Automated Detection of Wave Velocities in Concrete Bridge Decks

by

David Prosper

Submitted to the Department of Civil and Environmental Engineering in partial fulfillment of the requirements for the degree of Master of Science in Civil and Environmental Engineering at the Massachusetts Institute of Technology May 1998

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Abstract

The assessment of the structural conditions of aging large infrastructure systems, such as bridges, roads and pipelines, plays a key role on the management of such systems. Still, commonly used non-destructive evaluation techniques for concrete are conducted on a relatively slow point by point basis, and involve high time and cost requirements. We need therefore, more efficient ways of scanning and analyzing concrete structures. Wave Stress Methods show potential to this effect.

In this thesis, we first analyze the impact of a steel pellet on to a pavement. As this is a common way of generating stress pulses for the non-destructive evaluation of concrete decks. Second, we present three simple ways to estimate wave velocities in concrete. These methods use the concepts of wave time of arrival, time of delay and phase delays. The first two concepts correspond to the time domain, while the third one to the frequency domain. Third, we study the propagation of a transient pulse in an elastic half space and draw similarities with the case of an elastic plate. In the last chapter, we propose a technique based in non-linear least squares fitting to estimate velocities and identify surface waves in a plate.

We apply the velocity estimation techniques to some empirical signals and compare the results of the different methods. We also apply the curve fitting technique to the same empirical signals. In general, the estimates are consistent, and different techniques give similar results.

Thesis Supervisor: Eduardo Kausel
Title: Professor of Civil and Environmental Engineering
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Chapter 1. Wave propagation in plates

In this chapter we describe some basic concepts of the propagation of waves in an infinite elastic plate. We focus on those aspects that are indispensable to understand the chapters that follow. A more detailed description of the concepts here may be found in textbooks such as [1] and [2].

We make two principal assumptions regarding the plate

- It is made of an homogeneous isotropic elastic linear material. Thus, its behavior is defined by only two elastic constants; say, the elastic modulus $E$ and the Poisson ratio $\nu$.
- It is unbounded; so, there are no edge effects.

1.1. Elastic waves in a plate

Any perturbation within an elastic body propagates through it in the form of elastic waves. In the case of a plate, there may be three different types of waves:

*Dilatational waves (P-waves):* For this type of waves, the particle motion is parallel to the direction of propagation of the wave, and only tensile and compression stresses are generated. This type of wave if the fastest of the three, and for an infinite elastic solid, its velocity depends on the elastic constants and density of the material, namely

$$C_p = \sqrt{\frac{E \cdot (1 - \nu)}{(1 + \nu) \cdot (1 - 2 \cdot \nu) \cdot \rho}}$$

(1)

*Transverse waves (S-waves):* For this type of waves, the particle motion is transverse to the direction of propagation of the wave, and only shear stresses are generated. Its velocity is

$$C_s = \sqrt{\frac{E}{2 \cdot (1 + \nu) \cdot \rho}} = C_p \cdot \sqrt{\frac{1 - 2 \cdot \nu}{2 - 2 \cdot \nu}}$$

(2)
Surface waves (R-waves): These types of waves occur only in bodies with free boundaries, and their effects are confined to regions close to the free surfaces. They were first investigated by Lord Rayleigh and are also known as Rayleigh waves. The particle motion next to the surface is elliptical and retrograde with respect to the direction of propagation; below a depth of approximately \(0.192 \cdot \lambda\) (\(\lambda\) is the wavelength), the motion reverses direction, and decreases exponentially in amplitude. The velocity of this type of waves, which is slightly smaller than that of S-waves, may be approximated by

\[
C_R = C_S \cdot \frac{(0.87 + 1.12 \cdot v)}{(1 + v)}
\]  

(3)

If the perturbation is a normal impact on the surface of the plate (i.e. the top surface) the three types of waves mentioned above exist. The R-wave propagates from the impact source along a circular wavefront on the surface. The P and S-waves propagate along hemispherical wavefronts into the body. Furthermore, when the P and S-waves reach other boundaries, they both reflect and convert into other waves.

![Figure 1: Elastic waves in a plate](image)

Because the R-waves are two-dimensional (they are confined to the surface), as opposed to the P-waves and S-waves, which are three-dimensional (body waves), the energy associated with the R-waves does not attenuate as fast as the energy associated with the body waves. In a plate this is only part true, as a plate itself is nearly a two-dimensional body.
1.2. Direct and reflected waves

A disturbance on the plate surface will quickly spread through it in the form of elastic waves. For any point of the plate some waves will reach it in a straight line from the source, which are direct waves. Other waves will reach the point after reflecting once or more on the boundaries of the plate and these are the reflected waves.

![Figure 2: Direct and reflected wave-paths for points of the surface](image)

1.3. Wave reflections in a free boundary. Snell's law

When an elastic wave impinges on a free boundary, it is converted into two waves on reflection, namely a P-wave and a S-wave. This is a distinguishing characteristic of the reflection of waves in solids, known as mode conversion. In some cases, surface waves may also develop.

We define a Cartesian space XYZ such that the free boundary is the surface at $y = 0$, and consider a plane P-wave impinging on it. For a plane wave, all particles in the wavefront have the same displacement; thus, the motion at a point $\mathbf{x}^T = [x, y, z]$ is given by

$$\mathbf{u} = \mathbf{n} \cdot \mathbf{u}(t - \mathbf{n} \cdot \mathbf{x}/c_p) \quad (2)$$

where $\mathbf{n}$ is a unit vectors parallel to the direction of propagation, $\mathbf{u}(\cdot)$ is any function of time and $c_p$ is the P-wave speed. For simplicity, we choose the XY plane as the plane of incidence; so, $\mathbf{n}^T = [n_x, n_y, 0]$. The angle of incidence $\theta_p$ is the angle between the direction of propagation and the perpendicular to the boundary, such that $\sin \theta_p = n_x$. 
We have then two reflected plane waves

\[ \mathbf{u} = \mathbf{n} \cdot A \cdot u(t - \mathbf{n} \cdot \mathbf{x}/c_p) \]
\[ \mathbf{v} = \mathbf{p} \cdot B \cdot u(t - \mathbf{m} \cdot \mathbf{x}/c_s) \]  

(3)

where \( \mathbf{n} \) and \( \mathbf{m} \) are unit vectors parallel to the direction of propagation of the reflected P-wave and S-wave respectively, \( \mathbf{p} \) is a unit vector perpendicular to \( \mathbf{m} \), \( c_s \) is the S-wave speed and \( A \) and \( B \) are constants. This is clearly a plane-strain problem and all displacements will be confined to the plane of incidence; thus, \( \mathbf{n}^T = [n_x, n_y, 0] \) and \( \mathbf{m}^T = [m_x, m_y, 0] \).

The boundary condition on the surface requires that the incident and reflected waves must have the same variation with \( x \) and \( t \). This is satisfied if

\[ \frac{n_x}{c_p} = \frac{\tilde{n}_x}{c_p} = \frac{\tilde{m}_x}{c_s} \]  

(4)

Defining the angles of emergence \( \tilde{\theta}_p \) and \( \tilde{\theta}_s \) for the reflected waves in the same way as the angle of incidence, we have

\[ \frac{\sin \theta_p}{c_p} = \frac{\sin \tilde{\theta}_p}{c_p} = \frac{\sin \tilde{\theta}_s}{c_s} \]  

(5)
which is Snell's law for waves in solids.

If we consider instead a shear wave polarized in the plane of incidence propagating in a direction \( \mathbf{m} \), we have

\[
\begin{align*}
\sin \theta_s &= \sin \bar{\theta}_\rho = \sin \bar{\theta}_s \\
\frac{c_S}{c_P} &= \frac{c_S}{c_S}
\end{align*}
\]  

(6)

Now, if \( \sin \theta_s > c_s/c_p \), then \( \bar{\theta}_\rho \) and the expression for the reflected P-wave are complex. The angle \( \theta_s = \arcsin(c_s/c_p) \) is referred to as the critical angle. For angles of incidence above this value, the reflected P-wave ceases to be an outgoing plane wave. Instead, we have a plane wave travelling parallel to the boundary.

Finally, if the incident wave is a shear wave polarized in a plane perpendicular to the plane of incidence, the incident wave reflects completely with not mode conversion, as in the case of acoustic waves.

1.4. Time of flight

The time of flight of a wave is the time it takes a wave to travel from the source to the point under consideration.

For the direct waves, it equals the distance to the source divided by the wave speed. For a point on the surface of the slab at a distance \( d \) from the source, we have \( t = d/c \), where \( c \) is the corresponding wave velocity.

After the P and S-waves reflect once on the face of the slab opposite to the point of impact, and because of the mode conversion, we have 4 waves: the P-P, P-S, S-P and S-S wave. The times of flight of the P-P and S-S are easy to calculate due to the symmetry on reflection, it is \( t = \sqrt{d^2 + 4 \cdot h^2}/c \), where \( h \) is the slab thickness.

Calculation of the times of flight of the other two waves is not straightforward. From Snell's law and the geometry, we have
\[
\theta_p = a \sin \left( \frac{c_p}{c_s} \cdot \sin \theta_s \right) \quad \text{(Snell's law)}
\]

\[
\theta_p = a \tan \left( \frac{d}{h} - \tan \theta_s \right) \quad \text{(Geometry)}
\] (7)

Taking \(d\) as a parameter in eq. (7), we can solve for the angles of incidence and reflection (\(\theta_p\) and \(\theta_s\) for the P-S wave, \(\theta_s\) and \(\theta_p\) for the S-P wave).

Once we know these angles, the time of flight is given by

\[
t = h \cdot \left( \frac{\sec \theta_p}{c_p} + \frac{\sec \theta_s}{c_s} \right)
\]

(8)

Figure 4 shows the time of flight for a plate 24 cm. thick, Poisson’s ratio of 0.25 and S-wave velocity of 2000 m/s, for distances up to 1.5 m.

![Figure 4: Times of flight in a plate](image)
Chapter 2. Impact of pellet onto slab

2.1. Elastic collision between two spheres

In textbooks on the theory of elasticity (e.g. Timoshenko), the elastic collision between two elastic spheres is referred to as the Hertz contact problem. While the solution method presented in such textbooks is based on quasi-static idealizations that neglect wave propagation processes within the spheres, it can be argued that when the duration of collision is greater than the fundamental period of either sphere, such processes do not significantly affect the variation of contact forces with time. We consider again this problem to assess both the duration of impact and the impact forces associated with a metallic spherical pellet impinging onto a flat surface, such as a pavement.

Consider two elastic, homogeneous spheres moving in opposite directions along the straight vertical line connecting their centers, which undergo collision at some moment in time (Figure 5). The spheres' instantaneous positions are then measured with respect to their locations when first contact is made.

