τ^2\) for the perturbed eigenvalues, where \(α_j ∈ R\) and \(β_j(τ)\) is an ordinary power series in \(τ\) for \(|τ| ≤ T_0\); and
\( u^{(j)}(\tau) = u_j + v_j \tau + w_j(\tau)\tau^2 \) for the perturbed eigenfunctions, where \( u_j, v_j \in C^\infty(M) \) and \( w_j(\tau) \in C^\infty(M) \) is analytic with respect to the \( C^S \)-norm on \( C^\infty(M) \) for \( |\tau| \leq T_0 \). Also we will only need linear perturbations \( \rho(\tau) = \rho + \sigma \) where \( \sigma \in C^\infty(M) \).

Differentiating the equation (1) with respect to \( \tau \) and setting \( \tau = 0 \) yields

\[
(L + \rho + \lambda)v_j + (\sigma + \alpha_j)u_j = 0 \quad j = 1, \ldots, h \tag{2}
\]

From this it follows that

\[
(\sigma + \alpha_j)u_j \perp \text{ker} (L + \rho + \lambda) \quad j = 1, \ldots, h
\]

implying

\[
f(\sigma + \alpha_j)u_j u_k = 0 \quad j, k = 1, \ldots, h
\]

and

\[
-f\sigma u_j u_k = \alpha_j \delta_{jk} \tag{3}
\]

where \( \delta_{jk} \) is the Kronecker delta.
Also \( v_j = Q_\lambda [(\sigma + \alpha_j)u_j] + \sum_{k=1}^{h} c_{jk}u_k \) \hspace{1cm} (4)

\[ = Q_\lambda [\sigma u_j] + \sum_{k=1}^{h} c_{jk}u_k \]

for some \( c_{jk} \in \mathbb{R} \), where \( Q_\lambda \) is the pseudo-differential inverse of the operator \( L + \rho + \lambda \) (see 4.3).
5. Operators with Simple Eigenvalues

I. Openness

(5.1) Theorem. (Continuity of the Eigenvalues).

Let \( \rho, \rho' \in C^\infty(M) \) be two forcing functions,
\( \lambda_n \) and \( \lambda_n' \) the n-th eigenvalues of the operators
\( L+\rho \) and \( L+\rho' \) respectively. Then:
\[
|\lambda_n' - \lambda_n| \leq |\rho' - \rho|.
\]

\( \textbf{Pf:} \) Let \( \Lambda(f) = -\int f(Lf) - \int \rho f^2 \) and \( \Lambda'(f) = -\int f(Lf) - \int \rho' f^2 \)
be the quadratic forms of the minimax principle (4.2),
corresponding to the operators \( L+\rho \) and \( L+\rho' \) and
define \( \Lambda_n(f_1, \ldots, f_{n-1}) \) and \( \Lambda'_n(f_1, \ldots, f_{n-1}) \) relative
to \( \Lambda(f) \) and \( \Lambda'(f) \) respectively. Then
\[
|\Lambda'(f) - \Lambda(f)| = |\int (\rho' - \rho) f^2| \\
\leq |\rho' - \rho| \int f^2 \\
= |\rho' - \rho| \quad \text{if} \quad ||f|| = 1
\]

Let \( r = |\rho' - \rho| \), Hence
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\[ \Lambda(f) - r \leq \Lambda'(f) \leq \Lambda(f) + r \quad \text{for} \quad ||f|| = 1 \quad (*) \]

Let \( \{f_1, \ldots, f_{n-1}\} \) be a linearly independent set of \( C^\infty \) functions. By taking the minimum of (\(^*\)) over all \( f \in C^\infty(M) \) with \( ||f|| = 1 \) and \( f \perp f_1, \ldots, f_{n-1} \), we get

\[ \Lambda_n(f_1, \ldots, f_{n-1}) - r \leq \Lambda_n'(f_1, \ldots, f_{n-1}) \leq \Lambda_n(f_1, \ldots, f_{n-1}) + r \]

Similarly by taking the maximum over all linearly independent sets \( \{f_1, \ldots, f_{n-1}\} \) of \( C^\infty \) functions on \( M \) and applying (1) of the minimax principle, we get

\[ \lambda_n - r \leq \lambda_n' \leq \lambda_n + r \]

hence \( |\lambda_n' - \lambda_n| \leq r = |\rho' - \rho| \) as required.

(5.2) Proof of Theorem I: \( A_n \) is open in the \( C^S \) topology on \( C^\infty(M) \) for all \( n \).

Let \( \rho \in A_n \) be given and let \( \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} \leq \ldots \) be the eigenvalues of \( L + \rho \); the first \( n \) are simple. Let
δ = \min \{ \lambda_{j+1} - \lambda_j : j = 1, \ldots, n \}; \; \delta > 0. \] Let \( U = \{ \rho' \in C^\infty(M) : |\rho' - \rho|_g < \delta/2 \} \). \( U \) is an open nbd of \( \rho \) in the \( C^s \) topology. Let \( \{ \lambda_j' \} \) be the eigenvalues of \( L + \rho' \) for \( \rho' \in U \). Then, by (5.1),

\[ |\lambda_j' - \lambda_j| \leq |\rho' - \rho| \leq |\rho' - \rho|_g < \delta/2 \quad \text{all } j. \]

Claim \( \lambda_j' \neq \lambda_{j+1}' \) for \( j = 1, \ldots, n \). We have

\[
\begin{align*}
\delta & \leq \lambda_{j+1} - \lambda_j \leq |\lambda_j' - \lambda_{j+1}'| + |\lambda_j' - \lambda_{j+1}'| + |\lambda_j' - \lambda_j'|
\end{align*}
\]

\[
< \delta/2 + |\lambda_j' - \lambda_{j+1}'| + \delta/2
\]

\[
= \delta + (\lambda_{j+1}' - \lambda_j')
\]

implying \( \lambda_{j+1}' - \lambda_j' > 0, \; j = 1, \ldots, n \). Hence the first \( n \) eigenvalues of \( L + \rho' \) are simple if \( \rho' \in U \), so \( U \subset A_n \), implying \( A_n \) is open.
II. Density

We want to show that by perturbing the forcing function slightly, we can break up a multiple eigenvalue into eigenvalues with lower multiplicities. Then we can start with the smallest eigenvalue and break up the multiple eigenvalues one at a time until they are simple. We use the perturbation theorem (see 4.5 and 4.6 for notation).

Assume $\lambda$ is an eigenvalue of multiplicity $h$ of $L+\rho$ and that $\rho(\tau) = \rho + \tau \sigma$ is a perturbation of $\rho$. We have expressions

$$\lambda^{(j)}(\tau) = \lambda + \tau \alpha_j + \tau^2 \beta_j(\tau) \quad j = 1, \ldots, h$$

$$u^{(j)}(\tau) = u_j + \tau v_j + \tau^2 w_j(\tau)$$

for the perturbed eigenvalues and eigenfunctions, valid for $|\tau| \leq T_0$. 
(5.3) **Proposition** There exists $\sigma \in C^\infty(M)$ such that for the perturbation $\rho(\tau) = \rho + \tau \sigma$, at least two of the $\alpha_j$ are distinct.

**Pf.** Let $N = \ker (L + \rho + \lambda)$. $N$ is a vector space of dimension $h$ contained in $C^\infty(M)$, so $N$ has an inner product $\langle f, g \rangle = \int fg$. There is a quadratic form on $N$ given by $(f, g) \rightarrow -\int \sigma fg$, so there exists a self-adjoint linear operator $G_\sigma : N \rightarrow N$ such that $\langle G_\sigma f, g \rangle = -\int \sigma fg$, for all $f, g \in N$. Furthermore, given any orthonormal basis $\{f_1, \ldots, f_h\}$ of $N$, the matrix of $G_\sigma$ with respect to this basis has $\langle G_\sigma f_j, f_k \rangle$ as its $jk$-th entry; so:

(1) Given $\sigma$, and the orthonormal basis $\{u_1, \ldots, u_h\}$ of unperturbed eigenfunctions, we have $\langle G_\sigma u_j, u_k \rangle = -\int \sigma u_j u_k = \alpha_j \delta_{jk}$ (by 4.6, eqn (3)). So the matrix of $G_\sigma$ is diagonal in this basis; and its diagonal entries and hence its eigenvalues are $\alpha_1, \ldots, \alpha_h$.

(2) Fix a basis $\{f_1, \ldots, f_h\}$. Let $j \neq k$ and take $\sigma = f_j f_k$. Then the matrix of $G_\sigma$ has $-\|f_j f_k\|^2$. 
in its \( jk \)th place. Therefore \( G_\sigma \) is not a multiple of the identity so it must have at least two distinct eigenvalues. Hence for the perturbation \( \rho(\tau) = \rho + \tau \sigma \) with this \( \sigma \), at least two of the \( \alpha_j \) must be distinct by (1).

(5.4) Proof of Theorem II. \( A_n \) is dense in \( A_{n-1} \).

Let \( \rho \in A_{n-1} \) and let \( \varepsilon_0 > 0 \) be small enough so that (i) \( |\rho' - \rho|_s < \varepsilon_0 \) implies \( \rho' \in A_{n-1} \) and (ii) \( \lambda_n \) is the only eigenvalue of \( L + \rho \) in the interval \( [\lambda_n - \varepsilon_0, \lambda_n + \varepsilon_0] \). Let \( h \) be the multiplicity of \( \lambda_n \).

