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Continuous Time Stopping Games with Monotone Reward Structures

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To appear in "Mathematics of Operations Research"

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ABSTRACT

We prove the existence of a Nash equilibrium of a class of continuous time stopping games when certain monotonicity conditions are satisfied.

*This is an extensively revised version of the first half of a paper entitled "Continuous Time Stopping Games" by the authors. They would like to thank conversions with Michael Harrison and Andreu Mas Colell. Three anonymous referees and seminar participants at Princeton University and Yale University provided useful comments. Remaining errors are of course the authors'.

1 Introduction

The game theoretic extension of the optimal stopping theory in the discrete time framework was initiated by Dynkin [1969] in an analysis of a class of two-person zero-sum stopping games. The extensions of Dynkin's work include Chaput [1974], Elbakidze [1976], Neveu [1975], and Ohtsubo [1986] in discrete time and Bensoussan and Friedman [1974], Bismut [1977, 1979], Chaput [1974], Kiefer [1971], Krylov [1971], Lepeltier and Maingueneau [1984], and Stettner [1982] in continuous time.

Non-zero-sum stopping games have been studied by Mamer [1986] and Ohtsubo [1986] in discrete time and by Bensoussan and Friedman [1977], Morimoto [1986], Nagai [1987], and Nakoulima [1981] in continuous time.

The widespread and potential use of stopping games can be found in economics, finance, and management science. Examples include the entry and exit decisions of firms, job search, optimal investment in research and development, and the technology transition in industries. (See Fine and Li [1986], Mamer and McCardle [1987], Reinganum [1982] etc.) In fact, many stochastic dynamic games in which each player's strategy is a single dichotomous decision at each time can be formulated as a stopping time problem.

The existing literature on continuous time non-zero-sum stopping games mentioned above, with the exception of Morimoto [1986], uses stochastic environments that have the Markov property. Morimoto [1986] considers cyclic stopping games. The purpose of this paper is to provide an existence theorem for Nash equilibria for a class of non-zero-sum non-cyclic stopping games in a non-Markov environment. We basically extend the discrete time analysis of Mamer [1987] to a continuous time setting. Some properties of a symmetric Nash equilibrium are also characterized.

The rest of this paper is organized as follows. In Section 2 we formulate an N-person continuous time non-zero-sum stopping game. Reward processes are optional processes that may be unbounded and can take the value $-\infty$ at $t = +\infty$. A martingale approach is adopted in Section 3 to show the existence of optimal stopping policies of players under fairly general conditions. The existence of a Nash equilibrium in games with monotone payoff structures is proved in Section 4 by using Tarski's lattice theoretic fixed point theorem. We show in the same section that, for a symmetric stopping game, there always exists a symmetric equilibrium when the reward processes satisfy a monotone condition. Moreover, a symmetric equilibrium, when it exists, must be unique when reward processes satisfy another monotone condition and are separable in a sense to be made precise. We discuss two duopolistic exit games in Section 5

to demonstrate the two kinds of monotonicity conditions posited in Section 4. The reward processes of these two exit games also satisfy the separability condition mentioned above.

2 The formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with an increasing family of sub-sigma-field of $\mathcal{F}$, or a *filtration*, $\mathbf{F} = \{\mathcal{F}_t; t \in \Re_+\}$. We shall denote the smallest sigma-field containing $\{\mathcal{F}_t; t \in \Re_+\}$ by $\mathcal{F}_\infty$ and assume that $\mathbf{F}$ satisfies the *usual conditions*:

1. *right continuity*: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$; and

2. *complete*: $\mathcal{F}_0$ contains all the P-null sets.

We interpret each $\omega \in \Omega$ to be a complete description of the state of the world. The filtration $\mathbf{F}$ models the way information about the true state of the world is revealed over time. In a discrete time finite state setting, a filtration can be thought of as an event tree.

Let $\overline{\Re}_+$ denote the extended positive real line. An *optional time* T is a function from Ω into $\overline{\Re}_+$ such that

$$\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t \qquad \forall t \in \Re_+,$$

where $\Re_+$ denotes the positive real line. An optional time T is *finite* if $P\{T < \infty\} = 1$. An optional time T is said to be bounded if there exists a constant $K \in \Re_+$ such that $P\{T \leq K\} = 1$.

Let T be an optional time. The sigma-field $\mathcal{F}_T$, the collection of events *prior to* T, consists of all events $A \in \mathcal{F}_\infty$ such that

$$A \bigcap \{T \leq t\} \in \mathcal{F}_t \quad \forall t \in \Re_+.$$

A process X in this paper is a mapping $X : \Omega \times \overline{\Re}_+ \mapsto \overline{\Re}$ measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\overline{\Re}_+)$, the product sigma-field generated by $\mathcal{F}$ and the Borel sigma-field of $\overline{\Re}_+$. A process X is said to be *adapted*, if $X(t)$ is measurable with respect to $\mathcal{F}_t$ $\forall t \in \overline{\Re}_+$. The *optional sigma-field*, denoted by $\mathcal{O}$, is the sigma-field on $\Omega \times \overline{\Re}_+$ generated by adapted processes having right-continuous paths (see, e.g., Chung and Williams [1983]). A process is optional if it is measurable with respect to $\mathcal{O}$. Naturally, any adapted process with right-continuous paths is optional. It is also known that any optional process is adapted (see, e.g., Chung and Williams [1983]). For any process X, we write $X^+(t) \equiv \max[X(t), 0]$ and $X^-(t) \equiv \max[-X(t), 0]$. It is clear that $X(t) = X^+(t) - X^-(t)$. For an optional time T, we will use $X(T)$ to denote the random variable $X(\omega, T(\omega))$.