![Diagram of elastic spheres](image)

**Figure 5:** Collision of elastic spheres
Denote with indices $j=1,2$ the upper and lower spheres, respectively. We then define the following symbols:

- $R_j =$ radii
- $\rho_j =$ mass densities
- $m_j = \frac{4}{3} \pi \rho_j R_j^3 =$ masses
- $u_j =$ upper sphere’s downward, and lower sphere’s upward displacements
- $u = u_1 + u_2 =$ total deformation (relative displacement)
- $\nu_j =$ Poisson’s ratios
- $G_j =$ shear moduli
- $E_j = 2G_j (1 + \nu_j) =$ Young’s moduli
- $C_s = \sqrt{G_j / \rho_j} =$ shear wave velocities
- $f_j = \left(1 - \nu_j^2\right) / E_j = \frac{1}{2} \left(1 - \nu_j\right) / G_j =$ flexibilities
- $\sigma_z =$ maximum contact pressure
- $P =$ total contact force
- $a =$ radius of contact area
- $V =$ relative impact velocity

Following Timoshenko, the contact pressure has a semi-spherical distribution whose maximum value $\sigma_z$ is

$$\sigma_z = \frac{3P}{2\pi a^2} \quad (9)$$

On the other hand, the radius of the contact area depends on the deformation, and is given by the expression

$$a = \sqrt[3]{\frac{3 P(f_1 + f_2) R_1 R_2}{4 \left(R_1 + R_2\right)}} \quad (10)$$

Finally, the distance by which both spheres approach each other (i.e. the total deformation) is given by
\[ u = \sqrt[3]{\frac{9}{16} \frac{P^2 (f_1 + f_2)^2 (R_1 + R_2)}{R_1 R_2}} \] (11)

From Newton's law, the dynamic equilibrium equations are \( m_1 \ddot{u}_1 = -P \) and \( m_2 \ddot{u}_2 = -P \).

It follows that

\[ \ddot{u} = \ddot{u}_1 + \ddot{u}_2 = -P \left( \frac{1}{m_1} + \frac{1}{m_2} \right) = -\frac{P(m_1 + m_2)}{m_1 m_2} = -\frac{P}{m} \] (12)

in which the effective mass \( m \) is defined as

\[ m = \frac{m_1 m_2}{m_1 + m_2} \] (13)

From eq. (11), we have immediately

\[ u^3 = \frac{9}{16} \frac{(f_1 + f_2)^2 (R_1 + R_2)}{R_1 R_2} P^2 = n^2 P^2 \] (14)

with

\[ n = \frac{3(f_1 + f_2)}{4} \sqrt{\frac{R_1 + R_2}{R_1 R_2}} \] (15)

From eq. (12) and (7), it follows that

\[ mn \ddot{u} + u^{3/2} = 0 \] (16)

Multiplying this equation by \( du \) and considering the identity \( \ddot{u} du = \frac{1}{2} d\dot{u}^2 \), we obtain

\[ \frac{1}{2} nm \dot{u}^2 + u^{3/2} du = 0 \] (17)

Integration of this equation with initial conditions \( u_0 = 0 \) and \( \dot{u}_0 = V \) produces

\[ \frac{1}{2} nm(u^2 - V^2) + \frac{2}{3} u^{5/2} = 0 \] (18)
In particular, the maximum deformation occurs when the velocity is zero. Hence,

\[ u_{\text{max}} = \left( \frac{5}{4} \frac{n}{m} \right)^{2/5} V^{4/5} \]  

(19)

On the other hand, eq. (18) can also be written as

\[ \frac{du}{dt} = \sqrt{V^2 - \frac{4u^{5/2}}{5mn}} = V \sqrt{1 - \left( \frac{u}{u_{\text{max}}} \right)^{3/2}} \]  

(20)

Defining \( \xi = u / u_{\text{max}} \), this expression can be integrated to obtain the contact time as

\[ t_c = \frac{2u_{\text{max}}}{V} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^{5/2}}} = 2.943 \frac{u_{\text{max}}}{V} \]  

(21)

The factor 2 accounts for the fact that contact time is twice the time needed to reach maximum deformation. Finally, we introduce the dimensionless ratios:

\[ r = \frac{R_1}{R_2}, \quad \mu = \frac{\rho_1}{\rho_2}, \quad \gamma = \frac{G_1}{G_2} \]  

(22)

with which eq. (13) and (15) can be written as

\[ m = m_1 \frac{1}{(1 + \mu r^3)} = \frac{4\pi R_1^3}{3(1 + \mu r^3)} \]  

(23)

\[ n = \frac{3}{8} \frac{1}{G_1 \sqrt{R_1}} (1 + r)^{1/2} \left[ 1 - \nu_1 + \gamma (1 - \nu_2) \right] \]  

(24)

Hence, combining eq. (19), (23) and(24), the maximum displacement is

\[ \frac{u_{\text{max}}}{R_1} = \left( \frac{V}{C_{s1}} \right)^{4/5} \left\{ \frac{5\pi (1 + r)^{1/2}}{8 (1 + \mu r^3)} \left[ 1 - \nu_1 + \gamma (1 - \nu_2) \right] \right\}^{2/5} \]  

(25)
We turn next to the determination of the variation of forces and displacements with time. From eq. (21), if \( x = \frac{u}{u_{\text{max}}} \) is a specific intermediate value of the normalized displacement, the time associated with that displacement is

\[
\frac{2.943 t}{t_c} = \int_0^x \frac{d\xi}{\sqrt{1 - \xi^{5/2}}} = I(x) \quad (26)
\]

To evaluate the integral \( I(x) \), we make first a change of variable \( \xi = \sin \varphi \). Noting that \( \sqrt{1 - \xi^{5/2}} = \cos \varphi \) and \( d\xi = \frac{4 \cos \varphi d\varphi}{5 \sin^{3/5} \varphi} \), we have

\[
I(x) = \frac{4}{5} \int_0^{\arcsin x^{1.25}} \frac{d\varphi}{\sin^{3/5} \varphi} = \frac{4}{5} \int_0^{\arcsin x^{1.25}} \left( \frac{\varphi}{\sin \varphi} \right)^{1/5} \frac{d\varphi}{\varphi^{1/5}} \quad (27)
\]

The term in parenthesis in the last integral changes only slowly in the interval \( 0 < \varphi < \pi/2 \) (with values ranging from 1 to about 1.09) An excellent approximation for this term, with absolute error smaller than 0.001, is given by the least squares approximation

\[
\left( \frac{\varphi}{\sin \varphi} \right)^{1/5} = 1 + 0.0302 \varphi^2 + 0.00496 \varphi^3 \quad (28)
\]

Substituting this expression in eq. (27), the integral can then be evaluated analytically, which results in

\[
I(x) = \varphi^{0.8} \left[ 1 + 0.008629 \varphi^2 + 0.001044 \varphi^3 \right]_{\arcsin x^{1.25}}^{0} \quad (29)
\]

In particular, the upper limit \( x=1 \) corresponds to \( \varphi=\pi/2 \) and \( I(1) = 1.4715 = (0.5)(2.943) \), which agrees with the coefficient used in eq. (21) and (26). Application of this method for intermediate values of the normalized displacement \( x \) produces the time history shown in Figure 6.

Also shown in this figure in dashed lines is the approximate curve
\[
\frac{u}{u_{\text{max}}} = \sin^{16/15} \pi \frac{t}{t_c} 
\]

(30)

which could prove advantageous in some cases. It should be added, though, that the extremely close fit of this equation is somewhat deceiving: if we took the derivative of this equation with respect to time, the resulting expression would contain a term in the fifth root of the sine function, which at \( t=0 \) begins at zero and not at the initial velocity \( V \); however, it rises quickly to that number.

Finally, we consider the variation of contact forces with time. From eq. (7) and (24),

\[
P = \frac{1}{n} u^{3/2} \\
\]

\[
P_{\text{max}} = \frac{1}{n} u^{3/2}_{\text{max}} 
\]

(31)

from which it follows immediately that

\[
\frac{P}{P_{\text{max}}} = \left( \frac{u}{u_{\text{max}}} \right)^{3/2} 
\]

(32)

\[
\begin{array}{c}
\text{Variation of displacement with time} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Variation of force with time} \\
\end{array}
\]

\[0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \\
0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \\
\]

\[
\begin{array}{c}
\text{True solution} \\
\text{Approximation} \\
\end{array}
\]

**Figure 6:** Relative displacement and contact force

This is a parametric representation of the force \( P \) with time \( t \), which can be evaluated from the numerical solution for \( u \) presented previously. The time history for the contact force thus obtained is shown in Figure 6; also shown is the very close approximation
\[
\frac{P}{P_{\text{max}}} = \sin^{8/5} \pi \frac{t}{t_c}
\]  

(33)

which once more can prove advantageous in practical applications. Of course, \(P_{\text{max}}\) in this equation would be obtained from eq. (31).

It remains to verify the accuracy of the approximation, eq. (33), by computing the total momentum transferred during collision. By definition, it must equal the impulse

\[
I = \int_0^t P \, dt = P_{\text{max}} \int_0^t \sin^{8/5} \pi \frac{t}{t_c} \, dt = \frac{P_{\text{max}} t_c}{\pi} \int_0^\pi \sin^{8/5} \theta \, d\theta
\]

\[
= \frac{P_{\text{max}} t_c}{\pi} \left[ \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}ight) \right] = P_{\text{max}} t_c \frac{1}{\sqrt{\pi}} \Gamma(1.3) = P_{\text{max}} t_c \frac{0.89747}{\sqrt{\pi}} = 0.54365 P_{\text{max}} t_c
\]

(34)

Taking into consideration eq. (19), (21) and (31), this expression reduces to

\[
I = (0.54365) \cdot (2.943) \cdot (5/4) \, m \cdot V = 1.99995 \, m \cdot V
\]

(35)

which agrees almost perfectly with the exact theoretical result, namely \(2mV\).

2.2. Inelastic collision

Actual conditions during impact will deviate from the ideal equations presented previously because the colliding bodies will exhibit nonlinear effects, including plastic deformations. Among the results of such inelastic effects will be a substantial increase in the contact time, and therefore, of the spectral characteristics of the impact force (the impulse). Since a rigorous solution of this non-linear problem can be exceedingly difficult, we will not attempt to obtain formal results here; instead, only a rough upper bound for the contact time will be derived.