Claim: if \( h > 1 \), then for all \( \varepsilon : 0 < \varepsilon < \varepsilon_0 \), there exists \( \rho' \in C^\infty(M) \) such that \( |\rho' - \rho|_s < \varepsilon \) and \( \lambda_n' \) has multiplicity \( < h \). By applying the claim repeatedly, we can obtain a sequence \( \rho = \rho_1, \ldots, \rho_h \) of forcing functions so that \( |\rho_{j+1} - \rho_j| < \varepsilon/h \) (\( j = 1, \ldots, h-1 \)) and the \( n \)-th eigenvalue of \( L + \rho_j \) has multiplicity \( \leq h - j + 1 \).
Thus \( |\rho_h - \rho| < \varepsilon \) and the first \( n \) eigenvalues of \( L+\rho_h \) are simple, proving the theorem. Now we prove the claim:

1. By (5.3), we can choose \( \sigma \in C^\infty(M) \) such that if 
\[ \lambda^{(j)}(\tau) = \lambda + \alpha_j \tau + \beta_j(\tau) \tau^2 \]
are the perturbed eigenvalues corresponding to the unperturbed eigenvalue \( \lambda = \lambda_n \) and the perturbation \( \rho(\tau) = \rho + \tau \sigma \), then at least two of the \( \alpha_j \) are distinct. Then there exists \( T_1 > 0 \) such that for \( 0 < |\tau| < T_1 \), at least two of the \( \lambda^{(j)}(\tau) \) are distinct.

2. Since \( \lambda \) is the only eigenvalue of \( L+\rho \) in \( [\lambda-\varepsilon, \lambda+\varepsilon] \), there exists \( T_2 > 0 \) such that \( 0 < |\tau| < T_2 \) implies \( \lambda^{(1)}(\tau), \ldots, \lambda^{(h)}(\tau) \) are the only eigenvalues of \( L+\rho(\tau) \) in \( (\lambda-\varepsilon, \lambda+\varepsilon) \) (counting multiplicity) - see thm. 4.5 (d). If in addition \( |\tau| < T_1 \), at least two of these are distinct, hence they must all have multiplicity \( < h \).

3. Let \( \tau = \frac{1}{2} \min \{ T_1, T_2, \varepsilon/|\sigma|_s \} \) and let \( \rho' = \rho(\tau) \). Then \( |\rho' - \rho|_s = |\tau| |\sigma|_s < \varepsilon \). By (5.1), \( |\lambda'_n - \lambda_n| < \varepsilon \), so by (2), \( \lambda'_n \) must be one of the \( \lambda^{(j)}(\tau) \) and have multiplicity \( < h \).
6. **Generic Eigenfunctions I: Openness**

The object of this section is to prove theorem III, that by perturbing the forcing function a small amount, the property of having generic eigenfunctions, i.e. Morse with non-nodal critical points, is stable.

Throughout this entire section and the next we consider one eigenvalue $\lambda$ of $L+p$ at a time. We always assume that all the eigenvalues $\lambda \leq \lambda_n$ are simple. If $\lambda = \lambda_n$, then corresponding to a perturbed forcing function $p'$ we get a sequence of eigenvalues $\lambda_1' < \ldots < \lambda_n' < \lambda_{n+1}' < \ldots$. If $p' = p(\tau)$, then (for small $\tau$) $\lambda_n' = \lambda_n(\tau)$ is the perturbed eigenvalue discussed in the perturbation theorem. Thus there is no confusion; the perturbed eigenvalue corresponding to the $n$-th eigenvalue of the unperturbed operator is the $n$-th eigenvalue of the perturbed operator and it is simple. So we will talk about the "corresponding eigenvalue" $\lambda'$ or $\lambda(\tau)$ and the "corresponding eigenfunctions."

(6.1) Let $p \in A_n$ and let $\lambda$ be one of the first $n$ eigenvalues of $L+p$. Let $\delta_0 > 0$ be such that
\[ |\rho' - \rho|_s < \delta_0 \] implies \( \rho' \in \Lambda_n \). Let

\[ u \in \ker (L + \rho + \lambda), \quad ||u|| = 1. \] Assume

\[ s \geq t + \lfloor v/2 \rfloor - 1. \]

**Theorem:** For all \( \varepsilon > 0 \), there exists \( \delta : 0 < \delta < \delta_0 \) such that if \( |\rho' - \rho|_s < \delta \) and if \( \lambda' \) is the eigenvalue of \( L + \rho' \) corresponding to \( \lambda \) of \( L + \rho \), there exists \( u' \in \ker (L + \rho' + \lambda') \) with \( ||u'|| = 1 \) and

\[ ||u' - u||_t < \varepsilon. \]

**Proof:**

I. **Discussion.** Let \( \rho' \in C^\infty(M) \) be arbitrary except that \( |\rho' - \rho|_s < \delta_0 \). Let \( \lambda' \) be the corresponding eigenvalue and let \( u' \) be an eigenfunction of \( L + \rho' \) with eigenvalue \( \lambda' \) and \( ||u'|| = 1 \). (Note that there are only two such eigenfunctions, \( u' \) and \( -u' \), since \( \ker (L + \rho' + \lambda') \) is one-dimensional.) We have

\[ Lu + \rho u + \lambda u = 0 \]

\[ Lu' + \rho' u' + \lambda' u' = 0 \]
Subtractracting and letting \( v = u' - u \) gives

\[
Lv + \rho'u' - \rho u + \lambda'u' - \lambda u = 0
\]

\[
\iff Lv + (\rho'-\rho)u' + \rho v + (\lambda'-\lambda)u' + \lambda v = 0
\]

\[
\iff (L + \rho + \lambda)v = [(\rho-\rho') + (\lambda-\lambda')]u'
\]

Let \( r = (\rho-\rho') + (\lambda-\lambda') \) and let \( w = ru' \). Then there exists \( \eta \in \mathbb{R} \) such that

\[
v = Q_\lambda(w) + \eta u \tag{1}
\]

where \( Q_\lambda \) is the pseudo-differential inverse of \( L + \rho + \lambda \) (by 4.3 and \( \dim \ker (L + \rho + \lambda) = 1 \)). To show \( |v|_t \) is small, we will use (1) and show that both \( |\eta| \) and the appropriate Sobolev norms of \( Q_\lambda w \) are small. Since \( v = u' - u \), we have

\[
u' = Q_\lambda(w) + (1 + \eta)u \tag{2}
\]

(2) is a decomposition of \( u' \) into components in and
orthogonal to \( \ker (L + \rho + \lambda) \), so by the Pythagorean theorem,

\[
1 = ||Q_w||^2 + (1 + \eta)^2
\]
since \( ||u|| = ||u'|| = 1 \). Letting \( \theta = ||Q_w|| \), we have

\[
\eta^2 + 2\eta + \theta^2 = 0
\]

If \( \theta < 1 \), there are two distinct real roots

\[
\eta_1 = -1 + \sqrt{1 - \theta^2} \\
\eta_2 = -1 - \sqrt{1 - \theta^2}
\]

so \(-2 < \eta_2 < 1 < \eta_1 \leq 0\). \( \eta_1\eta_2 = \theta^2 \) implies that \( |\eta_1| = \theta^2 / |\eta_2| < \theta \) since \( \theta < 1 < |\eta_2| \). Finally, if \( \eta = \eta_2 \) in (1), we are not about to show \( |\eta| \) small, since \( |\eta_2| > 1 \). However, in this case we have
Thus we have shown: there exists \( u' \in \ker (L + \rho' + \lambda') \) such that (2) holds with \( \eta = \eta_1 \). We will assume this in part II.

II. **Estimates.** \( c_1, c_2, \) etc., will denote various positive constants which never depend on \( \rho', \lambda' \) or \( u' \). We have, for all \( s \),

\[
|r|_s \leq |\rho' - \rho|_s + |\lambda' - \lambda| \leq 2|\rho' - \rho|_s \tag{3}
\]

\[
||Q_\lambda w||_{s+2} \leq c_1 ||w||_s \quad \text{by (4.4)}
\]

\[
\leq c_2 |r|_s ||u'||_s \quad \text{by lemma 2, (4.1) (4)}
\]

\[
\leq 2c_2 ||u'||_s |\rho' - \rho|_s \quad \text{by (3)}
\]
For $s = 0$, since $||u'|| = 1$, this implies

$$\theta = ||Q_\lambda w|| \leq ||Q_\lambda w||_2 \leq c_2 |r|_s$$

so if $|r|_s < 1/c_2$, i.e. $|\rho' - \rho|_s < 1/2c_2$, then $\theta < 1$

and $|\eta_1| \leq \theta < 2c_2 |\rho' - \rho|_s$ (6)

Claim that for all $s$, there is a constant $K_s > 0$ such that $||u'||_s \leq K_s$. This follows by induction on even $s$.

For $K_0 = 1$ and

$$||u'||_s \leq ||Q_\lambda (w) + (1 + \eta_1)u||_s$$

$$\leq ||Q_\lambda (w)||_s + (1 + |\eta_1|)||u||_s$$

$$\leq c_3 ||w||_{s-2} + 2||u||_s$$ (by 4.4 and $|\eta_1| < 1$)

$$\leq c_3 |r|_{s-2}||u'||_{s-2} + 2||u||_s$$ (4.1, lemma 2)

$$\leq c_3 (1/c_2)K_{s-2} + 2||u||_s = K_s$$ (by (5) and the induction hypothesis)
Finally, using Sobolev's lemma, the claim and (4), we have

\[ |Q_\lambda w|_t \leq c_4 |Q_\lambda w|_{s+2} \leq c_5 |\rho'-\rho|_s \quad \text{for } s \geq t + \lceil \nu/2 \rceil - 1 \quad (7) \]

where \( c_5 = 2c_2c_4K_s \).

