Motion Based Design: Solution Algorithms to the Inverse Problem with Applications to Seismic Design

by

Carlos Mario Gallegos

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Abstract

A general solution algorithm to the inverse equilibrium problem is derived. The algorithm finds stiffness coefficients for a given loading and a given set of displacment constraints. Optimization techniques are employed for the statically indeterminate case and for the case of partial displacement specification. The applicability of the method to seismic design of buildings is demonstrated. An inverse eigen-problem is solved for a stiffness distribution that produces a desired fundamental mode shape. The stiffness is then scaled based on the magnitude of the excitation with the use of response spectra. Emphasis is placed on stiffness proportional damping, which can be generated from the physical placement of dampers.

Thesis Supervisor: Jerome J. Connor
Title: Professor of Civil and Environmental Engineering

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I dedicate this thesis to my grandmothers, Corina V. Duran and Lucia Lopez. You provided me with a stable cultural environment and a unique perspective on life. I have always drawn and will continue to draw upon your wise words of wisdom in times of need.

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Chapter 1

Introduction

Historically, the driving constraint on structural design has been strength. This is perhaps the most crucial constraint on the design, for if a structure fails, the consequences can be economically devastating and life threatening. The design methodology for structures has thus evolved into strength based design. Serviceability constraints are then checked after the design is near completion.

In some cases, however, serviceability constraints drive the design process. Buildings that satisfy strength constraints may cause human discomfort and equipment malfunction. In the case of seismic excitation, limits on the amount of deformation are necessary to prevent inelastic yielding. The use of sophisticated equipment requires limits on the amount of acceleration and deflection. Human discomfort requires limits on velocity and acceleration. The work by Connor and Klink [2] contains an excellent discussion of when serviceability constraints control the design versus strength constraints. An indepth discussion on serviceability limit states under wind loading can be found in the article by Griffis [5].

A comprehensive design methodology for motion constraints has been developed by Connor and Klink [2]. Numerous strategies are presented to limit the amount of deflection and acceleration a structure may experience. This thesis continues upon the work of these two authors.

The first part of the thesis is more theoretical in nature. It presents solution algorithms for the problem we have called the "inverse problem". In short, we solve for stiffness distributions given displacment constraints and loading, rather than solving for displacements given stiffness and loading. Shear beams and trusses are discussed in detail, although the method is applicable to other types of structures. Where necessary, optimization techniques are implemented.

The second part of the thesis is more practical. It presents a rational design methodology for limiting the amount of damage a structure may incur during seismic excitation by use of large-scale linear viscous dampers. The design procedure involves modifying an inverse algorithm presented in the first part for dynamic excitation. The particulars of this part are discussed in the introduction for part II.

1.1 Strength Based Design

In a strength based design, the objective is to design a structure that complies with strength constraints. Typically, the geometry of the structure is defined and trial member sizes are selected. The stresses can be computed by solving the following equilibrium equations for displacements:

$$\mathbf{P} = \mathbf{K}\mathbf{U}. \tag{1.1}$$

The stresses are checked to see if they comply with the strength constraints, and if not the members are re-sized accordingly. By iterating on equation 1.1, an acceptable design may be achieved. The design is then checked to see if it complies with motion constraints. The underlying assumption is that in most cases, if the structure satisfies the strength constraints, it will also satisfy the motion constraints. If not, the designer may arbitrarily scale up the stiffness to satisfy the motion constraints. As motion constraints on structures become more stringent and increase in number, the designer will need a new approach to deal with the increased complexity. Motion Based Design addresses these issues.

1.2 Motion Based Design

The inverse problem algorithms presented in this thesis present a new mathematical approach to designing structures which are subject to several displacement constraints. Rather than using the traditional form of equation 1.1 to calculate displacements for given stiffnesses, we rewrite the system of equations to solve for unknown stiffness coefficients given displacment constraints and a design loading.

The basis of the inverse algorithms presented in this thesis, is a transformation of equation 1.1 into the following form that can be solved by Gaussian elimination,

$$\mathbf{P} = \mathbf{B}\mathbf{K}_{\mathbf{u}}. \tag{1.2}$$

The displacements and any other pertinent information are transformed into a matrix, $\mathbf{B}$ and the unknown member sizes are transformed into a vector of unknown stiffness coefficients,$\mathbf{K}_{\mathbf{u}}$.

Chapter 2

Formulation of Inverse Equilibrium Equations

Generating an inverse algorithm is applicable to any structural system for which a stiffness matrix can be generated. The algorithm is based on the direct stiffness method and is used to generate equilibrium equations in the inverse form, i.e. with the stiffness coefficients as the unknowns. The algorithm can be broken down into two steps. The first step involves forming the displacement matrix $\mathbf{U_m}$. The second step involves formation a matrix containing the geometrical information related to the element stiffness matrices, $\mathbf{K_g}$ and a vector $\mathbf{K_u}$, which contains the unknown stiffness coefficients.

2.1 Formation of Displacement Matrix

The equilibrium equations for a typical structure are

$$\mathbf{P} = \mathbf{KU} \tag{2.1}$$

where:

- $\mathbf{P}$ is the load vector and of size $n \times 1$.
- $\mathbf{K}$ is the stiffness matrix and of size $n \times n$, for a structural problem it is symmetric

and non-singular.

- $\mathbf{U}$ is the displacement vector and of size $n \times 1$.
- n is the number of unknowns in the system of equations.

The first step in the inverse algorithm involves transforming the matrix $\mathbf{K}$ and vector $\mathbf{U}$ into a displacement matrix $\mathbf{U_m}$ and a stiffness vector $\mathbf{K_t}$.

$$\mathbf{P} = \mathbf{U_m}\mathbf{K_t}. \tag{2.2}$$

where

- $\mathbf{P}$ is still of order $n \times 1$.
- $\mathbf{K_t}$ contains the entries of the stiffness matrix and is of order l
- $\mathbf{U_m}$ is the displacement matrix and of size $n \times l$.
- $l = 0.5n(n+1)$ (number of elements in upper triangular portion of $\mathbf{K}$)

The initial system of equations looks like:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} \tag{2.3}$$

In order to transform $\mathbf{K}$ into a vector, we need to determine the number of independent stiffness entries. This is given by the number of entries in the upper triangle of $\mathbf{K}$, since it is a symmetric matrix. For an $n \times n$ system of equations, there will be l unknown stiffness entries..

The size of the matrix $\mathbf{U_m}$ will be $n \times l$. This matrix can be filled with the help of a counter matrix, **CM**, which mimics the nature of a symmetric matrix. Integers fill out the upper triangle of $\mathbf{K}$ and are then placed in the corresponding lower triangle such that **CM** is symmetric . In order to help place stiffness coefficients later, we form a vector that records the indices of the stiffness coefficients, **IV**. The following are examples for 2×2 and 3×3 matrices:

$$\mathbf{CM} = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}, \qquad \mathbf{IV} = \begin{bmatrix} 11 \\ 12 \\ 22 \end{bmatrix}$$

$$\mathbf{CM} = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_4 & k_5 \\ k_3 & k_5 & k_6 \end{bmatrix}, \qquad \mathbf{IV} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 22 \\ 23 \\ 33 \end{bmatrix}.$$

By looking at equation 2.3, we can see that **U** is multiplied by every row of **K**. Hence, in $\mathbf{U_m}$ all displacements will be present in each row, and in the positions specified by the integers in the rows of the counter matrix. All other elements will be zero. For 2×2 and 3×3 matrices, we will have the following:

$$\begin{bmatrix} u_1 & u_2 & 0 \\ 0 & u_1 & u_2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\begin{bmatrix} u_1 & u_2 & u_3 & 0 & 0 & 0 \\ 0 & u_1 & 0 & u_2 & u_3 & 0 \\ 0 & 0 & u_1 & 0 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix}$$

2.2 Formation of the Stiffness Matrix and Vector

In the previous section, we formed the displacement matrix, $\mathbf{U_m}$, and a temporary vector, $\mathbf{K_t}$, which holds the entries of the stiffness matrix $\mathbf{K}$. The aim of this section is to expand the vector $\mathbf{K_t}$ into a matrix containing geometrical information $\mathbf{K_g}$ and a vector, $\mathbf{K_u}$, containing the unknown stiffness coefficients. By geometrical information, we mean the member orientation and length. By stiffness coefficients, cross sectional properties are meant. In the case of beams the stiffness coefficient is given by $K_u = EI$ and in the case of trusses, the stiffness coefficient is given by $K_u = EA$. The formation of the matrix $\mathbf{K_g}$ is based on the direct stiffness method. Thus, the method is applicable to any element for which a stiffness matrix can be generated.

The dimensions of the matrix $\mathbf{K_g}$ are $l \times m$, where m is given by the number of elements that compose the strucutre, i.e. the number of unknown stiffness coefficients. Each column corresponds to a specific stiffness coefficient, and the rows of each column correspond to the displacements. Let us fill in this matrix element by element.

For each element, we form the element stiffness matrix. Degrees of freedom which correspond to zero displacements can be deleted at this stage. We form an index vector for each element, $\mathbf{IV_e}$, similar to the index vector, $\mathbf{IV}$, we formed for the matrix $\mathbf{K_t}$. For each element, we fill out the corresponding column of $\mathbf{K_g}$. By matching the index of $\mathbf{IV_e}$ to $\mathbf{IV}$, we can find the appropriate row to place each entry of the stiffness matrix. Table 2.1 summarizes the procedure for writing equilibrium equations in the inverse form. Examples of deriving these equations and methods of solving them are presented in the next chapter.

Table 2.1: Inverse Algorithm

Step #	Action
1	Form $\mathbf{U_m}$ and $\mathbf{IV}$.
2	Loop over elements (for $i = 1$ to m). Form Element Stiffness Matrices and $\mathbf{IV_e}$. Place coefficient in column i, and row for which $\mathbf{IV_e}$ and $\mathbf{IV}$ match, place unknown stiffness coefficient in row i of $\mathbf{K_u}$.
3	Multiply $\mathbf{U_m}$ and $\mathbf{K_g}$ to obtain $\mathbf{B}$. The inverse equilibrium equation is $\mathbf{P} = \mathbf{BK_u}$

Chapter 3

Complete Displacement Specification

This chapter deals with solution methods to the inverse problem when displacement constraints are specified for each degree of freedom. There are two general cases for this type of problem. The first and more trivial is the case of statically determinate structures. The second, which requires the use of optimality criterion, is the case of statically indeterminate structures.

3.1 Statically Determinate Structures

For the case of statically determinate structures, the number of unknown stiffness coefficients, m , is equal to the number of degrees of freedom, n. If $m < n$, the structure will contain kinematic degrees of freedom, or will be initially unstable. If $m > n$, the structure is statically indeterminate. The size of the matrix $\mathbf{B}$ is n x n for a statically determinate structure. Thus, equation 1.2 can be solved using Gaussian Elimination. If the specified displacements are feasible, positive solutions exist.

The following example illustrates the procedure for a cantilever shear beam. Solution algorithms to the inverse problem for shear beams have been presented by Nakamura and Yamane [6]. The algorithm was presented for constant mass distribution. Connor and Klink [2] presented a solution for arbitrary mass distribution in the form of a summation equation. The matrix formulation presented here is much simpler and the methodology more general.

Example

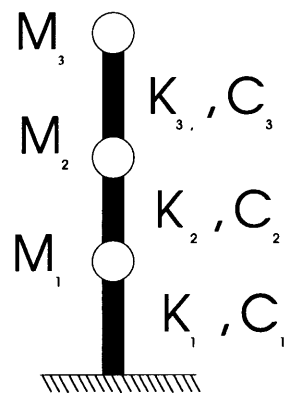

Figure 3-1: Shear Beam Model

Consider the cantilever shear beam shown in Figure 3-1. The desired deformation profile is given by $\mathbf{U}_{des}^T = \{\ 0.025\ 0.050\ 0.075\}$ (m). The loading is given by $\mathbf{P}^T = \{19{,}600\ 19{,}600\ 19{,}600\}$ (N). Find the stiffness distribution that produces the desired deformation profile.