We consider an N player *stopping game*. Players are indexed by $i = 1, 2, \dots, N$. The payoffs of this game are described by a family of optional *reward processes*:

$$z_i(\omega, t; T_{-i}); t \in \Re_+, i = 1, 2, \dots, N,$$

where T_{-i} runs through all $(N-1)$-tuple of optional times. Interpret $z_i(\omega, t; T_{-i})$ to be the payoff that player i receives in state ω when his strategy is T_i and $T_i(\omega) = t$, if T_{-i} are the strategies employed by players other than i.

Note that the value of a reward process "at infinity" specifies the payoff to a player if he never stops. There are at least two possibilities depending on the nature of the game. First, if a player does not receive any reward when he never stops, we simply set $z_i(\omega, +\infty; T_{-i}) = 0$. Second, if the value at time t of reward process represents accumulated payoffs a player receives from time 0 to time t if he does not stop until time t, then a natural selection of the value at infinity is

$$z_i(\omega, +\infty; T_{-i}) = \limsup_{t \to +\infty} z_i(\omega, t; T_{-i}).$$

For an example of the second case, see Section 5. In the above two cases, it is easily verified that the reward processes as defined on $\overline{\Re}_+$ are optional and their values at infinity are measurable with respect to $\mathcal{F}_\infty$.

The following assumptions on reward processes will be made throughout this paper:

Assumption 1 *For any $(N-1)$-tuple optional times T_{-i} and any bounded optional time T_i we have $E[z_i(T_i; T_{-i})] > -\infty$. Moreover, there is a martingale*[1] *$m = \{m(t); t \in \overline{\Re}_+\}$ such that $m(\omega, t) \geq z_i(\omega, t; T_{-i})$ for all $(\omega, t) \in \Omega \times \overline{\Re}_+$.*

Assumption 2 *The reward process $z_i(\omega, t; T_{-i})$ is upper-semi-continuous on the right and quasi-upper-semi-continuous on the left.*[2]

Assumption 1 ensures that

$$-\infty < \sup_{T \in \mathbf{T}} E[z_i(T; T_{-i})] < +\infty$$

[1] The process $m = \{m(t); t \in \overline{\Re}_+\}$ is a martingale if it is adapted to $\{\mathcal{F}_t; t \in \overline{\Re}_+\}$, $E[|m(t)|] < \infty$ for all $t \in \overline{\Re}_+$, and $E[m(t)|\mathcal{F}_s] = m(s)$ *a.s.* for all $t \geq s$. Note that since m is defined on the extended positive real line, it is uniformly integrable. Thus $E[m(\tau)|\mathcal{F}_T] = m(T)$ *a.s.* for optional times $\tau \geq T$.

[2] An optional process $x = \{x(t); t \in \overline{\Re}_+\}$ is upper-semi-continuous from the right if for every optional time τ we have $\limsup_{n \to \infty} x(\tau + 1/n) \leq x(\tau)$ on the set $\{\tau < \infty\}$. Similarly, an optional process $x = \{x(t); t \in \overline{\Re}_+\}$ is said to be quasi-upper-semi-continuous on the left if for each optional time τ and any sequence of optional times $\{\tau_n; n = 1, 2, \dots,\}$ increasing to τ, $\limsup_{n \to \infty} x(\tau_n) \leq x(\tau)$.

for all T_{-i}, which we will formally state in Theorem 1, where $\mathbf{T}$ denotes the collection of all optional times. The usefulness of Assumption 2 will become clear later. Note that a reward process satisfying Assumption 1 can be unbounded from above and from below, and can take the value $-\infty$ at $t = +\infty$.

Given the strategy of his opponents T_{-i}, the objective of player i is to find an optional time T_i that solves the following program:

$$\sup_{T \in \mathbf{T}} E[z_i(T; T_{-i})]. \tag{1}$$

If such an optional time T_i exists, we will sometimes say T_i is a best response to T_{-i}.

A *Nash equilibrium* of the stopping game is an N–tuple of optional times $(T_i; i = 1, 2, \ldots, N)$ such that T_i solves (1) for $i = 1, 2, \ldots, N$.

3 Existence of best responses

In this section we will show that there always exists a solution to (1). This solution will also be said to be a "best response." Our analysis closely follows that of Thompson [1971]. In Thompson [1971], players are allowed to use only the finitely-valued optional times. Here, however, the admissible strategy space is composed of all the optional times. Our generalization of Thompson's analysis is quite straightforward. For completeness of this paper we shall provide some details of this generalization.

