Machine learning: lecture 5

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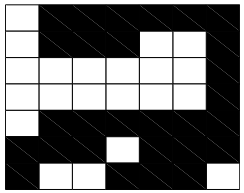
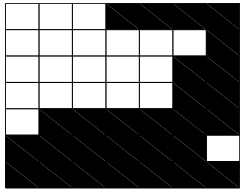
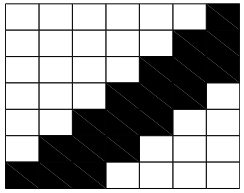
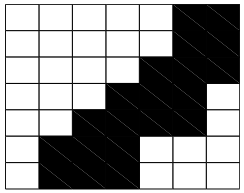
Topics

- Classification
  - regression approach to classification
  - elementary decision theory
  - Fisher linear discriminant
  - Generative probabilistic classifiers
  - discriminative classifiers: logistic regression

Classification

Example: digit recognition (8x8 binary digits)

binary digit	actual label	target label in learning
	"2"	1
	"2"	1
	"1"	0
	"1"	0
...	...	

Classification via regression

- We ignore the fact that the output is binary (e.g., $0/1$) rather than a continuous variable

  Given a linear regression function

$$f(\mathbf{x}; \mathbf{w}) = w_0 + w_1 x_1 + \ldots + w_d x_d$$

  we minimize the squared difference between the predicted output (continuous) and the observed label (binary):

$$J_n(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^{n} \left(y_i - f(\mathbf{x}_i; \mathbf{w}) \right)^2$$

- How do we classify any new example $\mathbf{x}$?

Classification via regression cont'd

$$f(\mathbf{x}; \mathbf{w}) = w_0 + w_1 x_1 + \ldots + w_d x_d$$

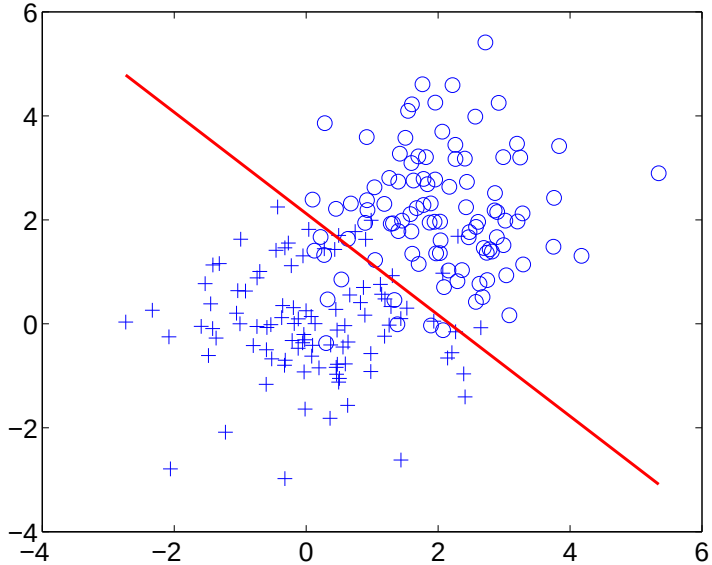
Any new (test) example $\mathbf{x}$ can be classified according to

$$\text{label} = 1 \text{ if } f(\mathbf{x}; \mathbf{w}) > 0.5, \text{ and label} = 0 \text{ otherwise}$$

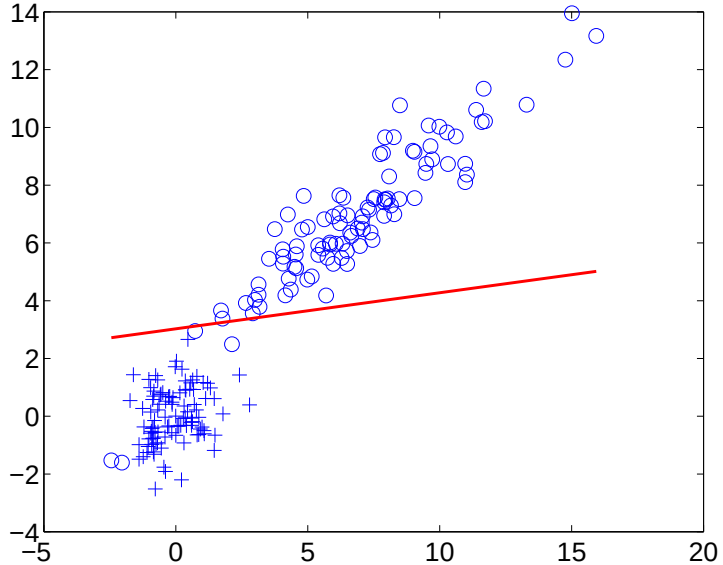
where $f(\mathbf{x}; \mathbf{w}) = 0.5$ defines the *decision boundary*.

