A MODEL PERFORMANCE INDEX IMPROVEMENT

by

LUIZ CLAUDIO GOULART COUTINHO MARQUES

B.S., Military Institute of Engineering (Brazil)
(1969)

SUBMITTED IN PARTIAL FULFILLMENT
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at the

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Luiz Claudio Goulart Coutinho Marques

Submitted to the Department of Aeronautics and Astronautics
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for the Degree of Master of Science.

ABSTRACT

An investigation of the quality of matching between a pre-
selected model and systems chosen through the lower limit for-
mulation was carried out. From this analysis, a new functional
relationship of the states was developed which not only permits
one to select analytically the optimum value for lower limit
but allows the direct selection of the optimum set of free
parameters. The capabilities of the new formulation were tested
through a series of examples, in which we tried to focus the
case in which zeroes show up in the transfer functions, a situ-
ation not properly handled by any previous work.

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LIST OF PRINCIPAL NOMENCLATURE AND NOTATION

Symbol	Definition
n	order of the system
m	number of system zeros
a_i	system characteristic polynomial coefficient of s^i
b_i	system transfer function numerator polynomial coefficient of s^i
y	system output variable
u	system input variable
y_{ss}	steady state value of the output variable
x	system transient response variable
s	laplace operator
$(\cdot)$	indicates derivative with respect to time
$\underline{x}_o$	system's pseudo initial condition vector
$\underline{x}$	system's transient response vector
$\underline{\tilde{x}}$	system's extended transient response vector
$\underline{a}$	system's coefficient vector
$\underline{\tilde{a}}$	system's extended coefficient vector
MPI	Model Performance Index
$(\ )^1$	indicates transpose of a vector
$\| \underline{x}^1(t)\underline{\tilde{\alpha}}$	indicates the length of $\underline{x}^1(t)\underline{\tilde{\alpha}}$
r	pseudo initial condition weighting factor
$\underline{\alpha}$	model's coefficient vector
$\underline{\tilde{\alpha}}$	model's extended coefficient vector
$\underline{\hat{x}}_{mo}$	model's extended pseudo initial condition vector

Symbol	Definition
ℓ	order of model
I	Identity matrix
0	null matrix
$\tilde{W}$	extended pseudo initial condition weighting matrix
$d(t)$	instantaneous distance between trajectories
PIC	Pseudo Initial Condition
$\hat{\underline{x}}$	desired response state vector
$\hat{\underline{x}}_o$	desired pseudo initial condition vector
$\Delta\underline{x}$	error state vector
A	system's coefficient matrix
$\hat{A}$	model's coefficient matrix
$\Delta\underline{x}_o$	initial error state vector
$(\ )^T$	transpose of a matrix
$i(t)$	error input excitation function
α_i	model's characteristic polynomial of s^i
β_i	model's transfer function numerator polynomial coefficient of s^i
p	indicates a functional relationship of a particular set of free parameters
$G(s)$	the closed loop transfer function of a system
$\hat{G}(s)$	the closed loop transfer function of a model
$\tilde{G}(s)$	a modified transfer function in Palsson's approach
LL	a particular value of lower limit
$A(t)$	area under $d(t)$ curve

CHAPTER ONE

INTRODUCTION

1.1 Historical Summary of Problem Area

Wiener (1) and Kolmogorff (2) are the first formulation of filter design as an optimization problem. Their problem is referred to as the "free-configuration" that selects the filter from the class of all possible linear filters that minimizes the mean squared error between the actual and desired signals. Later Hall (3) and Phillips (4) formulated the "fixed-configurations," using a mean squared error Performance Index. In designing, the lef the characteristic frequences, time constants and gains unspecified, to be determined by the optimization process. Phillips derived a procedure for evaluating the mean squared error over an infinite time interval as an explicit nonlinear function of the free design parameters. The only work is to select the free parameter values that correspond to a minimum point of the function.

Hall and Phillips's method was that the resulting design could allow excessive signal magnitudes within the system that may exceed the range of assumed linearity or saturate. Newton, Gould and Kaiser (5) proposed constraining any signal magnitude by adjoining the mean squared value of that signal to the original Performance Index by a Lagrange Multiplier. The

augmented Performance Index can be evaluated as a function of the free design parameters and Lagrange Multiplier using tabulated integrals. The design selected is the one in which the values of the free parameters minimize this augmented Performance Index while requiring the signal magnitude constraint to be satisfied.

Aizerman (6) proposed a new concept for representing the desired system response within a Performance Index. Rather than using the squared error between the desired and actual response, Aizerman used a linear combination of the square of the actual transient response and its derivatives. The relative weighting of these squared variables in the Performance Index was chosen so that the absolute minimum value of the Performance Index would correspond to a system design with a transient response identical to the desired response. In general, the absolute minimum value can be obtained if one has complete freedom in selecting the closed-loop system design. Since the feedback configuration and design parameter values are usually constrained due to practical requirements, it is not usually possible to obtain the desired response identically. However, minimizing Aizerman's Performance Index would tend to force the system's response to be similar to the desired response, at least for a certain class of system and types of desired response. Aizerman's concept could provide a significant computational advantage in the optimization process over model-referenced performance indices in that the

model's response never enters the computational problem.
Aizerman's Performance Index is rather limited in application
but represented a distinctly different philosophy in analyti-
cal design.

1.2 Previous Work done by Rediess and Palsson

In 1968, Rediess (7) developed a new concept for Model
Performance Index. It is based on a geometrical representa-
tion for matching a dynamical system or the model by another
actual system. In other words, it is an interpretation of the
relationship between the transfer function representation of
a system and the geometrical representation by its character-
istic plane and Pseudo Initial Condition vector. The basic
form of the resulting "Model Performance Index" is the same as
that of quadratic function frequently appearing in modern con-
trol theory. When the Model Performance Index is minimized,
the system response becomes closer to or identical to the spe-
cified model response. By response is meant not only the
displacement response but also the velocity, acceleration, etc.
Since the Model Performance Index not only weights the dis-
placement but also their higher derivatives, because the zeros
in the system transfer function effect the error response,
Rediess added a quadratic term in the initial error state to
the Performance Index.

By choosing models whose excess of polies over zeros is
no greater than that of the system, Palsson (8) apparently

eliminated the need for initial errors in the Model Performance Index. He also defined the "error excitation source" in both state space and transfer function representation, and found that MPI is the infinite integration of squaring this scalar error excitation source.

Chang Ho, in 1976, developed the concept of the lower limit which consists in beginning the integration in an instant larger than zero. This formulation allows the selection of a set of parameters with good conditions of matching but it is a completely arbitrary process.

1.3 Purpose of this Work

The purpose of this work was to obtain a way in which one could select the lower limit value of Ho's formulation without the present trial and error procedure. Besides, we should establish a way of quantifying the quality of matching, which is an aspect not treated in previous works. During this analysis, we were able to develop a functional relationship of the states which led to a new performance index which proved to have very definite advantages over the previous formulations.

CHAPTER TWO

REVIEW AND ANALYSIS OF REDIESS, PALSSON AND HO'S WORKS

In order to allow a better understanding of the problem in hand, of the difficulties found and of the different formulations tried for its solution, we are going to follow briefly the works done by Rediess, Palsson and Ho. Where necessary, specific examples will be presented and comments added in a suitable way to clarify the reader.

2.1 Rediess Formulation

Herman Rediess (ref. 7), in 1968, based in a geometric "criterium" for the approximation of a dynamic system (the model) by another (the actual system), developed a new performance index which allows a direct interpretation in terms of a certain model of the desired response. This index has received the title "Model Performance Index" or "Model P.I."

Let a system be described by the differential equation below

$$y^n(t) + a_{n-1}y^{n-1}(t) + \ldots + a_2\ddot{y}(t) + a_1y(t) + a_0y(t) =$$

$$b_m u^m(t) + \ldots + b_1\dot{u}(t) + b_0u(t) \qquad (2-1)$$

where: y is the output

u is the input

$m \leqslant n-1$

initial condition assumed to be zero

The corresponding transfer function is

$$\frac{y(s)}{\mu(s)} = \frac{b_m s^m + \ldots + b_1 s + b_o}{s^n + a_{n-1} s^{n-1} + \ldots a_2 s^2 + a_1 s + a_o} \qquad (2-2)$$

If we assume the existence of a finite steady state value for the output, y_{ss}, in response to a step input, one can write the transient portion of the response as

$$x(t) = y(t) - y_{ss}.$$

The laplace transform of the transient response can now be obtained from (2-2), in the form below:

$$x(s) = \frac{b_m s^{(m-1)} + \ldots b_3 s^2 + b_2 s + b_1}{s^{(n)} + a_{(n-1)} s^{(n-1)} + \ldots + a_2 s^2 + a_1 s + a_o} +$$

$$\frac{\frac{bo}{ao}\left(s^{(n-1)} + a_{n-1} s^{(n-2)} + \ldots a_. s + a_1 \right)}{s^n + a_{n-1} s^{n-1} + \ldots a_2 s^2 + a_. s + a_o} \qquad (2-4)$$

Defining a set of hypothetical initial conditions here called "pseudo initial conditions," capable of producing a response identical to that of a unit step, made possible the inclusion of the zeros of the tranfer function in the index to be developed. Therefore, the transient response become described by the homogeneous differential equation

$$x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \ldots a_1 \dot{x}(t) + a_o x(t) = 0 \qquad (2-5)$$

subject to the pseudo initial conditions

$$x(o) = x_o$$

$$\dot{x}(o) = \dot{x}_o$$

$$\vdots$$

$$x(o) = x_o$$

These initial values are then determined in a way that assures the already mentioned equivalence and are given by

$$x_o = - b_o/a_o$$

$$x_o = \begin{vmatrix} 0 & \text{for } i > m \\ \\ b_1 - \displaystyle\sum_{j=n-m}^{n-i-1} a_{j+1} x_o^{(j)} & \text{for } i = 1,2,\ldots m \end{vmatrix} \qquad (2.6)$$

In this expression, the summation is zero when $n-m < i-1$.