Assuming that the contact stresses have reached the yielding limit \(\sigma_y\) and remain constant during impact, that the contact area is \(A_c\), and that the duration of collision is \(t_c\), we then
have from the principle of conservation of momentum \( mv = A_c \sigma_y t_c \). It immediately follows that

\[
t_c \leq \frac{mv}{A_c \sigma_y}\quad (36)
\]

### 2.3. Impact of spherical pellet onto pavement

We consider next the particular case of a spherical pellet impinging onto a flat pavement, which ostensibly can be regarded as a sphere of infinite radius, that is \( R_2 = \infty \). In this case \( r=0 \), and eq. (25) simplifies to

\[
\frac{u_{\text{max}}}{R_1} = \left( \frac{V}{C_{s1}} \right)^{4/5} \left\{ \frac{5\pi}{8} \left[ 1 - \nu_1 + \frac{G_1}{G_2} (1 - \nu_2) \right] \right\}^{2/5}\quad (37)
\]

Eq. (21) and (35) can be used to assess the impact of the pellet onto either a concrete or an asphalt pavement. In particular, we consider the following data:

<table>
<thead>
<tr>
<th>Pellet</th>
<th>Pavement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Copper</td>
</tr>
<tr>
<td>( R ) [mm.]</td>
<td>2.73</td>
</tr>
<tr>
<td>( \rho ) [kg/m(^3)]</td>
<td>8900</td>
</tr>
<tr>
<td>( C_s ) [m/s]</td>
<td>2325</td>
</tr>
<tr>
<td>( G ) [Gpa]</td>
<td>48.110</td>
</tr>
<tr>
<td>( \nabla )</td>
<td>1/3</td>
</tr>
</tbody>
</table>

**Table 1:** Geometric and elastic properties of pavement and pellet

Hence, for the concrete

\[
u_{\text{max}} = 6.20 \cdot \left( \frac{V}{C_{s1}} \right)^{4/5} \text{ mm.}\quad t_c = 7.84 \cdot \left( \frac{C_{s1}}{V} \right)^{1/5} \mu s.
\]

and for the asphalt
\[ u_{\text{max}} = 9.42 \left( \frac{V}{c_{s1}} \right)^{4/5} \text{mm.} \quad t_c = 11.93 \left( \frac{c_{s1}}{V} \right)^{1/5} \mu s. \]  

(39)

The previous four expressions are plotted in Figure 7, as a function of the impact velocity.

A rough upper bound for the contact time can be obtained by means of eq. (36), which assumes full plastification for the duration of the collision. This requires estimating the contact area, which cannot exceed the cross-section of the pellet, and an effective yield stress, which can be expressed in terms of the effective modulus and the yield strain. We have then

\[ \sigma_y = \frac{E}{1-\nu^2} \varepsilon_y = \frac{2G}{1-\nu^2} \varepsilon_y \quad \quad A_c \leq \pi R_1^2 \]  

(40)

so that

\[ t_c \leq \frac{4}{3} (1-\nu) \rho_1 \pi R_1^2 V \frac{\pi R_1^2 \varepsilon_y}{2 \rho C_s^2 \pi R_1^2 \varepsilon_y} = \frac{2}{3} (1-\nu) \frac{\rho_1 V}{\rho} \frac{R_1}{C_s C_s} \]  

(41)

During impact of the pellet onto the concrete surface, both the pellet and the concrete will undergo inelastic effects. The effective shear wave velocity \( C_s \) must then lie between that of concrete and copper, and so must also the effective mass density \( \rho \).

![Graph of Maximum Indentation into Pavement and Contact Time](image)

**Figure 7**: Maximum indentations and contact times
Chapter 3. Wave velocities estimation

The application of non-destructive techniques based on stress waves to concrete requires accurate estimates of the velocity of propagation of the waves. This information is useful in two ways:

- The velocities of the stress waves within an elastic medium are closely related to its mechanical properties: elastic modulus and Poisson’s ratio. Exceptionally low velocities may indicate excessive cracking within the concrete or poor condition in general.

- The velocity of the P-wave may be used to estimate the thickness of concrete walls and slabs. Measuring the time that it takes for the P-wave to travel from one side to the other, reflect and return, we obtain the thickness as the product of the time and the estimated velocity. This is the foundation for the Impact Echo method [3], which is a non-destructive technique used to measure thicknesses and to locate structural flaws in concrete structures.

In our estimation of wave velocities, we assume that the R-wave dominates the vibrations measured on the surface of the concrete slab. We base this assumption on two facts: the energy associated with the R-wave is more than double the sum of the energies associated with the P- and S-waves, and the geometric attenuation of the R-wave is less than for the other two waves. Miller and Pursey [7] studied the relative amplitudes of P, S and R waves at some point in the neighborhood of the impact location and showed that the energies associated with each wave are 7, 26 and 67% of the total, respectively. Furthermore, R-waves are two-dimensional waves, since they travel near the surface, whereas the P and S waves are three-dimensional body waves; hence, the energy associated with the R-waves does not decay as fast with distance from the source as the energy associated with the body waves. Thus, the further we measure from the impact point, the more the R-wave will dominate. However, this effect is partly counteracted by the fact that as the waves attenuate with distance, it is more difficult to distinguish the...
signal from the background noise. Thus, the significance of the noise increases with distance, decreasing the readability of the signal.

There is another important property that makes the use of R-waves appealing in the estimation of wave velocities. While P-waves and S-waves in plates are dispersive, the R-wave is not, at least not when the wavelengths considered are shorter than the thickness. Thus, all the harmonics composing the R-wave travel at essentially the same speed, which means that the shape of the R-wave pulse is preserved as it travels along the surface, so its time of arrival can be used to measure wave velocities. This, of course, is only true for a homogeneous medium.

We present three methods to estimate the R-wave velocity. The first two operate in the time domain, while the third method relies on the phase angles of the Fourier transforms of the measures. This method requires working with the unwrapped phase angles, and is computationally more expensive than the first two; however, it is included here for added robustness. On the other hand, the velocities for the P-wave and S-wave are back-calculated using relations from the theory of elasticity.

3.1. R-wave velocity estimation based on times of arrival (TOA)

3.1.1. Theoretical approach

The time $t$ that it takes a wave propagating in an elastic medium at constant speed $c_R$ to travel a distance $d$ is $t = d / c_R$. If we impact the surface of an elastic body and record the arrival of the R-wave produced by this impact $t$ seconds later at a distance $d$ from the impact, from the equation above, we can calculate the propagation velocity for the wave. Because the velocity in a homogeneous medium is constant, if we record the traveling times for several receivers at various distances on the surface and display these times against the known distances, then ideally all points should lie on the same straight line. The slope of this line is the inverse of the R-wave velocity, namely $c_R$.

The previous statements still hold when the arrival times are measured relative to a time other than the instant of impact. Indeed, let $t_i$ be time at which the impact occurs. If the
TOA of the waves at various distances from the impact is \( t_A \), then the associated travel times will be \( (t_A - t_1) = d / c_R \). Since the impact time is the same for all the measurements, we again have a straight line \( t_A = d / c_R + t_1 \), with slope \( 1 / c_R \). Thus, if we know the exact impact location relative to the sensors, we can also calculate the time when that impact was applied. The intersection of the straight line with the time axis \( (d = 0) \) yields the desired value of \( t_1 \).

### 3.1.2. Practical considerations

Inaccuracies in measurements, discretization and signal analysis procedures lead to errors in TOA estimations. Frequently, a graphic display of the TOA will not align exactly on the same straight line, so linear regression is required to fit the data, using for this purpose the Least Squares Method.

Our aim is to find the best fit to the data with the straight line

\[
t_A = \frac{d}{c_R} + t_1 \tag{42}
\]

The sum of the square of the errors is thus

\[
S = \sum_i \left( t_A(d_i) - t_A \right)^2 = \sum_i \left( \frac{d_i}{c_R + t_1} - t_A \right)^2 \tag{43}
\]

where \( t_A \) are the actually measured TOAs, and the estimated TOAs are given by substituting the distances \( d_i \) in eq. (1). In the error calculation, we only consider uncertainties in the time variables, as the exact distances are known. Of course, should the exact location of the impact be unknown, then we would not be able to determine the impact time.

To minimize the squared error, we differentiate it with respect to the two unknown parameters \( c_R \) and \( t_1 \), and set the partial derivatives equal to zero.
\[
\frac{\partial S}{\partial (1/c_R)} = \sum_i 2 \cdot \left( \frac{1}{c_R} \cdot d_i + t_i - t_{A_i} \right) \cdot d_i = 0
\] (44)

\[
\frac{\partial S}{\partial t_i} = \sum_i 2 \cdot \left( \frac{1}{c_R} \cdot d_i + t_i - t_{A_i} \right) = 0
\] (45)

Finally, solving the system of eq. (3) and (4), we obtain expressions for the estimation of the wave velocity and impact time. If \( n \) is the total number of signals, then

\[
\frac{1}{c_R} = \frac{n \cdot \sum_i t_{A_i} \cdot d_i - \sum_i t_{A_i} \cdot \sum_i d_i}{n \cdot \sum_i d_i^2 - \sum_i d_i \cdot \sum_i d_i}
\] (46)

\[
t_i = \frac{\sum_i t_{A_i} - \frac{1}{c_R} \cdot \sum_i d_i}{n}
\] (47)

The error for a given signal recorded at a distance \( d_i \) from the impact and with an estimated TOA \( t_{A_i} \) is

\[
\varepsilon = t_A(d_i) - t_{A_i} = \frac{d_i}{c_R} + t_i - t_{A_i}
\] (48)

3.1.3. Estimation of TOA

There are two aspects we must consider when choosing a technique for the estimation of TOAs, namely the shape of the pulse being detected and the signal to noise ratio (SNR). In the application considered, the shape of the signals to be detected is unknown. It results from a mix of direct and reflected waves that overlap in time with the R-wave and confound the shape of the latter. Because the various wave components travel at different speeds, this mixing will vary from one receiver location to another. Moreover, the shape of R-wave itself depends on how the impact force changes with time, which is also unknown. Fortunately, the SNR for the R-wave is high, which facilitates its TOA estimation.
The TOA of a pulse with respect to some arbitrary time origin will be defined here as the time when the rising phase of the pulse exceeds a certain threshold value, expressed as a fraction of the pulse amplitude. Thus, the TOA is independent of the amplitude of the pulse. Standard methods of determining the TOA of an unknown pulse are adaptive thresholding (ATH) and double differentiation (DD) [4]. ATH determines the maximum amplitude of the signal and defines a threshold value based on this maximum (say some percentage of it). It takes as TOA the time when the signal exceeds this threshold value. DD takes the TOA as the time when the slope of the leading edge of the signal attains a maximum.

We combine both methods for the estimation of the TOAs. We first determine the highest amplitude of the signal to process, and we define a threshold value as a percentage of this maximum. We then look for the time when the signal first exceeds this threshold. Because the signal is the result of the combination of pulses propagating with different speeds along multiple paths in the concrete, and because of the sometimes high background noise, we cannot guarantee complete similarity between the signals recorded at different distances from the source. Thus, an equal ratio between the signal at a certain time and the maximum amplitude does not necessarily imply equal location with respect to the arrival of the R-wave.

To improve the estimate, we look backward from this estimate to the closest maximum of the slope, or equivalently, to the time when the second difference changes sign.

The second central difference of the signal \( x_n = \{x_1, x_2, \ldots, x_N\} \) at the position \( j \) is calculated as

\[
\ddot{x}_j = x_{j+2} - 2 \cdot x_j + x_{j-2} \quad (49)
\]

In principle, this equation should be divided by the square of the time increment, but since the signals are recorded at a constant rate, we need not consider the time increment between samples. To locate the exact time where the second difference changes sign, we interpolate linearly between consecutive samples.
The following figures show the results of applying this technique to signals for a concrete deck that were computed numerically. Clearly, since these signals are noise-free, the estimation is expected to be better than estimations with real data, but if the method were to fail for these ideal conditions, then it would surely fail for actual recordings.