Classification via regression cont'd

- This is not optimal...



sometimes good

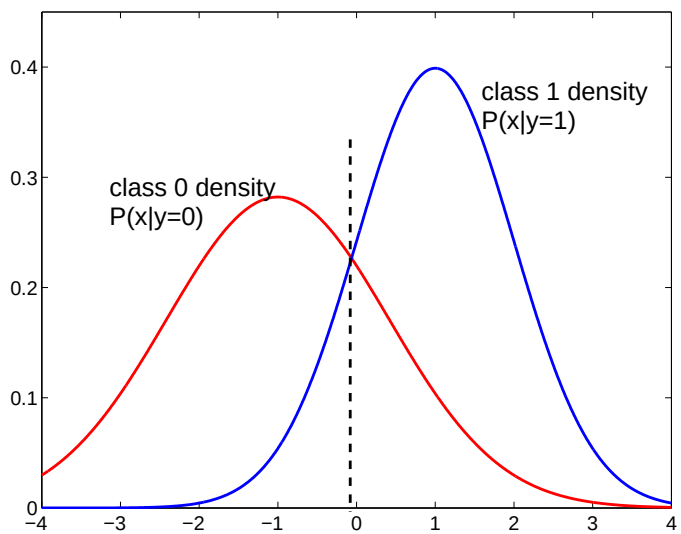


sometimes bad

A bit of decision theory

- Suppose we know the distribution of examples in each class, i.e., we know the class-conditional densities $p(x|y=0)$ and $p(x|y=1)$.

  How do we decide (optimally) which class a new example x' should belong to?



The optimal decisions in the sense of the lowest possible miss-classification error are based on the log-likelihood ratio

$$y = 1 \text{ if } \log \frac{p(x'|y=1)}{p(x'|y=0)} > 0$$

and $y = 0$ otherwise

Decision theory cont'd

- When the examples fall more often in one class than another, we have to modify the decision rule a bit:

$$y = 1 \text{ if } \log \frac{p(x'|y=1)P(y=1)}{p(x'|y=0)P(y=0)} > 0$$

  and $y = 0$ otherwise

- More generally, the *Bayes optimal decisions* are given by

$$y' = \arg\max_{y=0,1} \{ p(x'|y)P(y) \} = \arg\max_{y=0,1} \{ P(y|x') \}$$

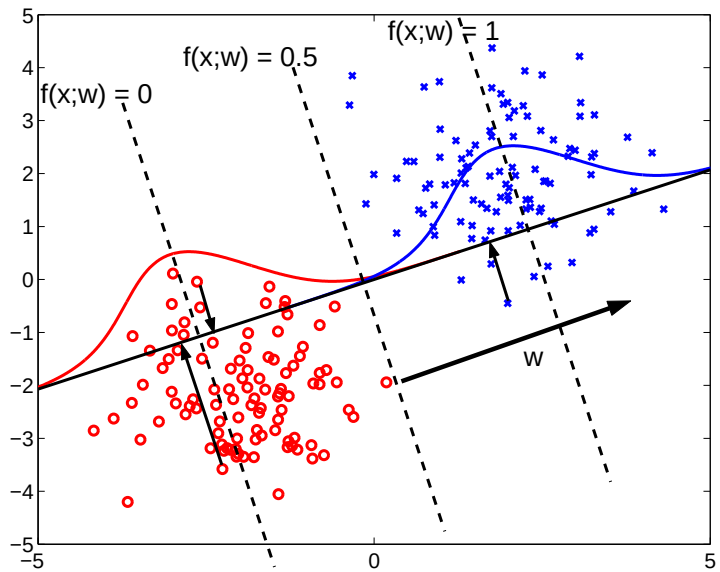
  (this is optimal only if we have the correct densities and prior frequencies)

Linear regression and projections

- Evaluating any linear regression function (here in 2D)

$$f(\mathbf{x};\mathbf{w}) = w_0 + w_1 x_1 + w_2 x_2 = w_0 + \mathbf{x}^T \vec{\mathbf{w}}$$