Note that Fakeev [1970] and Maingueneau [1976/77] also analyzed optimal stopping problems by allowing non-finite optional times in a non-Markov setting. Fakeev [1970] required that $E[\sup_t z_i^-(t; T_{-i})] < \infty$ and Maingueneau [1976/77] worked with reward processes that are of *class D*, that is, random variables $\{z_i(T; T_{-i}); T \in \mathbf{T}\}$ are uniformly integrable. Assumptions 1 and 2 admit processes that lie outside the framework of Fakeev [1970] and Maingueneau [1976/77].

We proceed first with some definitions.

Definition 1 *A regular supermartingale $\{X(t); t \in \overline{\Re}_+\}$ is an optional process so that for any bounded optional time T,*

$$E[X^-(T)] < \infty \tag{2}$$

and for all optional times $S \geq T$, $E[X(S)]$ is well-defined and

$$E[X(S)|\mathcal{F}_T] \leq X(T) \quad a.s. \tag{3}$$

Remark 1 *Our definition of a regular supermartingale is a generalization of Merten [1969] to include values at infinity. Mertens [1969, Theorem 1] has shown that almost all of the paths of a regular supermartingale have right and left limits and are upper semi-continuous from the right. The regular supermartingale defined in Definition 1 also has the same characteristics.*

An optional process Y is said to lie above the optional process W, denoted by $Y \geq W$, if $Y(\omega,t) \geq W(\omega,t)$ for all (ω,t) outside a subset of $\Omega \times \overline{\Re}_+$ whose projection on Ω is of P-measure zero.

A regular supermartingale Y is the *minimum regular supermartingale (MRS)* lying above an optional process W if $Y \geq W$ and if $X \geq W$ for any other regular supermartingale X, then $X \geq Y$.

The following proposition shows that, for any $(N-1)$-tuple of optional times T_{-i}, there exists a MRS lying above the reward process $\{z_i(t;T_{-i}); t \in \overline{\Re}_+\}$.

Proposition 1 *For any $(N-1)$-tuple of optional times T_{-i}, there exists a MRS, denoted by $\{Y_i(t;T_{-i}); t \in \overline{\Re}_+\}$, lying above the reward process $\{z_i(t;T_{-i}); t \in \overline{\Re}_+\}$. The MRS $Y_i(T_{-i})$ is right-continuous on the set $\{(\omega,t) \in \Omega \times \Re_+ : Y_i(\omega,t;T_{-i}) > z_i(\omega,t;T_{-i})\}$. Moreover, fix $\epsilon > 0$ and $t \in \Re_+$, and let*

$$\tau_i \equiv \inf\{s \geq t : z_i(s;T_{-i}) \geq Y_i(s;T_{-i}) - \epsilon\}.$$

Then $P\{\tau_i < \infty\} = 1$ and

$$E[Y_i(\tau_i;T_{-i})|\mathcal{F}_t] = Y_i(t;T_{-i}) \quad a.s.$$

PROOF Put

$$z_i^M(t;T_{-i}) \equiv \max[z_i(t;T_{-i}), -M] \quad \forall t \in \overline{\Re}_+.$$

The process $z_i^M(T_{-i})$ is bounded above by a martingale by Assumption 1 and is bounded below by M. Dellacherie and Meyer [1982, Appendix I, Theorem 22] shows that there exists a MRS $Y_i^M(T_{-i})$ lying above $z_i^M(T_{-i})$. Now define

$$Y_i(t;T_{-i}) \equiv \operatorname{ess\,inf}\{Y_i^M(t;T_{-i}); M = 1,2,\ldots\}$$

for all $t \in \overline{\Re}_+$.

We claim that $Y_i(T_{-i})$ is the MRS lying above $z_i(T_{-i})$. First, it is easy to see that $Y_i(T_{-i})$ is optional. Second, we show that $Y_i(T_{-i})$ is a regular supermartingale. Since $Y_i(T_{-i})$ lies above $z_i(T_{-i})$, for any bounded optional time T, $E[|Y_i(T;T_{-i})|] < \infty$ by Assumption 1. Next let T

and S be two optional times with $T \le S$. By Assumption 1, $E[Y_i(S;T_{-i})]$ is strictly less than $+\infty$. Moreover,

$$\begin{aligned} E[Y_i(S;T_{-i})|\mathcal{F}_T] &\le E[Y_i^M(S;T_{-i})|\mathcal{F}_T] \\ &\le Y_i^M(T;T_{-i}) \quad a.s. \text{ for all } N = 1,2,\ldots \end{aligned} \tag{4}$$

where the first inequality follows from the definition of $Y_i(T_{-i})$ and the second inequality follows from the fact that $Y_i^M(T_{-i})$ is a regular supermartingale. Relation (4) implies that

$$E[Y_i(S;T_{-i})|\mathcal{F}_T] \le \operatorname{ess\,inf}_M Y_i^M(T;T_{-i}) = Y_i(T;T_{-i}) \quad a.s.$$

Thus $Y_i(T_{-i})$ is a regular supermartingale.

The rest of the assertion follows from similar arguments of Lemmas 2.2 and 4.2-4.5 of Thompson [1971]. ∎

Now define an optional time

$$T_i(T_{-i}) \equiv \inf\{t \in \Re_+ : Y_i(t;T_{-i}) = z_i(t;T_{-i})\}.$$

The following is our main theorem of this section.

