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by

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Abstract

Let $\{X_n, n=0, \pm 1, \pm 2, \ldots\}$ be a stationary Gaussian stochastic process with means zero, variances one, and covariance sequence $\{r_n\}$. Let $M_n = \max\{X_1, \ldots, X_n\}$ and S_n = second largest $\{X_1, \ldots, X_n\}$. Limit properties are obtained for the joint law of M_n and S_n as n approaches infinity. A joint limit law which is a function of a double exponential law is known to hold if the random variables X_i are mutually independent. When M_n alone is considered Berman has shown that a double exponential law holds in the case of dependence provided either $r_n \log n \to 0$ or $\sum_{n=1}^{\infty} r_n^2 < \infty$. In the present work it is shown that the above conditions are also sufficient for the convergence of the joint law of M_n and S_n. Weak convergence properties of the stochastic processes $M_{[nt]}$ and $S_{[nt]}$ with $0 < a \leq t < \infty$ are also discussed.

1. Introduction. This paper extends and simplifies a theorem obtained by the author in section 4 of [6]. The reader is assumed to have some acquaintance with those results.

Let $\{X_n, n=0, \pm 1, \pm 2 \ldots\}$ be a discrete parameter stationary Gaussian stochastic process, characterized by expectation, and covariance function, respectively:

$$(1.1) \quad \begin{aligned} &E(X_n) \triangleq 0, \\ &E(X_i\ X_{i+n}) \triangleq r_n,\ r_0 \triangleq 1. \end{aligned}$$

This paper treats some of the limit properties of the random variables

$$M_n = \max\ \{X_1, \ldots, X_n\},$$

$$S_n = \text{second largest}\ \{X_1, \ldots, X_n\}.$$

A double exponential limit law is known to hold for M_n if the random variables X_i are mutually independent, that is $r_n \triangleq 0$, $n \neq 0$. Berman [2] has shown that the same law holds in the case of dependence provided either

$$(1.2) \quad r_n \log n \to 0, \text{ or}$$

$$(1.3) \quad \sum_{n=1}^{\infty} r_n^2 < \infty.$$

The author [8] has shown that the processes $\{M_{[nt]}, S_{[nt]}\}$, properly normalized, and with $0 < a \leq t < \infty$, converge weakly in the Skorohod space $D^2[a,\infty]$ when the Gaussian sequence is strong-mixing and

(1.4) $r_n \log n = O(1)$.

The limit law is the same as that which occurs in the independent case.

Condition (1.4) is weaker than (1.2) but we imposed the strong-mixing condition. In many cases strong-mixing is difficult to verify and it is natural in view of Berman's work to see if the weak convergence results mentioned above hold when the strong-mixing assumption is dropped and just (1.2) or (1.3) is assumed. The purpose of this paper is to show that this is, in fact, true. The reader is referred to [7] for some examples of why it is of interest to consider the joint distribution of M_n and S_n. A more extensive discussion of the maxima of stationary Gaussian processes is contained in [4].

2. Some Properties of Gaussian Distributions. Let (r_{ij}) be a $k \times k$ symmetric positive definite matrix with 1's along the diagonal, and let $\phi_k(x_1,\ldots,x_k;\ r_{ij},\ 1 \le i < j \le k)$ be the k-dimensional Gaussian density function with mean vector 0 and covariance matrix (r_{ij}); ϕ_k is a function of the x's and the $k(k-1)/2$ parameters r_{ij}. Define:

$$(2.1)\quad \tilde{Q}_k(c,d,\alpha,\{r_{ij}\})$$

$$= \int_{-\infty}^{c} dx_1 \cdots \int_{-\infty}^{c} dx_{\alpha-1} \int_{d}^{\infty} dx_\alpha \int_{-\infty}^{c} dx_{\alpha+1} \cdots \int_{-\infty}^{c} dx_k \cdot \phi_k(x_1,\ldots,x_k;\{r_{ij}\}).$$