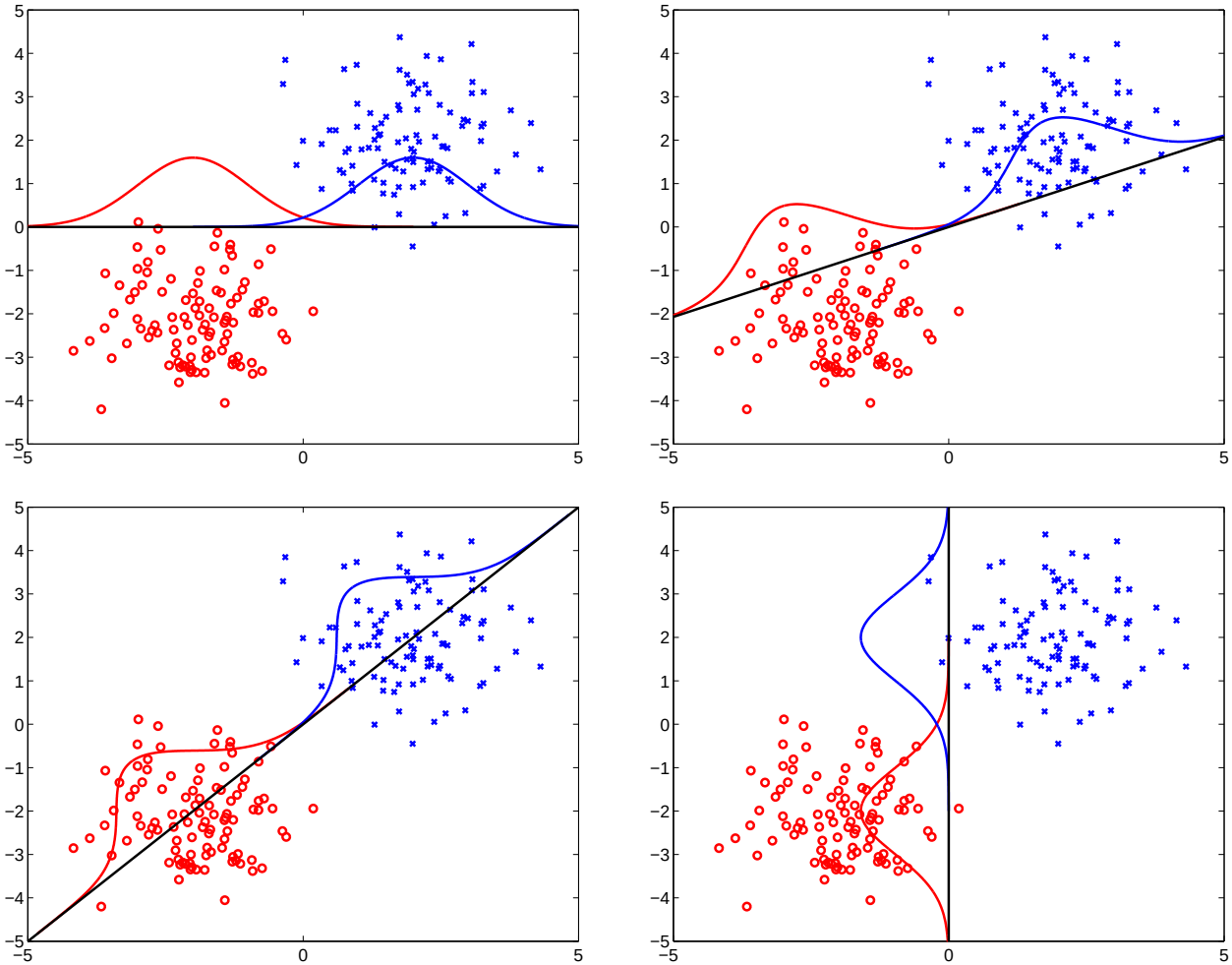
amounts to projecting the points $\mathbf{x} = [x_1 \ x_2]^T$ to a line parallel to $\vec{\mathbf{w}}$.



Since we are primarily interested in how the points in the two classes are separated by this projection, we can temporarily forget the bias/offset term w_0.