III. The proof: Choose \( \delta < \min \{ \delta_0, 1/2c_2, \varepsilon/(c_5 + 2c_2|u|_t) \} \) and let \( |\rho'-\rho|_s < \delta \), with \( \lambda' \) the eigenvalue corresponding to \( \lambda \). By the concluding remark in part I, there exists \( u' \in \ker (L + \rho' + \lambda') \) with \( \|u'\| = 1 \) and

\[ u' = Q_\lambda(w) + (1 + \eta)u, \text{ where: } w = ru', \theta = \|Q_\lambda w\| \]

(and \( \theta < 1 \) since \( \delta < 1/2c_2 \), by (5)) and

\[ \eta = -1 + \sqrt{1 - \theta^2}. \]

Then

\[ |u' - u|_t = |Q_\lambda w + au|_t \]

\[ \leq |Q_\lambda w|_t + |a||u|_t \]

\[ \leq c_5 |\rho'-\rho|_s + 2c_2|u|_t|\rho'-\rho|_s \]

\[ \leq (c_5 + 2c_2|u|_t)\delta \]

(by (6), (7))

\[ < \varepsilon. \]
(6.2) Proposition. If \( U \subseteq M \) is an open set containing all the nodal critical points of \( f \in C^\infty(M) \), then there exists \( \delta > 0 \) such that if \( |f - g|_1 < \delta \), \( g \in C^\infty(M) \), then all the nodal critical points of \( g \) lie in \( U \) also.

**Pf:** Since \( |\ldots|_1 \) and \( |\ldots|_{(1)} \) are equivalent, it suffices to show this for \( |\ldots|_{(1)} \). Recall that \( p \in M \) is a nodal critical point if and only if \( f_{(1)}(p) = 0 \). (See 2.10 and 3.1 for notation.) Let \( V_\delta = \{ p \in M : f_{(1)}(p) < \delta \} \). **Claim:** (1) if \( |f - g|_{(1)} < \delta \), then all the nodal critical points of \( g \) lie in \( V_\delta \).

(2) There exists \( \delta > 0 \) such that \( V_\delta \subseteq U \). (1) and (2) prove the proposition. Proof of (1): If \( p \) is a nodal critical point of \( g \), then \( g_{(1)}(p) = 0 \). Then

\[
f_{(1)}(p) \leq (f - g)_{(1)}(p) + g_{(1)}(p)
\]

\[
\leq |f - g|_{(1)} < \delta , \text{ hence } p \in V_\delta
\]

Proof of (2): Let \( \delta = \min \{ f_{(1)}(p) : p \in M - U \} \). For
p ∈ M - U, we have \( f_1(p) > 0 \) since all the nodal critical points of \( f \) lie in \( U \). But \( f_1 \) is a continuous function, hence it attains its minimum on the non-empty compact set \( M - U \). Thus \( δ > 0 \). Finally, \( p \notin U \) implies \( f_1(p) ≥ δ \) and \( p \notin V_δ \). Hence \( V_δ \subset U \).

(6.3) **Lemma** (Morse). \{ \( f ∈ C^∞(M) : f \) is Morse\} is open and dense in the \( C^s \) topology on \( C^∞(M) \) if \( s ≥ 2 \).

(6.4) **Proof of Theorem III:** \( B_n \) is open in the \( C^s \) topology on \( C^∞(M) \), for all \( n \), provided \( s ≥ \lceil ν/2 \rceil + 1 \).

The proof is by induction on \( n \). Since \( B_0 = C^∞(M) \) it is true for \( n = 0 \). Assume \( B_{n-1} \) is open in \( C^∞(M) \).

For \( ρ ∈ B_n \subset B_{n-1} \cap A_n \), there exists \( δ_0 > 0 \) such that if \( |ρ' - ρ|_s < δ_0 \), then \( ρ' ∈ B_{n-1} \cap A_n \). Let \( u_n ∈ ker (L + ρ + λ_n) \), \( ||u_n|| = 1 \). \( ρ ∈ B_n \) means that \( u_n \) is a Morse function, so by (6.3) there is an \( ε' > 0 \) such that if \( |f - u_n|_2 < ε' \), then \( f \) is a Morse function.
Also \( \rho \in B_n \) implies \( u_n \) had no nodal critical points, so by proposition (6.2), there exists \( \varepsilon'' > 0 \) such that \( |f - u_n|_1 < \varepsilon'' \) implies \( f \) had no nodal critical points. The hypothesis \( s > [v/2] + 1 \) enables us to apply theorem (6.1) with \( t = 2 \). Let 
\( \varepsilon = \min(\varepsilon', \varepsilon'') \). There exists \( \delta : 0 < \delta < \delta_0 \) such that 
\( |\rho' - \rho|_S < \delta \) implies (1) \( \rho' \in A_n \cap B_{n-1} \) and 
(2) there exists \( u'_n \in \ker (L + \rho' + \lambda'_n) \) such that 
\( ||u'_n|| = 1 \) and \( |u'_n - u_n|_2 < \varepsilon \). Therefore \( u'_n \) is Morse and had no nodal critical points. Since 
\( \ker (L + \rho' + \lambda'_n) \) is one dimensional, every non-zero function in it has these properties, so \( \rho \in B_n \).

Thus \( \{ \rho' \in C^\infty(M) : |\rho' - \rho|_S < \delta \} \subset B_n \), so \( B_n \) is open.
7. **Generic Eigenfunctions II: Density**

Finally we show that the set of forcing functions yielding operators with generic eigenfunctions is dense. First we show that by a small linear perturbation we may eliminate the nodal critical points, which are isolated and whose structure we know from section 2C. Then we will show that we can make all the non-nodal critical points non-degenerate.

We assume throughout this section that $\dim M = 2$. This is used in two places: in prop. 7.1 we use the corollary of (2.13) to assert that the nodal set has measure zero; in theorem 7.5 we use (2.13) to assert $|du(x)| \geq K|x - a|^{k-1}$ if $a$ is a zero of order $k$ of an eigenfunction $u$. Neither of these statements involves dimension 2 although we have used complex variables to prove them.

(7.1) **Proposition.** Let $u \in \text{ker } (L + \rho + \lambda)$, with $\lambda$ simple. If $a \in M$ is a zero of $u$, there exists $\sigma \in C^\infty(M)$ such that $v(a) \neq 0$, where $u(\tau) = u + \tau v + \tau^2 w(\tau)$ is the perturbed eigenfunction
corresponding to $\rho(\tau) = \rho + \tau \sigma$.

**Pf.** By contradiction. Let $a \in M$, $u(a) = 0$. Given a perturbation $\rho(\tau) = \rho + \tau \sigma$, $\exists \eta \in \mathbb{R}$ such that

$$v = Q_\lambda(\sigma u) + \eta u$$

by remarks (4.6) and the fact that $\lambda$ is simple; $Q_\lambda$ is the pseudo-differential inverse of $L + \rho + \lambda$.

$$v(a) = [Q_\lambda(\sigma u)](a) \quad \text{since} \quad u(a) = 0$$

$$= \int K_\lambda(a, y) \sigma(y) dy$$

where $K_\lambda$ is the kernel of $Q_\lambda$ (see 4.3). If $v(a) = 0$ for all $v$ of the form $(*)$, we have

$$0 = \int K_\lambda(a, y) \sigma(y) u(y) dy \quad \text{for all } \sigma \in C^\infty(M),$$

By (4.3), $y \to K_\lambda(a, y)$ belongs to $L_p(M)$, where

$$p = \begin{cases} 2 & \text{if } v = 2 \\ \frac{v-1}{v-2} & \text{if } v > 2 \end{cases}$$
Choose \( q \) so that \( L_q \) is dual to \( L_p \), i.e. \( \frac{1}{q} + \frac{1}{p} = 1 \).

Since \( C^\infty(M) \) is dense in \( L_q(M) \), we have

\[
0 = \int_M K_\lambda(a,y)\sigma(y)u(y)dy 
\]

for all \( \sigma \in L_q(M) \). Therefore \( y \to K_\lambda(a,y)u(y) \) is zero in \( L_p(M) \), so it vanishes almost everywhere. Since \( \{ x : u(x) = 0 \} \) has measure zero by the corollary to (2.13), we conclude that \( K_\lambda(a,y) = 0 \) almost everywhere, and hence \( y \to K_\lambda(a,y) \) is zero in \( L_p(M) \).

Thus if \( f \in C^\infty(M) \) and \( g(x) = \int K_\lambda(x,y)f(y)dy \), we must have \( g(a) = 0 \). Letting \( N \) be the orthogonal complement of \( N = \ker (L + \rho + \lambda) \) in \( L_2(M) \), we have : \( g \in N \perp \cap C^\infty(M) \) implies \( g(a) = 0 \) (for \( Q_\lambda \) maps \( C^\infty(M) \) onto \( N \perp \cap C^\infty(M) \)). But any element of \( N \) is a constant multiple of \( u \), so vanishes at \( a \). Since \( C^\infty(M) = N \oplus N \perp \cap C^\infty(M) \), \( g(a) = 0 \) for all \( g \in C^\infty(M) \), which is impossible.
(7.2) Lemma. Let \(a_1, \ldots, a_m \in M\). Given \(m\) functions \(f_1, \ldots, f_m\) such that \(f_1(a_i) \neq 0\), there exists a function \(g\) which is a linear combination of the \(f_i\), with \(g(a_i) \neq 0\) for \(i = 1, \ldots, m\).