The integral from d to ∞ will always be on the α^{th} dummy variable and we assume that $0 < c \le d$. The partial derivative with respect to $r_{h\ell}$ is obtained by the method of Slepian [5]:

$$(2.2)\quad \frac{\partial \tilde{Q}_k}{\partial r_{h\ell}}$$

$$= \int_{-\infty}^{c} \cdots \int_{d}^{\infty} \cdots \int_{-\infty}^{c} \phi_k(x_1,\ldots,x_{h-1},c,x_{h+1},\ldots,x_{\ell-1},c,x_{\ell+1},\ldots,x_k) \prod_{\substack{j\neq h \\ j\neq \ell}} dx_j$$

when $h \neq \alpha$, $\ell \neq \alpha$, $h \neq \ell$, and

$$(2.3)\quad \frac{\partial \tilde{Q}_k}{\partial r_{\alpha\ell}}$$

$$= - \int_{-\infty}^{c} \cdots \int_{-\infty}^{c} \phi_k(x_1,\ldots,x_{\alpha-1},d,x_{\alpha+1},\ldots,x_{\ell-1},c,x_{\ell+1},\ldots,x_k) \prod_{j\neq\alpha,\, j\neq\ell} dx_j$$

with a corresponding exoression when $h \neq \alpha$ and $\ell = \alpha$. If the upper limits of integration in (2.2) are replaced by $(\infty,..,\infty)$ then the value of the integral is increased. Now integrate k-3 variables from $-\infty$ to $+\infty$ to obtain

$$(2.4)\quad \frac{\partial \tilde{Q}_k}{\partial r_{h\ell}}(c,d,\alpha,\{r_{ij}\}) \leq \int_d^{\infty} \phi_3(c,c,x_\alpha;\ \Sigma(h,\ell,\alpha))dx_\alpha$$

where

$$\Sigma(h,\ell,\alpha) \triangleq \begin{pmatrix} 1 & r_{h\ell} & r_{h\alpha} \\ r_{\ell h} & 1 & r_{\ell\alpha} \\ r_{\alpha h} & r_{\alpha\ell} & 1 \end{pmatrix}.$$

We note that if the limits of integration in (2.3) are replaced by $(\infty,...,\infty)$ then

$$(2.5)\quad \left|\frac{\partial \tilde{Q}_k}{\partial r_{\alpha\ell}}\right| \leq \phi_2(c,\ c;\ r_{\alpha\ell}) = (2\pi)^{-1}(1-r_{\alpha\ell}^2)^{-\frac{1}{2}}\exp[-c^2/(1+r_{\alpha\ell})]$$

Since $\{X_n\}$ is a stationary process, r_{ij} is a function of the difference $j - i$, $i < j$; we write $r_{j-i} = r_{ij}$. The function $\tilde{P}_k$ is defined as

$$\tilde{P}_k(c,d,\alpha,r_1,...,r_{k-1}) = \tilde{Q}_k(c,d,\alpha,\{r_{ij}\});$$

the partial derivatives are given by the chain rule as

$$\partial \tilde{P}_k / \partial r_j = \sum_{\ell - h = j} \partial \tilde{O}_k / \partial r_{h\ell}.$$

Let the sequences $\{a_n\}$ and $\{b_n\}$ be defined as

$$a_n = (2 \log n)^{-\frac{1}{2}}$$

(2.6)

$$b_n = (2 \log n)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi).$$

It is known (cf. [3]) that when $r_n \triangleq 0$, $n \neq 0$

$$\lim_{n\to\infty} P\{M_n \le a_n x + b_n\} = \exp(-e^{-x}) \triangleq G(x)$$

for all x.

Both (1.2) and (1.3) imply that $r_n \to 0$; therefore, there exists a positive number δ such that

$$\sup_n |r_n| = \delta < 1.$$

Define: $\delta(n) = \sup_{k \ge n} |r_k|$, $q_n = n^\beta$, $\delta_n = \delta(q_n/2)$ where $0 < \beta < (1-\delta)^2/2(1+\delta)$.