The form of equation (2.5) suggested an extention of the state space in one dimension to $(n+1)$ order. In this space, referred as the "extended state space," the equation might be interpreted as defining a hyper-plane perpendicular to the vector $\tilde{\underline{a}}$, that's to say

$$\tilde{\underline{x}} \; \tilde{\underline{a}} = 0$$

where $\tilde{\underline{x}} = (x,\dot{x},\ldots x^{(n-2)}, x^{(n-1)}, x^{(n)})$ \qquad (2.7)

$$\tilde{\underline{a}} = (a_o, a_1, \ldots a_{n-2}, a_{n-1}, 1)$$

The trajectory of $\tilde{\underline{x}}$ as a function of time, which is the same as the transient response and its first n derivatives, is completely contained in this hyper-plane. Any other system which might generate a trajectory also contained in this hyper-plane, would differ from the first one only by the pseudo initial

conditions. We then conclude that the hyper-plane contains the trajectories of all systems possessing the same characteristics equation. In other words, a linear invariant system is completely described by this plane and corresponding set of pseudo initial conditions. Rediess called this plane the "characteristic plane."

We must stress that the representation of a system by its characteristic plane and pseudo initial conditions does not bring any new information from the mathematical point of view. However, it brings a very useful way for visualizing the process of approximation of a system by another, using the Model Performance Index.

If both system and model could be represented by its characteristic planes and pseudo initial conditions, it would then be possible to establish a geometrical "criterium" for evaluation of its approximation. The Model Performance Index developed by Rediess is one of these criteria. Its basic form is a generalized measure of the distance between system's trajectory and model's characteristic plane, as can be seen below.

$$\text{MPI} = \int_{0}^{\infty} \left(\frac{\|\tilde{\chi}(t)\tilde{\underline{\alpha}}\|}{\|\tilde{\underline{\alpha}}\|}\right)^2 dt \tag{2.8}$$

where: $\tilde{\underline{\alpha}} = (\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_{n-2}, n-1' 1)$ and refers to the characteristic plane of the model. The subscript ' means a transpose matrix or vector.

The vector $\underset{\sim}{\underline{a}}$ is a function of design parameters and the equation (2.5) allows one to determine for each set of these parameters, the vector $\underset{\sim}{\underline{x}}(t)$.

In the case in which system and model do not have zeroes in their transfer function, Rediess proved that a necessary and sufficient condition for a perfect matching of the trajectories was the perfect matching of the characteristic planes. However, the functional relationship $\underset{\sim}{\underline{a}} = \underset{\sim}{\underline{a}}(p)$ in most cases rules out the possibility of a perfect matching of trajectories. In this case, however, it is possible to consider that the best matching condition corresponds to the closest proximity of system's trajectory and model's characteristic plane, which means the lowest possible value for Model Performance Index.

In the case in which zeroes do show up in the transfer function, Rediess realized that, to obtain a correct solution, it would be necessary to add to the performance index a term representing the difference between the pseudo initial conditions of model and system. Without this correction, we might get very close characteristic planes but with trajectories well apart. With this modification, the performance index becomes:

$$MPI = r \left\| \underset{\sim}{\underline{x}}_0 - \underset{\sim}{\underline{x}}_{mo} \right\|_{\tilde{W}}^2 + \int_0^\infty \left(\frac{\left\| \underset{\sim}{\underline{x}}'(t) \underset{\sim}{\underline{\alpha}} \right\|}{\left\| \underset{\sim}{\underline{\alpha}} \right\|} \right)^2 dt \qquad (2.9)$$

Where: $\left\| \ \right\|_{\tilde{W}}^2$ indicates here $(\underset{\sim}{\underline{x}}_0 - \underset{\sim}{\underline{x}}_{mo}) \tilde{W} (\underset{\sim}{\underline{x}}_0 - \underset{\sim}{\underline{x}}_{mo})$

and $\tilde{W}$ is a square, positive matrix $\left[-\dfrac{I}{O}\dfrac{O}{I}-\right]$

of order ℓ $(\ell \leq n)$.

In this expression, the scalar quantity 'r' must be selected by the designer in order to establish the relative weight between the matching of the pseudo initial conditions and of the characteristic planes.

This last comment should be examined with care. Would it be enough to make 'r' equal to 1 in (2.9)? The experience has shown that this is not enough and, to make some points clear, we reproduce here the example 3-3 of ref. (7).

Example

Consider the system shown in the figure below, where k_1 and k_2 are free design parameters. The objective here is to select k_1 and k_2 in a way that the closed loop step response become close to that of the model.
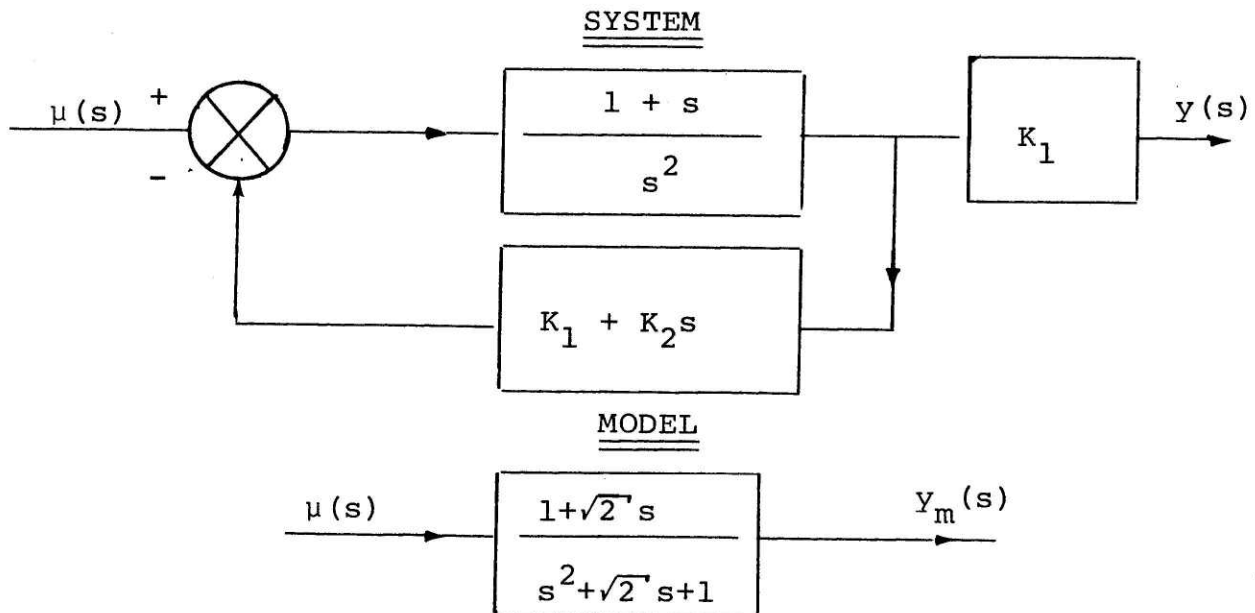
The autonomous representation of the transient response in closed loop form is

$$\ddot{x} + \left(\frac{k_1 + k_2}{1 + k_2}\right) \dot{x} + \left(\frac{k_1}{1 + k_2}\right) x = 0$$

The corresponding model performance index is

$$MPI = r\left(\frac{k_1}{1 + k_2} - \sqrt{2}\right)^2 + \frac{1}{4} \int_0^\infty (x + 2\dot{x} + \ddot{x})^2 \, dt \qquad (2.10)$$

The values of the free parameters which minimize the MPI are presented, for two values of the scalar 'r', in Table 1 below:

TABLE 1

NUMERICAL RESULTS OF PROBLEM 3.3

r	x_1	x_2	$\tilde{\underline{a}}$	$\tilde{\underline{x}}'_o$
0.0025	1.59	0.48	1.08/ 1.40/ 1.0	-1.0/ 1.08/ -0.45
0.25	1.82	0.32	1.38/ 1.60/ 1.0	-1.0/ 1.38/ -0.86

Figures (1) and (2) reproduce the solutions for these two values of 'r', respectively.

Some comments

   a) The reader probably has noted that Rediess included in (2.10) only the initial value of the first order state. This was done based on the fact that W was a square matrix of order '1' (2 in this case). Apparently Rediess did not pay attention to the fact that we are dealing with pseudo initial conditions and the second order states must be accounted for. In other

words, W should be a 1+1 matrix. In this particular example,
the second order states are different.

b) We are going to use, as an ultimate measure of quality
of matching, the instantaneous distance between trajectories,
here referred as d(t). Table 2 below summarizes these results,
for t=0.

TABLE 2

States Model System(r=.0025) d(t)comp. System(r=.25) d(t)comp.

States	Model	System(r=.0025)	d(t)comp.	System(r=.25)	d(t)comp.
x_0	-1.0	-1.0	0	-1.0	0
$\dot{x}_0$	1.41	1.08	0.33	1.38	0.03
$\ddot{x}_0$	-1.0	-.45	-0.55	-0.86	-0.14

Notice that d(0) is 0.6414 and 0.1432 for 'r' 0.0025 and 0.25,
respectively. These results confirm, now based on the new tri-
dimensional figure of merit, that 'r' = 0.25 gives the best
result, at least for the set of values under scrutiny.

c) If we observe the behavior of the integral term in
(2.10) for t=0, we would see that this term is in fact intro-
ducing an excessive weight for the PIC values in MPI calcula-
tions. Being more specific, the 0th order state which, by any
logical reasoning, ought not contribute to the index (at t=0),
is being fully charged (we recall that, at t=0, x_0 and x_{mo} are
equal). And more, the first order state receives a weight of
1.4142 which in terms of our figure of merit is distorting the
picture of d(0).

These facts suggest that the scalar 'r' is effectively

releasing the weight unduly introduced through the integral term. This conclusion is important and, as will be shown during the present work, has its counterpart in the 'lower limit' mechanism introduced by Ho in Palsson's formulation.