For the numerical simulation we used the thin layer method [5], which we applied to a homogeneous concrete slab of 24 cm thickness. We computed the displacements of points at distances of 0.061, 0.244, 0.61 and 0.701 m. from the source, and subjected the system to an impact of 0.02 ms duration. We chose the mass density as well as Poisson’s ratio of the concrete to be 2,400 kg/m$^3$ and 0.25, respectively. With these data, the theoretical S-wave velocity is 2 m/ms, while the R-wave velocity may be estimated as:

$$C_R = C_S \cdot \frac{(0.87 + 1.12 \cdot \nu)}{(1 + \nu)} = 1.84 \text{ m/ms.} \quad (50)$$

Using the TOAs technique, we estimate a velocity for the R-wave of 1.8383 m/ms.

Figure 8: Estimates of TOA for numerical simulation at various distances
3.2. **R-wave velocity estimation based on time delay estimation (TDE)**

3.2.1. *Theoretical approach*

An alternative approach to estimate the velocity of the R-wave is based on TDE techniques. Once again, we build upon the predominance of the R-wave over both the noise and other types of waves. This time, we also stress the non-dispersive character of the R-wave.

Consider two sensors: sensor 1 at close range from the impact and sensor 2 at a somewhat larger distance. At either sensor, we measure the signal $x^1, x^2$, which include the R-wave, the P and S-waves, and the noise. Because of the non-dispersive character of the R-
wave, we expect the R-wave at sensor 2 to be a scaled and delayed replica of that recorded at sensor 2. So, if we regard the P and S-waves as noise, we have

\[ x'(t) = r(t) + n'(t) \]

\[ x^2(t) = A \cdot r(t - D) + n^2(t) \] (51)

where \( A \) is the attenuation (scaling) factor and \( D \) is the delay between signals.

If we can estimate the delay between the signals \( D \), the ratio distance between sensors to delay gives an estimate of the R-wave velocity.

\[ c_R = \frac{d}{D} \] (52)

In this alternative approach, we look and compare in pairs the signals at different sensors, while in the previous approach we considered individually each of the signals. Thus, for every pair, we obtain an estimate of the velocity. Because we have 4 signals, we obtain 6 individual estimates.

Finally, we can take the velocity estimate as the mean value of these individual estimates. A slightly more elaborate alternative is to make a linear regression, as for the TOA technique.

An advantage of the TDE approach is that along with the delay, we obtain an estimate of how similar the signals are, which in turn gives us indications about the validity of the delays thus estimated. A clear disadvantage is that TDE techniques are computationally more expensive than the simple TOA method used previously.

### 3.2.2. Estimation of TDE

There exist several TDE techniques to estimate delays in the arrival of signals. A classical TDE technique consists of identifying the maximum value of the cross correlation function (CC) between the reference and the delayed signal [6]. The CC of two functions, \( x \) and \( y \), is defined as
\[ R_{xy}(\tau) = \frac{1}{T} \int_{-T}^{T} x(t) \cdot y(t + \tau) \cdot dt \]  
\hspace{1cm} (53)

where \( T \) is the time window of the signal analyzed.

In practical cases, we work with discrete representations of continuous signals. Thus, we have two discrete time series \( x^1_n = \{x^1_1, x^1_2, \ldots, x^1_N\} \) and \( x^2_n = \{x^2_1, x^2_2, \ldots, x^2_N\} \), consisting of \( N \) values each, sampled at time intervals \( \Delta t \). The CC becomes then

\[ R^{12}_r = \frac{1}{N} \sum_{j=0}^{N} x^1_j \cdot x^2_{j+r} \]  
\hspace{1cm} (54)

The total duration of the recorded signal is \( T = (N - 1) \cdot \Delta t \).

Instead of using eq. (54) directly, we normalize the CC, such that it has values only between \(-1\) and \(1\). The final equation is

\[ NR^{12}_r = \frac{R^{12}_r}{\sqrt{R^{11}_0 \cdot R^{22}_0}} \]  
\hspace{1cm} (55)

Although the signals are discrete, the CC as expressed in eq. (53) is really continuous. It appears reasonable to apply interpolation to obtain the closest value to the actual maximum of the CC. We could use elaborate interpolation methods such as trigonometric interpolation by means of Fourier Transforms; however, this would be computationally expensive. Instead, we approximate the CC near its maximum by a convex parabola, as proposed by Jacovitti et Al [6]. The final estimate becomes

\[ D = \frac{\Delta t}{2} \cdot \frac{NR^{12}_{M+1} - NR^{12}_{M-1}}{NR^{12}_{M+1} - 2 \cdot NR^{12}_M + NR^{12}_{M-1}} + (M - 1) \cdot \Delta t \]  
\hspace{1cm} (56)

where \( \Delta t \) is the sampling time step and \( M \) is the index for which the discrete CC is a maximum.
We apply this approach to the same synthetic displacements shown previously (Table 2), and estimate an R-wave velocity of 1.8357 m/ms. This result is not as good as the one obtained by TOA.

<table>
<thead>
<tr>
<th>Distance sensor to source [m]</th>
<th>Distance sensor to source [m]</th>
<th>Distance between sensors [m]</th>
<th>correlation between signals</th>
<th>Delay estimate [ms]</th>
<th>Raleigh speed [m/ms]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0610</td>
<td>0.2440</td>
<td>0.1830</td>
<td>0.8999</td>
<td>0.1004</td>
<td>1.8231</td>
</tr>
<tr>
<td>0.0610</td>
<td>0.6100</td>
<td>0.5490</td>
<td>0.7386</td>
<td>0.2995</td>
<td>1.8332</td>
</tr>
<tr>
<td>0.0610</td>
<td>0.7010</td>
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<td>0.6944</td>
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</tr>
<tr>
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<td>0.3660</td>
<td>0.7644</td>
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</tr>
<tr>
<td>0.2440</td>
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<td>0.4570</td>
<td>0.7230</td>
<td>0.2485</td>
<td>1.8390</td>
</tr>
<tr>
<td>0.6100</td>
<td>0.7010</td>
<td>0.0910</td>
<td>0.8969</td>
<td>0.0493</td>
<td>1.8455</td>
</tr>
</tbody>
</table>

**Mean:** 1.8357
**Standard deviation:** 0.0001

**Table 2:** Delay and R-wave velocity estimates by TDE

![Image](image.png)

**Figure 11:** Normalized cross correlation with time increments of 0.001 ms.

### 3.3. R-wave velocity estimation based on phase delays (PD)

#### 3.3.1. Theoretical approach

The displacements at the boundary of a half space due to an harmonic plane R-wave propagating parallel to the X-axis are

\[ v = A \cdot \cos(\omega \cdot (x/c_R - t)) \]  \hspace{1cm} (57)
where $A$ is a constant, $\omega$ is the angular frequency and $c_R$ is the wave speed. So, given 2 different locations, say $x_1$ and $x_2$, the corresponding displacements will be

$$
\begin{align*}
v_1 &= A \cdot \cos \omega \cdot \left( \frac{x_1}{c_R} - t \right) \\
v_2 &= A \cdot \cos \omega \cdot \left( \frac{x_1}{c_R} + \Delta x/c_R - t \right)
\end{align*}
$$

(58)

where $\Delta x = x_2 - x_1$. Eq. (58) shows a delay between both signals, equal to $\Delta \theta = \omega \cdot \Delta x/c_R$, where we know the frequency $\omega$ and the distance between control points $\Delta x$. Thus, if we can measure the phase delay between signals, we can estimate the wave speed as

$$
c_R = \frac{\omega \cdot \Delta x}{\Delta \theta}
$$

(59)

For a more general wave shape, we can always decompose it into harmonics by means of the Discrete Fourier Transform (DFT) and apply eq. (59) to each of the components of the DFT.

Furthermore, if the half space is homogeneous, the R-wave is not dispersive ($c_R$ is constant regardless of frequency); so, if we plot the phase delays as a function of frequency, we should obtain a straight line passing through the origin and with slope $\Delta x/c_R$.

3.3.2. R-wave behavior in a plate

Next, we should ask ourselves whether the theory above also applies to a plate. At this point, we recall that the R-wave is confined to the region close to the boundary. It is shown that given a harmonic R-wave, the motion for points deeper than its wavelength, $\lambda = c_R/f$, is negligible [9]; so, the material below this depth does not affect the propagation of the R-wave. From this, we conclude that for wavelengths smaller than the thickness of the plate, the R-wave propagates as in the half space. For a thickness of 25 cm. and a wave speed of 2000 m/s., for instance, we will have a minimum frequency of 8 kHz; thus, we should take into account only frequencies above this value.
The maximum frequency we should consider will depend on the impact; we should reject frequencies for which the impact contains little energy. Maximum values around 60 kHz seem reasonable for the case of a pellet impact. Normally, by looking at the unwrapped phases, it is easy to read a sensible maximum.

### 3.3.3. Discrete Fourier Transform

Any angle may be considered as the sum of two quantities: its principal value, which is bounded between $\pm\pi$, and a multiple of $2\cdot\pi$. Common DFT algorithms compute only the principal values of the phases; so, they turn the phase that otherwise would be a continuous function of frequency, into a discontinuous one. Nevertheless, to apply the concepts above, we must compute the continuous phases.

The process of computing the continuous phase from the DFT is called phase unwrapping, and the result is the unwrapped phase. To do the phase unwrapping, we will use a method proposed by Tribolet [10].

Finally, it must be noted that phase unwrapping is a lengthy process and it is the main reason why this method of estimating velocities is computationally expensive.

![Figure 12: Wrapped vs. unwrapped phase](image)

3.3.4. Practical considerations

In summary, the steps required to estimate velocities based on phase delays are
- Compute DFT of the signals
- Unwrap the phases of the DFT
- Take the signals in pairs and subtract the phases.
- Fit the resulting functions to straight lines using, for instance, least squares linear regression
- Estimate velocities by dividing the distance between sensors by the slope of the fitted lines
- Find the average of the individuals estimates

It may be worthwhile to apply some type of time window before computing the DFT. Because the DFT is periodic, non-zero values at the edges of the signals will produce high frequencies in the DFT.

We apply this approach to the same synthetic displacements above. As for the TDE technique, with four control locations we can compute 6 estimates of the velocity. Considering only frequencies between 20 and 80 kHz, we estimate an R-wave velocity of 1.8411 m/ms (Table 3).

<table>
<thead>
<tr>
<th>Distance sensor to source [m]</th>
<th>Distance sensor to source [m]</th>
<th>Distance between sensors [m]</th>
<th>Raleigh speed [m/ms]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0610</td>
<td>0.2440</td>
<td>0.1830</td>
<td>1.8537</td>
</tr>
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<td>0.0610</td>
<td>0.6100</td>
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</tr>
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<td>0.6400</td>
<td>1.8415</td>
</tr>
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<td>0.3660</td>
<td>1.8370</td>
</tr>
<tr>
<td>0.2440</td>
<td>0.7010</td>
<td>0.4570</td>
<td>1.8367</td>
</tr>
<tr>
<td>0.6100</td>
<td>0.7010</td>
<td>0.0910</td>
<td>1.8354</td>
</tr>
</tbody>
</table>

**Table 3:** R-wave velocity estimates by phase delays
3.4. Signal improvement by digital filtering

There are numerous reasons why discrete signals contain some error or noise. For instance, the physical processes we measure (vibrations in a plate) are generally continuous in nature and the time discretization introduces some error; in the measuring process, we round off the measurements to a certain precision; in addition, there is always some background noise during the recording of the signals.