**Pf.** By induction. Trivial for \(m = 1\). Suppose true for \(m - 1\). There exists a linear combination \(h\) of \(f_1, \ldots, f_{m-1}\) such that \(h(a_i) \neq 0\) for \(i = 1, \ldots, m-1\).

If \(h(a_m) \neq 0\), you're done with \(g = h\). If \(h(a_m) = 0\), let \(\theta\) be chosen so that

\[
0 < |\theta| < \min \{|h(a_i)|/2|f_m(a_i)|: i=1,\ldots,m-1 \text{ and } f_m(a_i) \neq 0\}
\]

Let \(g = h + \theta f_m\). Then \(|g(a_i)| = |h(a_i) + \theta f_m(a_i)|\)

\[
\geq |h(a_i)| - |\theta| |f_m(a_i)|
\]

\[
\geq |h(a_i)|/2 > 0
\]

and \(g(a_m) = h(a_m) + \theta f_m(a_m) = \theta f_m(a_m) \neq 0\).

(7.3) Proposition. Let \(u \in \ker (L + \rho + \lambda)\), \(\lambda\) simple and let \(a_1, \ldots, a_m\) be zeros of \(u\). There exists
σ ∈ C^∞(M) such that if \( u(τ) = u + vτ + w(τ)τ^2 \) is the perturbed eigenfunction of \( ρ(τ) = ρ + τσ \), then \( v(a_i) ≠ 0 \) for all \( i = 1, \ldots, m \).

**Pf:** By prop. (7.1), there exist \( σ_1, \ldots, σ_m ∈ C^∞(M) \) such that if \( u_j(τ) = u + v_jτ + w_j(τ)τ^2 \) is the perturbed eigenfunction corresponding to \( ρ_j(τ) = ρ + σ_jτ \), we have \( v_j(a_j) ≠ 0, j = 1, \ldots, m \). Furthermore

\[
v_j = Q_λ[σジュ] + η_ju \tag{*}
\]

for some \( η_j ∈ R \). By lemma (7.2), there exists a function \( v' \) which is a linear combination

\[
v' = \sum_{j=1}^{m} θ_jv_j \text{ and satisfies } v'(a_j) = 0 \text{ for all } j = 1, \ldots, m. \]

Let \( σ = \sum_{j=1}^{m} θ_jσ_j \) and let \( ρ(τ) = ρ + τσ \).

If \( u(τ) = u + τv + τ^2w(τ) \) is the perturbed eigenfunction of \( L + ρ(τ) \),
v = Q_\lambda[\sigma u] + \eta u \quad \text{for some } \eta \in \mathbb{R}

= Q_\lambda[\sum_{j=1}^{m} \theta_j \sigma_j u] + \eta u

= \sum_{j=1}^{m} \theta_j Q_\lambda[\sigma_j u] + \eta u

= \sum_{j=1}^{m} \theta_j (v_j - \eta_j u) + \eta u \quad \text{by (\ast)}

= \sum_{j=1}^{m} \theta_j v_j + \eta' u \quad \text{where } \eta' = \eta - \sum_{j=1}^{m} \theta_j \eta_j

= v' + \eta' u

Since the \( a_i \) are all zeros of \( u \), we have

\( v(a_i) \neq 0 \), all \( i = 1, \ldots, m \).

(7.4) Proposition. Suppose that \( u \in C^\infty(M) \) and for \( |\tau| < T \), \( v(\tau) \) is analytic in \( \tau \) with respect to the \( C^1 \) norm on \( C^\infty(M) \); let \( u(\tau) = u + \tau v(\tau) \). Let \( U \) be a coordinate nbd of \( a \in M \) and suppose that for all \( \tau : |\tau| \leq T \) and all \( x \in U \),
73.

(i) \(|u(x)| \leq K|x - a|^k\)

(ii) \(|du(x)| \geq K'|x - a|^{k-1}\) for \(K, K' > 0, k\) an integer \(\geq 2\)

(iii) \(|v(\tau)(x)| \geq J\) for some \(J > 0\) independent of \(\tau, x\)

Then there exists \(T' > 0\) such that \(0 < |\tau| \leq T'\) implies \(u(\tau)\) has no nodal critical points in \(U\).

**Pf:** If \(x \in U\) is a critical point of \(u(\tau)\), then

\[
0 = d[u(\tau)](x) = du(x) + \tau d[v(\tau)](x)
\]

Then \(|x - a|^{k-1} \leq \frac{1}{|K'|} |du(x)|\) by (ii)

\[
= \frac{1}{K'} |\tau| |d[v(\tau)](x)|
\]

\[
\leq c|\tau| \text{ where } c = \frac{1}{K'} \sup \{|dv(\tau)|: \tau \leq T\}
\]

\[
|u(\tau)(x)| = |u(x) + \tau v(\tau)(x)|
\]

\[
\geq |\tau| |v(\tau)(x)| - |u(x)|
\]

\[
\geq |\tau|J - K|x - a|^k \text{ by (i) and (iii)}
\]

\[
\geq |\tau|J - K(c|\tau|)^{k/k-1}
\]

\[
\geq |\tau|J/2 \text{ provided } |\tau| \leq T' = \frac{1}{c} (J/2Kc)^{k-1}
\]

\[
> 0 \text{ if } \tau \neq 0.
\]
(7.5) **Theorem.** Assume that \( p \in A_n \) and let \( \lambda \leq \lambda_n \) be an eigenvalue of \( L + \rho \). For all \( \varepsilon > 0 \), there exists \( \rho' \in C^\infty(M) \) such that \( |\rho' - \rho|_g < \varepsilon \) and if \( \lambda' \) is the eigenvalue of \( L + \rho' \) corresponding to \( \lambda \) of \( L + \rho \), then \( u' \in \text{ker} (L + \rho' + \lambda') - \{0\} \) implies \( u \) has no nodal critical points.

**Pf:** We will take \( \rho' = \rho(\tau) \) for a linear perturbation \( \rho(\tau) = \rho + \tau \sigma \) to be chosen and some \( \tau \). Let \( u \in \text{ker} (L + \rho + \lambda), \quad ||u|| = 1 \). There are a finite number of nodal critical points \( a_1, \ldots, a_m \) of \( u \) and coordinate nbds \( V_i \) of \( a_i \), \( i = 1, \ldots, m \), on which

\[
|u(x)| \leq K_i |x - a_i|^{-k_i} \quad \text{and} \quad |du|(x) \geq K_i' |x - a_i|^{-k_i-1}
\]

for \( K_i, K'_i > 0 \) and \( k_i \geq 2 \) an integer (see section 2C). Choose \( \sigma \in C^\infty(M) \) so that \( v(a_i) \neq 0 \) for \( i = 1, \ldots, m \) (prop 7.3). Look at each critical point \( a_i \). Let \( W_i = \{x \in V_i : |v(x)| > |v(a_i)|/2\}. \) **Claim:** there exists \( T_i > 0 \) such that if \( |\tau| < T_i \), then

\[
|v(\tau)(x)| > |v(a_i)|/4 \quad \text{for all} \quad x \in W_i \quad (v(\tau) = v + \tau w(\tau)).
\]
Let $S_i = \sup \{|w(\tau)| : |\tau| \leq T_0\}$, ($T_0$ is essentially the radius of convergence of $u(\tau)$; see 4.5). If $S_i = 0$, let $T_1 = T_0$ and the claim is immediate. If $S_i \neq 0$, let $T_1 = |v(a_i)|/4S_i$. Then for $|\tau| < T_1$ and $x \in W_i$,

$$|v(\tau)(x)| \geq |v(x)| - |\tau||w(\tau)(x)|$$

$$\geq \frac{1}{2}|v(a_i)| - T_1S_i$$

$$\geq \frac{1}{2}|v(a_i)| - \frac{1}{4}|v(a_i)|$$

$$= \frac{1}{4}|v(a_i)|$$

proving the claim. Thus the conditions of prop 7.4 are satisfied [with $U = W_i$, $a = a_i$, $T = T_1$, $k = k_i$, $K = K_i$, $K' = K'_i$, $J = |v(a_i)|/4$], so there exists $T'_1 > 0$ such that if $0 < |\tau| < T'_1$, $u(\tau)$ has no nodal critical points in $W_i$.

Now let $T' = \min \{T'_1, \ldots, T'_m\}$ and $W = \bigcup_{i=1}^m W_i$.

For $0 < |\tau| < T'$, $u(\tau)$ has no nodal critical points
in $W$. By prop. 6.2, there exists $\delta > 0$ such that if $|u(\tau) - u|_1 < \delta$, then all the nodal critical points of $u(\tau)$ lie in $W$. Since $u(\tau)$ is analytic with respect to the $C^1$ norm on $C^\infty(M)$, there is $T'' > 0$ such that $|\tau| < T''$ implies $|u(\tau) - u|_1 < \delta$. Now let $T = \min \{T', T'', \varepsilon/|\sigma|_S\}$ and let $\rho' = \rho(\tau)$, $u' = u(\tau)$ for $0 < |\tau| < T$. (i) $|\rho' - \rho|_S = |\rho + \tau \sigma - \rho|_S = |\tau||\sigma|_S < \varepsilon$; (ii) $|\tau| < T''$ implies that all the nodal critical points of $u(\tau)$ lie in $W$; (iii) $|\tau| < T'$ implies that there are no nodal critical points of $u(\tau)$ in $W$. Hence $u' = u(\tau)$ has no nodal critical points at all, nor does any non-zero multiple of it, so the theorem is proved.