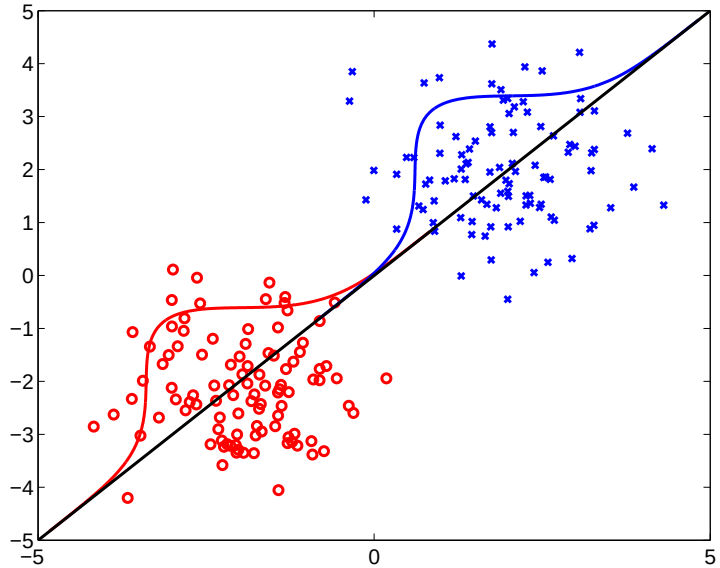
Beyond regression

- By varying the lines (or $\vec{\mathbf{w}}$) we get different levels of separation between the classes



Fisher linear discriminant

- We find a direction $\vec{\mathbf{w}}$ in the input space such that the projected points become "well-separated".



- Some notation:

  class 0: n_0 samples, mean μ_0, covariance Σ_0

  class 1: n_1 samples, mean μ_1, covariance Σ_1

Fisher linear discriminant cont'd

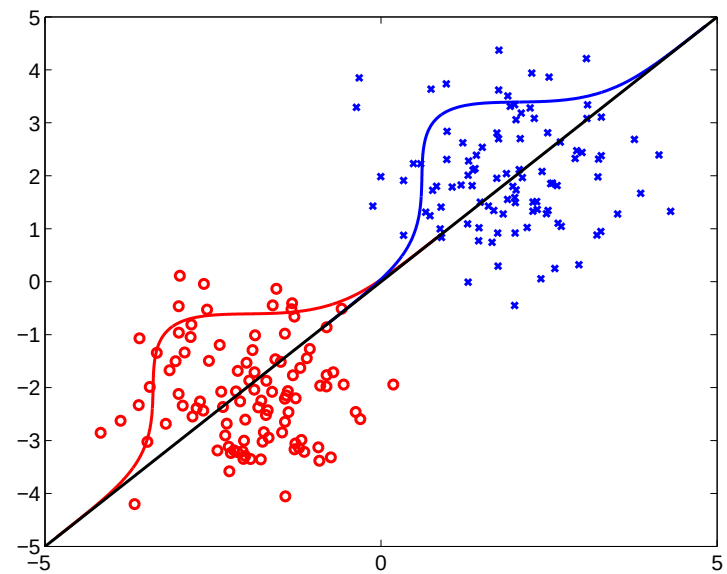
- Estimation criterion: we find $\vec{\mathbf{w}}$ that maximizes

$$J_{Fisher}(\vec{\mathbf{w}}) = \frac{(\text{Separation of projected means})^2}{\text{Sum of within class variances}} = \frac{(\vec{\mathbf{w}}^T\mu_1 - \vec{\mathbf{w}}^T\mu_0)^2}{\vec{\mathbf{w}}^T(n_1\Sigma_1 + n_0\Sigma_0)\vec{\mathbf{w}}}$$