In order to summarize this presentation, we can say that Rediess' method works pretty well in the case in which there are no zeroes in the transfer functions. When these zeroes show up, the expression (2.9) of MPI can be applied still with success, but the value of the scalar 'r' is selected by trial and error, which is a drawback.

2.2 Palsson's Formulation

In 1971, Thorgeir Palsson (ref. 8) introduced his concept of system's error equation. Through the representation of the transient response by the matrix equations below

$$\dot{\underline{x}} = \underline{A}\underline{x}$$
$$\underline{x}(0) = \underline{x}_o \tag{2.11}$$

and the desired response by the equations

$$\dot{\hat{\underline{x}}} = \hat{\underline{A}}\,\hat{\underline{x}}$$
$$\hat{\underline{x}}(0) = \hat{\underline{x}}_o \tag{2.12}$$

Palsson defined the error equation in the following form

$$\Delta\dot{\underline{x}} = \hat{\underline{A}}\,\Delta\underline{x} + \Delta\underline{A}\underline{x}$$
$$\Delta\underline{x}(0) = \Delta\underline{x}_o \tag{2.13}$$

where: $\Delta\underline{x} = \underline{x} - \hat{\underline{x}}$

$\Delta\underline{x}_o = \underline{x}_o - \hat{\underline{x}}_o$

$\Delta A = \underline{A} - \hat{\underline{A}}$

This last equation is valid only for the case where system and model are of same order.

We observe that the homogeneous part of equation (2.13) is identical to the model's equation, with the state of the model replaced by the state of the error. The particular term is expressed only in terms of the states of the system and, being so, the response of the error might very well be imagined as shown in Fig. (3).

Even considering that, in practical situations, the model is of lower order than the system, no comments will be made here on the modifications necessary to take care of the case. Such details would lengthen without reason this work and, when needed, the very symbology will alert the reader for the proper case. We suggest here, to those interested, ref. (2).

The error state equation allows a very simple and objective interpretation of the Model Performance Index, as we will see in the sequel.

In the case of system without zeroes in the transfer function, we can see from expression (2.6) that all the states, except the 0th order, are null for t=0, since in this case m=0. Moreover, this state is equal to the system static sensitivity with a minus signal applied. Therefore, if we assume that system and model have the same value for static sensitivity, the initial state of the error will be zero. We recall that this assumption is reasonable since in most practical situations, the steady state output error due to a unit step

will be zero by a project requirement.

The error equation, written in detailed form, is

$$
\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \\ \vdots \\ \Delta \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_0 & -\alpha_1 & & \cdots & & -\alpha_{n-1} \end{bmatrix} + \begin{bmatrix} \Delta x_1 & 0 & \cdots & 0 \\ \Delta x_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ (\Delta x_n) & (\alpha_0 - a_0) & \cdots & (\alpha_{n-1} - a_{n-1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

$$(2.14)$$

where: $\underline{a}^T = (a_0, a_1, \ldots a_{n-1})$

$\underline{\alpha}^T = (\alpha_0, \alpha_1, \ldots \alpha_{n-1})$

are the coefficients of the characteristic equations of system and model respectively.

There are two potential sources of excitation to the error, as can be seen in Fig. (3). First, any initial error will result in an error response. For the case in which the model and system do not have zeroes in the tranfer functions, this effect does not need to be considered, since the initial state of the error is zero as already seen. The second source is represented by the scalar

$$
i(t) = \sum_{i=0}^{n-1} (\alpha_i - a_i) x_{i+1} = (\underline{\alpha} - \underline{a})^T \underline{x}
$$

As the matrix of the model, A, is a constant, the response of the error can only be influenced if we change the sources of excitation of the error. It is clear, for instance, that the response of the error is zero for all values of t when i(t) is

identically null, since there is no other perturbation to the error equation in this case. Therefore, one obvious manner to reduce the error between the response of the system and the desired response would be a minimization of some measure of the excitation to the error equation which is a scalar when the equations are written in phase variable form.

One of these measures is

$$J = \int_0^\infty \underline{x}^T (\underline{\alpha} - \underline{a})(\underline{\alpha} - \underline{a})^T \underline{x} \, dt \qquad (2.15)$$

It should be noticed that the Index is still a function of the free parameters, since $\underline{a} = \underline{a}(p)$.

In the case in which $l < n$, we can define $\underline{\tilde{\alpha}}^T = (\alpha_o, \alpha_1, \alpha_2, \ldots \alpha_{l-1}, 1, 0, \ldots 0)$ in such a form that the Index will take the form below:

$$J = \int_0^\infty \underline{x}^T \underline{\tilde{\alpha}} \, \underline{\tilde{\alpha}}^T \underline{x} \, dt \qquad (2.16)$$

For the case in which zeroes appear in the transfer functions, Palsson suggested an approach that would eliminate the need for including the error initial states in the performance index. He had previously proved the (in the simpler case) validity of the expression below.

$$i(s) = (\frac{G(s)}{\hat{G}(s)} - 1) \, u(s) \qquad (2.17)$$

Writing this expression explicitly for the case in which we have zeroes

$$i(s) = (\frac{s^{\ell} + \alpha_{\ell-1} s^{\ell-1} \cdots \alpha_1 s + \alpha_0}{\beta_k s^k + \cdots \beta_1 s + \beta_0} \times \frac{b_m s^m + \cdots b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} - 1)u(s) \quad (2.18)$$

If we define a new system whose transfer function contains the zeroes of the model as poles of the system (which means added to the regular poles of the system) in such a way that

$$\tilde{G}(s) = \frac{\beta_0 (b_m s^m + b_1 s + b_0)}{\beta_k s^k + \cdots \beta_1 s + \beta_0)(s^n + a_{n+1} s^{n-1} + \cdots + a_0)} \quad (2.19)$$

then it would be possible to write expression (2.18) in the same form we had for expression (2.17). Therefore, Palsson shows that it was possible to reach the same performance index as in expression (2.15), without having to add any term representing the initial states, as Rediess did.

To summarize, Palsson's method works pretty well in the case with no zeroes. In the other case, however, if the difference in the initial states was big enough, his mechanism did not work as he thought. It may, at most, be considered satisfactory if and only if there is no emphasis on the behavior around t=0. In other words, since we accept as unavoidable a certain difference during the initial portion of the transient response. In any way, as Palsson's formulation had some conceptual advantages and as its form really facilitates the computation of the performance index, the efforts then became focused in how to reach a weighting of the initial states similar to Rediess' method.

2.3 Ho's Formulation

In 1967, Chang Ho, after applying Palsson's method to a
series of examples in which, due to the particular arrangement
of zeroes of model and system, the error input excitation func-
tion presented a strong initial spike (Fig. 4), which contri-
buted heavily to the performance index, suggested that this
undesired behavior could be screened out the calculations if
we began the integration in an instant larger than zero. This
method would certainly produce lower performance indexes and
it might very well allow the selection of a set of parameters
with better conditions of matching. The experience confirmed
this suggestion, but the process of choosing the lower limit
is still a trial and error one, as in Rediess' formulation.

To summarize the discussion up to this point, we can say
that the present difficulties in Model Performance Index Devel-
opment can be traced to the existence of zeroes in the transfer
functions. Both Rediess' method and Palsson's modified lower
limit method are arbitrary, a fact that renders the minimiza-
tion algorithm only partially useful. Both present limitations
as follows:

a) Model and system must have the same static sensitivity,
which is a reasonable hypothesis, as seen before.

b) The model should have an excess of poles over zeroes
lower or at most equal to that of system.

This last restriction was raised with the idea that, in
mathematical terms, there is no difference in which transfer

function is associated to the model and which is associated to the system.

The reader certainly has noticed the fact that we constantly used words as "good matching" when speaking about trajectory closeliness. And certainly has noticed also that no effort was made to quantify such statements. The search of a function which could be used as a yardstick of the quality of matching is the departure point of the present work.

CHAPTER THREE

THEORETICAL DEVELOPMENT

The most important difficulty affecting Ho's formulation is the process of selecting the lower limit, which is entirely arbitrary. Palsson's performance index has no usefulness now that the index decreases monotonically as we increment the lower limit. It can be verified that this course of action can even lead to trajectory matchings which can be considered very bad. Example 3 (ref. 11), was solved for different values of lower limit. Table 3 below presents a summary of these results.

TABLE 3

Lower limit values	MPI	Matching quality evaluation
0.00	1.39876	Bad
0.01	0.556044	Bad
0.02	0.181143	Good
0.03	0.048856	Bad
0.05	0.040300	Bad

NOTE: Figures 5 and 6 present displacement and first derivative for this example. From these figures one could say that lower limit 0.02 certainly is better than the others but can not say a word to answer the question "Is it the best?" nor to try a classification of quality.

In the examples 2a and 2b of ref. (11), Ho tried to show that displacement token by itself was not enough to judge the merits of two values of MPI and matching conditions associated

with it. Ho plotted the first and second derivatives in order
to have a better appreciation which, we must say, was very far
from being a clear cut one.

Extending these ideas, we could say that as we have the
trajectories in a three-dimensional extended state space (in
this example), it would be interesting to see how these tra-
jectories match one another directly in the extended space,
instead of each state at a time.

Following this line of thought, it was proposed a plot-
ting of the trajectories using a tri-dimensional plotter in
the IPC (Calcomp   ), which is valid for any second order
model and particularly suited for the analysis of Ho's example
three. Some of the principal points will be aborded here in
a brief manner.

Basically, a cataloged procedure allows the plotting (in
Calcomp Plotter) of any bi-dimensional array, interpreting the
value of each element of the array as the "Z" component and
the subscripts as the components "X" and "Y." This procedure
is particularly suited for plotting surfaces, the functional
relationships of which could be considered as well behaved in
the interval of definition of "X" and "Y." The plotter is an
off-line device, which means that we cannot control the para-
meters of the plotting subroutine during the picture producing
process.