Furthermore, in the actual signals we analyzed, we detected the presence of a predominant resonant frequency around 40 kHz, which matches the main natural frequency of the sensors. This resonance is likely related to the recording instruments and not to the plate itself.

To minimize these undesired errors, we apply a low pass filter to the signals before applying TOA or TDE techniques. We aim to minimize the negative effect of

- the high frequency noise
- the resonance from the sensors

Unfortunately, we also may lose some valid information.

3.4.1. Definition of the low pass filter

We applied a linear non-recursive filter in the time domain, as follows
\[ y_n = \sum_{K=-N}^{N} c_K \cdot x_{n-K} \]  \hspace{1cm} (60)  

where \( x_n \) is the \( n \)th sample of the original signal and \( y_n \) the resulting \( n \)th sample after applying the filter. In eq. (60), we see that the filter is non-recursive, as it includes in the right hand only terms from the original signal.

The coefficients \( c_k \) are defined as

\[
\begin{align*}
c_0 &= 4 \cdot f_{Max} \\
c_k &= \frac{\sin\left(\frac{\pi \cdot K}{N}\right)}{\left(\frac{2}{\pi \cdot K} \cdot \sin(2 \cdot \pi \cdot f_{Max} \cdot K)\right)} \hspace{0.5cm} (N \geq K > 0) \\
c_{-K} &= c_K
\end{align*}
\]  \hspace{1cm} (61)

where \( f_{Max} \) is half the bandwidth of the filter and \( N \) defines the width of the filter in the time domain. We define the filter in the frequency domain between 0 and 0.5, where 0.5 represents the Nyquist frequency. Thus, \( f_{Max} \) must belong to this interval, too.

The values \( f_{Max} \) and \( N \) must be chosen based on which frequencies we want to stop and how wide we want the transition band of the filter (Figure 14).

**Figure 14:** Fourier series for the low pass filter
3.5. Experimental results

We estimated R-wave velocities using all three methods for two groups of signals provided by a local company, NDT Engineering:

Tank: This group was recorded on the slab of a concrete tank and included 208 sets of 4 signals each. The whole group was contained in 4 binary files: Line1.bin, Line2.bin, Line3.bin and Line4.bin. The 3 last signals within every set were recorded at distances 0.1524, 0.4572 and 0.762 m. from sensor 1 (the sensor recording the first signal).

Bridge: This group was recorded on the concrete-deck of a bridge and included 988 sets of 4 signals each. The whole group was contained in 7 binary files: 4a.bin, 13.bin, 19.bin, 26a.bin, 34.bin, 36.bin and 39.bin. The signals were recorded at distances 0.061, 0.244, 0.610 and 0.701 m. from the impact location.

In both cases, the waves were induced by the impact of steel pellets on the concrete.

The results presented include the 208 sets from the tank data, and 143 sets from the bridge data (file 26a.bin). The quality of the signals from the bridge was poor compared to the signals from the tank. This poor quality translated in a large dispersion in the estimated velocity distribution.
3.5.1. R-wave velocity estimation based on times of arrival (TOA)

For every set of four signals we computed a linear regression, from which we estimated the corresponding R-wave velocity.

We computed the minimized sum of squared errors, eq. (43), and used its square root as a measure of the quality of the velocity estimate.

We set the parameters of the filter to:

<table>
<thead>
<tr>
<th>Group</th>
<th>Tank</th>
<th>Harrison</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\text{Max}}$</td>
<td>$15kHz \over 125kHz = 0.12$</td>
<td>$15kHz \over 125kHz = 0.12$</td>
</tr>
<tr>
<td>$N$</td>
<td>15</td>
<td>12</td>
</tr>
</tbody>
</table>

Those values seemed to give the best results. The sampling step for the signals was four microseconds, which gave a Nyquist frequency of 125 kHz.

The following figures summarize the results. They show both, the distribution of the velocity estimates and of the error for the Tank and the Bridge separately. We applied the method to both groups of signals with and without filtering, as it is shown.
Figure 16: Distributions of R-wave velocity estimates by TOA

Figure 17: Distributions of R-wave velocity estimates by TOA with filtering

Figure 18: Distributions of minimized sums of squared errors
Figure 19: Distributions of minimized sums of squared errors

Figure 20: Estimated velocities vs. quality measure of the estimates
3.5.2. R-wave velocity estimation based on times of delay (TDE)

We tested only the signals from the tank. For every set of four signals, we considered 6 pairs of signals, computed 6 times of delay, and estimated the corresponding velocities. We estimated the R-wave velocity for each set both, as the mean value of the 6 individual estimates and using linear regression. In the first case, we used the standard deviation of the individual estimates as a measure of the quality of the velocity estimates. For the second case, we used the square root of the sum of the squared errors. The parameters of the filter were as indicated previously.

In order to compare the error measure of the linear regression with the TOA results, we divided the error measurement by 1.5 (4/6).
Figure 21: Distributions of R-wave velocity estimates by TDE

Figure 22: Distributions of variances

Figure 23: TDE estimation with regression
3.5.3. *R-wave velocity estimation based on phase delays (PD)*

We tested only 30 signals from the tank. Because the enormous computational effort involved and the significant dispersion in the results, we deemed unnecessary to test more signals. As for the TDE technique, we used the standard deviation as a measure of the quality of the velocity estimates.

We did not filter the signals in this case. And, we did try time windowing before computing the DFT, but we did not observe clear improvement.

In the following Table, we compare the results with the corresponding ones using TDE.
When applied to the numerical simulation, the 3 methods yield excellent estimates, with errors of 0.1% for the TOA, 0.25 for the TDE and 0.05% for the PD. However, while the single TOA estimates adjust almost perfectly to a straight line, the TDE and PD methods show some dispersion in the single estimates that when averaged give the final velocity estimate. Small deviations in the maximum peak of the CC, due to the P and S-waves, may cause the dispersion in the TDE single estimates.

### Table 4: PD vs. TDE velocity estimates for first 30 shots in file Line1.bin

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### 3.6. Conclusions

When applied to the numerical simulation, the 3 methods yield excellent estimates, with errors of 0.1% for the TOA, 0.25 for the TDE and 0.05% for the PD. However, while the single TOA estimates adjust almost perfectly to a straight line, the TDE and PD methods show some dispersion in the single estimates that when averaged give the final velocity estimate. Small deviations in the maximum peak of the CC, due to the P and S-waves, may cause the dispersion in the TDE single estimates.
When applied to the signals of the Tank, TOA gives the least disperse distribution of velocities. PE shows more disperse results than TDE. This fact indicates less sensitivity to noise for the TOA method than for the other 2.

When using TDE, linear regression of the individual estimates gives a less disperse, thus better, distribution than simply averaging.

Filtering improves greatly the estimation process by TOA, while worsens the TDE estimates. The application of the low pass filter gives a less disperse distribution of velocities and decreases the error for the TOA estimates, namely the sum of the squared errors of the linear regression. The same filter leads to opposite results when the TDE method is used.

The TOA method requires the least computational effort, and this effort is mostly due to the filtering process. The TDE method is considerably more expensive. The PD method demands excessive computational time due to the phase unwrapping.

A significant smaller computational cost and less dispersion in the estimates encourage the use of the TOA method over the other two. Due to its computational cost, we do not recommend the use of the PD method.

The quality of the signals from the Bridge was very poor, which makes hard to draw any conclusion from their results.
Chapter 4. Analytical solution for the Half-space

Two types of waves cause the displacements of any point of the surface of a plate due to a normal impact load: direct and reflected waves. The direct waves are those which travel directly from the impact location to the observation point, while the reflected waves reach that point after one or more reflections at the boundaries of the plate.

In this chapter, we present the analytical solution for the half space problem, which depends only on the elastic properties of the material and the distance to the source. This solution provides us with a reference model of the direct waves that we can use to interpret measured signals in a plate.

This is of course an imperfect model, because in a plate the direct waves are mixed with the reflected ones. There are, however, two facts that allow us to use such a model:

- For points close to the impact, the path for the direct ways is much shorter than the one for the reflected waves. Hence, the direct waves will arrive well in advance of the first reflected waves.

- The energy content of the R-wave, which is a direct wave, is very high if compared to the other waves. Therefore, the early part of the signal rises clearly above the noise and the other waves in the signal.

We will use these facts to identify the contributions of the direct R-wave to the signals, which will help to estimate the parameters of the model, namely the elastic properties of the plate.

4.1. Normal step load on a half space

Given the impulse response function of a linear system, its convolution with any transient load yields the response of the system to that load. Thus, if we consider a load applied at the origin on the surface of a half space \( f(r,t) = f(t) \cdot \delta(r) \), the response for any surface point at a distance \( r \) from the origin is given by
\[ u(r,t) = \int_{-\infty}^{t_o} f(t) \cdot h(r,t_0 - t) \cdot dt = \int_{-\infty}^{t_o} f(t_0 - t) \cdot h(r,t) \cdot dt = f * h \] (62)

where \( h(r,t) \) is the response at the observation point due to a unit impulse applied at the origin. If instead of the impulse response, we know the unit step response function \( g(r,t) \), we can substitute \( h(r,t) = \frac{\partial g(r,t)}{\partial t} \) into eq. (62). Moreover, since there is no load applied for negative times, there is no response at negative times (the system is causal). We have then

\[ u(r,t) = \int_{0}^{t_o} f(t_0 - t) \cdot \frac{\partial g(r,t)}{\partial t} \cdot dt \] (63)

Substituting

\[ f(t_0 - t) \cdot \frac{\partial g(r,t)}{\partial t} = \frac{\partial (f(t_0 - t) \cdot g(r,t))}{\partial t} - \frac{\partial f(t_0 - t)}{\partial t} \cdot g(r,t), \] (64)

into eq. (63), we obtain

\[ u(r,t) = \int_{0}^{t_o} \frac{\partial (f(0,t_0 - t) \cdot g(r,t))}{\partial t} \cdot dt - \int_{0}^{t_o} \frac{\partial f(0,t_0 - t)}{\partial t} \cdot g(r,t) \cdot dt \] (65)

\[ = [f(0,t_0 - t) \cdot g(r,t)]_{0}^{t_o} - \int_{0}^{t_o} \frac{\partial f(0,t_0 - t)}{\partial t} \cdot g(r,t) \cdot dt \]

Because at time \( t = 0 \) both the force and the step response are zero, we have finally

\[ u(r,t) = -\int_{0}^{t_o} \frac{\partial f(t_0 - t)}{\partial t} \cdot g(r,t) \cdot dt \] (66)

Pekeris [11] studied the case of a normal step load applied at a point on the surface of the half space. Assuming a Poisson ratio of 0.25, which is a reasonable value for rock, he found analytical expressions for the radial and vertical displacements on the surface of the half space. In this work, we are only interested in the vertical displacements. For a point distant \( r \) from the origin, the vertical displacement varies with time according to the following expressions
where \( F \) is the magnitude of the step load, \( \mu \) is the shear modulus of the material and
\[ \gamma = \frac{\sqrt{3} + \sqrt{3}}{2} \] is a dimensionless variable, related to time and distance by the expression \( \tau = \frac{c_s \cdot t}{r} \). The constant \( c_s \) is the shear wave velocity.