Theorem 1 *$T_i(T_{-i})$ is a best response to T_{-i} for player i. Moreover,*

$$E[Y_i(0;T_{-i})] = \sup_{T \in \mathbf{T}} E[z_i(T;T_{-i})]$$

and $E[|Y_i(0;T_{-i})|] < \infty$.

PROOF Fix $n > 0$ and let

$$\tau_n = \inf\{s \ge 0 : Y_i(s;T_{-i}) \le z_i(s;T_{-i}) + \frac{1}{n}\}.$$

By Proposition 1, τ_n is finite $a.s.$ and

$$E[Y_i(\tau_n;T_{-i})] = E[Y_i(0;T_{-i})]. \tag{5}$$

By the hypothesis that $z_i(T_{-i})$ is upper-semi-continuous on the right (Assumption 2), the fact that $Y_i(T_{-i})$ is right-continuous on the set $\{Y_i(s;T_{-i}) > z_i(s;T_{-i})\}$ (Proposition 1), and (5) we have

$$\begin{aligned} E[z_i(\tau_n;T_{-i})] &\ge E[Y_i(\tau_n;T_{-i})] - \frac{1}{n} \\ &= E[Y_i(0;T_{-i})] - \frac{1}{n} \\ &\ge \sup_{T \in \mathbf{T}} E[z_i(T;T_{-i})] - \frac{1}{n}. \end{aligned}$$

Letting $n \to \infty$ and noting that

$$\sup_{T \in \mathbf{T}} E[z_i(T; T_{-i})] \geq E[z_i(\tau_n; T_{-i})] \quad \forall n,$$

we get

$$\sup_{T \in \mathbf{T}} E[z_i(T; T_{-i})] = E[Y_i(0; T_{-i})]. \tag{6}$$

Moreover, Assumption 1 implies that $E[|Y_i(0; T_{-i})|] < \infty$. These are the second and the third assertions.

Next put $\tau' \equiv \lim_n \tau_n$. The limit is well-defined since τ_n is increasing in n. It is clear that $\tau' \leq T_i(T_{-i})$. We have

$$\begin{aligned} E[Y_i(\tau'; T_{-i})] \geq E[z_i(\tau'; T_{-i})] &\geq \lim_{n \to \infty} E[z_i(\tau_n; T_{-i})] \\ &= E[Y_i(0; T_{-i})] \\ &\geq E[Y_i(\tau'; T_{-i})], \end{aligned}$$

where the second inequality follows from the hypothesis that $z_i(T_{-i})$ is quasi-upper-semi-continuous on the left. It then follows that

$$E[z_i(\tau'; T_{-i})] = E[Y_i(\tau'; T_{-i})] = E[Y_i(0; T_{-i})].$$

Recall from above that $E[|Y_i(0; T_{-i})|] < \infty$. Therefore it must be that $z_i(\tau'; T_{-i}) = Y_i(\tau'; T_{-i})$ *a.s.* and $\tau' = T_i(T_{-i})$. By (6), it follows that $T_i(T_{-i})$ is a best response to T_{-i}. This is the first assertion. ∎

The following propositions give some properties of a best response which we will find useful in Section 4.

Proposition 2 *Fix T_{-i}. Let τ_i be a best response for player i and let $Y_i(T_{-i})$ be the MRS lying above $z_i(T_{-i})$. Then*

$$Y_i(\tau_i; T_{-i}) = z_i(\tau_i; T_{-i}) \quad a.s., \tag{7}$$

and hence

$$T_i(T_{-i}) \leq \tau_i \quad a.s. \tag{8}$$

PROOF By the fact that $Y_i(T_{-i})$ is a regular supermartingale, τ_i is a best response, and $Y_i(T_{-i})$ lies above $z_i(T_{-i})$, we have, almost surely,

$$E[Y_i(\tau_i; T_{-i})] \leq E[Y_i(0; T_{-i})] = E[z_i(\tau_i; T_{-i})] \leq E[Y_i(\tau_i; T_{-i})],$$

where the equality follows from the second assertion of Theorem 1. Hence

$$E[Y_i(0;T_{-i})] = E[Y_i(\tau_i;T_{-i})] = E[z_i(\tau_i;T_{-i})].$$

It then follows from $Y_i(T_{-i}) \geq z_i(T_{-i})$ *a.s.* and the last assertion of Theorem 1 that

$$Y_i(\tau_i;T_{-i}) = z_i(\tau_i;T_{-i}) \quad a.s.$$

This is (7).

By the definition of $T_i(T_{-i})$, we then have $T_i(T_{-i}) \leq \tau_i$ *a.s.*, which is (8).

∎

Note that (7) says that any best response τ to T_{-i} must be the random times at which $Y_i(T_{-i})$ meets $z_i(T_{-i})$. Since $T_i(T_{-i})$ by definition is the first time that $Y_i(T_{-i})$ is equal to $z_i(T_{-i})$, $T_i(T_{-i})$ is the minimum best response for player i, given that his opponents play T_{-i}. This is (8).

The following proposition shows that a best response τ_i chosen by player i at time 0 continues to be optimal at any optional time S as long as at S player i has not stopped according to τ_i.