In our specific case, in which "X," "Y" and "Z" corres-
pond to $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ respectively, due to the sinusoidal

behavior of these functions, we have some additional complica-
tions, as explained below:

a) as "x" and "y" are not monotonically increasing or de-
creasing functions of time, the program must provide a way of
transforming these coordinates into an array. In other words,
the program must sort them to be monotonically increasing or
decreasing functions which can then be used by the plotter.

b) as "x" and "y" are interpreted as subscripts of an
array, they must be converted into positive, integer quantities
before they can be used by the plotter. This means a certain
distortion in the picture which, in this case, did not change
the qualitative result.

c) the plotting had to be done using the "histogram mode,"
since the "point to point mode" (an ideal option in plottings)
would blur the picture. This means that we are going to see
cylindric surfaces instead of curves in the extended state
space, making the task of interpretation somewhat difficult.
This difficulty has played an important role in the example
under scrutiny, and we had to plot each curve at a time, super-
imposing the pictures afterwards in order to evaluate the
results.

To summarize, in terms of usefulness as a project tool,
this approach is limited to second order models, has low effi-
ciency since the program must be adjusted to each case (we
recall that some of the parameters in the plotting subroutine
must be closely controlled), and the program is expensive even

number of points.

Now considering the results for this example, we can see from Fig. 7, 8, 9, 10, which represent the trajectories for values of lower limit equal to 0.00, 0.01, 0.02 and 0.03 that LL = 0.02, represents the better matching condition, at least for the set of values under scrutiny (we recall that this approach does not provide a method for finding "the best" curve). It is interesting to note that for LL = 0.00 the initial distance between trajactories is very small, while the whole system curve is very far from the model's one. If we observe the behavior for other values of LL, we can see that the initial distance is increasing as we increase LL and, at the same time, we obtain trajectories which are much closer on the whole. This fact is very important because it shows that the lower limit mechanism is different from Rediess' weighting term (we recall that in Rediess' formulation we made the initial distance close in order to get better matching condition).

Due to the difficulties found in the previous approach, we began to consider an alternate solution which consisted in a direct measurement of the distance between trajectories and plotting against time (two dimensions). This was done for the same set of lower limit values and the resultant curves are presented in Fig. 11.

There are some interesting points in this figure which deserve a comment:

a) As we increase the value of the lower limit, in all

curves (except the one for LL = 0.01), the initial distance d(0) assume monotonically increasing values. This confirms that the initial states are not being made closer through the lower limit mechanism (in the form that the initial quadratic term of Rediess used to work); this unexpected result led us to the conclusion that the lower limit method has no correlation with Rediess' formulation.

b) As we increase the values of lower limit, we observe that all the curves cut the one corresponding to LL = 0.00 in an instant of time t = t which is each time lower. In other words, it is possible to develop a qualitative relationship as a function of lower limit in the following form:

$$\tau(LL) \leq \tau(LL + 1)$$

where LL is a general value for the lower limit. Besides, it can be seen that the curve corresponding to (LL+1) is kept essentially below the one corresponding to (LL), for all $t > \tau$.

c) Once more we can see that there exists a trade-off between the behaviors in low and high frequency. This fact was indicated by the tri-dimensional plotting, as we have seen before. In other words, as we increase the value of lower limit, we observe that the state of the initial error becomes bigger and bigger, and, at the same time, a better matching condition becomes evident in the low frequency range. This trade-off suggests that some generalized measure of d(t) may go through a minimum while we change the value of the lower limit.

d) The curves corresponding to LL = 0.00 and LL = 0.01 present a very curious behavior in their initial portion -- they grow for a certain time, going even beyond d(0). This fact suggests that Palsson's formulation would have been applying an excessive weight to the pseudo initial conditions (we recall that we reached a similar conclusion when we made the analysis of the integral term of Rediess). Then, the method of lower limit would be releasing this undesired situation and allowing the minimization algorithm to select a better matching condition.

e) It is not possible to tell, by inspection, that a particular curve represents a better matching condition than any other curve. That is to say, even having a function as d(t), which was selected in rather logical grounds, as the ultimate measure of matching quality, it is necessary to find some other generalized function of d(t) or some other function which can describe more effectively the quality of matching.

Following the line exposed in comments c) and e) above, we decided to try the area under the curve d(t) as a generalized measure of matching quality. We calculated this function for the curves corresponding to LL = 0.00, 0.01, 0.02, 0.03 and 0.05. The results are presented in Table 4 below. As can be seen, the area in fact goes through a minimum and this minimum corresponds to LL = 0.02, a condition better than any of the values under scrutiny, as we have seen before.

TABLE 4

Area under curve d(t) for some values of lower limit

Lower limit	Area
0.00	3.331800
0.01	3.156600
0.02	1.058634
0.03	1.212800
0.05	2.078300

It should be noticed that we have only a small sample of the universe of possible values, a fact which keeps us from using the term "best possible condition," at least until now. It should be stated in this point that the function area under d(t) in fact should be submitted to a formal minimization algorithm. This task, however, can't be made within the time allocated to this work.

In any way, considering the actual matching condition for LL = 0.02, particularly in terms of the 2nd order state ("Z" coordinate in Fig. 9), it seems that we have still some margin for improvement. In order to explore this situation, we have chosen an inverted way, which consists in plotting A(t) against LL (supposing that a nice functional relationship really exist between them) (Fig. 12), guessing the LL value which corresponds to the lower value of A(t) and the final use of the present algorithm to verify the true value of A(t). These results are presented below.

TABLE 5

Verification of the behavior of A(t)
for LL = 0.0225

Lower limit	Area under d(t)
0.00	3.331800
0.01	3.156600
0.02	1.058634
0.0225	2.3890
0.03	1.212800
0.05	2.078300

This unexpected result deserves some comments:

a) It is clear that our previous assumption on the be-havior of the pair A(t)-LL is wrong, that is to say, this functional relationship is far from being smooth, a fact which render our forecast useless.

b) It is very interesting to note that a small variation in LL had brought such a big departure from optimality. For comparison, we present below the values of the free parameters for LL = 0.02 and 0.0225:

	LL = 0.02	LL = 0.0225
Parameter 1	0.39067	0.51603
Parameter 2	0.79100	0.83061

At this point, as the parameters for LL = 0.225 were obtained using a different minimization algorithm, some extra runs were tried which have indicated 0.49699 and 0.76628 as the new parameters for LL = 0.02. These values are far from the previous ones but MPI(s) are quite the same. Within

the new situation, A(t) would then select the set correspond-
ing to LL = 0.03. This seems to show that there is no specific
correlation between A(t) and LL. However, it does not render
the new function invalid -- it only stresses the necessity for
submitting A(t) to a minimization algorithm.

In another effort to check the capabilities of the new
function, we did a follow-up of an actual computer run (which
still uses Palsson's formulation through a minimization algo-
rithm), calculating A(t) for each set of parameters. The
results are presented below:

TABLE 6

Computer Follow-up

Parameter 1	Parameter 2	MPI value	Area under d(t)
0.44056	0.86849	0.193162	2.1395
0.43596	0.83349	0.191412	2.1201
0.42387	0.83579	0.187253	2.0801
0.40862	0.81246	0.183488	2.0475
0.39384	0.79423	0.181372	1.0768
0.39067	0.79100	0.181143	1.0586

These results show that A(t) can in fact be used for
selecting directly the best set of free parameters, which in
turn means that the new function A(t) can be used to define a
new form of Model Performance Index.

CHAPTER FOUR

CONCLUSION

As a conclusion, it is perfectly safe to say that we developed a new performance index based in a generalized measure of the distance between the trajectories of a particular system and a chosen model, in the form below:

$$MPI = \int_O^\infty \sqrt{(\underline{x}(t) - \underline{x}_{mo}(t))^2} \, dt \qquad (3.1)$$

In the case in which l<n, or the model is of an order lower than that of the system, one must project the extended state space of the system over the one of the model, and expression (3.1) is still applicable.

The new formulation has some advantages, if compared to the previous work, as summarized below:

a) Eliminate, as seen, the necessity to use a trial and error method in order to include the effect of the states of the initial error which was a major drawback in the previous formulations.

b) As there is no difference, in mathematical terms, between model and system, we have raised the restriction about the excess of poles over zeroes, which was common to the other formulations. The artifice used until now to take care of the case in which model had a greater poles over zeroes excess consisted in an inversion of the roles played by the transfer

functions of model and system. When the system possessed right half plane zeroes, a case which is very common in lateral control systems of airplanes, this idea brought a divergent performance index (since i(p) would have right half plane poles) and, therefore, useless.