The various intervals are associated with the arrival of the P-wave, S-wave and R-wave respectively. At the arrival of the R-wave (\( \tau = \gamma \)) there is a singularity, and the solution at this point shows a discontinuity.

Pekeris' solution provides us one half of eq. (66). Then, we must define a reasonable time history for the load, and use it to compute the response of the plate.

### 4.2. Time history of the load

We can idealize the impact of a spherical pellet onto a plate as a particular case in the elastic collision of two spheres, by making the radius of one of the spheres equal to infinity. The elastic collision of two spheres is known as the Hertz problem and it predicts a time history for the contact force that can be approximated as
\[
f(t) = \begin{cases} 
F \cdot \sin^2 \left( \pi \cdot \frac{t}{t_c} \right), & 0 \leq t \leq t_c \\
0, & \text{anywhere else}
\end{cases}
\]  
(68)

where \( t_c \) is the duration of the load or contact time, and \( F \) is the maximum amplitude.

Unfortunately, if we substitute this load and Pekeris’ solution into eq. (66), the resulting expression cannot be integrated analytically. While, we can always evaluate the convolution numerically, it is easier and faster to use a polynomial approximation to the sine pulse. To this effect, we substitute the half sine squared with the following 4th order polynomial, which closely approximates the half sine squared (Figure 24).

\[
f(t) = P_4(t) = F \cdot \frac{64}{t_c^4} \left( 0.25 \cdot t^4 - \left( \frac{t_c}{2} \right)^2 \cdot t^3 + \left( \frac{t_c}{2} \right)^4 \cdot t^2 \right)
\]  
(69)

\[\text{Half sine squared vs. Polynomial load}\]

\[\text{Half sine squared vs. Polynomial load}\]

\[\text{Figure 24: Polynomial approximation of a half sine squared load}\]

4.3. The direct waves analytical model: Solution of the convolution integral

Having defined both the load and the step response expressions, we have all we need to develop the analytical model for the vertical displacements on the surface of a half space caused by the impact of a spherical pellet. We need to substitute these expressions into eq. (66) and calculate the resulting integral.
From our polynomial approximation of the load, we have

\[ f_r(t_o - \tau) = \frac{\partial f(t_o - t)}{\partial t} = \left( \frac{64 \cdot F}{t_c \cdot \tau_c^3} \right) \left( F_3 \cdot \tau^3 + F_2 \cdot \tau^2 + F_1 \cdot \tau + F_0 \right) \]  

(70)

in which \( \tau \) is defined as for eq. (67) and

\[
\begin{align*}
F_0 &= -\tau_o^3 + 1.5 \cdot \tau_c \cdot \tau_o^2 - 0.5 \cdot \tau_c^2 \cdot \tau_o \\
F_1 &= 3 \cdot \tau_o^2 - 3 \cdot \tau_c \cdot \tau_o + 0.5 \cdot \tau_c^2 \\
F_2 &= 1.5 \cdot \tau_c - 3 \cdot \tau_o \\
F_3 &= 1
\end{align*}
\]  

(71)

Pekeris' solution for an observation point at a distance \( r \) from the source can be expressed in the general form

\[ g_i(\tau) = g(r, t) = A + \sum_i \frac{B_i}{\sqrt{\tau^2 - C_i}} \]  

(72)

Where \( A, B_i \) and \( C_i \) are independent of \( \tau \).

Introducing eq. (70) and (72) and \( dt = \frac{r}{c_s} \cdot d\tau \) into eq. (66), we obtain

\[
\begin{align*}
u_r(\tau) &= u(r, t) \\
&= -\int_0^{t_o} \frac{\partial f(t_o - t)}{\partial t} \cdot g(r, t) \cdot dt \\
&= -\int_0^{t_o} f_r(\tau_o - \tau) \cdot g_1(\tau) \cdot \frac{r}{c_s} \cdot d\tau \\
&= -\frac{r}{c_s} \sum_i \int_0^{t_o} \frac{P_i(\tau)}{\sqrt{\tau^2 - C_i}} \cdot d\tau
\end{align*}
\]  

(73)

in which \( P_i(\tau) \) are polynomials of various orders. The solution to integrals of this type can be obtained in mathematical reference books.
The evaluation of these integrals is laborious and tedious, and special attention must be paid at the transitions of the intervals in Pekeris' equations. Hence, we will skip the details and present the final results only. These are

\[
u(r,t) = \begin{cases} 
U_0(\tau) - U_0(m), & \tau < 1 \\
U_1(\tau) - U_1(m), & 1 \leq \tau < \gamma \land m > 1 \\
U_1(\tau) \cdot U_1(1) + U_0(1) - U_0(m), & 1 \leq \tau < \gamma \land m \leq 1 \\
U_2(\tau) - U_2(m), & \gamma \leq \tau < \gamma + 1 \land m > \gamma \\
U_2(\tau) \cdot U_2(1) + U_1(1) \cdot U_1(m), & \gamma \leq \tau < \gamma + 1 \land 1 < m \leq \gamma \\
U_2(\tau) - U_2(1) \cdot U_1(1) + U_0(1) \cdot U_0(m), & \gamma \leq \tau < \gamma + 1 \land m \leq 1
\end{cases}
\]

where

\[
m = \min\left(\frac{1}{\sqrt{3}}, \tau - \tau_c\right)
\]

\[
U_0(\tau) = G_0 \cdot \left\{6 \cdot R_0(\tau) + \sum_{i=1}^{3} R_i(\tau)\right\}
\]

\[
U_1(\tau) = 2 \cdot G_0 \cdot \left\{6 \cdot R_0(\tau) + R_3(\tau)\right\}
\]

\[
U_2(\tau) = 12 \cdot G_0 \cdot R_0(\tau)
\]

\[
R_0(\tau) = 0.25 \cdot F_3 \cdot \tau^4 + \frac{1}{3} \cdot F_2 \cdot \tau^3 + 0.5 \cdot F_1 \cdot \tau^2 + F_0 \cdot \tau
\]

\[
R_1(\tau) = -G_1 \cdot \left\{F_3 \cdot \frac{\sqrt{(\tau^2 - G_2)^3}}{3} + G_2 \cdot \sqrt{\tau^2 - G_2} + F_2 \cdot \left(\frac{\tau}{2} \cdot \sqrt{\tau^2 - G_2}\right) + \frac{G_2}{2} \cdot \ln\left(\tau + \sqrt{\tau^2 - G_2}\right) + F_1 \cdot \sqrt{\tau^2 - G_2} + F_0 \cdot \ln\left(\tau + \sqrt{\tau^2 - G_2}\right)\right\}
\]

\[
R_2(\tau) = -G_3 \cdot \left\{F_3 \cdot \frac{(G_4 - \tau^2)^3}{3} - G_4 \cdot \sqrt{G_4 - \tau^2} + F_2 \cdot \left(\frac{G_4}{2} \cdot \arcsin\left(\frac{\tau}{\sqrt{G_4}}\right)\right) - \frac{\tau}{2} \cdot \sqrt{G_4 - \tau^2} - F_1 \cdot \sqrt{G_4 - \tau^2} + F_0 \cdot \arcsin\left(\frac{\tau}{\sqrt{G_4}}\right)\right\}
\]
\[ R_3(\tau) = -G_5 \cdot \left\{ F_1 \cdot \left( \sqrt{\frac{\tau^2 - G_6}{3}} \right) + G_6 \cdot \sqrt{\tau^2 - G_6} \right\} + F_2 \cdot \left( \frac{\tau}{2} \cdot \sqrt{\tau^2 - G_6} \right) + \frac{G_6}{2} \cdot \ln(\tau + \sqrt{\tau^2 - G_6}) + F_1 \cdot \sqrt{\tau^2 - G_6} + F_0 \cdot \ln(\tau + \sqrt{\tau^2 - G_6}) \right\} \] 

\[ G_0 = \frac{1}{32 \cdot \mu \cdot \pi \cdot r} \cdot \left( \frac{64 \cdot F}{\tau_c^4} \right) \]

\[ G_1 = \sqrt{3} \]

\[ G_2 = 0.25 \]

\[ G_3 = \sqrt{3} \cdot \sqrt{3} + 5 \]

\[ G_4 = \frac{3 + \sqrt{3}}{4} \]

\[ G_5 = -\sqrt{3} \cdot \sqrt{3} - 5 \]

\[ G_6 = \frac{3 - \sqrt{3}}{4} \]

\[ F_0, \ F_1, \ F_2 \ and \ F_3 \ are \ defined \ in \ eq. \ (71); \ \gamma \ and \ \tau \ are \ defined \ in \ eq. \ (67) \ and \ \mu \ is \ the \ shear \ modulus \ of \ the \ material. \]

4.4. Validation of the analytical model

The last step is to test the suitability of our model. We wish to answer two questions: How good is the approximation of a half sine squared load with a polynomial, and how well does the solution for the half space fit the solution of the plate.

With these goals in mind, we first consider a half space with a Poisson ratio of 0.25, a S-wave velocity of 2000 m/s. and a density of 2400 Kg/m³ and a half sine squared load with duration 20 \( \mu s \). We then compute the vertical displacements 0.244-m. away from the impact by numerical discrete convolution, using a time step of 0.1 \( \mu s \).
**Figure 25:** Pekeris solution and differentiate of load (see eq. (66))

**Figure 26:** Comparison of half sin squared load vs. mathematical model (polynomial load)

The comparison of the results with the prediction of our model shows that the polynomial approximation yields excellent results.

Secondly, we compare our model with accurate numerical solutions for the plate. We consider a 24-cm-thick plate with the same properties as the half space above and compute the vertical displacements at distances 0.061, 0.244, 0.610 and 0.701 m. away from the impact using the thin layer method and a half sine squared load with duration 20 \( \mu s \). We use the same time step as above of 0.1 \( \mu s \).
For the two closest points to the impact, there are small and negligible differences due to numerical errors and different load shapes. For the furthest two points, reflected waves arrive before the direct R-wave, which invalidate the direct use of the model. Due to the similarities in the R-wave portion of the signals, the model may still be helpful at these locations.

In conclusion, for points close to the impact location, our mathematical model predicts the vertical displacements accurately. As to how close is close enough will depend on the thickness of the plate, which can be assessed easily by looking at the time of flight of the waves in the plate.

**Figure 27:** Displacements in a plate vs. displacement in a half space

**Figure 28:** Displacements in a plate vs. displacement in a half space
Chapter 5. Separation of direct and reflected waves

In previous chapters, we commented on the different waves propagating in an elastic plate and stressed the difference between direct and reflected waves. We also elaborated on how we can estimate the contribution of the direct waves by simply looking at the half space problem.

Our next objective is to separate the direct waves from the reflected waves. This separation will serve two purposes:

- To ease the analysis of the reflections in the plate, which are the base of the IE method used to measure the thickness of the plate and locate delaminations. Because the amplitude of the direct R-wave is considerably larger than that of the other waves (direct and reflected), its presence masks the reflected waves. Thus, the removal of the R-wave is an important step in the analysis of the reflections in the plate.