Following the procedure outlined in the previous section, we have:

Step 1

$$\mathbf{U_m} = \begin{bmatrix} 0.025 & 0.050 & 0.075 & 0 & 0 & 0 \\ 0 & 0.025 & 0 & 0.050 & 0.075 & 0 \\ 0 & 0 & 0.025 & 0 & 0.050 & 0.075 \end{bmatrix} \qquad \mathbf{IV} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 22 \\ 23 \\ 33 \end{bmatrix}$$

Step 2

$$\mathbf{K_1} = [\,1\,]k_1 \qquad \mathbf{IV_{e1}} = [\,11\,]$$

$$\mathbf{K_2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} k_2 \qquad \mathbf{IV_{e2}} = \begin{bmatrix} 11 \\ 12 \\ 22 \end{bmatrix}$$

$$\mathbf{K_3} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} k_3 \qquad \mathbf{IV_{e3}} = \begin{bmatrix} 22 \\ 23 \\ 33 \end{bmatrix}$$

Keep in mind, the columns of $\mathbf{K_g}$ correspond to the elements and the rows correspond to the entries defined by **IV**.

$$\mathbf{K_g} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 19,600 \\ 19,600 \\ 19,600 \end{bmatrix} = \begin{bmatrix} 0.025 & -0.025 & 0 \\ 0 & 0.025 & -0.025 \\ 0 & 0 & 0.025 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

We can solve for $\mathbf{K_u}$ using Gaussian elimination.

$$\mathbf{K_u} = \begin{bmatrix} 2,232,000 \\ 1,586,000 \\ 784,000 \end{bmatrix}$$

For the particular case of a cantilever shear beam, it can be shown that the equilibrium equations,

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 & \cdots & 0 \\ -k_2 & k_2+k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \tag{3.1}$$

can be written in the following general form,

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} u_1 & u_1-u_2 & \cdots & 0 \\ 0 & u_2-u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n-u_{n-1} \end{bmatrix} \begin{bmatrix} k_{u1} \\ k_{u2} \\ \vdots \\ k_{un} \end{bmatrix}. \tag{3.2}$$

3.2 Statically Indeterminate Structures

For the case of statically indeterminate structures, the number of unknown stiffness coefficients, m, is greater than the number of degrees of freedom, n. The size of the matrix $\mathbf{B}$ is $n \times m$. Thus, for a set of feasible specified displacements we have an infinite number of solutions. The dimension of the solution space is given by $m - n$. In order to arrive at a unique solution, we need to impose additional constraints on equation 1.2. Thus, a set of optimality criterion need to be defined and appropriate techniques for solving the resulting equations need to be implemented.

In the following example, the inverse equilibrium equations for a statically indeterminate truss are derived. The following subsections will then describe techniques used for arriving at a unique solution. The techniques will be illustrated on the same example.

Example

Consider the truss shown in Figure 3-2. The desired deformation is given by $\{U_{22}U_{41}U_{42}\}$ = {-0.0019 0.0012 -0.0087} (m). The loads corresponding to these nodes are given by $\mathbf{P}$ = {-100 50 -100} (kN). Find the stiffness distribution that achieves the desired deformations.

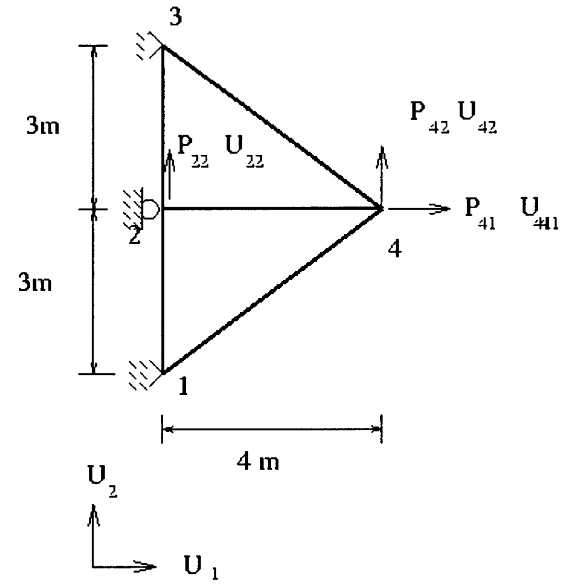

Figure 3-2: Truss for Example 1

<u>**Step 1**</u>

$$\mathbf{U_m} = \begin{bmatrix} -0.0019 & 0.0012 & -0.0087 & 0 & 0 & 0 \\ 0 & -0.0019 & 0 & 0.0012 & -0.0087 & 0 \\ 0 & 0 & -0.0019 & 0 & 0.0012 & -0.0087 \end{bmatrix} \qquad \mathbf{IV} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 22 \\ 23 \\ 33 \end{bmatrix}$$

<u>**Step 2**</u>

$$\mathbf{K_1} = [\,\tfrac{\cos^2\alpha_1}{l_1}\,]k_1 \qquad \mathbf{IV_{e1}} = [\,11\,]$$

$$\mathbf{K_2} = [\,\tfrac{1}{l_2}\cos^2\alpha_2\,]k_2 \qquad \mathbf{IV_{e2}} = [\,11\,]$$

$$\mathbf{K_3} = \begin{bmatrix} \frac{1}{l_3}\cos^2\alpha_3 & \frac{1}{l_3}\cos\alpha_3\sin\alpha_3 \\ \frac{1}{l_3}\cos\alpha_3\sin\alpha_3 & \frac{1}{l_3}\sin^2\alpha_3 \end{bmatrix} k_2 \qquad \mathbf{IV_{e3}} = \begin{bmatrix} 22 \\ 23 \\ 33 \end{bmatrix}$$

$$\mathbf{K_4} = \begin{bmatrix} \frac{1}{l_4}\sin^2\alpha_4 & -\frac{1}{l_4}\cos\alpha_4\sin\alpha_4 & -\frac{1}{l_4}\sin^2\alpha_4 \\ -\frac{1}{l_4}\cos\alpha_4\sin\alpha_4 & \frac{1}{l_4}\cos^2\alpha_4 & \frac{1}{l_4}\cos\alpha_4\sin\alpha_4 \\ -\frac{1}{l_4}\sin^2\alpha_4 & \frac{1}{l_4}\cos\alpha_4\sin\alpha_4 & \frac{1}{l_4}\sin^2\alpha_4 \end{bmatrix} k_4 \qquad \mathbf{IV_{e4}} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 22 \\ 23 \\ 33 \end{bmatrix}$$

$$\mathbf{K_5} = \begin{bmatrix} \frac{1}{l_5}\cos^2\alpha_5 & \frac{1}{l_5}\cos\alpha_5\sin\alpha_5 \\ \frac{1}{l_5}\cos\alpha_5\sin\alpha_5 & \frac{1}{l_5}\sin^2\alpha_5 \end{bmatrix} k_2 \qquad \mathbf{IV_{e5}} = \begin{bmatrix} 22 \\ 23 \\ 33 \end{bmatrix}$$

$$\mathbf{K_g} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{16}{125} & \frac{1}{4} & \frac{16}{125} \\ 0 & 0 & \frac{12}{125} & 0 & \frac{-12}{125} \\ 0 & 0 & \frac{9}{125} & 0 & \frac{9}{125} \end{bmatrix}$$

Step 3

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -0.633 & -0.633 & 0 & 0 & 0 \\ 0 & 0 & -0.682 & 0.300 & 0.989 \\ 0 & 0 & -0.511 & 0 & -0.742 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}$$

3.2.1 Stiffness Coefficient Specification

One method of introducing additional constraints on the statically indeterminate equilibrium equations is to specify additional stiffness coefficients. We need to specify $m-n$ coefficients to arrive at a unique solution. If less than $m-n$ stiffness coefficients are specified, the techniques (LS and MVLS) described in subsections 3 and 4 can be applied.

The method of stiffness coefficient specification is based on the fact that the designer may already have this information. Members of a certain size may be more economical. Other constraints, such as strength, may dictate these member sizes.

The designer may not specify the stiffness coefficients arbitrarily. The members specified must mathematically correspond to non-pivot columns in the resulting matrix obtained when **B** is reduced to row-echelon form. Physically, they are members you can remove without making the truss kinematic.

Example

For the truss in the previous example, the designer has sections with $EA = 70,000$ [kN] that are readily available.

When we reduce **B** to row echelon form, we obtain the following:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1.45 \\ 0 & 0 & 0 & 1 & 6.59 \end{bmatrix}$$

Thus, a feasible choice corresponds to members 2 and 5. By looking at the truss, we can conclude which are possible choices.

Feasible Choices	Non-feasible Choices
1 and 3	1 and 2
1 and 4	3 and 4
1 and 5	3 and 5
2 and 3	4 and 5
2 and 4	
2 and 5	

Let us choose members 1 and 3. The inverse equilibrium equations can be rewritten as:

$$\begin{bmatrix} -100 \\ 50 \\ -100 \end{bmatrix} - \begin{bmatrix} -0.633 \\ 0 \\ -0 \end{bmatrix} 70,000 - \begin{bmatrix} 0 \\ -0.682 \\ -0.511 \end{bmatrix} 70,000 = \begin{bmatrix} -0.633 & 0 & 0 \\ 0 & 0.300 & 0.989 \\ 0 & 0 & -0.742 \end{bmatrix} \begin{bmatrix} k_2 \\ k_4 \\ k_5 \end{bmatrix}$$

$$\begin{bmatrix} k_2 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 87,890 \\ 40,300 \\ 86,590 \end{bmatrix}$$

3.2.2 Minimum Weight Solution

Another possibility we can use to impose constraints on the equilibrium equations is a minimum weight solution. We can define a Lagrangian function for a minimum weight solution and invoke stationarity to solve for the unknown stiffness coefficients and Lagrangian multipliers. However, we will see that a minimum weight objective function results in a singular system of linear equations that cannot be solved.

Let the unknown stiffness coefficients be the cross sectional areas of the members. We can simply multiply the matrix, **B**, by the modulus of elasticity, E, to achieve this.

Our optimization problem can be stated as follows:
minimize the objective function

$$w = \mathbf{l}^{\mathbf{T}}\mathbf{K_u}\rho \tag{3.3}$$

subject to the following equality constraint

$$\mathbf{B}\mathbf{K_u} = \mathbf{P}. \tag{3.4}$$

where

- $\mathbf{l}$ is a vector containing the member lengths,
- $\mathbf{K_u}$ is a vector containing member areas,
- and ρ is the density of steel.

Using Lagrangian multipliers, we can formulate the following Lagrangian function:

$$L(\mathbf{K_u}, \lambda) = \mathbf{l}^{\mathbf{T}}\mathbf{K_u}\rho + \lambda^{\mathbf{T}}(\mathbf{B}\mathbf{K_u} - \mathbf{P}). \tag{3.5}$$

Note that the above equation is not quadratic in form. Thus, a minimum solution may not exist and in fact does not exist as will be shown.