Figure 1 Geometrical Representation for Example 3.3

(a) r = 0.0025

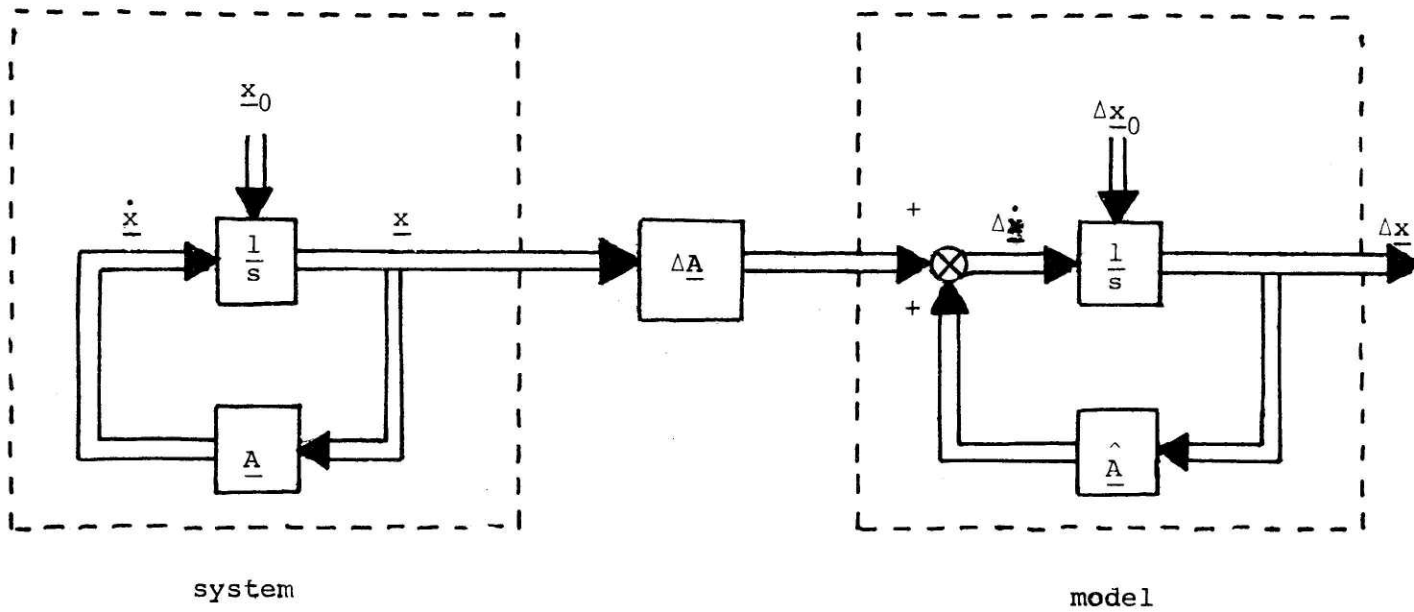$\ddot{\mathbf{x}}$

$\mathbf{x}$

$\dot{\mathbf{x}}$

$\ddot{\underline{x}}$

$\underline{x}$

$\dot{\underline{x}}$

$\bar{\underline{x}}_m(t)$

$\bar{\underline{x}}(t)$

$\bar{\underline{x}}_0$

$\bar{\underline{x}}_{m_0}$

$\underline{\alpha}$-plane

$\bar{\underline{a}}$-plane

37.

$\ddot{x}$

x

$\dot{x}$

$\underline{\bar{x}}_m(t)$

$\underline{\bar{x}}(t)$

$\underline{\bar{x}}_0$

$\underline{\alpha}$-plane

$\underline{\bar{a}}$-plane

$\underline{\bar{x}}_{m_0}$
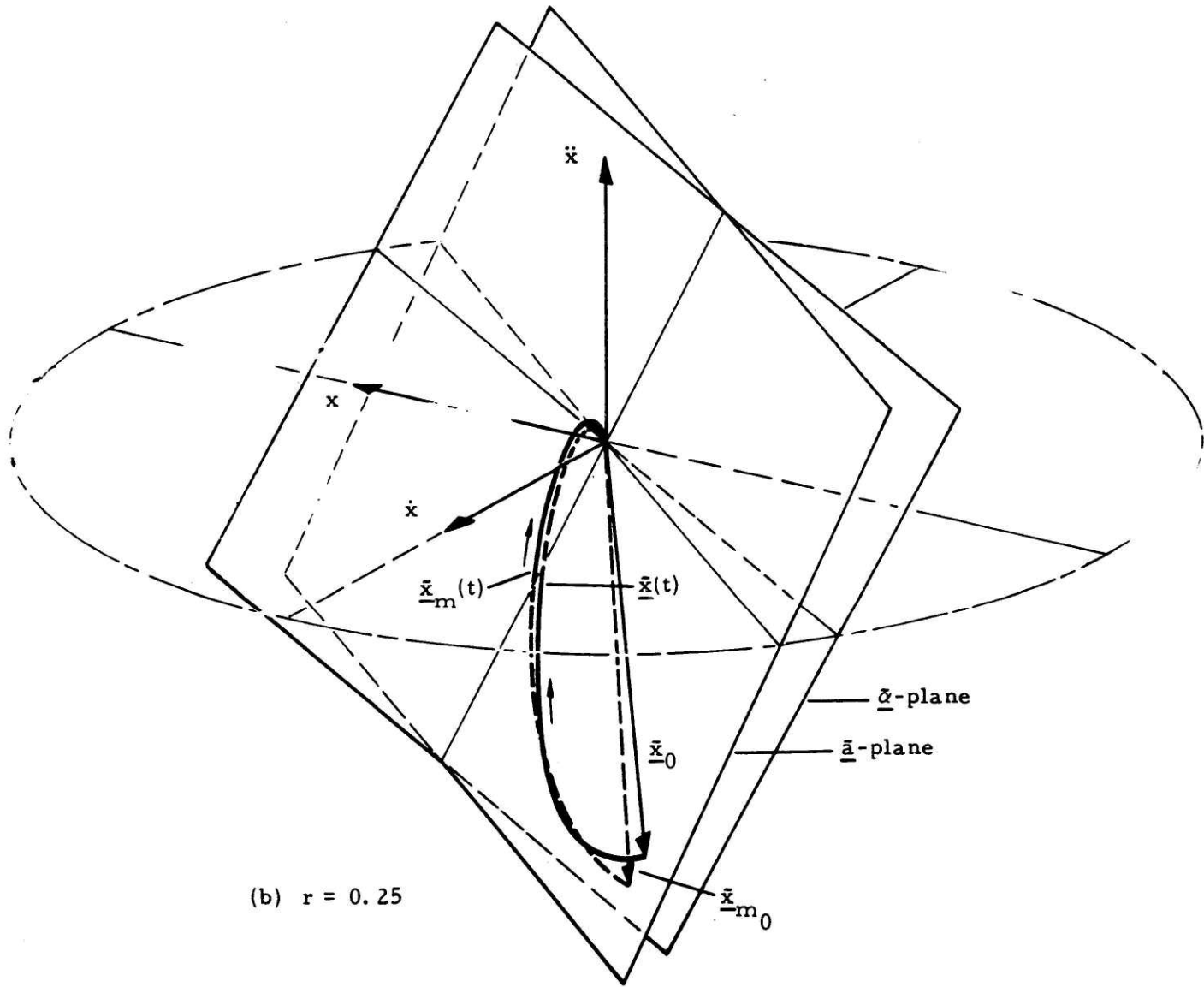
(b) r = 0.25