Proposition 3 *Let τ_i be a best response to T_{-i} and let S be an optional time. Then on the set $\{\tau_i \geq S\}$, τ_i is a solution to*

$$\sup_{\tau \in \mathbf{T}, \tau \geq S} E[z_i(\tau;T_{-i})|\mathcal{F}_S].$$

PROOF If the event $B \equiv \{\tau_i \geq S\} \in \mathcal{F}_S$ is of zero probability, there is nothing to prove. Suppose therefore that B is of a strictly positive probability.

Suppose that the assertion is false. Then there exists an optional time $\hat{\tau} \geq S$ a.s. such that $P(\hat{B}) > 0$, where

$$\hat{B} \equiv \{\omega \in B : E[z_i(\hat{\tau};T_{-i})|\mathcal{F}_S] > E[z_i(\tau_i;T_{-i})|\mathcal{F}_S].$$

Define

$$\tau^*(\omega) = \begin{cases} \tau_i(\omega) & \text{if } \omega \in \Omega \setminus \hat{B}; \\ \hat{\tau}(\omega) & \text{if } \omega \in \hat{B}. \end{cases}$$

It is easily verified that τ^* is an optional time and

$$E[z_i(\tau^*;T_{-i})] > E[z_i(\tau_i;T_{-i})],$$

a contradiction to the hypothesis that τ_i is a best response to T_{-i}. Hence the assertion follows.

∎

4 Nash equilibria when z_i's have a monotone structure

In this section we will show that there exists a Nash equilibrium when the reward processes of players satisfy certain monotone structures.

Consider the following assumptions:

Assumption 3 *There exists* $\hat{\Omega} \in \mathcal{F}$ *with* $P(\hat{\Omega}) = 1$ *and for* $\omega \in \hat{\Omega}$, t, $t' \in \overline{\Re}_+$ *and* $t > t'$, $z_i(\omega, t; T_{-i}) - z_i(\omega, t'; T_{-i})$ *is nondecreasing in* T_{-i}.

Assumption 4 *There exists* $\hat{\Omega} \in \mathcal{F}$ *with* $P(\hat{\Omega}) = 1$ *and for* $\omega \in \hat{\Omega}$, t, $t' \in \overline{\Re}_+$ *and* $t > t'$, $z_i(\omega, t; T_{-i}) - z_i(\omega, t'; T_{-i})$ *is nonincreasing in* T_{-i}.

Assumption 5 *There exists* $\hat{\Omega} \in \mathcal{F}$ *with* $P(\hat{\Omega}) = 1$ *and for* $\omega \in \hat{\Omega}$, t, $t' \in \overline{\Re}_+$ *and* $t > t'$, $z_i(\omega, t; T_{-i}) - z_i(\omega, t'; T_{-i})$ *is strictly decreasing in* T_{-i}.

Assumption 3 says that the longer his opponents stay in the game, the higher the "incremental" reward a player gets by staying in the game. For example, we often observe department stores and hotels gather in nearby locations. Being clustered together, they can generate a higher demand for their services such as convention business for hotels. Assumption 4 is the converse of Assumption 3: a player's incremental reward decreases the longer his opponent stay in the game. This is a more natural assumption for, say, oligopolists facing a fixed demand function for the industry as a whole. Assumption 5 is a strengthening of Assumption 4. Examples of games having reward processes that satisfy Assumption 3 or 4 can be found in Section 5.

Denoting by $\mathbf{T}^N$ the collection of N-tuple of optional times, we define $\Phi : \mathbf{T}^N \mapsto \mathbf{T}^N$ as

$$\Phi(T_1, \ldots, T_N) = (T_i(T_{-i}))_{i=1}^N.$$

By Theorem 1, Φ is well-defined. Since Φ specifies best responses for players, it will sometimes be referred to as a *reaction mapping*.

For two N-vectors of optional times $\tau = (\tau_i, \ldots, \tau_N)$ and $S = (S_i, \ldots, S_N)$, we denote by $\tau \geq S$ if $\tau_i \geq S_i$ *a.s.* for all $i = 1, 2, \ldots, N$. The mapping Φ is said to be *monotone increasing* if for any two N-vectors of Markov times $\tau \geq S$, $\tau' = \Phi(\tau)$ and $S' = \Phi(S)$ imply $\tau' \geq S'$. The mapping Φ is said to be *monotone decreasing* if for any two N-vectors of Markov times $\tau \geq S$, $\tau' = \Phi(\tau)$ and $S' = \Phi(S)$ implies $\tau' \leq S'$.

The following proposition shows that Φ is monotone increasing under Assumption 3 and is monotone decreasing under Assumption 4.