The optimal solution can be found by invoking stationarity of equation 3.5.

$$\frac{\partial L}{\partial K_{ui}} = 0, \qquad i = 1, \cdots, m \tag{3.6}$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \qquad j = 1, \cdots, n \tag{3.7}$$

These stationarity conditions yield the following two equations:

$$\mathbf{B}^{\mathbf{T}}\lambda = -\mathbf{l}\rho \tag{3.8}$$

$$\mathbf{B}\mathbf{K_u} = \mathbf{P}. \tag{3.9}$$

which can be written in the following matrix form:

$$\begin{bmatrix} \mathbf{0} & \mathbf{B^T} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K_u} \\ \lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{l}\rho \\ \mathbf{P} \end{bmatrix}. \tag{3.10}$$

Consider the first set of columns defnined by $\mathbf{0}$ and $\mathbf{B}$. Since $\mathbf{B}$ has at most a rank of n, we only have n linearly independent columns. The second set of columns defined by $\mathbf{B^T}$ and $\mathbf{0}$ are also of rank n. Hence, the rank of the above matrix is at most $2n$. This is less than the rank of $m + n$ required for the matrix to be nonsingular. We can conclude that a minimum weight objective function does not impose additional constraints on our underdetermined system of equilibrium equations.

3.2.3 Least Squares Solution

Another method for solving an underdetermined system equations is the least squares method. The optimal and unique solution is defined as the solution that minimizes the sum of the squares of the stiffness coefficients. We can state our objective as the following optimization problem:

Minimize the objective function

$$f(k) = \frac{1}{2}\mathbf{K_u^T}\mathbf{K_u} \tag{3.11}$$

subject to the following equality constraint :

$$\mathbf{BK_u} = \mathbf{P}. \tag{3.12}$$

Using Lagrangian multipliers, we can formulate the following Lagrangian function:

$$L(\mathbf{K_u}, \lambda) = \frac{1}{2}\mathbf{K_u^T}\mathbf{K_u} + \lambda^{\mathbf{T}}(\mathbf{BK_u} - \mathbf{P}). \tag{3.13}$$

Since the above equation is quadratic in form, a linear system of equations can be solved to find the minimum. The linear system of equations can be found by invoking the stationarity of the Lagrangian function.

$$\frac{\partial L}{\partial K_{ui}} = 0, \qquad i = 1, \cdots, m \tag{3.14}$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \qquad j = 1, \cdots, n \tag{3.15}$$

These stationarity conditions yield the following two equations:

$$\mathbf{K_u} + \mathbf{B^T}\lambda = \mathbf{0} \tag{3.16}$$

$$\mathbf{B}\mathbf{K_u} = \mathbf{P}. \tag{3.17}$$

which can be written in the following matrix form:

$$\begin{bmatrix} \mathbf{I} & \mathbf{B^T} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K_u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{P} \end{bmatrix}. \tag{3.18}$$

The Least Sqaures Solution is elegant mathematically. Physically however, it tends to eliminate the redundants of statically indeterminate structures. See the Table 3.1 which summarizes design results for each of the methods presented at the end of subsection 4. The following subsection develops a solution procedure which overcomes this.

3.2.4 Mean Valued Least Squares Solution

The mean valued least squares solution overcomes many of the shortcomings of the least squares solution. It does not eliminate any of the redundants in our indeterminate structure and minimizes the standard deviation of the stiffness coefficients about their mean value.

The mean value of stiffness coefficients can be defined as follows:

$$\bar{K_u} = \mathbf{E^T}\mathbf{K_u}\frac{1}{m} \tag{3.19}$$

where $\mathbf{E}$ is a vector of ones of dimension $m \times 1$. We define the following error vector which represents the amount of deviation from the mean for each stiffness coefficient,

$$\mathbf{e} = [\mathbf{E}\mathbf{E}^{\mathbf{T}}\frac{1}{m} - \mathbf{I}]\mathbf{K_u} \tag{3.20}$$

or simply

$$\mathbf{e} = \mathbf{C}\mathbf{K_u} \tag{3.21}$$

Our objective is to minimize the sum of the squares of the error terms subject to the equality constraint imposed by the equilibrium equations. We can define the following Lagrangian function which is quadratic in form. We can expect the solution of a linear system of equations to minimize this Lagrangian function.

$$L(\mathbf{K_u}, \lambda) = \frac{1}{2}\mathbf{e}^{\mathbf{T}}\mathbf{e} + \lambda^{\mathbf{T}}(\mathbf{B}\mathbf{K_u} - \mathbf{P}). \tag{3.22}$$

$$\frac{\partial L}{\partial K_{ui}} = 0, \qquad i = 1, \cdots, m \tag{3.23}$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \qquad j = 1, \cdots, n \tag{3.24}$$

These stationarity conditions yield the following two equations:

$$\mathbf{C}^{\mathbf{T}}\mathbf{C}\mathbf{K_u} + \mathbf{B}^{\mathbf{T}}\lambda = \mathbf{0} \tag{3.25}$$

$$\mathbf{B}\mathbf{K_u} = \mathbf{P}. \tag{3.26}$$

which can be written in the following matrix form:

$$\begin{bmatrix} \mathbf{C}^{\mathbf{T}}\mathbf{C} & \mathbf{B}^{\mathbf{T}} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K_u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{P} \end{bmatrix}. \tag{3.27}$$

3.2.5 Results

Results for each of the methods described in the previous section are given in Table 3.1.

- **SCS** ... Stiffness Coefficient Specification
- **LS** ... Least Square Solution
- **MVLS** ... Mean Valued Least Squares Solution

Table 3.1: Results for Various Techniques Used to Impose Additional Constraints on an Underdetermined Set of Equilibrium Equations

	SCS	**LS**	**MVLS**
K_{u1}	*70,000*	78,947	78,947
K_{u2}	87,890	78,947	78,947
K_{u3}	*70,000*	61,256	78,750
K_{u4}	40,300	570	79,723
K_{u5}	86,590	92,619	80,611

Chapter 4

Partial Displacement Specification

The limitation of the inverse methods described thus far has been that the designer must specify desired displacements at every degree of freedom. However, only a few displacements may be of importance, and hence the designer would like a method which deals with only partial displacement specification. For example, in the case of a bridge, vertical displacement of the deck may be of importance, while displacement at other nodes may not. In the case of a building type structure, interstory drift may be the driving constraint, but vertical displacements may not. This chapter presents a methodology for dealing with partial displacment specification. The example used in previous chapters is expanded upon to preserve continuity.

Example

Consider the truss in Figure 3-2. Assume that the only constraint on displacement is in the vertical direction at node four, $(U_{42} \equiv u_3) = -0.0087\,\mathrm{m}$. Find a set of stiffness coefficients that satisfies this constraint.

4.1 Objective Function

In order to develop some type of criterion for which we may develop a mathematical model, we need to clearly state our objectives. In the previous chapters, we developed a robust methodology to deal with both statically determinate and statically indeterminate structures. In both cases, the equilibrium equations,

$$\mathbf{P} = \mathbf{B}\mathbf{K_u}, \tag{4.1}$$

had to be satisfied. For the indeterminate case, it was necessary to define an objective function and introduce the equilibrium equations as equality constraints. The objective function was only a function of the unknown stiffness coefficients and the Lagrange multipliers, not displacements. Our objective function will now be a function of all three.

It is not possible to use our previous strategy of writing the unspecified displacements as a vector of unknowns, since equation 1.2 is a nonlinear function of these two variables i.e. we have multiplication of unknown displacements in $\mathbf{B}$ with unknown stiffness coefficients in $\mathbf{K_u}$. Thus, we clearly need to use some type of iteration strategy to solve for the unspecified displacement coefficients and the unknown stiffness coefficients.

If we specify a full set of displacements, we have a unique solution for $\mathbf{K_u}$. If we vary the values of the displacements, we also vary the values of $\mathbf{K_u}$. Thus, let us define a similar objective function for the case of partial displacement specification to the objective functions developed in the previous chapters. We have the objectives of a least squares solution and a mean valued least squares. Before we implement an optimization scheme, we need to know if our problem is convex.

4.2 Convexity

In Chapter 3, we defined a quadratic (i.e. convex) Lagrangian function for a Least Squares Solution and a Mean Valued Least Squares Solution, thus we could be sure a minimum solution existed. This section explores the issue of whether our Lagrangian function is still convex. Our problem is now a function of $\mathbf{u}, \mathbf{K_u}, \lambda$. Let us take the the Lagrangian function derived earlier for the mean valued least squares response.

$$f(\mathbf{u_u}, \mathbf{K_u}, \lambda) = L(\mathbf{u_u}, \mathbf{K_u}, \lambda) = \frac{1}{2}\mathbf{e}^{\mathbf{T}}\mathbf{e} + \lambda^{\mathbf{T}}(\mathbf{B}\mathbf{K_u} - \mathbf{P}) \tag{4.2}$$

A minimum of the above function occurs at

$$\frac{\partial L}{\partial K_{ui}} = 0, \qquad i = 1, \cdots, m \tag{4.3}$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \qquad j = 1, \cdots, n \tag{4.4}$$

$$\frac{\partial L}{\partial u_{uq}} = 0, \qquad q = 1, \cdots, r \tag{4.5}$$

where m is the number of unknown stiffness, n is the total number of degrees of freedom, and q is the number of unspecified displacements.

Invoking these stationarity conditions no longer results in a linear expression. (Refer to equation 3.22.) Thus we need to use an iterative strategy to solve this problem.

For the time being, assume we have an initial feasible full set of displacements. We can then calculate the set of unknown stiffness coefficients. If we perturb the displacements, we can compute the corresponding perturbations in the objective function. Using this information, we can obtain sensitivities and select a new set of displacements. We can use an unconstrained optimization scheme to iterate towards a minimum of the objective function.

In order to implement an unconstrained optimization scheme based on some sort of sensitivity analysis, we need to have some sort of idea about the qualitative nature of the objective function. Does a minimum solution exist i.e. is it convex? If so, are the stiffness coefficients that satisfy this solution positive?

The technique of meshing is used to visualize the qualitative behavior of our optimization problem. Ranges of values are specified for each unspecified displacement. In the case of a least squares criteria, (see equation 3.13), the solution space may be convex; however, the minimum of the objective function does not occur where all of

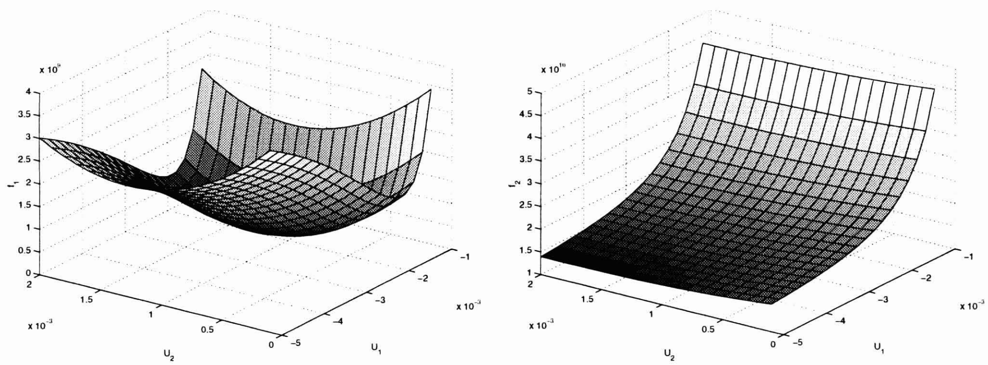

Figure 4-1: Mesh Plot for MVLS and LS

the stiffness coefficients are positive. See f_2 in Figure 4-1. Such a solution has no physical meaning and results in a stiffness matrix that is not positive definite. In the case of a mean valued least squares criteria, (see equation 4.2), the solution space is convex in the region where K_u is positive. See f_1 in Figure 4-1. Thus, we can use some sort of sensitivity analysis to find a meaningful solution to this set of equations.