Figure 2 Geometrical Representation for Example 3.3 Concluded

Figure 3   System Error Response

Fig. 4   Error Input Excitation Function

K·E 10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

46 1320

Figure 5   Step Response

K·E 10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

46 1320
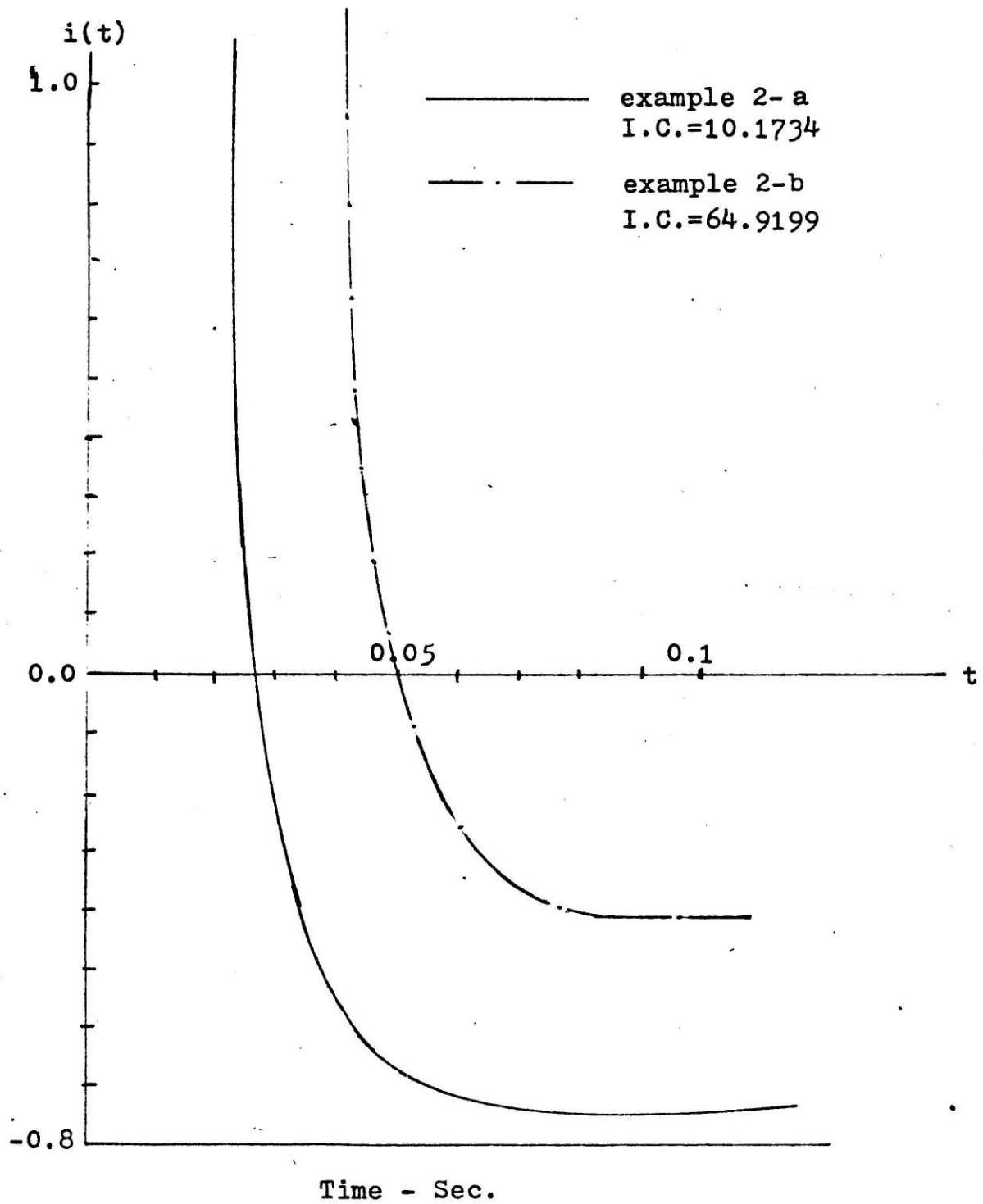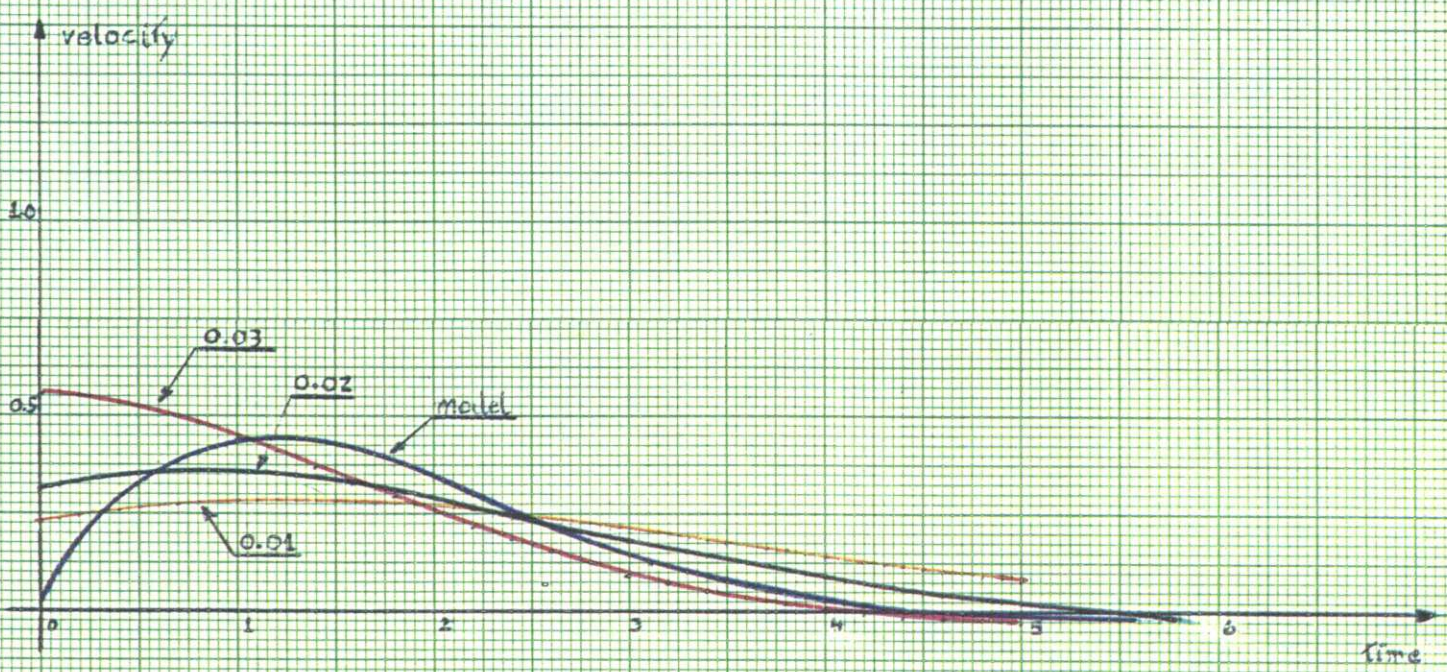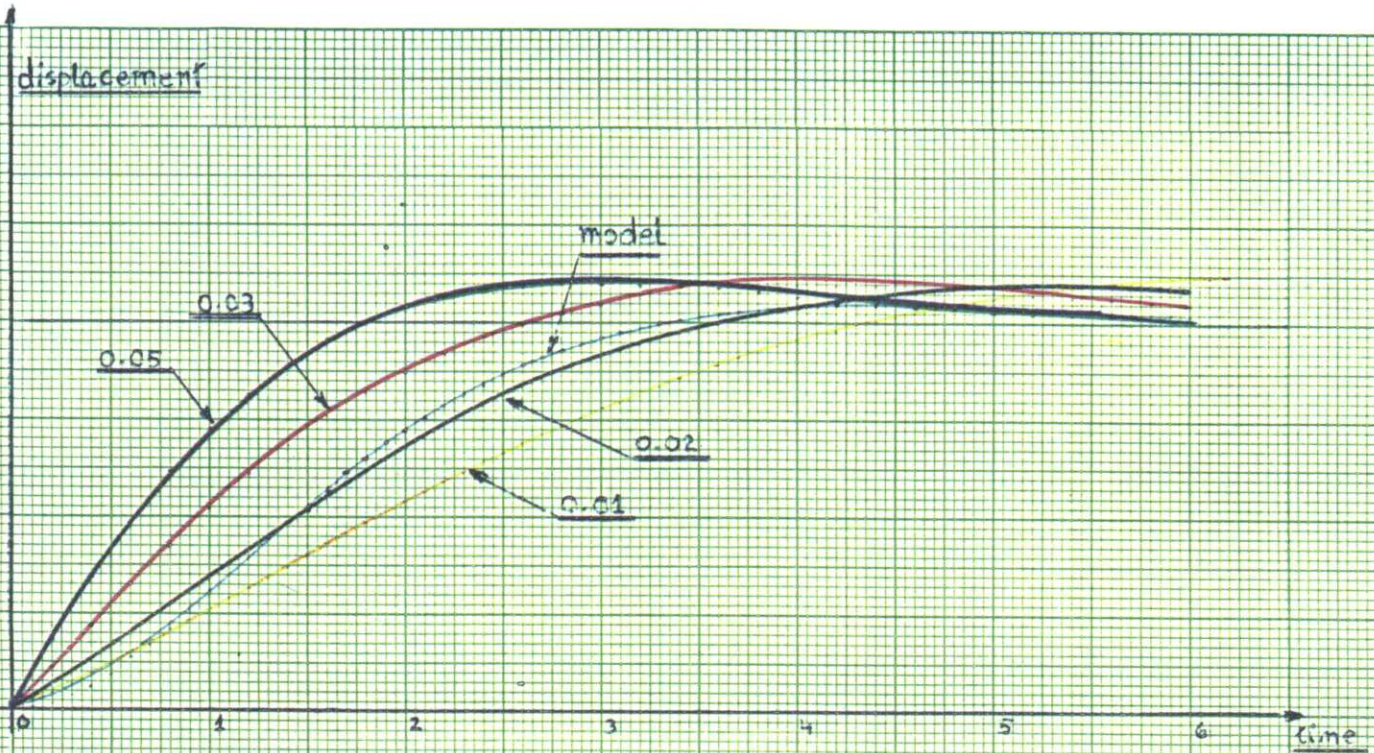
Figure 6   Velocity Response