- To help in the estimation of the elastic properties of the plate. The shape of the direct waves depends on the elastic properties of the plate, and its analysis may give us clues about them.

In this chapter, we propose a method to remove the R-wave and demonstrate how the determination of resonant frequencies of the plate can thereby be sharpened. To achieve this, we fit the mathematical model we developed previously to the empirical signals, using a non-linear least-squares curve-fitting algorithm. This allows us to estimate the parameters that control the direct waves in the plate. From here, we subtract the direct waves from the empirical signals.

5.1. Removing the R-wave

The application of a transient stress pulse on a plate generates P-waves and S-waves that reverberate between the top and bottom surface. For points close to the impact, the vertical displacement produced by the reflections is dominated primarily by the P-waves, and the travel distance for these waves is nearly twice the thickness.
For points close to the impact, this resonant condition causes amplitude peaks in the Fourier transform that correspond to multiple reflections of the P-wave. The frequency values at which these peaks appear correspond to the time it takes the P-wave to travel once, twice or more times across the thickness of the plate. Then, if we know the P-wave velocity, it is possible to measure the thickness of the plate. If there is a delamination, the first peak corresponds to the depth of this delamination.

In addition to the P-waves and S-waves we have the R-wave, which also shows on the Fourier transform. Because of the large energy content of the R-wave, it appears on top of the reflected waves in the frequency domain and hampers the reading of the resonant peaks.

Figure 29 shows vertical displacements computed at 61 mm. from the impact point in a plate 24 cm. thick. It shows how the peaks show clearly and are easier to read when the R-wave is removed.

![Vertical displacements and Fourier transform](image)

**Figure 29:** Main resonance peaks with and without the R-wave

### 5.2. Removal of the R-wave by curve fitting

The objective of fitting a curve to a set of recorded data is to find the particular function, out of a family of functions, that best fit the data, according to some specified criterion.

The empirical data depend on one or more independent variables and some parameters (elastic constants, thickness and others). Thus, the family of functions selected will be
dependent in the same independent variables and parameters. We must find a set of values for these parameters so as to best fit the data. Among the families of functions that may be considered in the approximations are polynomial, exponential and trigonometric functions.

Also, to fit these functions to the data, we must define the error criterion.

5.2.1. Collection of data

To remove the R-wave, we must focus attention on that part of the signal where we expect it. The direct P-wave is negligible and the S-wave overlaps with the R-wave for most of its length. For fitting purposes, we neglect that part of the signal that follows the R-wave. Furthermore, we only consider signals recorded close to the impact, because in the signals recorded far from the impact, the reflected waves arrive together with the R-wave and distort it, an effect which would prevent a good fitting.

The displacements for a point on the surface of the plate vary with time and with the distance to the impact. However, we already know for each signal the distance to the impact location; so, the only independent variable will be the time. Unfortunately, we do not know the exact time of impact, so the time origin is unknown.

To overcome this problem, we introduce a reference time \( t_o \) as a parameter into the fitting problem. Because, we know the exact time interval between measurements, which is the sampling step, we can express the time for measurement \( i \) as \( t_i = t_o + i \cdot \Delta t \), where \( \Delta t \) is the sampling step. In our curve fitting problem, \( t_o \) is one of the parameters that we must estimate; then, we can express the instant in time of every sample with respect to this reference time.

5.2.2. Family of fitting functions

In the previous chapter, we showed that the half space solution for vertical displacements on the free boundary models exactly the direct waves in a plate for points close to the impact location. Therefore, we select the mathematical model we developed previously as our fitting function.
To account for calibration errors in the instrument, we further introduce a constant term \( u_o \) to the mathematical model. Thus, we have a family of functions of the form

\[
\hat{u}(t, x) = u_o + f(t, t_o, t_c, F, c_s, r)
\]

where time \( t \) is the only independent variable, \( r \) is the distance to the impact location and \( x = [u_o, t_o, t_c, F, c_s] \) are the parameters we want to estimate. We assume we know the exact distance \( r \). The parameter \( t_o \) is the reference time defined above, \( t_c \) is the contact time, \( F \) is the maximum amplitude of the impact force and \( c_s \) is the S-wave velocity, which is a function of the elastic properties of the plate. We also estimate the constant \( u_o \).

Because the dependence of the displacements on the parameters is non-linear, we have a non-linear curve fitting.

**5.2.3. Error criterion**

We select as error criterion the Least Squares (LS) method: “Find the values of the constants (parameters) in the chosen equation that minimize the sum of the squared deviations of the observed values from those predicted by the equation” [13]. We have thus a non-linear LS fitting.

On the one hand, we have an experimental discrete time signal \( \mathbf{y}^T = [y_1, y_2, \ldots, y_n] \), where \( y_i \) is the displacement for time \( t_i = t_o + i \cdot \Delta t \). On the other hand, for a given set of parameters \( \mathbf{x} \), we can estimate the displacements using eq. (82), \( \hat{\mathbf{u}}^T = [\hat{u}(t_1, \mathbf{x}), \hat{u}(t_2, \mathbf{x}), \ldots, \hat{u}(t_n, \mathbf{x})] \). If we define the error as the deviation of the observed value from the predicted one, \( \varepsilon_i = y_i - \hat{u}(t_i, \mathbf{x}) \), we can express the error criterion as

\[
\min \bigg\{ Q(\mathbf{x}) = \sum_i (y_i - \hat{u}(t_i, \mathbf{x}))^2 \bigg\}
\]

We can write a more general expression of this same criterion, in matrix form, as


\[
\min \left\{ Q(x) = \varepsilon^T \cdot P \cdot \varepsilon \right\} \tag{84}
\]

where \( P \) is a real positive definite symmetric matrix. This expression allows us to apply different weight constants to the errors. If we know that some measurements are more reliable than others, we can weigh them accordingly. If we give the same weight to all the deviations, \( P \) becomes the identity matrix and we recover eq. (83). Another common procedure is to weigh the measurements according to their standard deviations.

\[
\min \left\{ Q(x) = \sum_i \left( \frac{y_i - \hat{y}(t_i, x)}{\sigma_i} \right)^2 \right\} \tag{85}
\]

For the time being, we consider the identity matrix \( I \) for \( P \) and assume the standard deviations for all the measurements to equal 1. This returns us to eq. (83).

### 5.3. Assumptions of the Least Square method

Before proceeding into the solution of non-linear LS problems, we should comment briefly on some of the assumptions of the LS method. It can be proved that if these conditions are satisfied, the LS method provides the maximum likelihood set of estimated parameters. If some of these conditions are not complied with, the results are not as robust, although they may still be useful.

These assumptions are

- The chosen mathematical model is the “correct” one. If it is not, the estimates of the parameters will be biased; that is, the estimates will not average out in the long run to the true value of the parameters. We show above that for points close to the impact this assumption is valid.

- The data are typical. We can not use the same problem formulation if, for instance, we have an asphalt layer. This would be a different situation, and the model may need to be redefined or at least revised. The same can be said if the surface of the concrete is poor or full of cracks.
• The errors in the measurements, vertical displacements in our case, are statistically uncorrelated. Each measurement is made up of a true value, \( \eta \), and a random error \( e \). If we consider the error in any two samples, the expectation of their product must be zero, \( E(e_i, e_j) = 0 \).

• The independent variable is known without error. This point is discussed above.

• The measurement errors follow a normal distribution with zero mean.

The last assumption is difficult to check. In general the error in the measurement will have two principal components. One is the error of the instrument when measuring. Very often in engineering practice, processes do approach a normal distribution, and it will be reasonable to assume that the error is normally distributed. The second component is the environmental noise, which rarely will approximate a normal process. The constant parameter \( u_o \) will include some of the low frequency noise. Regarding the medium and high frequency noise, we will assume it normally distributed for simplicity.

5.4. Iterative schemes to solve non-linear Least Square problems

Given a time signal \( \{y_1, y_2, \ldots, y_n\} \), we want to find the set of parameters \( x^T = [t_o, t_c, F, c_s] \) that minimizes the sum of the squared errors

\[
\min\left\{ Q(x) = \sum_i (y_i - \hat{u}(t_i, x))^2 \right\}
\]

The necessary condition for an extremum of the function \( Q(x) \) for a set \( x^* \) is the vanishing of the gradient.
To check if this extremum is actually a minimum, and not a maximum or a saddle point, we need to compute the Hessian matrix.

Except for very simple cases, eq. (87) can not be solved analytically. Instead, various iterative numerical methods already exist for the solution of this type of problem. Starting with an initial guess of the parameters, these iterative method progress towards the sought minimum by modifying the value of the parameters, according to certain criteria. These modifications must be such that the value of the function they want minimize must decrease and this is known as the stability condition. Different criteria for modifying the parameters lead to the different methods.

These methods will eventually converge to a local minimum and if the function \( Q(x) \) is not too complex and the initial guess in the parameters is reasonable, this minimum will likely be the global minimum as well. Unfortunately, it is usually hard, if not impossible, to confirm that we have attained the global minimum.

We may consider two main groups of methods: Universal minimizing methods and special minimizing methods [14].

5.4.1. Universal minimizing methods

There are many universal minimizing methods, which are based in various criteria. Of those we focus on the so-called gradient methods. For these methods, besides the function value, we need to know the gradient that may be calculated as

\[
\nabla Q(x) = -2 \cdot D^T \cdot P \cdot \epsilon 
\]

(88)
where $D$ is

\[
D = \begin{pmatrix}
\frac{\partial \hat{u}(t_1)}{\partial t_O} & \frac{\partial \hat{u}(t_1)}{\partial t_C} & \frac{\partial \hat{u}(t_1)}{\partial F} & \frac{\partial \hat{u}(t_1)}{\partial c_S} \\
\frac{\partial \hat{u}(t_2)}{\partial t_O} & \frac{\partial \hat{u}(t_2)}{\partial t_C} & \frac{\partial \hat{u}(t_2)}{\partial F} & \frac{\partial \hat{u}(t_2)}{\partial c_S} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \hat{u}(t_n)}{\partial t_O} & \frac{\partial \hat{u}(t_n)}{\partial t_C} & \frac{\partial \hat{u}(t_n)}{\partial F} & \frac{\partial \hat{u}(t_n)}{\partial c_S}
\end{pmatrix},
\]  

and $n$ is the number of data points to fit.

In some cases, analytical expressions for the gradient are accessible, in other cases, the gradient can be calculated only numerically.

The oldest gradient method is the method of “Steepest descent”. It is based in the fact that the negative gradient shows in the direction of the steepest descent of the function

\[
r = -\nabla Q(x)
\]  

(90)

So, if we make an infinitesimal step in the direction $r$, the value of the function $Q(x)$ decreases.

Because the gradient varies from point to point, this is only true for points very close to the point where we evaluate the gradient. If instead, we consider a finite step, the function may not decrease.

The parameters for iteration $k+1$ are calculated from the ones in iteration $k$, as

\[
\bar{x}^{k+1} = \bar{x}^k + \alpha \cdot r^k
\]

where $\alpha$, which defines the step size, is called the step-length factor.