4.3 Optimization Techniques

In this section, we discuss two methods of sensitivity analysis used for unconstrained optimization. First, Newton's method is discussed. Even though it is rarely implemented in practice, it forms the underlying theory for most optimization schemes based on sensitivity. Second, we discuss and implement a quasi-Newton algorithm that is much more efficient than Newton's method. The reader is referred to [7] for an in depth discussion of these and other optimization techniques.

4.3.1 Newton's Method

Newton's method forms the underlying theory for optimization schemes based on sensitivity analysis. Newton's method for solving systems of nonlinear equations has a high rate of convergence. However, it is expensive to use because it requires the computation of the Hessian matrix and the solution of a system of linear equations at each iteration step. All other optimization schemes are based on making some compromise on Newton's method, usually involving more iteration but less storage and computation for each iteration step.

Assuming our objective function has a minimum, we would like to apply the first order necessary condition for a local minimizer:

$$\nabla f(\mathbf{u_u}) = 0. \tag{4.6}$$

Newton's method solves the above equations by linearizing the above equation and iterating, which yields Newton's equations:

$$\mathbf{u_{k+1}} = \mathbf{u_k} + \mathbf{p_k} \tag{4.7}$$

where

$$\mathbf{p}_k = -\left[\nabla^2 f(\mathbf{u}_{\mathbf{u}k})\right]^{-1} \nabla f(\mathbf{u}_{\mathbf{u}k}). \tag{4.8}$$

4.3.2 Quasi-Newton Methods

There are many quasi-Newton methods available, but they are all based on approximating the Hessian $\nabla^2 \mathbf{f}(\mathbf{u_k})$ by another matrix $\mathbf{B_k}$ that is available at a lower cost. The method implemented here is a BFGS update with a backtracking line search. The complete algorithm is presented in Table 4.1.

The optimization strategy presented is one of the most popular used for solving systems of nonlinear equations. In order for the strategy to arrive at a sound solution, many of the particular parameters need to be adjusted. As mentioned previously, the solution space should be convex in the region where the optimization strategy is implemented. In the following subsections, the particulars of the optimization algorithm are discussed.

Table 4.1: Quasi-Newton Method with BFGS Update

Step 1	Specify some initial guess of the solution $\mathbf{u_{uo}}$ and an initial Hessian approximation $B_o = I$. Note: if we do not update B_o in iteration, we would be using the method of steepest descent.
Step 2	Use the norm of $\mathbf{s_k} = \mathbf{u_{u(k+1)}} - \mathbf{u_{uk}}$ to check for convergence.
Step 3	Solve $\mathbf{B_k p} = -\nabla \mathbf{f}(\mathbf{u_{uk}})$ for $\mathbf{p_k}$.
Step 4	Use a line search to determine α_k and $\mathbf{u_{u(k+1)}} = \mathbf{u_{uk}} + \alpha_k \mathbf{p_k}$.
Step 5	Compute $\mathbf{s_k} = \mathbf{u_{u(k+1)}} - \mathbf{u_{uk}}$ and $\mathbf{y_k} = \nabla \mathbf{f}(\mathbf{u_{u(k+1)}}) - \nabla \mathbf{f}(\mathbf{u_{uk}})$.
Step 6	Use the BFGS update formula to compute $\mathbf{B_{k+1}} = \mathbf{B_k} - \frac{(\mathbf{B_k s_k})(\mathbf{B_k s_k})^\mathbf{T}}{\mathbf{s_k^T B_k sk}} + \frac{\mathbf{y_k y_k^T}}{\mathbf{y_k^T s_k}}$
Step 7	Repeat Step 2

Initial Starting Point

In order to start off our optimization strategy and insure it iterates toward a minimum, we need a good initial guess for the starting point $\mathbf{u_o}$. A poor initial guess, may start the optimization algorithm in a region where stiffness coefficients found are negative. In order to avoid this, we can use initial estimates for $\mathbf{K_u}$. Displacement constraints are placed on the nodes of interest. (*Note that these constraints produce reactions at these nodes, but that is OK since we are only interested in finding an initial starting point.*) We then solve the equilibrium equations $\mathbf{P} = \mathbf{KU}$ for the unspecified displacements. We can then be sure that our initial starting point is in a region for which the stiffness matrix $\mathbf{K}$ is positive definite.

Convergence Criteria

In order to terminate the optimization algorithm, convergence criteria needs to be developed. Since our objective is to find those displacements for which the objective function is a minimum, we should use the displacements in our convergence criteria. If the displacement vectors $\mathbf{u_k}$ and $\mathbf{u_{k+1}}$ are essentially the same, we can conclude that the algorithm is in the vicinity of a minimum. Requiring the norm of the vector $\mathbf{s_k} \leq 10^{-8}$ has proven to be more than sufficient. For most engineering problems, accuracy above two decimal places is not necessary.

Derivative Calculations

The optimization algorithm requires that the gradient of our objective function be calculated. Since we have a set of feasible stiffness coefficients and the corresponding Lagrangian multipliers, we can calculate the gradient of the objective function at each iteration step.

Noting the following equality

$$\mathbf{B}\mathbf{K}_{\mathbf{u}} = \mathbf{K}\mathbf{U} \tag{4.9}$$

We can rewrite equation 3.22 as follows,

$$L(\mathbf{k}, \lambda) = \frac{1}{2}\mathbf{e}^{\mathbf{T}}\mathbf{e} + \lambda^{\mathbf{T}}(\mathbf{K}\mathbf{U} - \mathbf{P}). \tag{4.10}$$

The entries of the gradient are then given by

$$\left[\frac{\partial f}{\partial u_i}\right] = \left[\lambda^{\mathbf{T}}\mathbf{K}_i\right] \tag{4.11}$$

where the index i corresponds to the row number.

Line Search

With the gradient and the matrix B which provides us with an approximation to the Hessian, we can solve for a feasible search direction p_k. However, we must decide on the step length α. Too large of a step length may cause the algorithm to diverge into a region where the stiffness coefficients are meaningless. Too small of a step length will compromise the efficiency of the algorithm. Since we have an initial estimate of displacements, it is reasonable to assume that the maximum step length should be about one-tenth the order of the initial guess. Again we can use the norm of the displacement vectors to establish the step size.

$$\alpha_k = 2^{-i}\frac{||\mathbf{u}_{\mathbf{u}}||}{||\mathbf{p}_{\mathbf{k}}||} \qquad i = 3, \cdots \tag{4.12}$$

The algorithm for the line search starts with the initial value $i = 3$. A new value $\mathbf{u_u}$ is obtained and the objective function is evaluated at this new point. If the objective function is lower at the new point, the line search is terminated, otherwise the search continues.

4.3.3 Results

Consider the example presented at the beginning of the section. After 8 iterations, the results in Table 4.2 were obtained.

Table 4.2: Results for Truss in Example 1

	Initial Guess	Optimal Solution
K_{u1}	60,000	79,780
K_{u2}	60,000	79,790
K_{u3}	60,000	79,760
K_{u4}	60,000	79,820
K_{u5}	60,000	79,860
u_1	-0.0025	-0.00188
u_2	0.0016	0.00124
u_3	**-0.0087**	**-0.00870**

Part II

Optimal Seismic Design of Buildings Incorporating the Use of Linear Viscous Dampers

Chapter 5

Introduction

Motion based structural design is a performance based design paradigm which takes as its primary objective the satisfaction of performance criteria that involve constraints on motion. These constraints are established by considering the effect of motion on structural damage, non-structural damage, and human and equipment comfort. Structural damage usually depends on the magnitude and distribution of displacement, while comfort is related to peak velocity and acceleration. Under extreme loading, structural damage is the key performance measure for seismic design. Although design codes allow structures to experience damage due to inelastic deformation, the current trend is to reduce the "allowable" damage. This shift is being driven by the need to control the cost of repair and loss of service, so as to minimize the life cycle cost.

In motion based design, one is concerned with establishing the optimal distribution of structural stiffness and deployment of dissipation devices (dampers) and energy absorption devices (sacrificial structural elements) over the structure to meet the prescribed motion requirements. Providing adequate structural strength is treated as a design constraint. The structural parameters are determined from the design requirements on peak displacement and acceleration and the structure is then checked for strength. Conventional strength based methods generate initial estimates of the structural parameters using strength requirements based on factored loads, and then check whether the deformation and acceleration requirements are satisfied. Iteration is required with both approaches. however, with the increasing emphasis on controlling damage, i.e. limiting structural motion, it is of interest to explore the applicability of the motion based design approach for structures located in seismically active regions.

This part considers the case where the structural system is composed of two independent systems: (1) a primary structure that supports the vertical loading and also provides the lateral stiffness; (2) a set of viscous dampers that provide the energy dissipation mechanism for the earthquake loading.

The focus here is on building type structures where the dampers are incorporated in the lateral bracing schemes. The effectiveness of this concept depends on the ability of the primary structure to remain elastic during the motion resulting from a major seismic event, as well as the dampers not exceeding their energy dissipation capacity. Recent developments in high strength materials and viscous damper technology have made this concept more feasible, from both technical and economic perspectives.

The proposed design methodology for dynamic loading, such as seismic excitation, is based on the premise that the response can be confined to essentially a single fundamental mode, whose shape can be controlled by suitably distributing stiffness over the structure. Damping is deployed to minimize the contribution of higher modes. A combination of scaling of the stiffness and varying the modal damping ratio is used to regulate the amplitude of the fundamental mode response so as to satisfy the design requirements on displacement and deformation. By assigning costs to the stiffness and damping elements, one can asses the total cost and explore the trade-off between adjusting stiffness versus adjusting damping.

In what follows, we describe the application of the design paradigm to a shear-beam type building structure. Our design objective for motion is a linear distribution of lateral displacement over the height, which corresponds to uniform transverse shear deformation in each story. The design variables are the lateral stiffness and viscous damping parameters for each story.

The problem of establishing the stiffness distribution which produces a prescribed mode shape for a shear beam with uniform mass was examined by Nakamura and

Yamane [6]. Connor and Wada [3] extended the development to nonuniform beams and also included stiffness proportional damping. Later papers by Wada and Hwang [8] and Connor and Klink [1] dealt with the problem of suppressing the contribution of the higher modes by adjusting the stiffness distribution, and using a damping distribution based on the modified stiffness distribution.

This part presents a more general formulation for establishing the stiffness distribution. Emphasis is placed on damping matrices that result from physical placement of dampers in a structure, in particular stiffness proportional damping. Penzien and Wilson [9] have developed a method for creating damping matrices that produce specified modal damping ratios. However, in most cases the damping matrices generated are difficult if not impossible to physically construct. Unfortunately, damping matrices that are not $\mathbf{C}$ orthogonal require the use of complex eigenvectors. At the present time, the only method available for such damping matrices is design by simulation.

The design methodology presented in this part is described in detail in the following chapters. Appendix A contains three case studies which were used to verify the design methodology.

Chapter 6

Shear Beam Model

The design methodology developed is based on a shear beam model of a building, also known as the portal method. This is a reasonable model for buildings with an aspect ratio less than 5 [4]. The entire deformation of a structure is due to bending of the beams and columns. By taking the following assumptions about the behavior of a building, a highly indeterminate system can be reduced to a determinate one:

- All connections are moment resisting.
- Axial deformations can be neglected.
- Inflection points occur at mid-beam length and mid column height.

By summing the stiffness contribution of each panel over the floors, an equivalent shear beam model can be obtained. See figure 6-1

The beauty of the model is the fact that we can specify a shear deformation that is representative of the elastic limit due to bending for each subpanel. In such a manner, we can prevent damage due to inelastic deformation.