(7.6) Theorem. Suppose $\rho \in A_n$ and $\lambda \leq \lambda_n$ is an eigenvalue of $L+\rho$. Let $u \in \ker (L + \rho + \lambda)$, $u \neq 0$; assume $u$ has no nodal critical points. For all $\varepsilon > 0$, there is $\rho' \in A_n$ such that $|\rho' - \rho|_S < \varepsilon$, $\lambda$ is an eigenvalue of $L+\rho'$ and if $u' \in \ker (L + \rho' + \lambda) - \{0\}$, then all the critical points of $u'$ are Morse and non-nodal, i.e. $u'$ is generic.
Remark: In the proof, we choose a \( u' \) satisfying the required conditions and then show that there is a \( \rho' \in C^\infty(M) \) such that \( u' \) is an eigenfunction of \( L + \rho' \) with eigenvalue \( \lambda \). Also \( \rho' \) is sufficiently close to \( \rho \) if \( u' \) is close enough to \( u \).

\textbf{Pf:} It suffices to assume \( \epsilon \) is small enough so that \( |\rho' - \rho|_S < \epsilon \) implies \( \rho' \in A_n \).

I. Construction of \( u' \): Since \( u \) has no nodal critical points, there exists a nbd \( U \) of \( u^{-1}(0) \) such that \( u \) has no critical points in \( U \). For example, let \( \gamma = \inf \{|u(p)| : du(p) = 0\} \); then \( U = \{p \in M : |u(p)| < \gamma/2\} \) is such a nbd. Let \( V \) be a nbd of \( u^{-1}(0) \) such that \( V \subset U \) and let \( \psi \in C^\infty(M) \) be a bump function: \( 0 \leq \psi \leq 1, \, \psi = 0 \) on \( V \), \( \psi = 1 \) on \( M - U \).

Let \( \delta > 0 \). By the Morse lemma (6.3), there exists \( v \in C^\infty(M) \) such that \( |v - u|_{s+2} < \delta \) and \( v \) is a Morse function. Let \( u' = (1 - \psi)u + \psi v \). We have
(1) $u' = u$ on $\nabla$ (hence $u'$ has no critical points in $\nabla$)

(2) $u' = v$ on $M - U$ (hence $u'$ has only non-degenerate critical points in $M - U$)

(3) $|u' - u|_{s+2} = |\psi(v - u)|_{s+2} \leq c_1 |\psi|_{s+2} |v - u|_{s+2} \leq c_2 |v - u|_{s+2} \leq c_2 \delta$

(4) $|du'(p)| \geq |du(p)| - |d(u' - u)(p)|$

\[ \geq |du(p)| - |u' - u|_{(1)} \]

\[ \geq |du(p)| - c_3 |u' - u|_1 \]

\[ \geq |du(p)| - c_3 c_2 \delta \]

If $p \in \mathcal{U} - \nabla$, then $|du'(p)| \geq c_4/2 > 0$ where

$c_4 = \inf \{|du|(p) : p \in \mathcal{U} - \nabla\};$ provided $\delta < c_4/2c_2c_3$.

(hence $u'$ has no critical points in $\mathcal{U} - \nabla$)

(5) If $p \notin U$, $|u'(p)| \geq |u(p)| - |(u' - u)(p)|$

\[ \geq \gamma/2 - |u' - u| \geq \gamma/2 - \delta \]

\[ \geq \gamma/4 \quad \text{provided } \delta < \gamma/4 \]
(recall γ from definition of U). Therefore: u' has no zeros except in U, so \((u')^{-1}(0) = u^{-1}(0)\) and since \(u = u'\) in a nbd of this common zero set, u' has no nodal critical points either.

II. Construction of \(\rho'\): Next we show that u' is an eigenfunction of an operator \(L + \rho'\) with eigenvalue \(\lambda\).

The choice of \(\rho'\) is dictated by the fact that u' must satisfy the equation \((L + \rho' + \lambda)u' = 0\). Define

\[
\rho' = \begin{cases} 
\rho & \text{on V} \\
-\frac{L\rho' + \lambda u'}{u'} & \text{on } M - u^{-1}(0) 
\end{cases}
\]

These are \(C^\infty\) where defined and agree on the open overlap \(V - u^{-1}(0)\) since \(u' = u\) on \(V\) and \(u \in \ker (L + \rho + \lambda)\). Also \((L + \rho' + \lambda)u' = 0\) on \(M - u^{-1}(0)\) by definition, and on \(V\), the left hand side reduces to \((L + \rho + \lambda)u\), which is zero.

To estimate the size of \(|\rho' - \rho|\), let \(W\) be a nbd of \(u^{-1}(0)\) such that \(\bar{W} \subset V\) and let \(\phi \in C^\infty(M)\) be a bump function: \(0 \leq \phi \leq 1\), \(\phi = 0\) on \(\bar{W}\), \(\phi = 1\) on \(M - V\).
Then

\[ |\rho - \rho'|_S = |\phi(\rho - \rho') + (1-\phi)(\rho - \rho')|_S \]

\[ = |\phi(\rho - \rho')|_S \quad \text{since} \quad (1-\phi)(\rho - \rho') \equiv 0 \]

\[ \text{(for } \phi(x) \neq 1 \Rightarrow x \in V \Rightarrow \rho = \rho') \]

\[ = |\phi[\rho + \frac{Lu' + \lambda u'}{u'}]|_S \]

\[ = |(\phi/u')(L + \rho + \lambda)u'|_S \]

\[ \leq c_5 |\phi/u'|_S |(L + \rho + \lambda)u'|_S \]

\[ \leq c_5 |\phi/u'|_S |(L + \rho + \lambda)(u' - u)|_S \quad \text{since } u \in \ker (L + \rho + \lambda) \]

\[ \leq c_6 |\phi/u'|_S |u' - u|_{S+2} \]

**Claim:** There exists \( K > 0 \) such that \( |\phi/u'|_S \leq K \), with \( K \) depending only on \( \phi, u, s \). Assuming the claim, and using (3),

\[ (7) \quad |\rho - \rho'|_S \leq c_2 c_6 K \delta \]
[Note: \( \phi/u' \) is well-defined and \( C^\infty \) since \( \phi = 0 \) in a nbd of \( (u')^{-1}(0) \).]

To prove the claim, write

\[
|\phi/u'|_s = |\frac{\phi}{u'}[\phi + (1 - \phi)]|_s
\]

\[
\leq |\phi^2/u'|_s + |\phi(1 - \phi)/u'|_s
\]

\[
= |\phi^2/u'|_s + |\phi(1 - \phi)/u|_s
\]

\[
= |\phi^2/u'|_s + K_1
\]

(where we used the fact that \( \phi(1 - \phi)/u' = \phi(1 - \phi)/u \);
this follows from the fact that if \( x \notin \mathcal{V} \), then \( \phi(x) = 1 \)
and both sides are zero and if \( x \in \mathcal{V}, u' = u \))

Now

\[
|\phi^2/u'|_s \leq |\phi^2/u|_s + |\phi^2(u - u')/uu'|_s
\]

(since \( 1/u' = 1/u + (u - u')/uu' \))

\[
\leq K_2 + K_3|\phi/u|_s |\phi/u'|_s |u - u'|_s
\]

\[
\leq K_2 + K_4\delta|\phi/u'|_s
\]
Assuming $\delta < 1/K_4$ and combining these two estimates yields

$$|\phi/u'|_S \leq K_1 + K_2 + K_4\delta |\phi/u'|_S$$

$$\Rightarrow |\phi/u'|_S \leq K = (K_1 + K_2)/(1 - K_4\delta) \quad \text{and} \quad K > 0.$$

III. The proof: Let $\epsilon > 0$ be given, so that $|\rho' - \rho|_S < \epsilon$ implies $\rho' \in A_n$. Choose

$$\delta = \min \{c_4/2c_2c_3, \gamma/4, 1/K_4, \epsilon/c_2c_6K\}$$

Apply the procedures in part I to construct $u'$. (1) to (5) imply $u' \neq 0$ and all the critical points of $u'$ are Morse and non-nodal. Construct $\rho'$ as in part II. Then $u'$ is an eigenfunction of $L+\rho'$ with eigenvalue $\lambda$. Finally, our choice of $\delta$ and (7) imply $|\rho - \rho'|_S < \epsilon$, and since any other eigenfunction in $\ker (L + \rho' + \lambda)$ is a multiple of $u'$, they are all generic.
(7.7) Proof of Theorem IV: $B_n$ is dense in $B_{n-1} \cap A_n$ in the $C^s$ topology on $C^\infty(M)$.

Let $\rho \in B_{n-1} \cap A_n$. Claim there is a $\rho'' \in B_n$ with $|\rho'' - \rho|_s < \epsilon$. Choose $\epsilon' : 0 < \epsilon' < \epsilon$ such that $|\rho' - \rho|_s < \epsilon'$ implies $\rho' \in B_{n-1} \cap A_n$ since this is open. Then by theorem (7.5), there exists a $\rho' \in C^\infty(M)$ with $|\rho' - \rho|_s < \epsilon'/2$ and such that $u_n' \in \ker (L + \rho' + \lambda_n') - \{0\}$ has no nodal critical points. By theorem (7.6), there is a $\rho'' \in C^\infty(M)$, with $|\rho'' - \rho'|_s < \epsilon'/2$ and whose $n$-th eigenvalue is $\lambda_n'$, such that $u_n'' \in \ker (L + \rho'' + \lambda_n')$ implies that $u_n''$ is generic. Therefore $\rho'' \in B_n$ and $|\rho'' - \rho|_s \leq |\rho'' - \rho'|_s + |\rho' - \rho|_s < \epsilon'/2 + \epsilon'/2 = \epsilon' < \epsilon$. 