Proposition 4 *Φ is monotone increasing if Assumption 3 is valid and is monotone decreasing if Assumption 4 is valid.*

PROOF Suppose that Assumption 3 is satisfied. Choose two N-vectors of optional times $\tau \geq S$. Let $\tau' = \Phi(\tau)$ and $S' = \Phi(S)$. Suppose the set

$$A = \{\tau_i' < S_i'\}$$

is of strictly positive measure for some i. By Assumption 3 we know, almost surely,

$$[z_i(S_i'; \tau_{-i}) - z_i(\tau_i'; \tau_{-i})]1_A \geq [z_i(S_i'; S_{-i}) - z_i(\tau_i'; S_{-i})]1_A.$$

Taking conditional expectations with respect to $\mathcal{F}_{\tau_i'}$ on both sides of the above relation gives

$$\begin{aligned} 0 &\geq E[z_i(S_i'; \tau_{-i}) - z_i(\tau_i'; \tau_{-i})|\mathcal{F}_{\tau_i'}]1_A \\ &\geq E[z_i(S_i'; S_{-i}) - z_i(\tau_i'; S_{-i})|\mathcal{F}_{\tau_i'}]1_A \geq 0, \end{aligned}$$

where we have used the fact that $A \in \mathcal{F}_{\tau_i'}$ (see, e.g., Dellacherie and Meyer [1978, Theorem IV.56]), where the inequalities follows from Proposition 3. Thus

$$E[z_i(S_i'; S_{-i})|\mathcal{F}_{\tau_i'}] = z_i(\tau_i'; S_{-i}) \quad a.s. \text{ on the set } A. \tag{9}$$

Now define $\sigma_i \equiv S_i' \wedge \tau_i'$ *a.s.* It is easily checked that σ_i is an optional time and $\sigma_i \leq S_i'$, $\sigma_i \neq S_i'$ on a set of strictly positive measure. We claim that σ_i is a best response to S_{-i}. To see this, we note that

$$\begin{aligned} E[z_i(\sigma_i; S_{-i})] &= E[z_i(\tau_i'; S_{-i})1_A + z_i(S_i'; S_{-i})1_{A^c}] \\ &= E\left[E[z_i(S_i'; S_{-i})1_A + z_i(S_i'; S_{-i})1_{A^c}|\mathcal{F}_{\tau_i'}]\right] \\ &= E[z_i(S_i'; S_{-i})], \end{aligned}$$

where we have used (9) and where A^c denotes $\Omega \setminus A$. This is a contradiction to the fact that S_i' is the minimum best response to S_{-i}; see Proposition 2. Thus A must be of measure zero.

The rest of the assertion follows from similar arguments. ∎

The following is the first main theorem of this section:

Theorem 2 *There exists a Nash equilibrium of the stopping game under Assumption 3.*

PROOF Suppose that Assumption 3 is satisfied. From Proposition 4, Φ is a monotone increasing mapping from $\mathbf{T}^N$ to itself. Proposition VI.1.1 of Neveu [1975] implies that $(\mathbf{T}^N, \geq)$ is a complete lattice. It then follows from Tarski's fixed point theorem (see Tarski [1955]) that there exists a fixed point for Φ. It is easily verified that the fixed point is a Nash equilibrium.

∎

A stopping game is said to be *symmetric* if for any $(N-1)$-tuple of optional times T_{-i} we have $z_i(\omega, t; T_{-i}) = z_j(\omega, t; T_{-i})$ except on a subset of $\Omega \times \overline{\Re}_+$ whose projection to Ω is of measure zero. A Nash equilibrium $(T_i)_{i=1}^N$ is a symmetric equilibrium if $T_i = T_j$ *a.s.* for all i, j. Under Assumptions 3, the following theorem shows that, in a symmetric game, there always exists a symmetric equilibrium.

Theorem 3 *Suppose that Assumptions 3 are satisfied. Then there exists a symmetric Nash equilibrium in a symmetric stopping game.*

PROOF Let

$$D = \{T \in \mathbf{T}^N : T_i = T_j \; a.s. \forall i, j\}$$

and let F be the restriction of Φ to D. Note that for each $T \in D$, $\Phi_i(T_{-i}) = \Phi_j(T_{-j})$ *a.s.* for all i, j, where Φ_i denotes the i-th component of Φ. Therefore, F maps D into D. Arguments similar to the proof of Proposition 4 show that F is monotone increasing. It is also easily verified that D is a complete lattice. The assertion then follows from the Tarski's fixed point theorem. ∎

In the next theorem we specialize our model to the case where the number of players is equal to two. In this case, there exists a Nash equilibrium under Assumption 4.

Theorem 4 *Suppose that $N = 2$. Then there exists a Nash equilibrium under Assumption 4 for the stopping game.*

PROOF From Proposition 4, we know $\Phi : \mathbf{T}^2 \mapsto \mathbf{T}^2$ is monotone decreasing. Let $\Phi_i(T_{-i})$ be the i-th component of $\Phi(T)$. It is easily seen that Φ_i's are monotone decreasing. Consider the composite mapping $\Phi_1 \circ \Phi_2 : \mathbf{T} \mapsto \mathbf{T}$. This composite mapping is monotone increasing since Φ_1 and Φ_2 are monotone decreasing. It then follows from the Tarski's fixed point theorem (see Tarski [1955]) that there exists a fixed point of the composite mapping, that is, there exists $T_1 \in \mathbf{T}$ such that $\Phi_1(\Phi_2(T_1)) = T_1$. Hence $(T_1, \Phi_2(T_1))$ is a Nash equilibrium. ∎

Even in the two player case, there may not exist a symmetric Nash equilibrium for a symmetric game under Assumption 4; for an example of nonexistence see Huang and Li [1987].