By modifying the inverse algorithm presented in part I and using stiffness proportional damping, we can generate an appropriate shear rigidity distribution and damping distribution. The only parameters necessary are the height of the building, the mass per floor, and the desired first modal damping ratio. The following chapters describe this methodology in detail. The purpose of this chapter is to present the

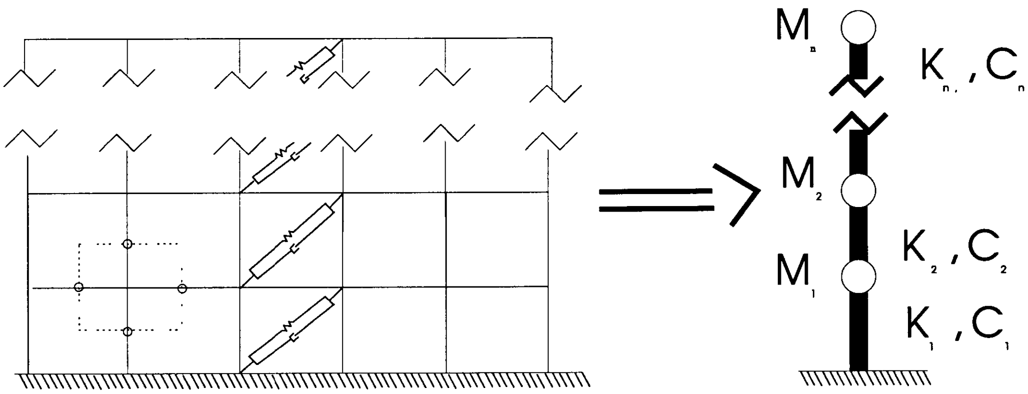

Figure 6-1: Shearbeam Model of a Building

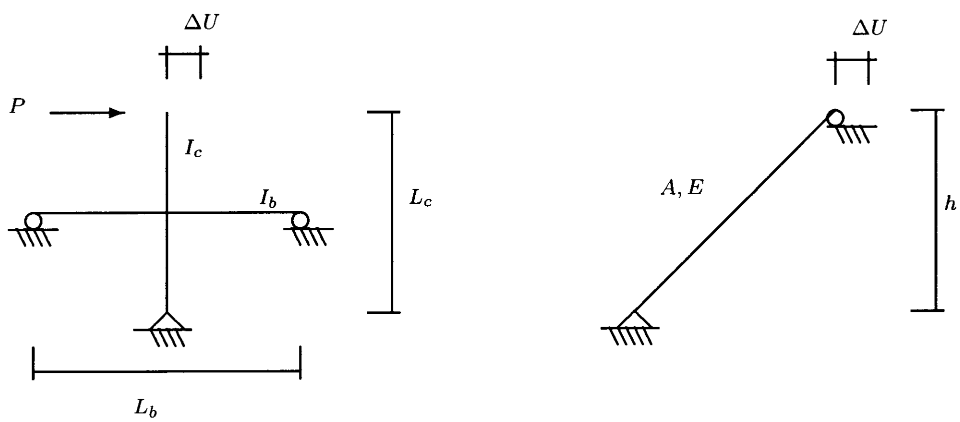

Figure 6-2: Subpanel Assembly

equations used to convert the shear stiffness and shear damping parameters to the actual member properties and sizes.

The task of the designer is to ensure that the structure supplies the required stiffness and damping distributions. The following equation is used to convert the stiffness contribution of bending members and bracing to a equivalent shear stiffness and vice versa. It is derived by considering a typical subpanel of the structure. (See Figure 6-2). It is assumed that the subpanel is symmetric with respect to the vertical and horizontal axes. A more general expression can be derived if so desired.

$$K_{shear_i} = \left\{ \left(\sum_j K_{subpanel}|_j \right) + \left(\sum_n (\frac{A_n E_n}{h_i}) \cos^2\theta_n \sin\theta_n \right) \right\}_i \Delta U \tag{6.1}$$

where

$$K_{subpanel} = \frac{12EI_cI_b}{L_c^2(L_cI_b + L_bI_c)} \qquad (interior \quad subpanel) \tag{6.2}$$

$$K_{subpanel} = \frac{12EI_cI_b}{h^2(2L_bI_c + I_bh)} \qquad (exterior \quad subpanel) \tag{6.3}$$

i is the floor number.

j is the number of subpanels per floor.

n is the number of braces per floor.

h is the story height

The following equation is used to convert the damping contribution of the bracing to a equivalent shear damping and vice versa.

$$C_{shear_i} = \left(\sum_n C_n \cos^2\theta_n \right)_i \Delta\dot{U} \tag{6.4}$$

In order to supply the required stiffness that will be generated by the algorithm presented in this part, the designer has three parameters he/she can adjust. The size of the columns and beams can be varied and if necessary, the designer can introduce a bracing scheme. For the case of damping, the designer needs to incorporate diagonal viscous dampers, so as to produce the required damping distribution.

The objective of our design is to prevent damage to the building in the form of inelastic deformation. This can be quantified by limiting the amount of interstory drift, or more precisely, the amount of shear deformation γ the structure undergoes. Typically, values of γ around $\frac{1}{200}$ represent the transition from elastic to inelastic deformation.

This value can be checked upon completion of design by considering the subpanel assembly in Figure 6-2. The maximum moment in the subpanel is given by

$$M_{max} = \frac{PL_c}{2} \tag{6.5}$$

Using the appropriate stiffness expressions given in equation 6.2 or equation 6.3 for the subpanels, we can calculate the loading at the top of each subpanel and hence the moment.

$$P_{subpanel} = K_{subpanel} \Delta U \tag{6.6}$$

Using the fact that the shear deformation is given by $\gamma = \Delta U / h$, and that the required section modulus is given by,

$$S \geq \frac{M_{max}}{\sigma_{allowable}}, \tag{6.7}$$

we can combine the above equations into the following equation that can be used to check that the beams and columns satisfy strength constraints:

$$S \geq \frac{1}{2\sigma_{allowable}} L_c^2 K_{subpanel} \gamma. \tag{6.8}$$

Similarly, the braces can be checked to ensure they satisfy the strength constraint by using the following equation,

$$\sigma_{brace} \leq \sigma_{allowable} \tag{6.9}$$

where

$$\sigma_{brace} = E \cos\theta \sin\theta \gamma. \tag{6.10}$$

Chapter 7

Stiffness Distribution

7.1 Designing for a State of Constant Shear Deformation

Our design strategy, is based on distributing the stiffness throughout the structure such that the fundamental mode has a prescribed shape. In the case of a shear beam, the desired state is uniform shear deformation and the corresponding displacement profile is linear. In order to obtain the stiffness distribution, one needs to solve the inverse eigenvalue problem.

Figure 6-1 defines the notation used to represent the various parameters and variables for a shear beam discretized as a lumped mass system. The complete set of n nodal equilibrium equations are written as:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{P} \tag{7.1}$$

where

$$\mathbf{U} = \{u_1, u_2, \cdots, u_n\}, \tag{7.2}$$

$$\mathbf{P} = -\mathbf{M}\mathbf{E}a_g \tag{7.3}$$

$$\mathbf{E} = \{1, 1, \cdots, 1\}, \tag{7.4}$$

$$\mathbf{M} = [m_i \delta_{ij}], \tag{7.5}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdots & 0 \\ -k_2 & k_2 + k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix} \tag{7.6}$$

$\mathbf{C}$ has a similar form as $\mathbf{K}$. Specializing equation 7.1 for undamped free vibration,

$$\mathbf{U} = \mathbf{\Phi} q e^{i\omega t} \tag{7.7}$$

$$\mathbf{P} = \mathbf{C} = \mathbf{0} \tag{7.8}$$

one obtains

$$\mathbf{K\Phi} = \omega^2 \mathbf{M\Phi} \tag{7.9}$$

Usually $\mathbf{M}$ and $\mathbf{K}$ are specified, and one determines the eigenvector $\mathbf{\Phi}$ and corresponding eigenvalue ω^2

The inverse eigenvalue problem is formulated as follows: given $\mathbf{M}, \mathbf{\Phi}$ and the loading, find $\mathbf{K}$ and ω. For this case, we specify $\mathbf{\Phi}$ to be a scaled version of the desired displacement profile and incorporate ω^2 in $\mathbf{K}$,

$$\mathbf{K}' = \frac{1}{\omega^2} \mathbf{K} \tag{7.10}$$

$$\mathbf{\Phi} = \bar{\mathbf{\Phi}} = \left\{ \frac{x_i}{H} \right\} \tag{7.11}$$

Equation 7.9 reduces to the following,

$$\mathbf{K}'\mathbf{\Phi} = \mathbf{M\Phi} \tag{7.12}$$

This choice of $\mathbf{\Phi}$ ensures that ω is the fundamental frequency. The problem reduces to solving equation 7.12 for the n scaled stiffness coefficients $\{k_1', k_2', \cdots, k_n'\}$

We define $\mathbf{K'}_{vec}$ as a vector containing the scaled stiffness parameters,

$$\mathbf{K'}_{vec} = \{k'_1, k'_2, \cdots, k'_n\} \tag{7.13}$$

Equation 7.12 can be written as,

$$\mathbf{\Phi}_{mat}\mathbf{K'}_{vec} = \mathbf{M\Phi} \tag{7.14}$$

where

$$\mathbf{\Phi}_{mat} = \begin{bmatrix} u_1 & u_1 - u_2 & \cdots & 0 \\ 0 & u_2 - u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n - u_{n-1} \end{bmatrix} \tag{7.15}$$

Given $\mathbf{\Phi}$ and $\mathbf{M}$ one can solve for the stiffness distribution, $\mathbf{K'}_{vec}$. The actual stiffness is determined by specifying ω and scaling $\mathbf{K'}$ according to:

$$\mathbf{K} = \omega^2\mathbf{K'} \tag{7.16}$$

Since ω is unknown at this point, we need to introduce a constraint on displacement and relate this constraint to stiffness, and finally to ω. Details of the stiffness calibration process for the case of seismic excitation are discussed in the next section.

7.2 Stiffness Calibration - Seismic Excitation

Our objective is to establish the magnitude of the stiffness and damping distributions such that the profile of maximum lateral displacement has the form

$$\mathbf{U}|_{max} = \mathbf{\Phi} q_{max} \tag{7.17}$$

where $\mathbf{\Phi}$ is the scaled fundamental mode vector (see equation 7.11) and q_{max} is the targeted displacement amplitude. We obtain a first estimate of the parameters by assuming the structure is responding in the fundamental mode, converting equation 7.1 to a modal form, and using a response spectrum to determine the peak modal displacement. Starting with

$$\mathbf{U} = \mathbf{\Phi} q \tag{7.18}$$

we transform equation 7.1 specialized for seismic excitation to

$$\ddot{q} + 2\xi_1\omega_1\dot{q} + \omega^2 q = -\Gamma a_g \tag{7.19}$$

where

$$2\xi_1\omega = \frac{\mathbf{\Phi^T C \Phi}}{\mathbf{\Phi^T M \Phi}} \tag{7.20}$$

$$\Gamma = \frac{\mathbf{\Phi^T M E}}{\mathbf{\Phi^T M \Phi}} \tag{7.21}$$

We include a subscript on ξ to denote that this quantity is the modal damping ratio for the first mode. The peak displacement for a specified seismic excitation is given by,

$$q|_{max} = \frac{1}{\omega}\Gamma S_v(\omega, \xi_1) \tag{7.22}$$

where $S_v(\omega, \xi_1)$ is the spectral velocity corresponding to that excitation.

We use an ensemble of design earthquakes to produce the design spectral velocity function $S_v(\omega, \xi)$. The earthquakes that compose the ensemble are selected such that the frequency content is representative of the design area. They are then normalized such that their maximum spectral velocity is equal to a reference value S_{vR} for a damping ratio of 0.02. This value should reflect the magnitude of excitation for which the structure is to be designed.