Clearly (1.2) implies that

(2.7) $\lim_{n\to\infty} \delta(n) \log n = 0$, and

(2.8) $\lim_{n\to\infty} \delta_n \log n = 0.$

3. Convergence Theorems. In this section we extend Berman's results to the joint laws of M_n and S_n.

Theorem 1. Let $\{X_n, n=0, \pm 1, \pm 2, \ldots\}$ be a stationary Gaussian sequence satisfying (1.1). If either

$$\lim_{n\to\infty} r_n \log n = 0$$

or

$$\sum_{n=1}^{\infty} r_n^2 < \infty$$

then

$$\lim_{n\to\infty} P\{M_n \le a_n x + b_n, S_n \le a_n y + b_n\}$$

$$= \begin{cases} G(y)\{1 + \log [G(x)/G(y)]\} & y < x \\ G(x) & y \ge x. \end{cases}$$

The following three lemmas will be needed in the proof. For convenience let $c_n = a_n y + b_n$ and $d_n = a_n x + b_n$, and to avoid technical details we will assume that n is so large that $c_n > 0$.

Lemma 1. Assume that the conditions of Theorem 1 are satisfied and $0 < \varepsilon \le \gamma_n \le 1$. Then

$$(3.1) \quad 1 - \Phi(b_n \gamma_n) = O(n^{-\gamma_n^2})$$

where $\Phi(\cdot)$ is the standardized Gaussian distribution function.

Proof. We shall use the following approximation for the tail of the standard normal distribution ([1], page 933):

$$(3.2) \quad 1 - \Phi(x) \leq (2\pi)^{-\frac{1}{2}} \frac{1}{x} \exp(-x^2/2) \qquad x > 2.2$$

Since $b_n\gamma_n \to \infty$ we can ignore the constraint $x > 2.2$.

To prove (3.1) we note that

$$(3.3) \quad b_n^2 = 2 \log n - \log\log n + O(1)$$

and then use (3.2) to obtain for large n

$$1 - \Phi(b_n\gamma_n) \leq K'n^{-\gamma_n^2}/\gamma_n((\log n)^{\frac{1}{2}(1-\gamma_n^2)} - o(1)).$$

The conditions on γ_n now imply that (3.1) holds for all n.

Lemma 2. If the conditions of Theorem 1 hold and

$$\gamma_n = (1 - 3\delta_n)/(1 + 2\delta_n) \text{ then}$$

$$(3.4) \quad \lim_{n\to\infty} n\, \phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j| = 0$$

and

$$(3.5) \quad \lim_{n\to\infty} n^2[1 - \Phi(b_n\gamma_n)]\phi_2(c_n, c_n, \delta_n) \sum_{j=q_n+1}^{n-1} |r_j| = 0.$$

Proof. The proof of (3.4) is contained in Theorem 3.1 of the paper by Berman [2]. We consider (3.5) first when (1.2) is assumed. Lemma 1 implies that

$$n[1 - \Phi(b_n\gamma_n)] \leq Kn^{1-\gamma_n^2}.$$

Now

$$1 - \gamma_n^2 = \delta_n\{(10 - 5\delta_n)/(1 + 2\delta_n)^2\} = \delta_n O(1)$$

and using (2.8) we have

$$n^{1-\gamma_n^2} = O(1)$$

which, when combined with (3.4), yields (3.5).

When (1.3) holds, the Schwarz inequality implies that (3.5) is less than

$$(3.6) \qquad n^2[1 - \Phi(b_n\gamma_n)]^2\phi_2^2(c_n,c_n,\delta_n)n^3 \sum_{j=q_n+1}^{n-1} r_j^2.$$

Finally we use Lemma 1 and substitute for $\phi_2(c_n,c_n,\delta_n)$ and γ_n so that (3.6) is dominated by

$$K''(\log^2 n)n^{3+\delta_n O(1)} \cdot n^{-4/(1+\delta_n)} \sum_{j=q_n+1}^{n-1} r_j^2$$

which tends to 0, completing the proof of Lemma 2.