Note that we only consider pure strategy equilibria in this paper. Our conjecture is that a symmetric equilibrium exists for a symmetric stopping game under Assumption 4 if mixed strategies are allowed. The following theorem, however, shows that if there exists a symmetric Nash equilibrium for a general N player symmetric game under Assumption 5 and if the reward processes satisfy a separability property, then this equilibrium must be the unique symmetric Nash equilibrium.

Theorem 5 *Suppose that Assumption 5 is satisfied and that there exists a symmetric Nash equilibrium in a symmetric stopping game. Then this Nash equilibrium is the unique symmetric equilibrium provided that reward processes satisfy the following condition of separability: If $\tau, S \in \mathbf{T}^{N-1}$ and $\tau = S$ a.s. on a set $B \in \mathcal{F}$, then almost surely*

$$z_i(\omega, t; \tau) = z_i(\omega, t; S)\, \forall t \in \overline{\Re}_+ \, \forall \omega \in B.$$

PROOF Let Ψ be a mapping from $\mathbf{T}^N$ to all the subsets of $\mathbf{T}^N$ that gives all the best responses to an element of $\mathbf{T}^N$. By Theorem 1, this mapping is well-defined. We claim that Ψ is monotone decreasing in the sense that for two optional times $\tau \geq S$ *a.s.* and $\tau' \in \Psi(\tau)$ and $S' \in \Psi(S)$ we have $\tau' \leq S'$. To see this, we suppose that the set

$$A = \{\tau_i' > S_i'\}$$

is of strictly positive probability for some i. By Assumption 5, we know almost surely,

$$\left(z_i(\tau_i'; \tau_{-i}) - z_i(S_i'; \tau_{-i})\right) 1_A < \left(z_i(\tau_i'; S_{-i}) - z_i(S_i'; S_{-i})\right) 1_A.$$

Taking conditional expectations with respect to $\mathcal{F}_{S_i'}$ gives

$$E\left[z_i(\tau_i'; \tau_{-i}) - z_i(S_i'; \tau_{-i}) | \mathcal{F}_{S_i'}\right] 1_A < E\left[z_i(\tau_i'; S_{-i}) - z_i(S_i'; S_{-i}) | \mathcal{F}_{S_i'}\right] 1_A,$$

where we have used the fact that $A \in \mathcal{F}_{S_i'}$ (see Dellacherie and Meyer [1978, Theorem IV.56]). The left side of the relation is nonnegative almost surely by Proposition 3. Hence the right side is strictly positive and is a contradiction to Proposition 3.

Now let $T^*, T' \in D$ and $T^* \neq T'$ be two symmetric Nash equilibria. We will show that this leads to contradiction by considering three cases.

Case 1: $T^* \leq T'$ a.s. Since T^* and T' are Nash equilibria, we must have $T^* \in \Psi(T^*)$ and $T' \in \Psi(T')$. The monotonicity of Ψ implies that $T^* \geq T'$ *a.s.*, hence $T^* = T'$ *a.s.*. This a contradiction.

Case 2: $T' \leq T^*$. Using the monotonicity of Ψ, we have a contradiction similar to the previous case.

Case 3: $P\{T' < T^*\} \in (0,1)$. Define $T = T^* \bigwedge T'$. Since D is a lattice, $T \in D$. Let $\hat{T} \in \Psi(T)$. By the monotonicity of Ψ and the fact that $T^* \in \Psi(T^*)$ and $T' \in \Psi(T')$, we know

$$\hat{T} \geq T' \, a.s. \text{ and } \hat{T} \geq T^* \, a.s.$$

Let $B = \{T' < T^*\}$. By Dellacherie and Meyer [1978, Theorem IV.56], $B \in \mathcal{F}_{T'}$. We have

$$\begin{aligned} 1_B E[z_i(T_i'; T_{-i}')|\mathcal{F}_{T_i'}] &= 1_B E[z_i(T_i'; T_{-i})|\mathcal{F}_{T_i'}] \\ &\leq 1_B E[z_i(\hat{T}_i; T_{-i})|\mathcal{F}_{T_i'}] \\ &= 1_B E[z_i(\hat{T}_i; T_{-i}')|\mathcal{F}_{T_i'}], \end{aligned}$$

where the first equality follows from the separability assumption, the inequality follows from Proposition 3, and the second equality follows again from the separability assumption. Since T_i' is a best response to T_{-i}' and since $\hat{T} \geq T'$, Proposition 3 implies that the above inequality must be an equality. That is

$$1_B E[z_i(T_i'; T_{-i}')|\mathcal{F}_{T_i'}] = 1_B E[z_i(\hat{T}_i; T_{-i}')|\mathcal{F}_{T_i'}] \; a.s. \tag{10}$$