Two methods for generating the design spectral velocity function are discussed here. Both methods are based on the ensemble average of normalized spectral velocity functions. The spectral velocity functions are generated from by scaling the accelogram records to the reference value S_{vR} for a damping ratio of 0.02. One method develops an analytical expression for a design spectral velocity function and is recommended for pedagogical purposes and for use in hand calculations. The second method develops a numerical solution and is recommended for use in software development since it produces more accurate results.

7.2.1 Analytical Formulation

The analytical spectral velocity function is generated for each damping ratio by assuming a bilinear log-log relationship between 0.1 and 0.6 seconds, and a constant value for $T > 0.6$ seconds. The maximum value of S_v is obtained by averaging the peak values of the individual normalized spectra. Similarly, the minimum value is the average of the spectral values at T = 0.1 seconds. Figure 7-1 shows normalized spectral velocity functions normalized such that $S_{vmax} = 1.2$ m/s for $\xi = 0.02$ - a representative value for a major seismic event. The accelograms corresponding to the earthquakes are also normalized based on these values.

Once accelograms are normalized, we can compute how the spectral velocity function varies with ξ by simply increasing this parameter and generating new spectral velocity functions. Spectral velocity functions for these "scaled" accelograms and different damping ratios are shown in Figure 7-2. Figure 7-3 illustrates how the value of Sv varies with ξ for $T > 0.6$ seconds.

The spectral velocity function based on the bilinear log-log relationship can be used to solve equation 7.22. Since, there is no unique solution, we generate a family of solutions by specifying different values of ξ_1. For each value of ξ, the computation reduces to iterating on the following equations:

$$\omega = \begin{cases} \frac{\Gamma S_v}{q_{max}} & \omega \leq 10.47 \text{ rad/sec} \\ \left[10^b (2\pi)^m \frac{\Gamma}{q_t}\right]^{\frac{1}{m+1}} & \omega > 10.47 \end{cases} \tag{7.23}$$

where

$$m = \frac{\log Sv_{max} - \log Sv_{min}}{\log T_{max} - \log T_{min}} \tag{7.24}$$

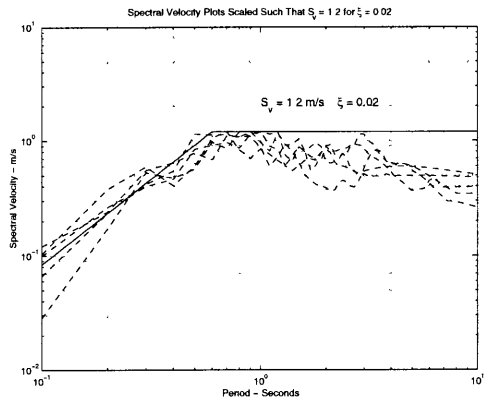

Figure 7-1: Normalized Spectral Velocity Plots

$$b = \log Sv_{max} - m \log T_{max}. \tag{7.25}$$

Given ω, one can scale the stiffness parameters and establish the system stiffness matrix, $\mathbf{K}$. Since our choice of $\mathbf{\Phi}$ has only positive elements, it follows that ω and $\mathbf{\Phi}$ are the fundamental eigenvector and eigenvalue for $(\mathbf{K}, \mathbf{M})$. The damping matrix, $\mathbf{C}$ is not defined at this time. We just have a single relation between $\mathbf{C}$ and ω_1 ξ_1 which follows from equation 7.20:

$$\mathbf{\Phi^T C \Phi} = (\mathbf{\Phi^T M \Phi})(2\xi_1\omega_1) \tag{7.26}$$

It remains to determine the elements of $\mathbf{C}$ the viscous damping parameters $c_1, c_2, c_3, \cdots, c_n$.

7.2.2 Numerical Formulation

The analytical expressions developed for the spectral velocity tend to be overly conservative. The reason being that we are attempting to generate an average design spectrum with two piecewise linear log-log plots. An improvement in design results can be obtained if an average spectral velocity function is generated by taking the ensemble average of the spectral velocity functions, instead of attempting to create an analytical expression. Equation 7.22 can be solved by iterating through the average

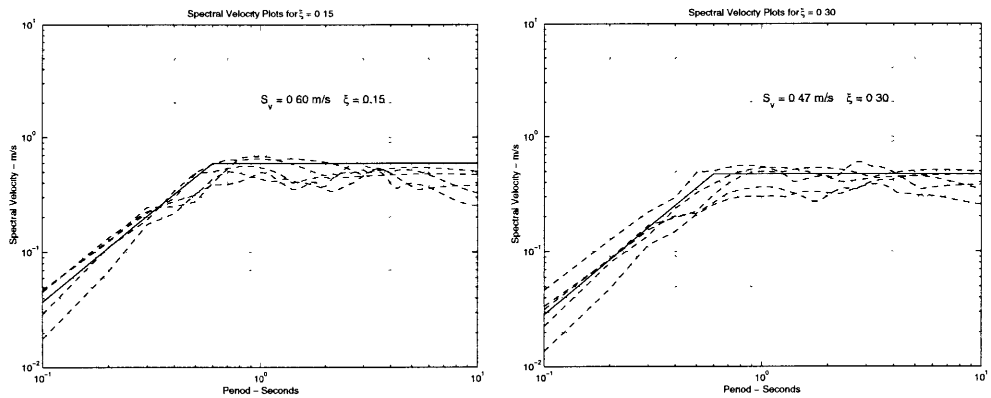

Figure 7-2: Spectral Velocity Plots for $\xi = 0.15$ and $\xi = 0.30$

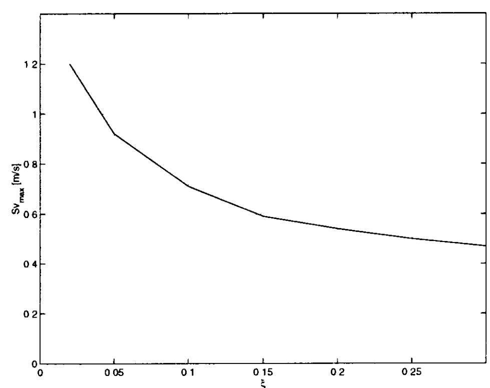

Figure 7-3: Sv_{max} vs. ξ_1

spectral velocity function and finding the value of ω and S_v that satisfy the equation for a given value of ξ. Figure 7-4 shows an average spectral velocity function generated for various values of ξ. For a given value of T, similar numerical functions to that of figure 7-3 can be used to interpolate for a given value of ξ. Figure 7-5 shows the improvement in results for a three story building. The reader is referred to appendix A for the particulars of the case study.

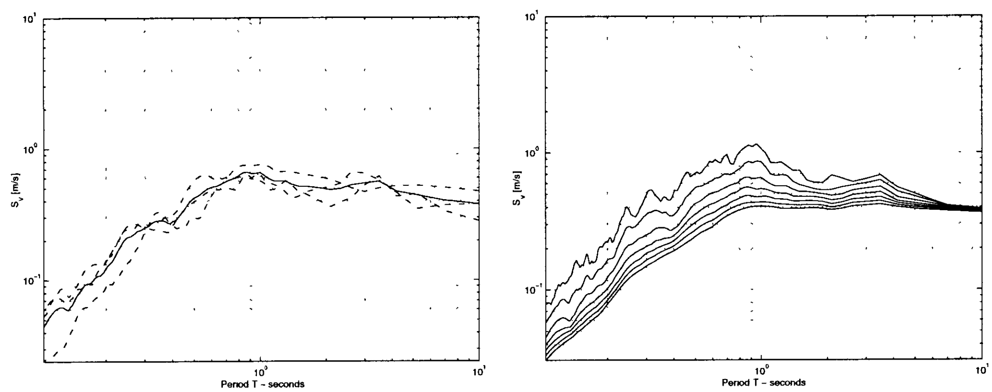

Figure 7-4: Average Spectral Velocity Functions

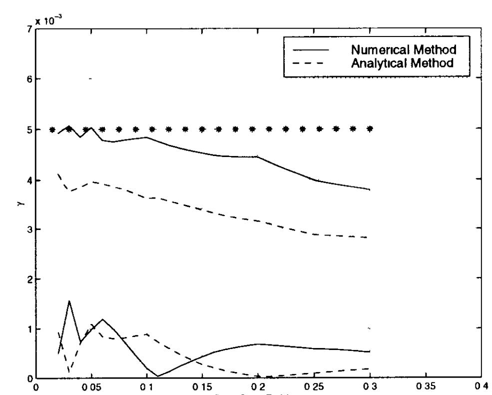

Figure 7-5: Ensemble Average Shear Deformation and Standard Deviation for a 3 Story Building

Chapter 8

Stiffness proportional damping

Stiffness proportional damping considers the elements of $\mathbf{C}$ to be scalar multiples of the corresponding elements in $\mathbf{K}$. One writes

$$\mathbf{C} = \alpha \mathbf{K} \tag{8.1}$$

which requires

$$c_i = \alpha k_i \qquad i = 1, 2, \cdots, n. \tag{8.2}$$

Substituting for $\mathbf{C}$ in equation 7.26 and noting

$$\omega_1^2 = \frac{\mathbf{\Phi^T K \Phi}}{\mathbf{\Phi^T M \Phi}}, \tag{8.3}$$

one obtains an expression relating α and ξ_1, ω_1.

$$\alpha = \frac{2\xi_1}{\omega_1} \tag{8.4}$$

Stiffness proportional damping uncouples the modal equilibrium equations based on an expansion in terms of the eigenvectors of $(\mathbf{K}, \mathbf{M})$. However, it introduces a constraint on the modal damping ratios,

$$\xi_i = \frac{\omega_i}{\omega_1} \xi_1 \tag{8.5}$$

Modal damping increases with modal number which is desirable, but we cannot adjust the individual viscous damping parameters, c_i so as to independently vary the modal damping coefficients.

At this stage of the design, the total amount of damping to place in the structure should be determined. This can be done by specifying the amount of modal damping, ξ_1, that should be placed on the first mode. The optimum value of ξ_1 can be determined by an economic analysis.

Cost Trade-Off Comparison

The stiffness and damping distributions can be computed as a function of the damping in the first mode. See figure A-3. One can see that as the damping increases linearly, the stiffness decreases nonlinearly. Such information could be used to calculate the most economical solution. At the present time, viscous dampers for use in commercial buildings are custom built. Prices are not available to make definitive cost comparisons. As the application of such dampers increases, we can expect their price to drop, thus making it feasible to have a high amount of damping in buildings.

Chapter 9

Conclusions

In this thesis, an algorithm for solving the inverse problem has been presented. The methodology is general and can be applied to any structure for which a stiffness matrix can be derived. For the case of statically determinate structures, a unique solution exists. For the case of statically indeterminate structures, it is necessary to formulate an optimization problem. The Mean-Valued Least Squares Solution is the method of choice because it ensures a statically indeterminate structure remains redundant.

Future work can be done on developing additional optimality criteria. Of interest is the case of bending members. Different penalties on the cross-sectional areas and moments of inertia would have to be imposed so as to produce typical sections.

For the method to produce meaningful results, a judicious choice of displacement constraints is necessary. The section on partial displacement constraints deals with this problem where the designer only specifies displacement constraints at nodes of interest. An algorithm is presented that selects the displacement constraints at the nodes that are not of interest. The algorithm is formulated as the solution to an optimization problem. A quasi-Newton method is implemented to solve the system of equations. The designer enters trial member sizes to start off the optimization problem. The traditional set of equilibrium equations $\mathbf{P} = \mathbf{KU}$ is solved for the unspecified displacements with displacement constraints at the nodes of interest.