Appendix I: Pseudo-Differential Inverses

Theorem (4.3) contains somewhat more information than the existence of a parametrix of an elliptic operator modulo operators of order $-\infty$, so we will give a derivation based on the standard theorems. For reference, see Seeley and Bers & Schechter.

Defn. An operator $P : C^\infty(M) \to C^\infty(M)$ has order $m$ if
\[ |Pu|_s \leq c_s |u|_{s+m}; \] order $-\infty$ means order $m$, for all $m$.

THEOREM I. If $P$ is an elliptic pseudo-differential operator of order $m$, then there exists a pseudo-differential operator $E$ of order $-m$ such that $EP = 1 - R$ and $PE = 1 - S$ where $R, S$ are operators of order $-\infty$.

THEOREM II. Let $P$ be a self-adjoint elliptic differential operator on $M$. For all $f \in L^2(M)$, there exists $u \in L^2(M)$ such that $Pu = f$ iff $f \perp \ker P$ in the $L^2$ inner product.

We will assume Theorems I and II.
85.

(1) Claim there exists a linear operator
\[ Q : L_2(M) \to L_2(M) \]
such that \[ QPu = u - \pi_N(u) \] and \[ Q\pi_Nu = 0 \], for all \( u \in C^\infty(M); \) \( \pi_N \) is projection on \( N = \ker P \).

By theorem II, we can define \( Q_0 : N^\perp \to N^\perp \) such that \( Q_0P = PQ_0 = I \) on \( N^\perp \). For theorem II says
\[ f \in N^\perp \Rightarrow \exists u \in L_2(M) \exists Pu = f. \]
Then \( v = \pi_{N^\perp}(u) \in N^\perp \)
and satisfies \( Pv = f \). If \( v' \) is any other element of \( N^\perp \) with \( Pv' = f \), then \( v - v' = N \cap N^\perp = \{0\} \).

Let \( Q_0f = v \); this is well-defined and satisfies
\[ Q_0P = PQ_0 = I \] on \( N^\perp \). Let \( Q = Q_0\pi_{N^\perp} \). Now

\[ QPu = Q_0\pi_{N^\perp}Pu = Q_0Pu = Q_0P(\pi_Nu + \pi_{N^\perp}u) \]

\[ = 0 + Q_0P\pi_{N^\perp}u = \pi_{N^\perp}u = u - \pi_Nu \]

Also \( Q\pi_N = Q_0\pi_{N^\perp}\pi_N = 0 \).

(2) Claim \( Q \) is a pseudo-differential operator of order \(-m\). Let \( E \) be the pseudo-differential
parametrix described in Theorem I. We will show

\[ Q = E + T \] where \( T \) is an operator of order \( -\infty \).

Write

\[ QP = 1 - \pi_N \quad \text{EP} = 1 - R \]

implying \( (Q - E)P = R - \pi_N \)

and \( 0 = EP\pi_N = \pi_N - R\pi_N \)

so \( (Q - E)P = R(1 - \pi_N) \)

Multiplying on the right by \( E \) and using \( PE = 1 - S \) gives

\[ R(1 - \pi_N)E = (Q - E)PE \]

\[ = (Q - E) - (Q - E)S \]

\[ \therefore T = (Q - E)S + R(1 - \pi_N)E \] is an operator of order \( -\infty \)

since the composition of an operator of order \( -\infty \) with an operator of finite order has order \( -\infty \), and \( R, S \) have
order $- \infty$. Also $Q = E + T$.

(3) It remains to show that we can write

$$Qf(x) = \int \int K(x,y)f(y)dy$$

where $y + K(x,y)$ is in $L_p(M)$ for

$$p = \begin{cases} 2 & \text{if } v = 2 \\ (v-1)/(v-2) & \text{if } v > 2 \end{cases}$$

By a partition of unity argument, it suffices to show that if $Q$ is a pseudo-differential operator on a bounded open set $U \subset \mathbb{R}^v$, then $Qf(x) = \int K(x,y)f(y)dy$ where $y + K(x,y)$ is in $L_p(U)$ for the appropriate $p$.

**Defn.** Let $\omega$ be a non-positive integer. $f$ is a **pseudo-homogeneous function** of degree $\omega$ if:

(1) for $\omega = 0$, $f(x) = c \log |x| + g(x)$, where $c \in \mathbb{R}$ and $g(x)$ is $C^\infty$ on $\mathbb{R}^v - \{0\}$, homogeneous of degree 0 and such that $\int_{|x|=1} g(x) = 0$. 
(ii) for \( \omega < 0 \), \( f \) is \( C^\infty \) on \( \mathbb{R}^\nu - \{0\} \) and is homogeneous of degree \( \omega \).

**Lemma.** Let \( f \) be a pseudo-homogeneous function of degree \( \omega \). For any bounded open set \( U \), \( f \in L^p(U) \) provided \( \omega p + \nu - 1 > 0 \) if \( \omega < 0 \).

**Pf.** Identify \( \mathbb{R}^\nu - \{0\} \) with \( \mathbb{R} \times S^{\nu-1} \) by \( x \leftrightarrow (r,\theta) \), where \( r = |x| \) and \( \theta = x/|x| \). Let \( c > 0 \) be such that \( U \subset [0,c] \times S^{\nu-1} \). Assume \( \omega < 0 \). Then

\[
\int_U |f(x)|^p \, dx \leq \int_{[0,c] \times S^{\nu-1}} |f(x)|^p \, dx
\]

\[
= \int_0^c \int_{S^{\nu-1}} |f(r,\theta)|^{p\nu^{\nu-1}} \, dr \, d\theta
\]

\[
= \int_0^c \int_{S^{\nu-1}} r^{\omega p + \nu - 1} |f(1,\theta)|^p \, dr \, d\theta
\]

since \( f \) homogeneous of degree \( \omega \leftrightarrow f(r,\theta) = r^\omega f(1,\theta) \)

\[
= \int_0^c r^{\omega p + \nu - 1} \int_{S^{\nu-1}} |f(1,\theta)|^p \, d\theta \, dr
\]

\[
= \int_0^c r^{\omega p + \nu - 1} \, dr \int_{S^{\nu-1}} |f(1,\theta)|^p \, d\theta
\]
The first integral in the product is finite provided \( \omega p + \nu - 1 \geq 0 \) and the second is an integral of a continuous function over a compact set.

If \( \omega = 0 \), observe that the function

\[
\begin{cases}
    r(\log r)^p & \text{for } r > 0 \\
    0 & \text{for } r = 0
\end{cases}
\]

is continuous at 0, for all \( p \).

Therefore \( (r,\theta) \to r|f(r,\theta)|^p \) is continuous on \( \mathbb{U} \), and so long as \( \nu \geq 2 \), we have

\[
\int_U |f(x)|^p \, dx = \int_U |f(r,\theta)|^p r^{\nu-1} \, dr \, d\theta < \infty.
\]

**THEOREM III.** Let \( U \subset \mathbb{R}^\nu \) be open. If \( Q \) is a pseudo-differential operator of order \( -m < 0 \), then

\[
Qf(x) = \int_U K(x,y)f(y) \, dy
\]

where (i) \( K(x,y) \) is everywhere defined and continuous if \( \nu < m \) and (ii) if \( \nu \geq m \), we have

\[
K(x,y) = \sum_{j=0}^{\nu-m} K_j(x,y) + K'(x,y)
\]
where $K_j(x,y)$ is $C^\infty$ for $y \neq x$ and for fixed $x$,

$$z \to K_j(x, x+z)$$

is a pseudo-homogeneous function of degree $j + m - v$; $K'$ is continuous everywhere.

We refer to Seeley, pp. 208-210 for details.

**Prop.** Let $U, Q$ be as above. If $U$ is bounded, then for fixed $x \in U$, the map $y \to K(x,y)$ is in $L_p(U)$, provided $p \leq (v-1)/(v-m)$ when $v > m$.

**Pf:** If $v < m$, there is no problem since $y \to K(x,y)$ is continuous. If $v \geq m$,

$$K(x,y) = \sum_{j=0}^{v-m} K_j(x,y) + K'(x,y)$$

(see above)

$K'(x,y)$ is continuous, and $z \to K_j(x, x+z)$ is pseudo-homogeneous of degree $j + m - v$. Now $y \to K(x,y)$ is in $L_p$ if each $y \to K_j(x,y)$ is and

$$\int_U |K_j(x,y)|^p dy = \int_U' |K_j(x, x+z)|^p dz$$
where $U' = \{ z : y = x + z \in U \}$. The second integral is finite by the lemma, provided $p(j + m - \nu) + \nu - 1 \geq 0$, $j = 0, \ldots, \nu - m$. It suffices to know $p(m - \nu) + \nu - 1 \geq 0$, which is satisfied for all $p$ if $m = \nu$ and yields the condition $p \leq (\nu - 1)/(\nu - m)$ if $\nu > m$.

Remark. Theorem (4.3) is now proved, since $m = 2$ in that case, and we have taken $p = (\nu - 1)/(\nu - 2)$ when $\nu > 2$. 
Appendix II. Perturbation Theory

Theorem 4.5 states that for perturbations of the forcing function depending linearly on a parameter $\tau$, the eigenvalues and eigenfunctions are convergent power series for small $\tau$. The purpose of this appendix is to prove theorem 4.5. The argument is almost identical to Theorem I, p. 57 in Rellich's book.

If $E, F$ are Banach spaces, let $B(E, F)$ denote the space of bounded linear operators from $E$ to $F$; $B(E, F)$ is a Banach space with the operator norm.