- The solution

$$\vec{\mathbf{w}} \propto (n_1\Sigma_1 + n_0\Sigma_0)^{-1}(\mu_1 - \mu_0)$$

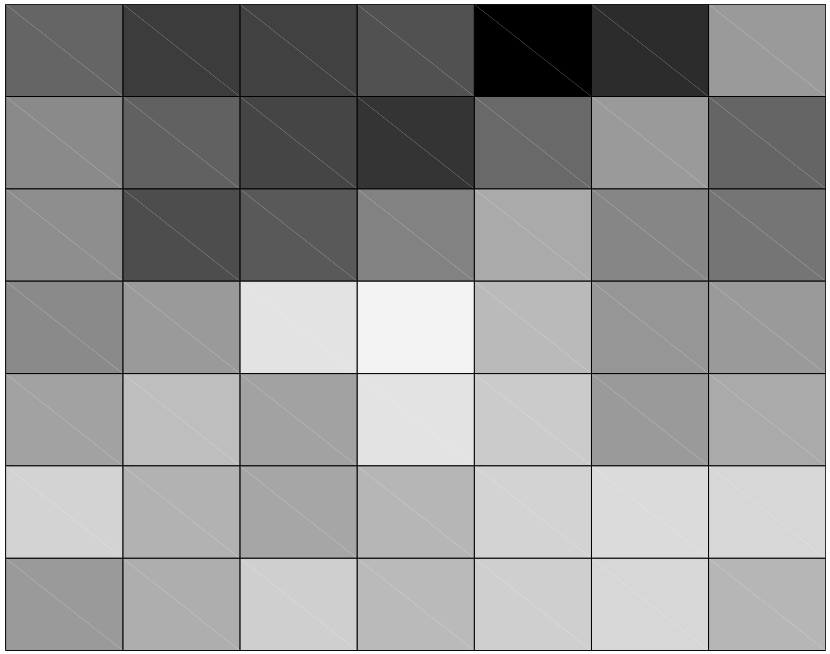
is *Bayes optimal* for two normal populations with equal covariances ($\Sigma_1 = \Sigma_0$)



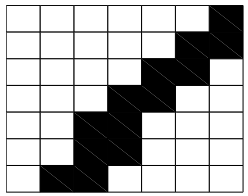
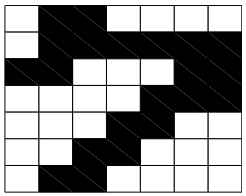
Fisher linear discriminant analysis: example

- Binary digits "1" versus "7"

c

$\vec{\mathbf{w}} =$ 

This is roughly speaking the elementwise matrix difference

$\vec{\mathbf{w}} \approx$  - 

Generative and discriminative classification

- To further refine our classification approach we can adopt one of two general frameworks:

1. Generative (model $p(\mathbf{x}|y)$)
   - directly build class-conditional densities over the multi-dimensional input examples
   - classify new examples based on the densities
2. Discriminative (only model $P(y|\mathbf{x})$)
   - only model decisions given the input examples; no model is constructed over the input examples

Generative approach to classification

- We can directly model each class conditional population with a multi-variate normal (Gaussian) distribution

$$\begin{aligned} \mathbf{x} &\sim N(\mu_1, \Sigma_1), \quad y = 1 \\ \mathbf{x} &\sim N(\mu_0, \Sigma_0), \quad y = 0 \end{aligned}$$

where

$$p(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\{ -\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \}$$

and $\mathbf{x} = [x_1, \ldots, x_d]^T$.

Mixture classifier: decisions

- Examples $\mathbf{x}$ are classified on the basis of which Gaussian explains the data better

$$\log \frac{p(\mathbf{x}|\mu_1, \Sigma_1)}{p(\mathbf{x}|\mu_0, \Sigma_0)} \begin{array}{rl} > & 0 \quad y = 1 \\ \leq & 0 \quad y = 0 \end{array}$$

  or, more generally, when the classes have different a priori probabilities, we use the *posterior probability*

$$P(y = 1|\mathbf{x}) \propto p(\mathbf{x}|\mu_1, \Sigma_1) P(y = 1)$$

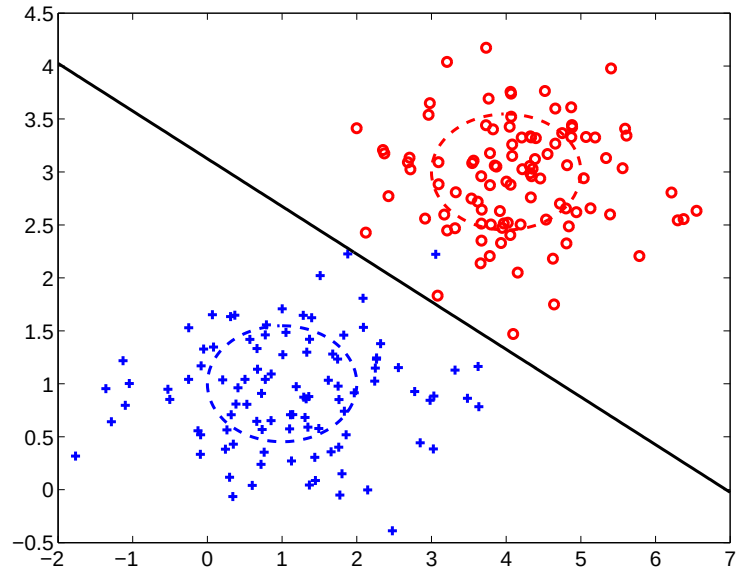
- The corresponding decision boundaries are

$$\log \frac{p(\mathbf{x}|\mu_1, \Sigma_1)}{p(\mathbf{x}|\mu_0, \Sigma_0)} = 0 \quad \text{or} \quad P(y = 1|\mathbf{x}) = 0.5$$