The second part of the thesis presents an application of the inverse problem to seismic design. Emphasis is placed on damping matrices that represent physical placement of dampers, in particular, stiffness proportional damping. The stiffness distribution is generated by solving an inverse eigen-problem. The stiffness distribution is calibrated based on an ensemble average response spectra, which are a function of damping ratio and frequency.

The earthquakes that compose the ensemble are chosen to reflect the frequency characteristics of the site. They are then scaled to a reference spectral velocity for $\xi = 0.02$ to reflect the magnitude of earthquake that is likely to be encountered. The damping distribution is scaled base on the choice of ξ. The results generated are very close to the design objectives and improve upon the results presented by Connor and Klink [2]. The case studies presented in Appendix A were used to verify the design methodology. The reader is referred to Appendix A for a discussion of the results.

Future work can be done on considering arbitrary damping distributions. This necessitates the use of a state space formulation. The eigenvalue problem involves complex eigenvalues and frequencies.

Appendix A

Case Studies - Seismic Design

Results for the case studies in Table A.1 are presented in this chapter. The buildings are to be designed for earthquake accelocgram records scaled such that they have a maximum spectral velocity of 1.2 m/s for a modal damping ratio of 0.02. The frequency content of the following three earthquakes is representative of the site selected:

1. El Centro
2. Northridge: Station 01 Component 090
3. Northridge: Station 03 Component 090.

Note that above a first mode damping ratio of 0.1, the shear deformation does not oscillate much. Thus, the design objectives converge to a value of about 80% of the design objective. See Figure A-1 and Figure A-2. At this modal damping ratio, increased damping has less of an effect on decreasing the stiffness. See Figure A-3. The frequency and periods of the first three modes for the case studies are given in Figure A-4. The reader can use this information to verify that our assumption on a fundamental mode response is correct by looking at the frequency content of the accelograms in Figures A-5, A-6, and A-7.

Table A.1: Case Studies

Case Study	Number of Stories	Story Height [m]	Mass/Story [kg]	γ_{obj}
1	3	4	10,000	0.005
2	6	5	10,000	0.005
3	9	5	10,000	0.005

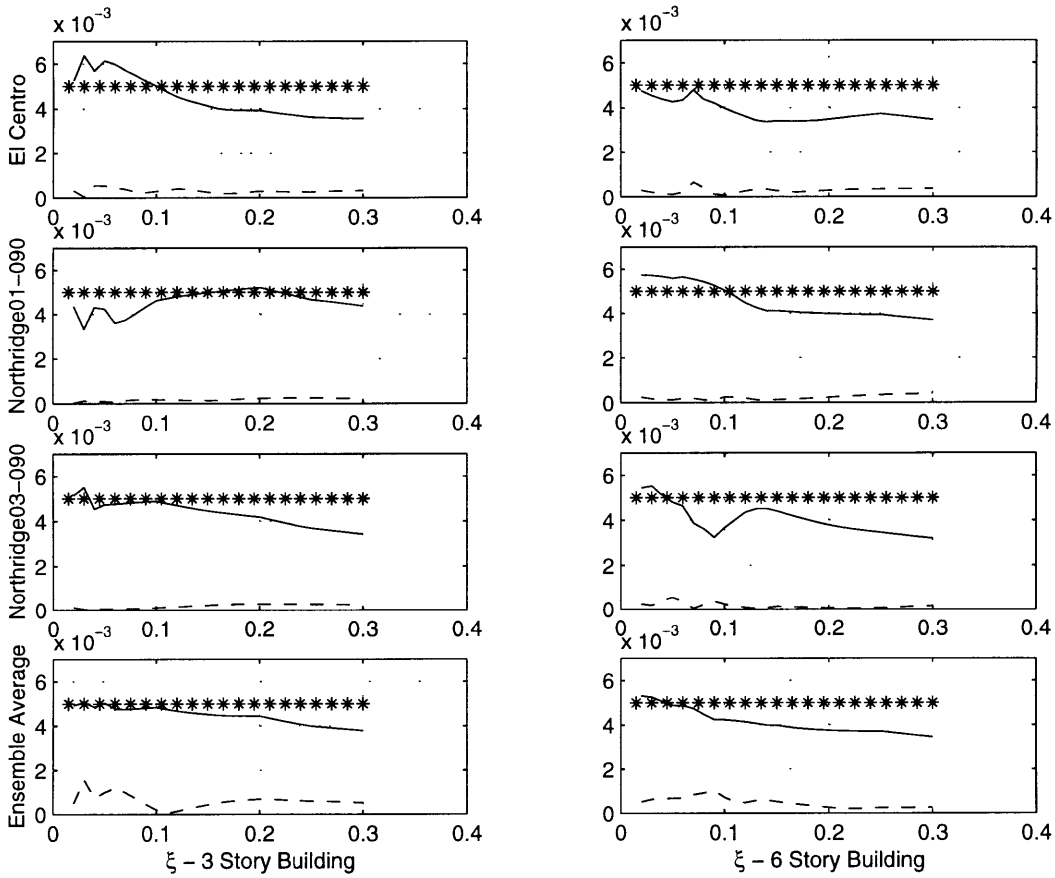

Figure A-1: Mean Story Shear Deformation and Standard Deviation for Design Earthquakes

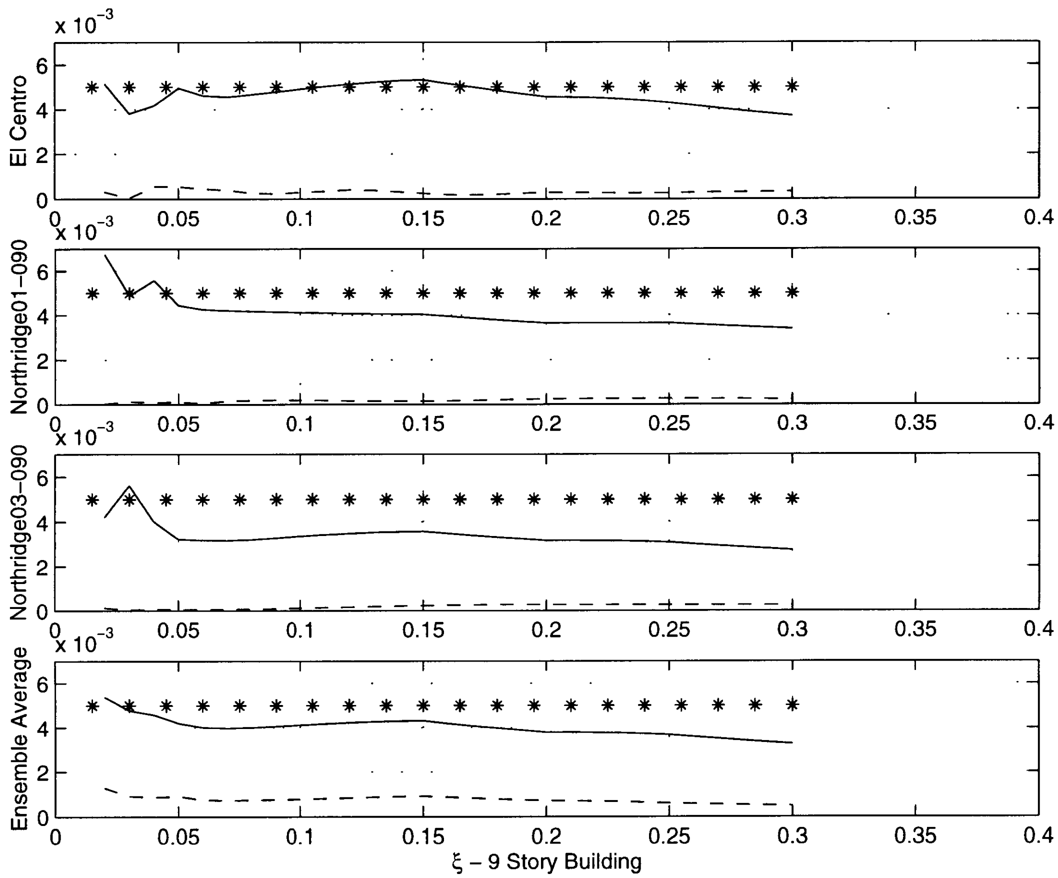

Figure A-2: Mean Story Shear Deformation and Standard Deviation for Design Earthquakes

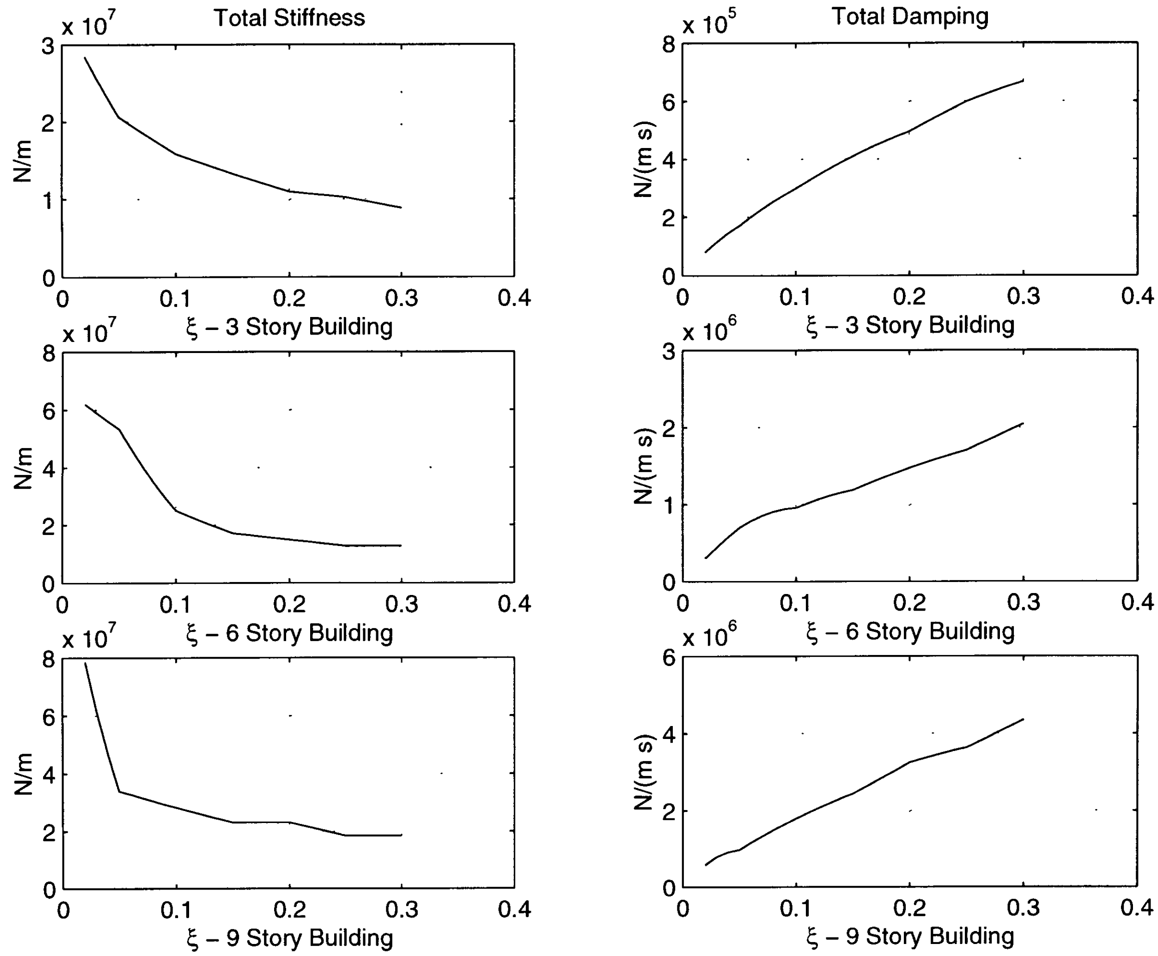

Figure A-3: Total Stiffness (N/m) and Total Damping (N/(m/s)) for Three Case Studies