HO 3 SYSTEMO.OO

VIEWING ANGLES: ·THETA = 60.00   PHI = 315.00

COORDINATE RANGES: X = 1   TO 120   BY 2
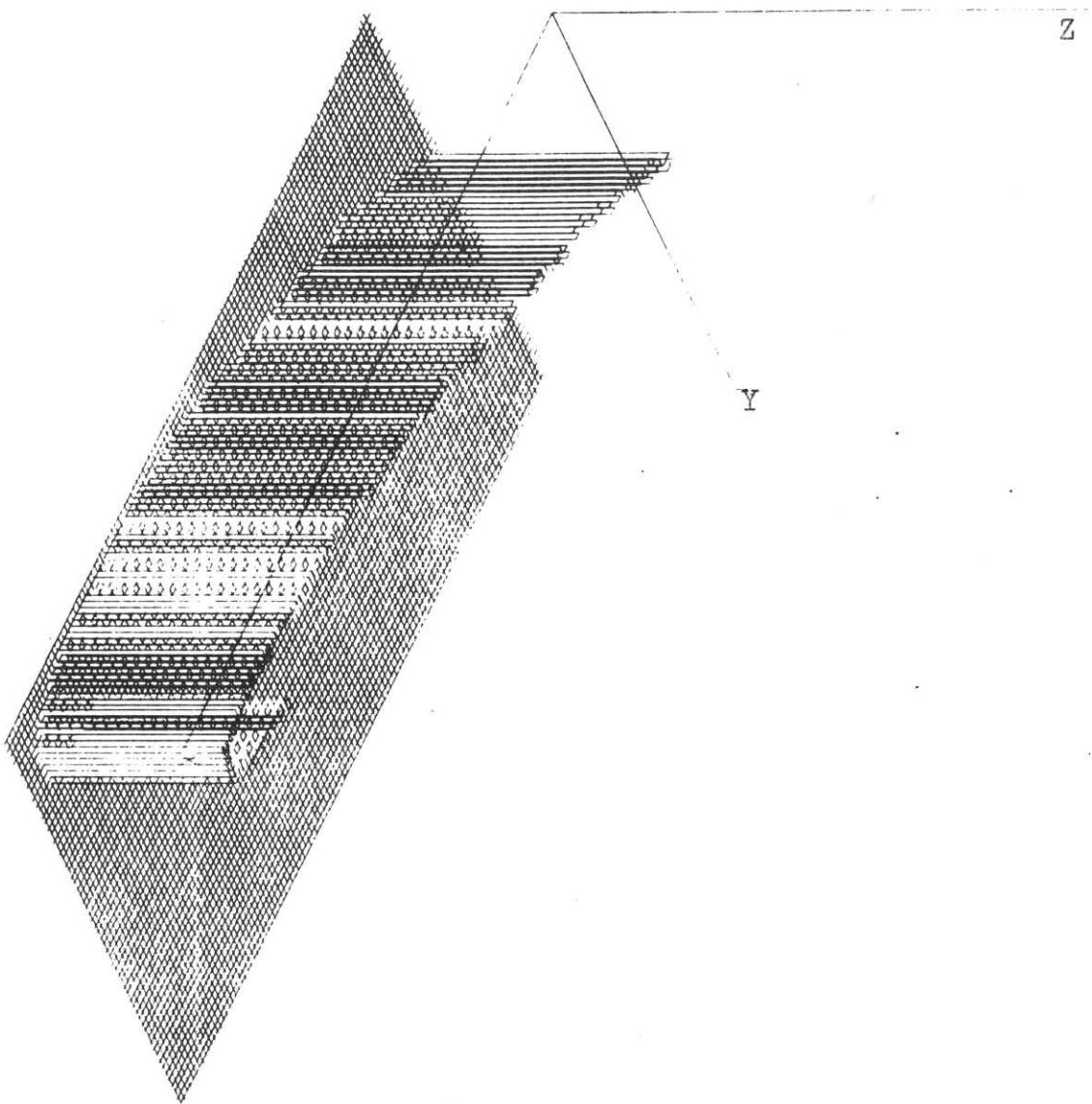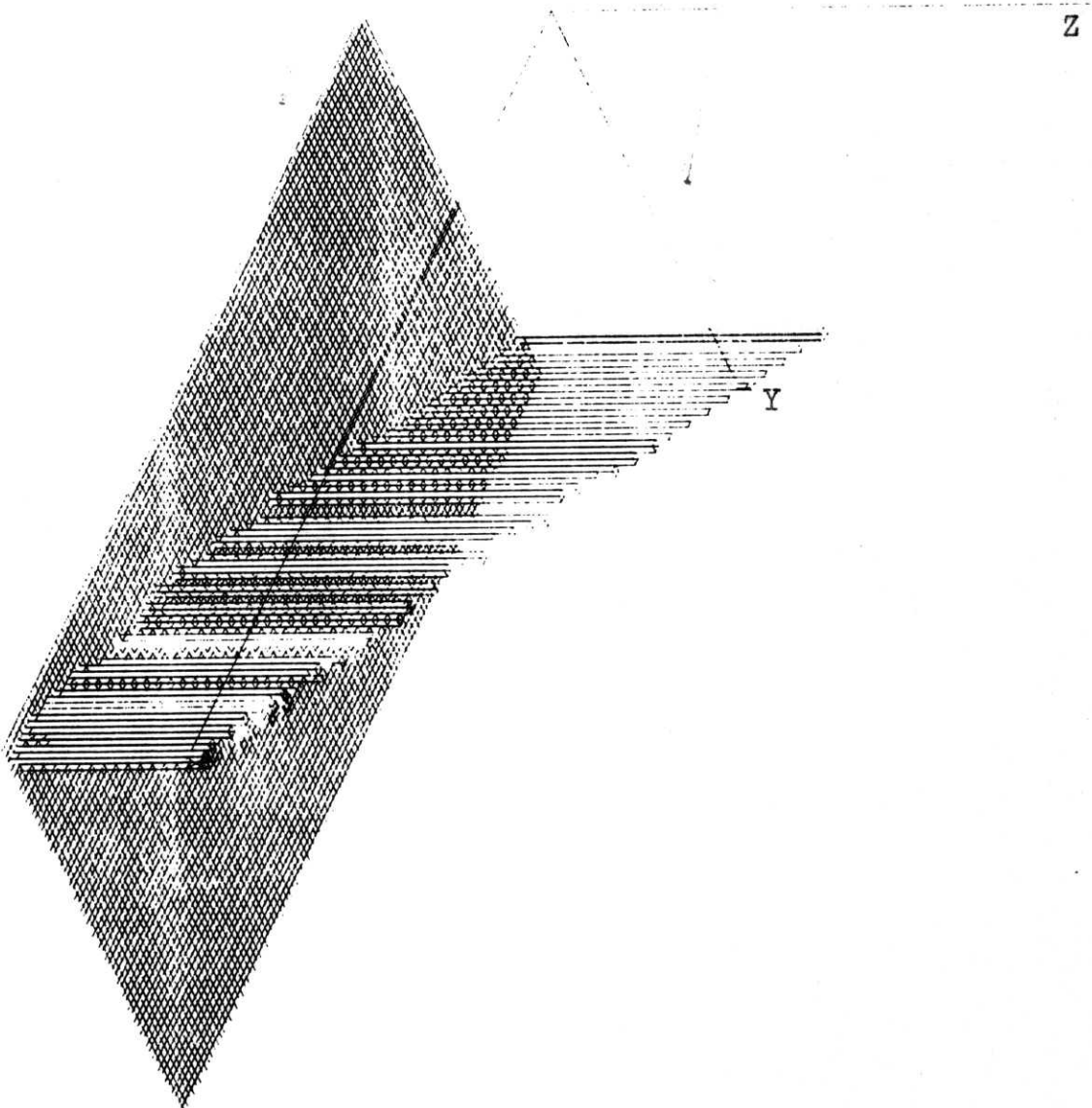
Y = 1   TO 60   BY 2

Z = -30   TO 100

Figure 7   HO-3 System 0.00

HO 3 SYSTEMO.01
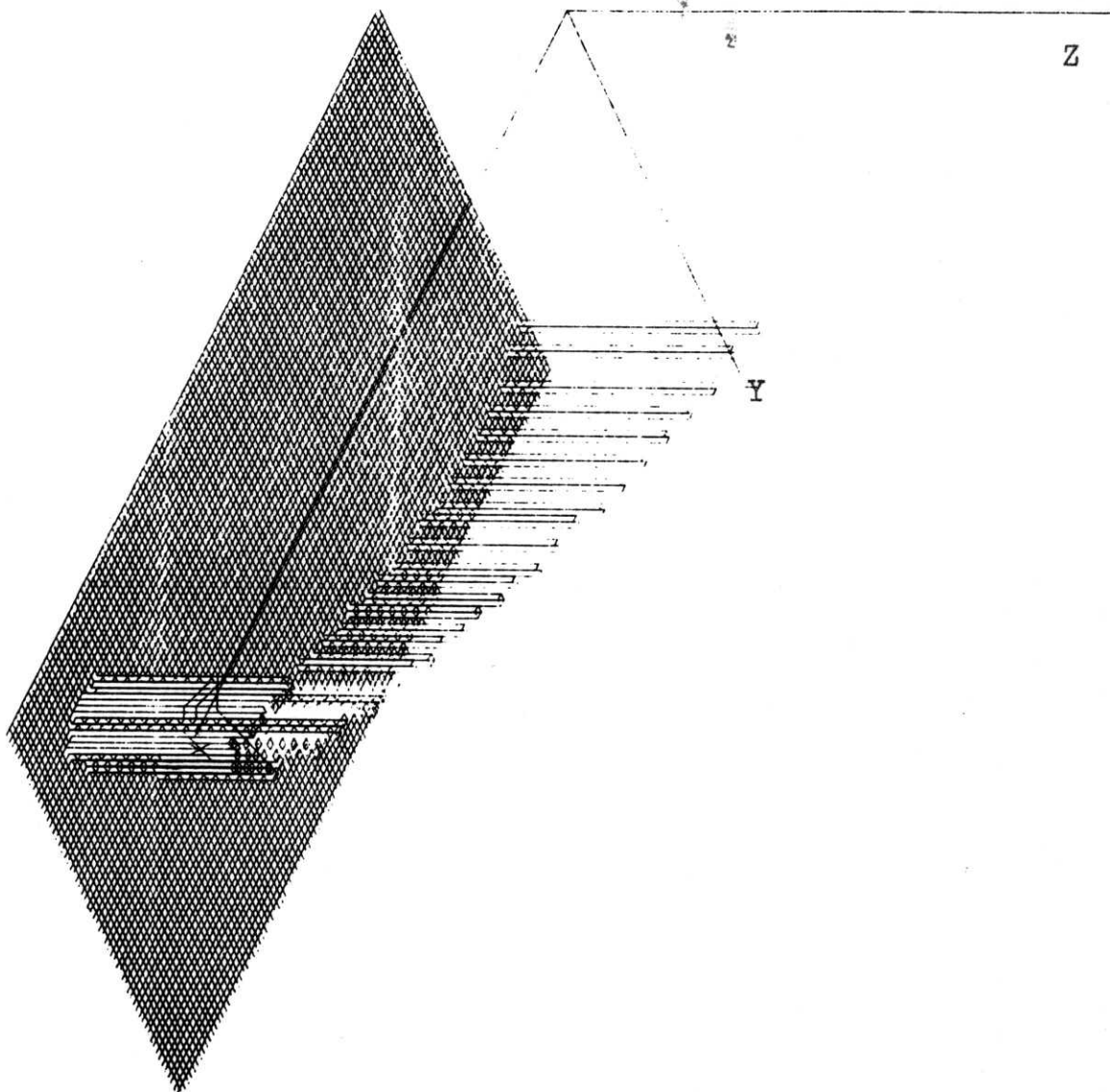
VIEWING ANGLES: THETA = 60.00   PHI = 315.00
COORDINATE RANGES: X = 1   TO 120   BY 2
                            Y = 1   TO 60   BY 2
                            Z = 30   TO 100

Z

Y

Figure 8   HO-3 System 0.01

HO 3 SYSTEMO.02

VIEWING ANGLES: THETA = 60.00   PHI = 315.00
COORDINATE RANGES: X = 1   TO 120   BY 2
                     Y = 1   TO 60   BY 2
                     Z = -30   TO 100