Now we define

$$T^\circ(\omega) = \begin{cases} T'(\omega) & \text{if } \omega \in B, \\ \hat{T}(\omega) & \text{if } \omega \in B^c, \end{cases}$$

where $B^c = \Omega \setminus B$. Since $B \in \mathcal{F}_{T^*} \subset \mathcal{F}_{\hat{T}}$, it is easily verified that T° is a optional time (see, e.g., Dellacherie and Meyer [1978, Theorem IV.53]). We claim that $T^\circ \in \Psi(T)$. To see this, we note that

$$\begin{aligned} E[z_i(T_i^\circ; T_{-i})] &= E[1_B z_i(T_i^\circ; T_{-i}) + 1_{B^c} z_i(T_i^\circ; T_{-i})] \\ &= E[1_B z_i(T_i'; T_{-i}) + 1_{B^c} z_i(\hat{T}_i; T_{-i})] \\ &= E[1_B z_i(\hat{T}_i; T_{-i}) + 1_{B^c} z_i(\hat{T}_i; T_{-i})] \\ &= E[z_i(\hat{T}_i; T_{-i})], \end{aligned}$$

where the third equality follows from (10). Hence we proved our claim that $T^\circ \in \Psi(T)$. This implies that $T^\circ \geq T^*$ $a, s.$, by the monotonicity of Ψ. But on the set B, $T^\circ = T' < T^*$, a contradiction. Therefore the set B must have a zero probability and $T' \geq T^*$, $T' \neq T^*$. This is Case 1. We thus conclude that $T' = T^*$ $a.s.$ and there is a unique Nash equilibrium. ∎

5 Two duopolistic exit games

In this section we will consider two continuous time duopolistic stochastic exit games. The first satisfies Assumption 3 and thus has a Nash equilibrium by Theorem 2. In addition, if this game is symmetric, there is a symmetric Nash equilibrium by Theorem 3. The second game satisfies Assumption 5 and therefore has a Nash equilibrium by Theorem 4.

Let the probability space and filtration be as specified in Section 2. There are two processes $X = \{X(t); t \in \overline{\Re}_+\}$ and $W = \{W(t); t \in \overline{\Re}_+\}$ adapted to $\mathbf{F}$. There are two firms in the market at the beginning of both games. These two firms are assumed to be risk neutral and are indexed by $i = 1, 2$. The riskless interest rate is a constant denoted by r. A firm's problem is to find an exit time in order to maximize the present value of its expected profits.

In the first game, let $\pi_{i1}(X(t))$ be the profit rate at time t for firm i when it is the only firm in the market and let $\pi_{i2}(W(t))$ be the profit rate at time t for firm i when there are two firms in the market. Suppose that $|\pi_{ij}|$ is bounded. We interpret X to be the industry demand when there is only one firm in the industry and W to be that when there are two firms in the industry. Assume that the demand is higher when there are two firms in the market. Thus it is natural to require that $\pi_{i2}(W(t)) \geq \pi_{i1}(X(t))$ for $i = 1, 2$. The reader can think about the example we gave in Section 4 on department stores and hotels clustering together to generate a higher demand. Given that its competitor exits at the optional time T, the reward process for firm i is

$$\begin{aligned} z_i(\omega, t; T) &= \int_0^{t \wedge T(\omega)} e^{-rs} \pi_{i2}(W(\omega, s)) ds + \int_{t \wedge T(\omega)}^{t} e^{-rs} \pi_{i1}(X(\omega, s)) ds, \ t \in \Re_+, \\ z_i(\omega, +\infty; T) &\equiv \limsup_{t \to +\infty} z_i(\omega, t; T). \end{aligned} \tag{11}$$

This reward process is absolutely continuous on $[0, +\infty)$ and adapted and therefore optional. It satisfies Assumptions 1, 2, and 3, and there exists a Nash equilibrium by Theorem 2. When $\pi_{1j}(\cdot) = \pi_{2j}(\cdot)$ for $j = 1, 2$, this is a symmetric game and there is a symmetric Nash equilibrium by Theorem 3.

Now consider the second game. Let $\pi_{ij}(X(t))$ be the profit rate firm i receives at time t when there are j firms in the market and when firm i is still in the market, where $j = 1, 2$. Assume that $|\pi_{ij}(\cdot)|$ is bounded. Also interpret $X(t)$ to be the demand. Unlike in the first game, the total demand here is independent of the number of firms in the market. Thus it is natural that a firm's profit is lower when it is a duopoly than when it is a monopoly. Hence we require $\pi_{i1}(X(t)) > \pi_{i2}(X(t))$.

Given that its competitor exits at the optional time T, the reward process of firm i is

$$\begin{aligned} z_i(\omega,t;T) &= \int_0^{t\wedge T(\omega)} e^{-rs}\pi_{i2}(X(\omega,s))ds + \int_{t\wedge T(\omega)}^{t} e^{-rs}\pi_{i1}(X(\omega,s))ds \; t\in\Re_+, \\ z_i(\omega,+\infty;T) &\equiv \limsup_{t\to+\infty} z_i(\omega,t;T). \end{aligned} \tag{12}$$

This reward process can be verified to be optional and satisfies Assumptions 1, 2, and 5. It then follows from Theorem 4 that there exists a Nash equilibrium of this duopolistic exit game. Readers interested in detailed analysis of this game and discussion on refinements of Nash equilibria when X is a Brownian motion are referred to Huang and Li [1987].

Finally, we remark that the reward processes defined in (11) and (12) satisfy the separability condition of Theorem 5.

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