<u>Lemma 3</u>. If the conditions of Theorem 1 hold then

(3.7) $$\sum_{\alpha=1}^{n} |\tilde{P}_n(c_n,d_n,\alpha,r_1,\ldots,r_{n-1}) - \tilde{P}_n(c_n,d_n,\alpha,r_1,\ldots,r_{q_n},0,\ldots,0)| \to 0.$$

<u>Proof</u>. By the law of the mean, there exist numbers r_i' between 0 and r_i, $i = q_n+1,\ldots,n-1$, such that

$$\tilde{P}_n(c_n,d_n,\alpha,r_1,\ldots,r_{n-1}) - \tilde{P}_n(c_n,d_n,\alpha,r_1,\ldots,r_{q_n},0,\ldots,0)$$

$$= \sum_{j=q_n+1}^{n-1} r_j(\partial\tilde{P}_n/\partial r_j)(c_n,d_n,\alpha,r_1,\ldots,r_{q_n},r'_{q_{n+1}},\ldots,r'_{n-1})$$

and therefore the sum in (3.7) is less than

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{\ell-h=j} |(\partial\tilde{Q}_n/\partial r_{h\ell})(c_n,d_n,\alpha,r_1,\ldots,r_{q_n},r'_{q_{n+1}},\ldots,r'_{n-1})|.$$

We now consider three cases:

(i) $\ell = \alpha$ or $h = \alpha$ (both cannot occur),

(ii) $|\ell - \alpha| > q_n/2$ and $|h - \alpha| > q_n/2$,

(iii) $|\ell - \alpha| \leq q_n/2$ or $|h - \alpha| \leq q_n/2$ (both cannot occur) $h \neq \alpha$ and $\ell \neq \alpha$.

In the first case (2.5) applies and

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{\substack{\ell-h=j \\ \ell=\alpha \text{ or} \\ h=\alpha}} |\partial\tilde{Q}_n/\partial r_{h\ell}| \leq 2n\phi_2(c_n,c_n,\delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

which goes to 0 with n by Lemma 2.

For the second case we use (2.4) so that

$$\left|\frac{\partial\tilde{Q}_k}{\partial r_{h\ell}}\right| \leq \int_d^\infty \phi_3(c_n,c_n,x_\alpha;\ \Sigma'(h,\ell,\alpha))dx_\alpha$$

where $\Sigma'(h,\ell,\alpha)$ contains some primed elements. Now we compute the conditional distribution of x_α given the first two variates, represented here by c_n. Thus

$$\int_d^\infty \phi_3(c_n,c_n,x_\alpha;\ \Sigma'(h,\ell,\alpha))dx_\alpha = \phi_2(c_n,c_n,r_{h\ell})(1 - \Phi((d_n - \mu_n)/\sigma_n))$$

with (suppressing the primes on the elements of Σ')

$$\mu_n = c_n(r_{h\alpha} + r_{\ell\alpha})/(1 + r_{h\ell})$$

$$\sigma_n^2 = (1 - r_{h\ell}^2 - r_{h\alpha}^2 - r_{\ell\alpha}^2 + 2r_{h\ell}r_{h\alpha}r_{\ell\alpha})/(1 - r_{h\ell}^2).$$

By assumption

(3.8) $\max\,(|r_{h\alpha}|,\ |r_{\ell\alpha}|,\ |r_{h\ell}|) \leq \delta_n$

and using this fact we obtain

$$(d_n - \mu_n)/\sigma_n \geq b_n(1 - 3\delta_n)/(1 + 2\delta_n)$$

provided n is taken so large that $1 - 3\delta_n > 0$. Summarizing we have

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{\substack{\ell-h=j \\ |\ell-\alpha|>q_n/2 \\ |h-\alpha|>q_n/2}} |\partial\tilde{Q}_n/\partial r_{h\ell}|$$

$$\leq n^2[1 - \Phi(b_n\gamma_n)]\phi_2(c_n,c_n,\delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

where $\gamma_n = (1 - 3\delta_n)/(1 + 2\delta_n)$. Applying Lemma 2 completes the proof.