The “Optimal gradient” method selects the step-length factor by minimizing the function $Q(x)$ in the direction of $r^k$. This way, in each step we have a one dimensional minimization problem.

Other methods modify the resizing direction, such that
If $R^K$ is a real positive definite matrix, for infinitesimal steps the function $Q(x)$ decreases.

The advantage of the gradient methods is that they always converge. The great disadvantage is that this convergence is often very slow.

5.4.2. Special minimizing methods

There is a group of special methods developed for the solution of non-linear LS problems that are based on approximations of the fitting function. The function, $\hat{u}(x)^T = \{\hat{u}(t_1, x), \hat{u}(t_2, x), \ldots \hat{u}(t_n, x)\}$ in our case, is expanded in a Taylor series and truncated after the linear term. Given a set of parameters $x^K$, the value of the function is approximated as

$$\hat{u}(x) = \hat{u}(x^K) + D^K \cdot (x - x^K)$$  \hspace{1cm} (92)$$

Imposing the condition of extremum and using the Taylor approximation, we have

$$\nabla Q(x) = 0$$
$$\nabla \left( (y - \hat{u}(x))^T \cdot P \cdot (y - \hat{u}(x)) \right) = 0$$ \hspace{1cm} (93)$$
$$-2 \cdot (D^K)^T \cdot P \cdot (\epsilon^K - D^K \cdot r^K) = 0$$

where $\epsilon^K = y - \hat{u}(x^K)$ and $r^K = x - x^K$.

The next step in the iteration is

$$x^{K+1} = x^K + \left[ (D^K)^T \cdot P \cdot D^K \right]^{-1} \cdot (D^K)^T \cdot P \cdot \epsilon^K$$ \hspace{1cm} (94)$$

The method given by this iteration scheme is known as the Gauss-Newton method.
The disadvantage of this simple scheme is that the stability condition is not guarantied. The Taylor approximation is only valid in the proximity of $x^K$ and the increments in the parameters may be such that $Q(x^{K+1}) > Q(x^K)$.

Substituting eq. (88) and $R^K = \frac{1}{2} \left[ (D^K)^T \cdot P \cdot D^K \right]^{-1}$ into eq. (94), we have

$$x^{K+1} = x^K - R^K \cdot \nabla Q(x^K)$$  \hspace{1cm} (95)

Eq. (95) shows that the Gauss-Newton method may be considered as an universal gradient method. It also shows that when the result of $(D^K)^T \cdot P \cdot D^K$, which is a first order approximation to the Hessian, is a singular matrix or close to it, we can expect an unstable behavior of the iteration. Unfortunately, this situation occurs rather frequently.

5.4.3. Levenberg-Marquardt method

Marquardt proposed a way to combine the steepest descent method and the Gauss-Newton method, based on an earlier suggestion by Levenberg.

In the steepest descent method, we calculate the increments in the parameters for iteration $k$ as

$$\delta x^K = x^{K+1} - x^K = -\alpha \cdot \nabla Q(x^K)$$ \hspace{1cm} (96)

In the Gauss-Newton method

$$\delta x^K = x^{K+1} - x^K = -\frac{1}{2} \left[ (D^K)^T \cdot P \cdot D^K \right]^{-1} \cdot \nabla Q(x^K)$$ \hspace{1cm} (97)

Marquardt proposed

$$\delta x^K = x^{K+1} - x^K = -\frac{1}{2} \left[ \lambda^K \cdot I + (D^K)^T \cdot P \cdot D^K \right]^{-1} \cdot \nabla Q(x^K)$$ \hspace{1cm} (98)
where \( I \) is the identity matrix and \( \lambda^K \) is a positive real number. The superindex \( K \) simply suggests that this value may change from step to step.

The addition of the diagonal matrix \([\lambda^K \cdot I]\) guarantees that the matrix 
\[
\left[ \lambda^K \cdot I + (D^K)^T \cdot P \cdot D^K \right]
\] is invertible. Furthermore, by the choice of a sufficiently large \( \lambda^K \), the stability condition is satisfied.

We can see that for large \( \lambda^K \) eq. (98) approaches the steepest descent method with a small step size; so, stability is guaranteed.

\[
\frac{1}{2} \left[ \lambda^K \cdot I + (D^K)^T \cdot P \cdot D^K \right]^{-1} \cdot \nabla Q(x^K) = -\frac{1}{2 \cdot \lambda^K} \cdot \nabla Q(x^K)
\] (99)

while, for small values of \( \lambda^K \) we readily see that eq. (98) approaches the Gauss-Newton method.

In each step of the iteration, we must check the stability condition. If it is not satisfied (the error increases), we increase the value of \( \lambda^K \) and recalculate the increments in the parameters. The disadvantage of this method is that sometimes we have to repeat the same step several times. Still, this method is significantly faster than the gradient methods.

5.5. Selected iterative scheme

For the solution of our particular problem, we select Marquart’s method. Figure 30 shows the steps of the iteration process. In the formulas of this chart, we have already substituted the identity matrix \( I \) for the weighting matrix \( P \).

\[
\left[ \lambda^K \cdot I + (D^K)^T \cdot D^K \right] \cdot \delta x^K = -(D^K)^T \cdot g^K
\] (100)
To calculate the increments in the parameters $\delta x^K$, we do not invert any matrix; instead we use Gauss decomposition to solve the system.

The resizing of the factor $\lambda^K$ is somehow arbitrary. We choose 10, but any other value can be used.

5.6. Initial values and convergence criterion

Once we have decided how to move from a set of parameters to a better one, we need to define how to choose the initial values for the parameters at the start of the iteration and also the convergence criterion to stop the iteration.

Figure 30: Iteration flowchart for Marquart’s method
5.6.1. Initial values

Depending on the function to minimize, the proper selection of the initial point can be a critical step. For smooth and very well behaved functions, we may reach the minimum even if we start the iteration far from it. Unfortunately, most of the time, that is not the case. A function may have various minimums, and if the starting point is ill selected, the iteration will probably converge to a different minimum (local) than the one we look for (global minimum).

For cases where there are 1 or 2 parameters to estimate, a plot of the corresponding curve or surface for one or two cases will give us an idea of the complexity of the function. With more parameters, however, is impossible to represent graphically the corresponding surface in the higher dimensional space. However, by fixing some of the parameters and plotting how the function changes with only 2 of them (Figure 31), we can get an idea of the function.

![Error function](image)

**Figure 31:** Error as a function of only two parameters

We select the initial values as follows:

- We estimate the S-wave velocity $c_s$ by TOA, as we explain above.
- We estimate the contact time $t_c$ as the duration of the R-wave. Because of the singularity for the step response at the arrival of the R-wave, the duration of the R-wave equals the contact time. Based on the shape of the analytical direct R-wave, we
identify the R-wave within the experimental signals and measured its approximate duration.

- We estimate the reference time as $t_0 = r/c_R$, where the distance $r$ is known and the R-wave velocity $c_R$ is approximated as a 90% of the S-wave velocity.
- To estimate $F$, we compute the signal with $F=1$ and look for the maximum amplitude. We then look for the maximum amplitude of the empirical signal and use the ratio of these maximum amplitudes as a first estimate of $F$.

**Figure 32: R-wave identification**

We use $u_0 = 0$ as a first estimate.

The estimations of $c_s$ and $t_c$ are critical, as they are the base to estimate $t_0$ and $F$. If these two values estimates are sound, it is very likely that we will converge to the correct minimum. Otherwise, we may converge to another minimum or even not converge at all.

**5.6.2. Convergence criterion**

We follow here the recommendations of W. H. Press et Al. [15]. They suggest stopping the iteration on the first or second occasion that the sum of the squared errors decreases by a negligible amount, either in absolute or relative terms. We chose our threshold value as a relative decrease of $5 \cdot 10^{-4}$.
5.7. Experimental results

We apply this curve fitting technique to the 208 sets of signals from the group Tank. Of the four signals contained in each set, we only consider the first two, which were measured approximately at 40 and 155 mm. away from the impact source respectively. The last two signals were too far to the impact.

5.7.1. Estimated parameters and improved frequency spectrum

During the recording of this signals, the pellets were shot by hand; so, the distance to the sensors was not always the same, but similar. To account for this, we include the distance $r$ as a new parameter to estimate, with initial guess of 40 mm. and 155 mm. respectively.

In the figures that follow, we present the estimated parameters. We also include, for the first 6 signals, the best fit, and the frequency spectra of the original signals and of the signals without the direct wave.
**Figure 33**: Adjustment parameters for sensor 1

**Figure 34**: Distance to source and force parameter for sensor 1

**Figure 35**: Velocities and contact time for sensor 1
Figure 36: Adjustment parameters for sensor 2

Figure 37: Distance to source and force parameter for sensor 2

Figure 38: Velocities and contact time for sensor 2
Figure 39: Best fit and frequency spectrum for shot 1 and sensor 1

Figure 40: Best fit and frequency spectrum for shot 2 and sensor 1

Figure 41: Best fit and frequency spectrum for shot 3 and sensor 1
Figure 42: Best fit and frequency spectrum for shot 4 and sensor 1

Figure 43: Best fit and frequency spectrum for shot 5 and sensor 1

Figure 44: Best fit and frequency spectrum for shot 6 and sensor 1
Figure 45: Best fit and frequency spectrum for shot 1 and sensor 2

Figure 46: Best fit and frequency spectrum for shot 2 and sensor 2

Figure 47: Best fit and frequency spectrum for shot 3 and sensor 2
Figure 48: Best fit and frequency spectrum for shot 4 and sensor 2

Figure 49: Best fit and frequency spectrum for shot 5 and sensor 2

Figure 50: Best fit and frequency spectrum for shot 6 and sensor 2
5.8. Conclusions

To actually evaluate the quality of the estimates that result from the curve fitting, we need accurate information about the tested slabs, the measuring instruments and other external conditions that may affect the recording of the signals; unfortunately, our information about these issues is scarce. However, we may still comment on the results.

The mathematical model fits well the R-wave; and, only in the initial part of it there seems to be minor lack of fitting.

The removal of the direct wave sharpens significantly the peaks in the frequency spectrum, as it is expected. This effect is larger for sensor 1 than for sensor 2. The spectra also show the predominance of the peak at 40 kHz, which corresponds to the resonance of the instruments.

The resonance of the instrument introduces displacements of the same order of magnitude of the direct waves, as can be seen from the ringing of the signals (Appendix). This effect limits the analysis of the other peaks, as well as the effectiveness of removing the direct waves.

The distributions of velocity estimates match those obtained previously on this study by other methods, namely TOA and TDE. However, the velocities estimated from sensor 1 are always smaller than those from sensor 2.

The distribution of contact times is similar for both sensors.

Because the pellets are shoot by hand, we know neither the exact distance from impact to sensors nor the distance from gun to slab surface. Thus, it is hard to comment on the other estimated parameters.

Despite all, the results seem promising and encourage further testing under controlled conditions.
Chapter 6. Appendix

Here, we include details of the estimation of wave velocities for some signals of Tank.

6.1. Estimation of wave velocity by TOA
6.2. Estimation of wave velocity by TDE
References