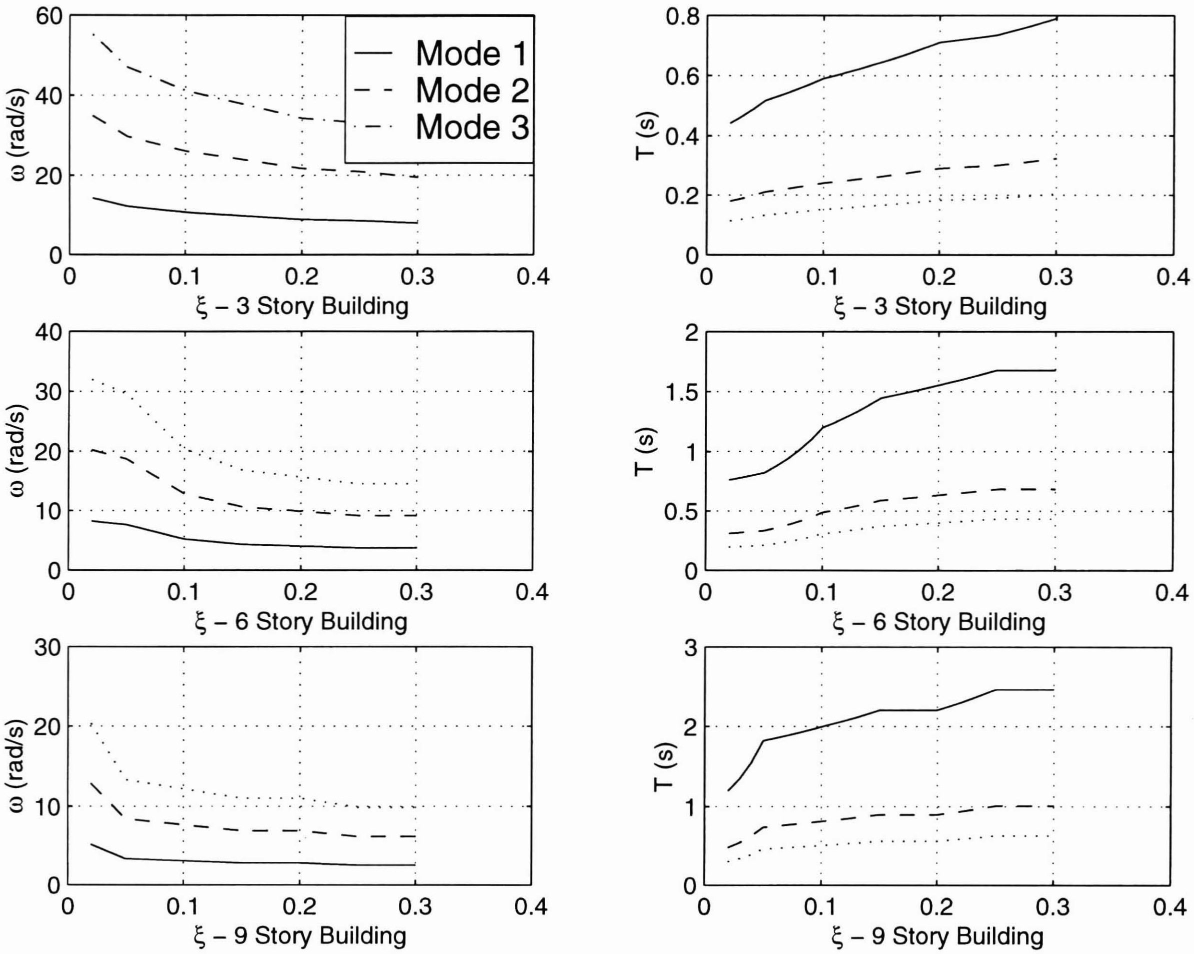

Figure A-4: Frequencies and Periods of First Three Modes for Three Case Studies

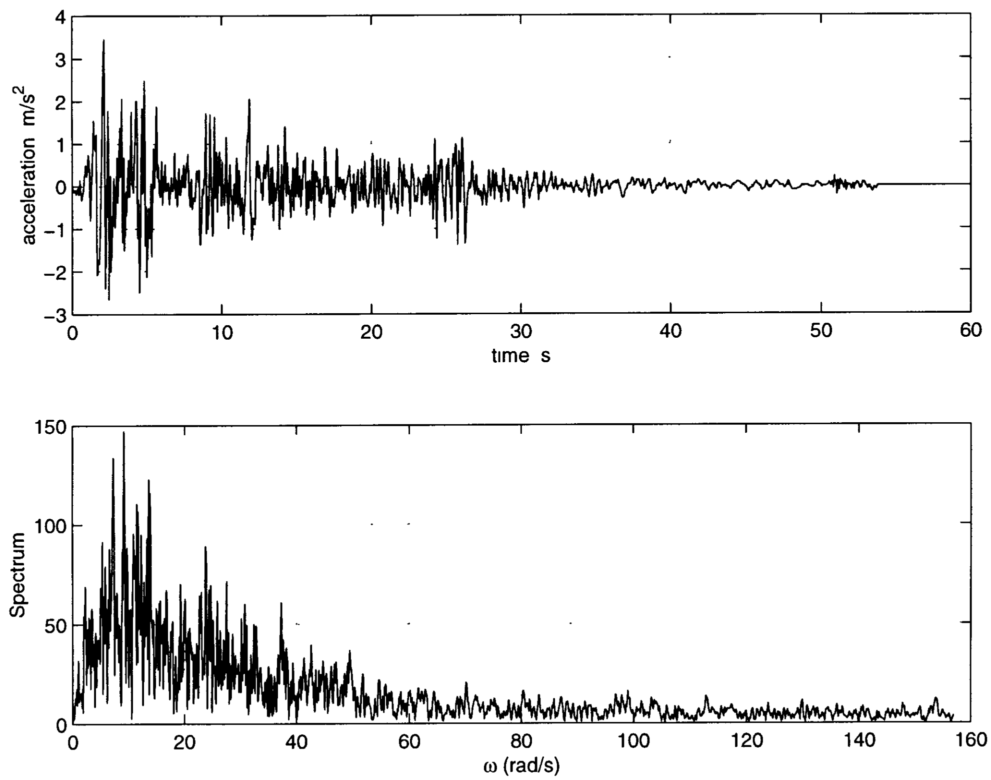

Figure A-5: Scaled Time History and Frequency Content of El Centro

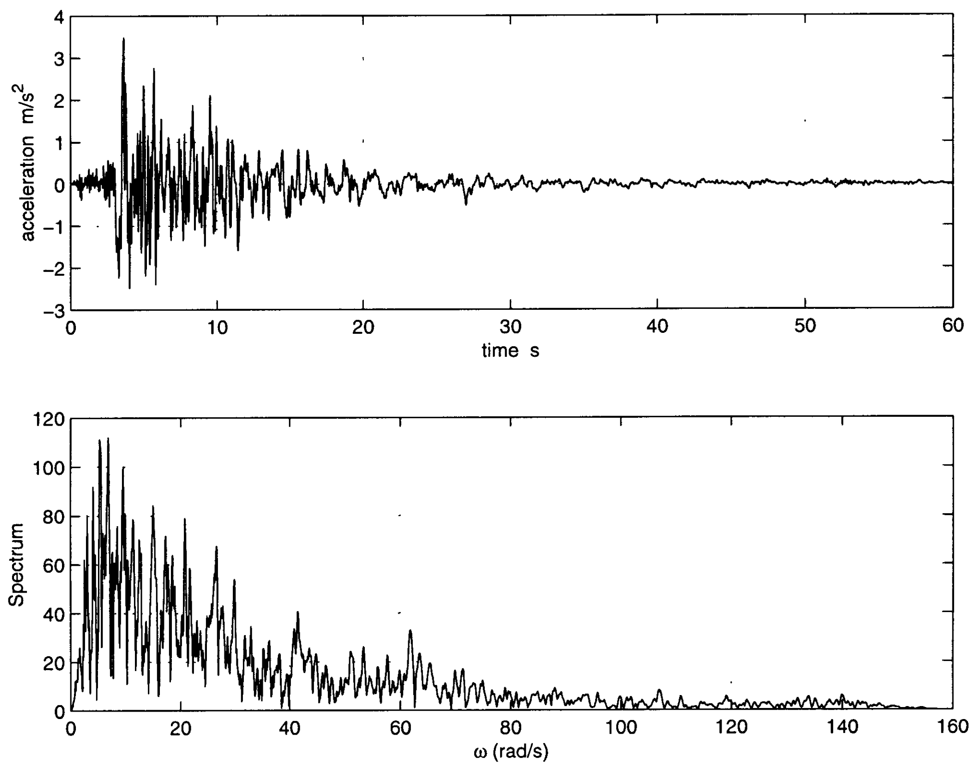

Figure A-6: Scaled Time History and Frequency Content of Northridge: Station 1 Component 090

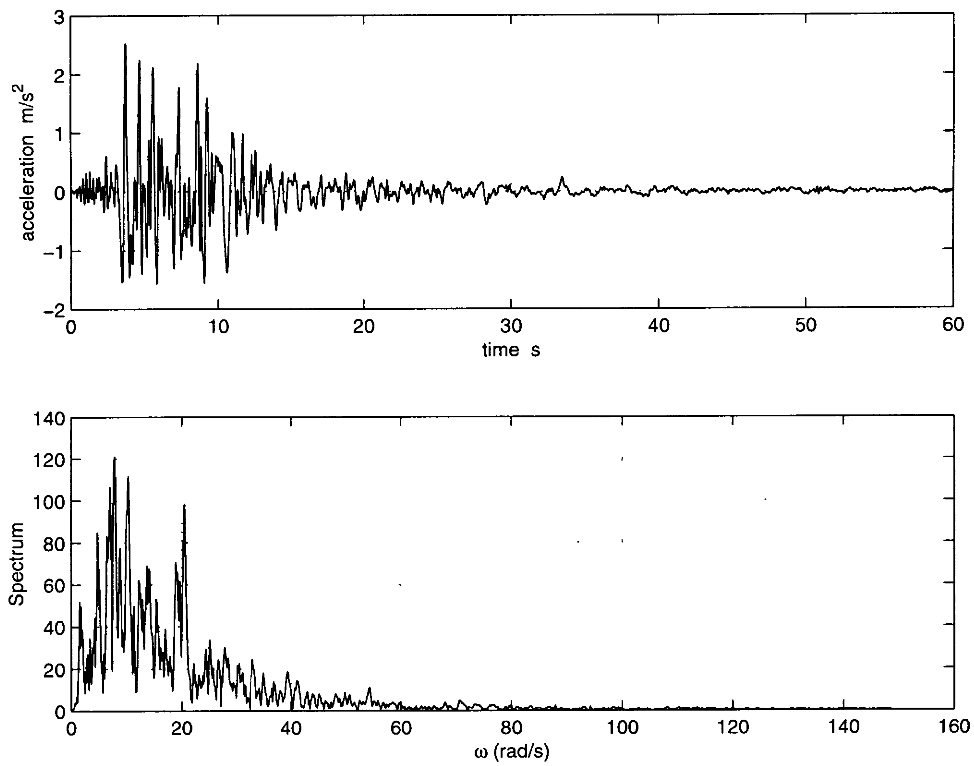

Figure A-7: Scaled Time History and Frequency Content of Northridge: Station 3 Component 090

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