The third case is similar to the second one except (3.8) is no longer satisfied. But either $|r_{\alpha\ell}| < \delta_n$ or $|r_{\alpha h}| < \delta_n$ and, of course, $|r_{h\ell}| < \delta_n$. Conditioning as before we obtain

$$\sum_{\alpha=1}^{n} \sum_{j=q_n+1}^{n-1} |r_j| \sum_{\substack{\ell-h=j \\ |\ell-\alpha|<q_n/2 \\ |h-\alpha|\leq q_n/2}} |\partial\tilde{Q}_n/\partial r_{h\ell}|$$

$$\leq n^{1+\beta}[1 - \Phi(b_n\hat{\gamma}_n)]\phi_2(c_n,c_n,\delta_n) \sum_{j=q_n+1}^{n-1} |r_j|$$

where $\hat{\gamma}_n = (1 - 2\delta_n - \delta)/(1 + 2\delta_n\delta)$. Lemma 2 shows that

$n\phi_2(c_n,c_n,\delta_n) \sum_{j=q_n+1}^{n-1} |r_j| \to 0$ and since $\beta < (1-\delta)^2/2(1+\delta)$, Lemma 1 implies

that $n^\beta[1 - \Phi(b_n\hat{\gamma}_n)] \to 0$. This completes the proof of Lemma 3.

Proof of Theorem 1. When $y \geq x$, Berman's result applies so we consider $y < x$. Then

$$P\{M_n \leq a_n x + b_n, S_n \leq a_n y + b_n\}$$

$$(3.9) \qquad = P\{M_n \leq c_n\}$$

$$+ \sum_{\alpha=1}^{n} (P\{X_\alpha > c_n; X_i \leq c_n, 1 \leq i \leq n, i \neq \alpha\}$$

$$- P\{X_\alpha > d_n; X_i \leq c_n, 1 \leq i \leq n, i \neq \alpha\}).$$

The first term in (3.9) converges to $G(y)$ by Berman's result. Each term in the sum of (3.9) is of the form treated in Lemma 3. Hence we need only find the limit of

$$(3.10) \quad \sum_{\alpha=1}^{n} \tilde{P}_n(c_n, c_n, \alpha, r_1, \ldots, r_{q_n}, 0, \ldots, 0) - \tilde{P}_n(c_n, d_n, \alpha, r_1, \ldots, r_{q_n}, 0, \ldots, 0).$$

This can be accomplished by using the proof developed in [6] for a strong-mixing sequence. For each n we are essentially considering a Gaussian sequence which is q_n-dependent. If

$$p_n = \frac{n - n^{1-\beta}}{n^{1-2\beta}} \quad , k_n = n^{1-2\beta}$$

then

(a) $k_n \to \infty$, $p_n \to \infty$

(b) $n/k_n p_n \to 1$, $n = k_n(p_n + q_n)$

and we may apply the block techniques of Theorem 1 of [6] provided we interpret $\alpha(\cdot)$ as a function with $\alpha(n) = 0$ if $n \geq q_n$. The q_n-dependence is used in place of strong-mixing. The only work that remains is to verify that

$$(3.11) \qquad \lim_n k_n \sum_{j=1}^{p_n-1} (p_n - j)P\{X_1 > c_n, X_{j+1} > c_n\} = 0.$$

When (1.2) holds the proof is exactly the same as the one used for Theorem 3 of [6]. If (1.3) holds then the proof of Theorem 3 of [6] again applies except that a new argument is needed to show that

$$(3.12) \qquad n \log n \sum_{[p_n^\alpha]+1}^{p_n} |r_j| \, n^{-2/(1+|r_j|)} \to 0.$$

By the Cauchy-Schwarz inequality, the square of (3.12) is dominated by

$$(\log n)^2 (p_n/n) n^{3-4/(1+\delta([p_n^\alpha]))} \sum_{j=[p_n^\alpha]+1}^{p_n} r_j^2$$

which clearly tends to 0. This completes the proof of Theorem 1.

4. Concluding Remarks. The weak convergence results of Theorem 2 of [6] are also valid. The convergence of the finite dimensional distributions of $(M_{[nt]} - b_n)/a_n$ and $(S_{[nt]} - b_n)/a_n$ can be proved in a manner similar to that given above. Even if just the one-dimensional process $M_{[nt]}$ is considered, it is necessary to verify the convergence of the second maximum since this is an essential part of the tightness proof given in Theorem 2 of [6]. We are able to use that tightness proof in this case because it depends on the form of the limit law for S_n and not on the strong-mixing property.

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