Lemma 1. Suppose that in a nbd of $\tau = 0$, $G(\tau)$ is an absolutely analytic function in $B(E, F)$ and $v(\tau)$ is analytic in $E$; then $G(\tau)v(\tau)$ is analytic in $F$ for small $\tau$. If $v(\tau)$ is absolutely analytic, then so is $G(\tau)v(\tau)$.

Lemma 2. If $G(\tau)$ is absolutely analytic in $B(E, F)$ for small $\tau$ and $G(0)$ is invertible, then for $\tau$ in a small nbd of 0, $G(\tau)$ is invertible and $[G(\tau)]^{-1}$ is absolutely analytic in $B(E, F)$ for small $\tau$.

Abstract Theorem (Statement)
Hypotheses: Let $E$ be a real Hilbert space with inner product $<\cdot,\cdot>$. Suppose $P$ is a symmetric, densely defined operator on $E$ and assume $\lambda$ is an eigenvalue of $P$ with multiplicity $h$. Assume there is a bounded linear operator $Q : E \to E$ such that $Q\pi_N = 0$ and $Q(P + \lambda) = 1 - \pi_N$, where $\pi_N$ is orthogonal projection onto $N = \ker (P + \lambda)$. Finally, let $R(\tau)$ be an absolutely analytic function in $B(E, E)$ for small $\tau$, self-adjoint for each $\tau$ and such that $R(0) = 0$. Let $P(\tau) = P + R(\tau)$.

Conclusions: There exist $h$ functions

$\lambda^{(1)}(\tau), ..., \lambda^{(h)}(\tau)$ analytic in $R$ for small $\tau$ and $h$ functions $u^{(1)}(\tau), ..., u^{(h)}(\tau)$ analytic in $E$ for small $\tau$, such that: (for $j = 1, ..., h$)

(a) $\lambda^{(j)}(0) = \lambda$;

(b) for each $\tau$, $u^{(j)}(\tau)$ is an eigenvector of $P(\tau)$ with eigenvalue $\lambda^{(j)}(\tau)$;

(c) for each $\tau$, \{u^{(1)}(\tau), ..., u^{(h)}(\tau)\} is an orthonormal set in $E$. 
The major step in the proof of this theorem is the following proposition, showing the existence of one pair of analytic eigenvalue and eigenvector.

**Proposition.** Hypotheses as in abstract theorem. There exists an analytic function \( \lambda(\tau) \) in \( R \) for small \( \tau \) and an analytic function \( u(\tau) \) in \( E \) for small \( \tau \) such that

(a) \( \lambda(0) = \lambda \)

(b) \( u(\tau) \) is an eigenvector of \( P(\tau) \) with eigenvalue \( \lambda(\tau) \)

**Pf:**

(1) **Derivation.** Before launching into the actual proof we will assume \( \lambda(\tau) \) and \( u(\tau) \) exist and show how they arise. Conclusion (b) says

\[
[P(\tau) + \lambda(\tau)]u(\tau) = 0
\]

\[
\Rightarrow [P + \lambda]u(\tau) = -[\lambda(\tau) - \lambda]u(\tau) \quad (1)
\]

\[
\Rightarrow u(\tau) = -Q([\lambda(\tau) - \lambda]u(\tau)) + v(\tau)
\]

where \( v(\tau) \in N = \text{ker} (P + \lambda). \)

\[
\Rightarrow \{1 + Q[\lambda(\tau) - \lambda]u(\tau) = v(\tau)
\]

\[
\Rightarrow u(\tau) = \{1 + Q[\lambda(\tau) - \lambda]\}^{-1}v(\tau) \quad (2)
\]
provided, of course, that \(1 + Q[R(\tau) + \lambda(\tau) - \lambda]\) is invertible; it will be if \(\tau\) is small enough. So we can solve for \(u(\tau)\) provided \(\lambda(\tau)\) and \(v(\tau)\) are known.

Now fix an orthonormal basis \(u_1, \ldots, u_h\) of \(N\). \(v(\tau)\) is given by specifying real analytic functions \(c_1(\tau), \ldots, c_h(\tau)\) with

\[
v(\tau) = \sum_{i=1}^{h} c_i(\tau) u_i
\]

By (1), \([R(\tau) + \lambda(\tau) - \lambda]u(\tau)\) is orthogonal to \(N\) since it is in the range of \(P + \lambda\). Writing this out and using (2) and (3) to get \(u(\tau)\), we have (for \(j = 1, \ldots, h\))

\[
0 = <[R(\tau) + \lambda(\tau) - \lambda]u(\tau), u_j>
\]

\[
= <[R(\tau) + \lambda(\tau) - \lambda][1 + Q[R(\tau) + \lambda(\tau) - \lambda]^{-1}v(\tau), u_j>
\]

\[
= \sum_{i=1}^{h} c_i(\tau) <[R(\tau) + \lambda(\tau) - \lambda][1 + Q[R(\tau) + \lambda(\tau) - \lambda]^{-1}u_i, u_j>
\]

We have \(n\) equations in \(n\) unknowns, so a non-trivial
solution  \( c_1(\tau), \ldots, c_n(\tau) \) exists if and only if

\[
0 = \det \left( \langle \left[ R(\tau) + \lambda(\tau) - \lambda \right] \left[ 1 + Q[R(\tau) + \lambda(\tau) - \lambda] \right]^{-1} u_1, u_j \rangle \right)
\]

If this last equation can be solved for \( \lambda(\tau) \), we can work these steps backwards and come up with \( u(\tau) \).
This is what will be done below.

**The proof.**

(2) Let \( F \) be the complexification of \( E \), i.e. \( F = E \otimes_{\mathbb{R}} \mathbb{C} \)
and a typical element of \( F \) is \( f_1 + f_2 \sqrt{-1} \) where \( f_1, f_2 \in \mathbb{E} \). \( F \) is a complex Hilbert space with inner product given by

\[
\langle f_1 + f_2 \sqrt{-1}, g_1 + g_2 \sqrt{-1} \rangle = \\
\langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle + \{ \langle f_2, g_1 \rangle - \langle f_1, g_2 \rangle \} \sqrt{-1}
\]

An analytic function on \( \mathbb{E} \) extends to a complex analytic function on \( F \); similarly for operators, and symmetry and boundedness are preserved.
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(3) Now, using these remarks, define

$$f_{ij}(\alpha, \tau) = \langle [R(\tau) + \alpha][1 + Q[R(\tau) + \alpha]]^{-1}u_i, u_j \rangle$$

for $\alpha, \tau$ in a small (complex) nbhd of 0. For $\alpha = \tau = 0$, $1 + Q[R(\tau) + \alpha] = 1$, so $(1 + Q[R(\tau) + \alpha])^{-1}$ exists and is analytic for small enough $\alpha, \tau$. [The extensions of our definitions and lemmas 1, 2 to more than one parameter and to complex parameters is easy.] Let

$$F(\alpha, \tau) = \det((f_{ij}(\alpha, \tau)))$$

(4) Suppose a complex-valued function $\alpha(\tau)$ satisfies $F(\alpha(\tau), \tau) = 0$ for small $\tau$ and $\alpha(0) = 0$. We will show that $\lambda(\tau) = \lambda + \alpha(\tau)$ is an eigenvalue of $P(\tau)$. This implies $\alpha(\tau)$ is real if $\tau$ is real, since $P(\tau)$ is symmetric. To see that $\lambda(\tau)$ is an eigenvalue, we argue as follows. Fix $\tau$. $F(\alpha(\tau), \tau) = 0 \Rightarrow c_1(\tau), \ldots, c_n(\tau) \in \mathbb{C}$, not all zero, such that

$$\sum_{j=1}^{h} f_{ij}(\alpha(\tau), \tau)c_j(\tau) = 0$$
Let \( v(\tau) = \sum_{j=1}^{h} c_j(\tau)u_j \)

and \( u(\tau) = \{1 + Q[R(\tau) + \alpha(\tau)]\}^{-1}v(\tau) \)

\( u(\tau) \) is a non-zero eigenvector of \( P(\tau) \) with eigenvalue \( \lambda(\tau) \), by reversing the steps in (1).

(5) \( \exists \) a real-analytic function \( \alpha(\tau) \), for small \( \tau \), such that \( \alpha(0) = 0 \) and \( F(\alpha(\tau), \tau) = 0 \). Since \( f_{ij}(\alpha,0) = \alpha \delta_{ij} \), \( F(\alpha,0) = \alpha^h \). By the Weierstrass preparation theorem,

\[
F(\alpha,\tau) = (\alpha^h + p_1(\tau)\alpha^{h-1} + \ldots + p_h(\tau))E(\alpha,\tau)
\]

where \( E(0,0) = 1 \) and \( p_j(\tau) \) is complex-analytic fcn of \( \tau \), \( j = 1, \ldots, h \). So solutions of \( F(\alpha,\tau) = 0 \) must be roots of \( \alpha^h + p_1(\tau)\alpha^{h-1} + \ldots + p_h(\tau) = 0 \). These roots are not a priori analytic functions of \( \tau \), but they can be written as power series in \( \tau^{1/h} \) [Puiseux series]. Let us fix the branch of \( \tau^{1/h} \) as the one which gives a real root if \( \tau \) is real and positive and write
\[ \alpha(\tau) = \sum_{i=1}^{h} a_i \tau^{i/h} \quad \text{for one root} \]

By (4), we know that \( \alpha(\tau) \) is real if \( \tau \) is real. This implies that only integral powers of \( \tau \) have non-zero coefficients. For if \( a_m \) is the first non-zero coefficient

\[ a(\tau) = \sum_{i=m}^{\infty} a_i \tau^{i/h} \]

so

\[ a_m = \lim_{\tau \to 0^+} a(\tau)/\tau^{m/h} \]