Mixture classifier: decision rule

- Equal covariances

$$\begin{aligned} \mathbf{x} &\sim N(\mu_1, \Sigma), \quad y = 1 \\ \mathbf{x} &\sim N(\mu_0, \Sigma), \quad y = 0 \end{aligned}$$

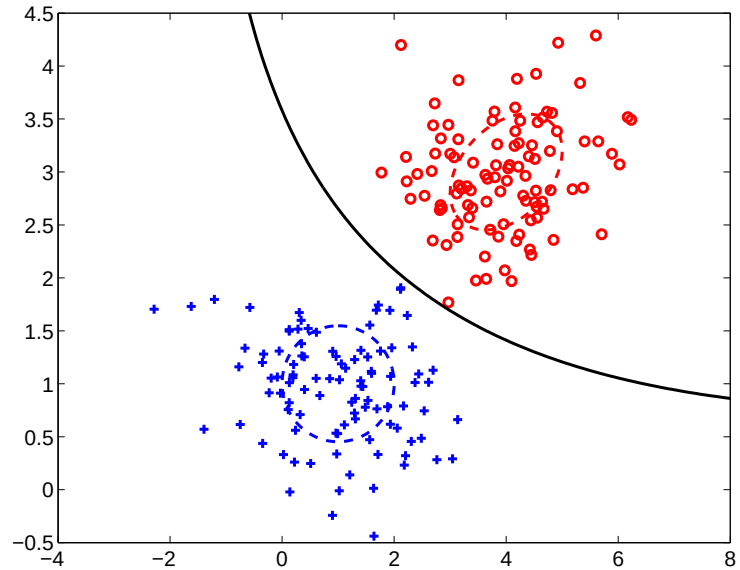


- The decision rule is *linear*

Mixture classifier: decision rule

- Unequal covariances

$$\begin{aligned} \mathbf{x} &\sim N(\mu_1, \Sigma_1), \quad y = 1 \\ \mathbf{x} &\sim N(\mu_0, \Sigma_0), \quad y = 0 \end{aligned}$$



- The decision rule is *quadratic*

Maximum likelihood estimation

- We can estimate the class conditional densities $p(\mathbf{x}|\mu, \Sigma)$ separately

  For a multivariate Gaussian model

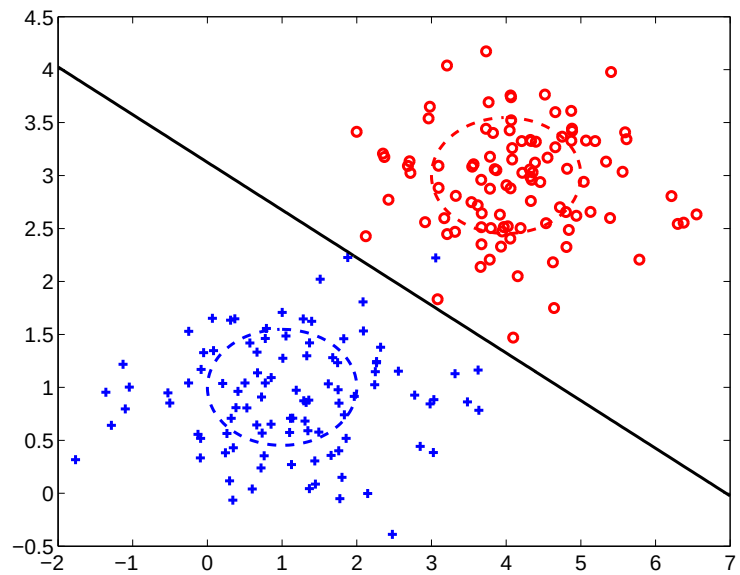
$$p(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\{ -\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \}$$

  the maximum likelihood estimates of the parameters based on a random sample $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ are given by the sample mean and sample covariance:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i, \quad \hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\mu})(\mathbf{x}_i - \hat{\mu})^T$$

Discriminative classification

- If we are only interested in the classification decisions, why should we bother with a model over the input examples?