Figure 9   HO-3 System 0.02

HO 3 MODEL

VIEWING ANGLES: THETA = 60.00 PHI = 315.00
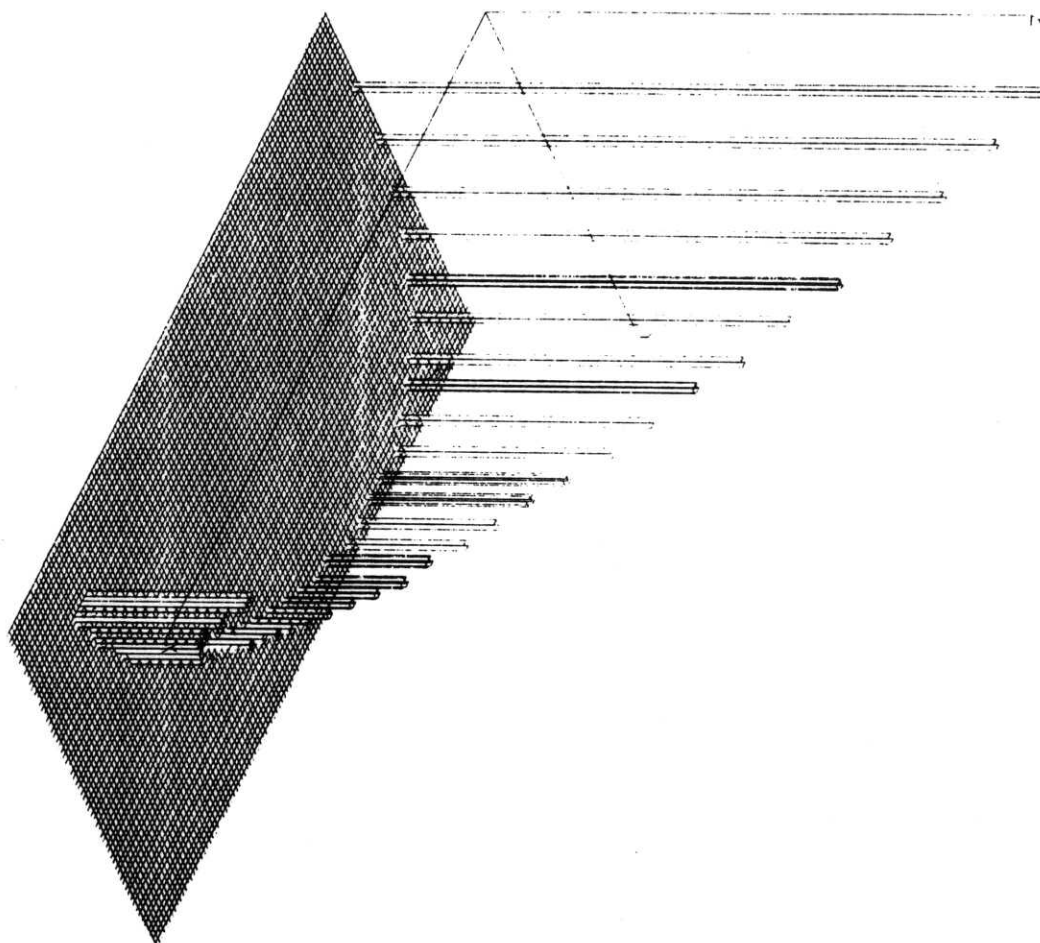COORDINATE RANGES: X = 1 TO 120 BY 2
Y = 1 TO 60 BY 2
Z = -30 TO 100

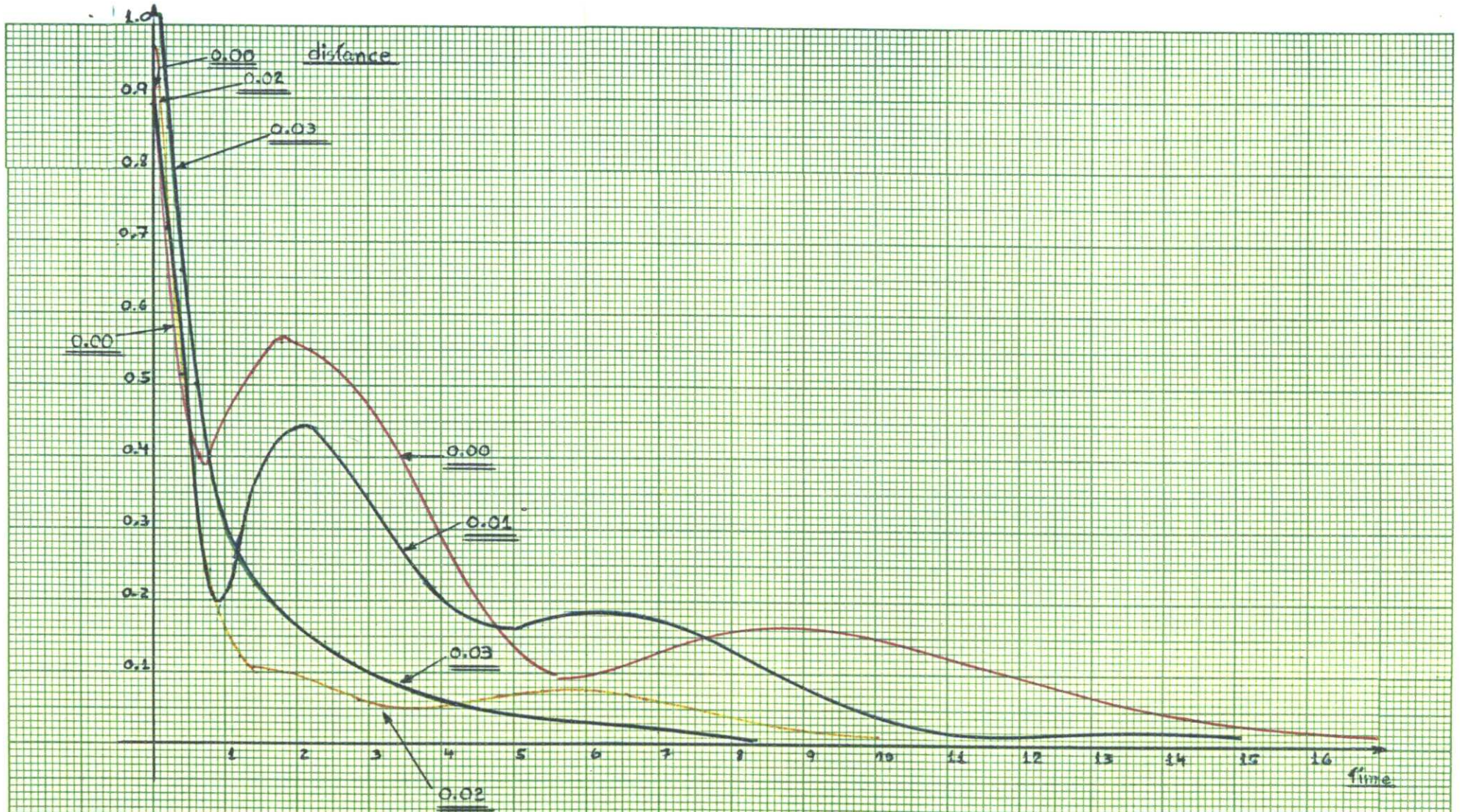**Figure 10 HO-3 Model**

K•E 10 X 10 TO ½ INCH   7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

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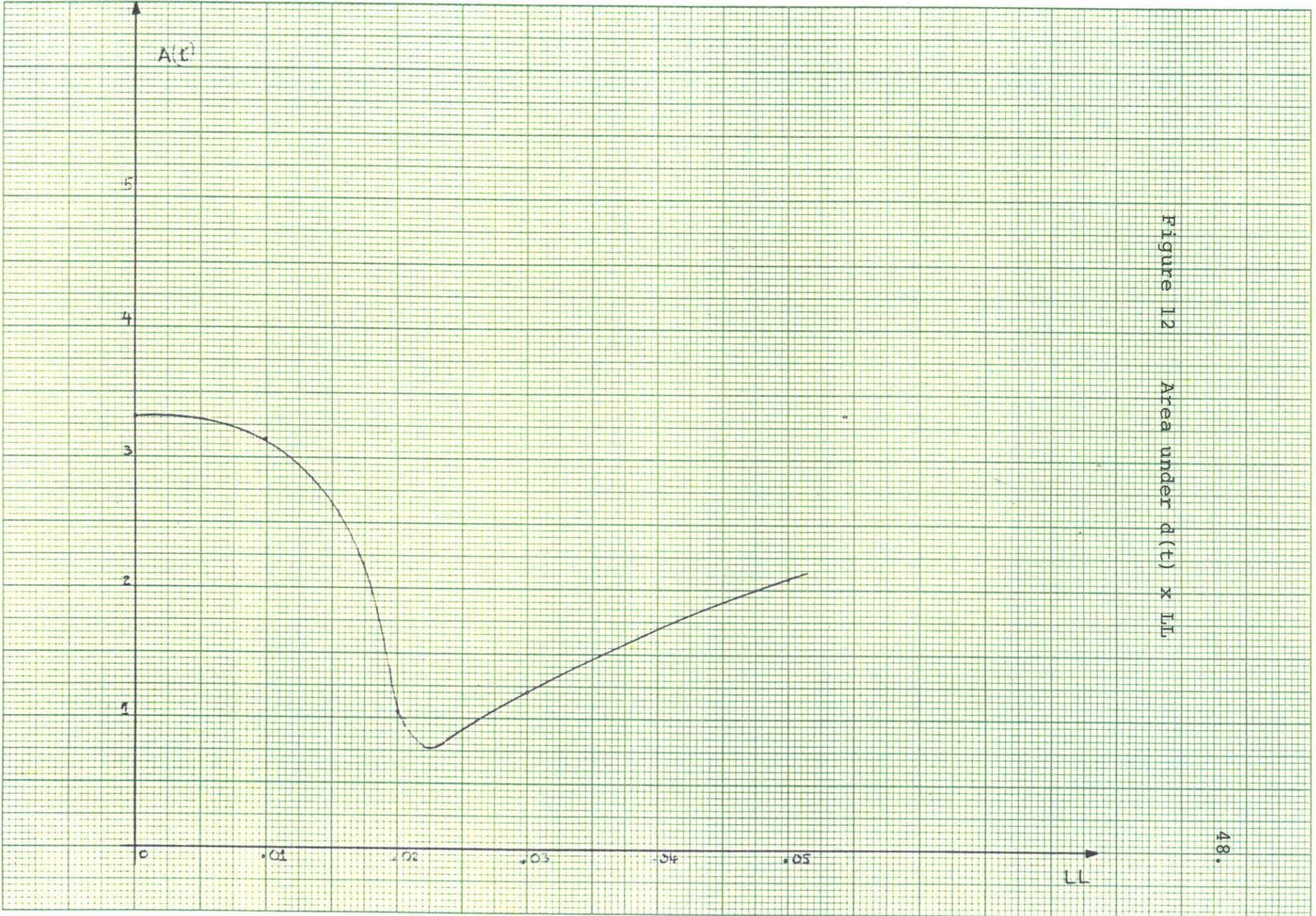Figure 11   Instantaneous Distance Between Trajectories for Some LL

Figure 12   Area under d(t) x LL

48.

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