The right hand side is real because \( a(\tau) \) is real and because of the branch we chose for \( \tau^{1/h} \). Also

\[ a_m (-1)^{m/h} = \lim_{\tau \to 0^-} a(\tau)/(-\tau)^{m/h} \]

is real, implying \((-1)^{m/h}\) real and \( m \) is an integral multiple of \( h \). This argument may be repeated for the higher coefficients. Thus the roots may be written as real-analytic functions of \( \tau \).
(6) In (4) and (5), we have constructed an eigenvalue 
\( \lambda(\tau) = \lambda + \alpha(\tau) \) of \( P(\tau) \) as a real-analytic function of \( \tau \). To get the eigenvector \( u(\tau) \), one repeats the steps in (4). \( \exists c_1(\tau), \ldots, c_h(\tau) \) such that

\[
\sum_{j=1}^{h} f_{ij}(\alpha(\tau), \tau)c_j(\tau) = 0
\]

Observe that for real \( \tau \), the matrix \( ((f_{ij}(\alpha(\tau), \tau))) \) is real and symmetric. By the finite dimensional analytic perturbation theorem the \( c_j(\tau) \) can be chosen to be analytic in \( \tau \) [see Rellich, p. 32] and if \( c_j(\tau) \) is not real, replace it by \( \text{Re} \ c_j(\tau) \) which is also a solution. Then, proceeding as before,

\[
v(\tau) = \sum_{j=1}^{h} c_j(\tau)u_j
\]

is analytic in \( \tau \) for real \( \tau \) and so is

\[
u(\tau) = \{1 + Q[R(\tau) + \alpha(\tau)]\}^{-1}v(\tau)
\]

by lemmas 1, 2. This completes the proof of the proposition.
Proof of Abstract Theorem: By the preceding proposition, we know that there is an eigenvector \( u^{(1)}(\tau) \) of \( P(\tau) \), analytic in \( E \) for small \( \tau \). We prove the existence of the other eigenvalues and eigenvectors satisfying (a), (b), (c) by induction on the multiplicity \( h \).

Define \( \pi(\tau) \) as the projection of \( E \) onto the subspace spanned by \( u^{(1)}(\tau) \), i.e. \( \pi(\tau)v = \langle v, u^{(1)}(\tau) \rangle u^{(1)}(\tau) \).

Let

\[
P'(\tau) = P(\tau) - \pi(\tau)
\]

Let \( u^{(1)}(0) = u_1 \) and let \( u_1, u_2, \ldots, u_h \) be an orthonormal basis for \( \ker (P + \lambda) \). Now for \( j = 2, \ldots, h \)

\[
P'(0)u_j = P(0)u_j - \pi(0)u_j = \lambda u_j
\]

so \( \lambda \) is an eigenvalue of \( P'(0) \) with multiplicity at least \( h - 1 \). Claim that the multiplicity is exactly \( h - 1 \).

If not, there would be an element \( v \in E \) orthogonal to \( u_2, \ldots, u_h \) and having norm 1, such that
Then $\lambda <v, u_\perp> = <P'(0)v, u_\perp>
= <v, P'(0)u_\perp>
= (\lambda - 1)<v, u_\perp>

=> <v, u_\perp> = 0$ and hence $P(0)v = \lambda v$. But this implies that the multiplicity of $\lambda$ is $> h$, a contradiction.

Applying the abstract theorem to the eigenvalue $\lambda$ of $P'(0)$ [by the induction hypotheses], there are analytic functions $\lambda^{(j)}(\tau)$ in $R$ and $u^{(j)}(\tau)$ in $E$ for $j = 2, ..., h$ such that $\lambda^{(j)}(0) = \lambda$, the $u^{(j)}(\tau)$ form an orthonormal set for each $\tau$, and

$[P'(\tau) + \lambda^{(j)}(\tau)]u^{(j)}(\tau) = 0$. In addition

$<u^{(1)}(\tau), u^{(j)}(\tau)> = 0$, $j = 2, ..., h$ by the following reasoning. We have

$0 = [P(\tau) + \lambda^{(1)}(\tau)]u^{(1)}(\tau)
= [P'(\tau) - \pi(\tau) + \lambda^{(1)}(\tau)]u^{(1)}(\tau)
= [P'(\tau) + (\lambda^{(1)}(\tau) - 1)]u^{(1)}(\tau)$
so \( u^{(1)}(\tau) \) is an eigenvector of \( P'(\tau) \) with eigenvalue \( \lambda^{(1)}(\tau) - 1 \). For small \( \tau \),

\[ \lambda^{(1)}(\tau) - 1 \neq \lambda^{(j)}(\tau) \]

for all \( j = 2, \ldots, h \), so that \( \langle u^{(1)}(\tau), u^{(j)}(\tau) \rangle = 0 \) since they are in different eigenspaces. In particular this implies \( \pi(\tau)u^{(j)}(\tau) = 0 \) and finally

\[ [P(\tau) + \lambda^{(j)}(\tau)]u^{(j)}(\tau) = [P'(\tau) + \lambda^{(j)}(\tau)]u^{(j)}(\tau) = 0 \]

**Proof of Theorem 4.5.** To get conclusions (1) to (3), we apply the abstract theorem just proved. Let

\[ E = L_2(M), \quad P = L + \rho, \]

and let \( Q \) be the pseudo-differential inverse of \( P + \lambda \) (see 4.3). Given an absolutely analytic function \( \rho(\tau) \) in \( (C^\infty(M), \| \ldots \|_s) \) for any \( s \), satisfying \( \rho(0) = \rho \), let \( R(\tau) = \rho(\tau) - \rho \). Then the abstract theorem gives conclusions (1), (2) and (3) of theorem 4.5 except that the \( u^{(j)}(\tau) \) are only analytic in \( L_2(M) \). We will show first that they are analytic in \( (C^\infty(M), \| \ldots \|_s) \) for any \( s \). Recall from the derivation (1) of the proposition that an
analytic eigenvector $u(\tau)$ of $P(\tau)$ with eigenvalue $\lambda(\tau)$ must satisfy

$$u(\tau) = - Q[P(\tau) + \lambda(\tau) - \lambda]u(\tau) + v(\tau)$$

where $v(\tau) = \sum_{j=1}^{h} c_j(\tau)u_j$ in terms of a fixed orthonormal basis $\{u_1, \ldots, u_h\}$ of $\ker (P + \lambda)$.

Since $\ker (P + \lambda) \subset C^\infty(M)$, $v(\tau)$ is analytic in $(C^\infty(M), |\ldots|_s)$, any $s$. Now if $u(\tau)$ is analytic in $H_s$ (some $s$) and $R(\tau)$ is absolutely analytic in $C^s(M)$, $\lambda(\tau)$ analytic in $R$, $[R(\tau) + \lambda(\tau) - \lambda]u(\tau)$ is analytic in $H_s(M)$ by lemma 1. Since $Q : H_s(M) \to H_{s+2}(M)$ is a bounded operator.

$Q[R(\tau) + \lambda(\tau)]u(\tau)$ is analytic in $H_{s+2}(M)$. We know $u(\tau)$ is analytic in $H_0(M) = L_2(M)$, so in fact $u(\tau)$ is analytic in every $H_s$-space by induction. By the Sobolev lemma, $u(\tau)$ is analytic in every $C^s$-space.
Secondly we must prove (d). We do this using the minimax principle (see 4.2). Let
\[ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \]
be the eigenvalues of
\[ L + \rho \]
aranged in increasing order. Assume
\[ \lambda_{n-1} < \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+h-1} < \lambda_{n+h}, \]
i.e. \( \lambda_n \) has multiplicity \( h \). Let \( I \) be an interval such that \( I \) contains only the eigenvalue \( \lambda_n \) of \( L + \rho \). Then there exists \( \delta > 0 \) such that \( I \subseteq (\lambda_{n-1} + \delta, \lambda_{n+h} - \delta) \).

Choose \( T > 0 \) such that \(|\tau| < T\) implies \(|\rho(\tau) - \rho| < \delta\).

Then by (5.1), \(|\tau| < T\) implies \(|\lambda_{n-1}(\tau) - \lambda_{n-1}| < \delta\)
and \(|\lambda_{n+h}(\tau) - \lambda_{n+h}| < \delta\), implying
\[ \lambda_{n-1}(\tau), \lambda_{n+h}(\tau) \notin I. \]
Hence \( L + \rho(\tau) \) has at most \( h \) eigenvalues in \( I \) counting multiplicity. But we have already shown that it has at least \( h \) eigenvalues \( \lambda^{(1)}(\tau), \ldots, \lambda^{(h)}(\tau) \). This completes the proof.
References


Biographical Sketch

The author was born in Brooklyn, New York on March 2, 1946 and lived in New York City until 1966. Interest in mathematics started in high school, where he was active on the school's math team and won several prizes in competitions. He earned an A.B. degree (magna cum laude) from Columbia College in 1966, and was elected to Phi Beta Kappa and won the mathematics prize in that year. While in college, he also worked part-time as an assistant mathematics editor for Schaum Publishing Company in New York. He started graduate work at M.I.T. in September 1966 on a National Science Foundation Graduate Fellowship and in September 1970 was appointed to a full-time instructorship in mathematics at Tufts University. Outside interests include recorder playing and music in general, and tennis and squash. His favorite activities are those relating to the mountains: hiking, cross-country skiing, snowshoeing and canoeing. He is married, and his wife Maril works at Somerville Hospital as a nurse.