- We could try to directly estimate the *conditional distribution* of labels given the examples or $P(y|\mathbf{x}, \theta)$ where $\theta = \{\mu_0, \mu_1, \Sigma_0, \Sigma_1\}$.

Back to the Gaussians... (1-dim)

- When the classes are equally likely *a priori*, the posterior probabilily of the label $y = 1$ given x is given by

$$
\begin{aligned}
P(y=1|x,\theta) &= \frac{p(x|\mu_1,\sigma_1^2)}{p(x|\mu_1,\sigma_1^2)+p(x|\mu_0,\sigma_0^2)} \\
&= \frac{1}{1+\frac{p(x|\mu_0,\sigma_0^2)}{p(x|\mu_1,\sigma_1^2)}} \\
&= \frac{1}{1+\exp\left\{-\log\frac{p(x|\mu_1,\sigma_1^2)}{p(x|\mu_0,\sigma_0^2)}\right\}}
\end{aligned}
$$

where $\theta = \{\mu_0, \mu_1, \sigma_1^2, \sigma_0^2\}$.

Form of the posterior

- Since the decision boundary is *linear* or *quadratic*, we know that

$$\log \frac{P(x|\mu_1, \sigma_1^2)}{P(x|\mu_0, \sigma_0^2)} = \begin{cases} w_0 + w_1 x, \text{ when } \sigma_1^2 = \sigma_0^2 \\ w_0' + w_1' x + w_2' x^2, \text{ otherwise} \end{cases}$$

for some coefficients $\mathbf{w}$.

When $\sigma_1^2 = \sigma_0^2$, we get

$$\begin{aligned} P(y = 1|x, \theta) &= \frac{1}{1 + \exp\left\{-\log \frac{p(x|\mu_1, \sigma_1^2)}{p(x|\mu_0, \sigma_0^2)}\right\}} \\ &= \frac{1}{1 + \exp\{-(w_0 + w_1 x)\}} \end{aligned}$$

Generalized linear models

- The posterior class probability $P(y = 1|\mathbf{x})$ can often be reduced to a *logistic regression model*

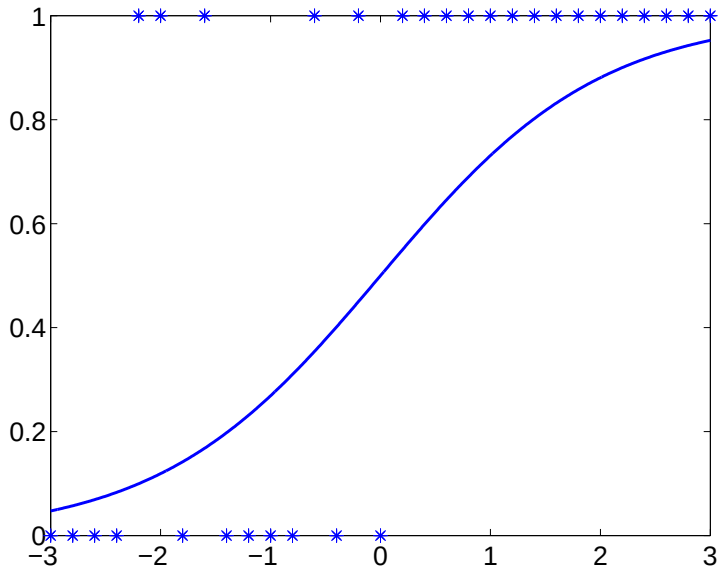
$$P(y = 1|\mathbf{x}, \mathbf{w}) = g\left(w_0 + w_1 x_1 + \ldots + w_d x_d \right)$$

with parameters $\mathbf{w}$.

Here the "squashing function"

$$g(z) = (1 + \exp(-z))^{-1}$$

that turns linear predictions into probabilities is known as the *logistic function*.



Fitting logistic regression models

- As in the case of linear regression models we can fit the logistic models using the maximum log-likelihood criterion

$$l(D; \mathbf{w}) = \sum_{i=1}^{n} \log P(y_i | \mathbf{x}_i, \mathbf{w})$$

where

$$P(y = 1 | \mathbf{x}, \mathbf{w}) = g\left(w_0 + w_1 x_1 + \ldots + w_d x_d \right)$$

(Note: although we can relate the resulting parameters to some class-conditional means and covariances, their values would be rather different from their values in the generative paradigm)