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CONTINUOUS-REVIEW POLICIES FOR A MULTI-ECHELON
INVENTORY PROBLEM WITH STOCHASTIC DEMAND

by

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ABSTRACT

This paper considers an inventory problem on a multi-stage, serial system. The demand for the end item is stochastic and stationary. The relevant costs include a fixed ordering cost and an echelon inventory holding cost for each stage, and a backorder cost for the end item. The objective is to find a continuous-review inventory control policy that minimizes the expected costs. We present and analyse an approximate cost model. This model is an extension to the traditional single-item, continuous-review inventory model that leads to a reorder point, reorder quantity policy. The nature of our approximation is identical to that for the traditional single-item model. The policies that we derive are also quite analogous to those for the single-item model.

August 27, 1982

1. Introduction

Most multi-echelon inventory systems have significant stochastic characteristics. Yet most of the inventory control systems in practice, such as materials requirements planning (MRP) systems, either ignore these stochastic elements or deal with them in an overly simplistic manner. Furthermore in the inventory literature there is very little theory that can be applied to these problems. In this paper we hope to make a small step at improving this theory. We consider a simple multi-echelon system, namely a serial system, for which we determine continuous-review control policies. We base this analysis on an approximate cost model that is a direct extension to the approximate cost model used for a single-item, continuous-review inventory problem. The resulting solution is quite analogous to that for the single-item model, that being the determination of a reorder point and a reorder quantity. In the remainder of this section we give a brief review of some relevant literature.

The most closely-related work is that by Clark and Scarf [3], [4]. In this work, they analyse both serial and assembly systems under a periodic review control system when the end item demand is uncertain. When a fixed ordering cost is assumed at all stages, their solution procedure only gives an approximate solution but does provide both upper and lower bounds on the cost for the optimal solution. Their solution method computes the optimal policy for each echelon separately, with successive echelons being linked by a penalty function that represents the cost of not being able to fill a replenishment request. Extensions to the Clark-Scarf approach are given by Schmidt and Nahmias [12] who characterize the optimal inventory policy for a simple assembly system and by Lambrecht et al. [8] who examine various safety stock policies for multi-echelon production/inventory systems.

An integral and important aspect of the Clark-Scarf studies is the use of echelon inventory in the control policies for the multi-echelon system. The echelon stock for a component is defined as the inventory of the component plus all of the inventory of downstream items that use or require the component (e.g. subassemblies or end items). In this paper we will rely heavily on the use of echelon stock in the control policy.

Additional analytical work on multi-echelon inventory systems has been done by Simpson [15] and by Hanssman [7], both of whom consider very specific cases. Simpson treats a serial system with no setup costs and with all shortages handled exogenously. He provides an analysis for a periodic-review, base stock control system. Hanssman extends this analysis for the case where the customer demand level is a function of the service level provided.

Finally there has recently been substantial effort attempting to determine appropriate safety stock policies for MRP control systems. (e.g. Carlson and Yano [2], [17], Meal [9], Miller [10], New [11], Whybark and Williams [16], Candea [1]) This work focuses on the questions of where to place safety stock in a complex production system and how much safety stock is necessary to provide adequate protection. This body of research includes both exploratory efforts at characterizing the proper policy (e.g. Miller [10], Whyback and Williams [16], Candea [1]) as well as detailed approaches to resolving some of the issues (e.g. Carlson and Yano [2], [17], Meal [9]).

2. Problem Description and Assumptions

The problem of interest is to find good inventory control policies for a multi-echelon production/inventory system with stochastic demand under continuous review. In Section 3 we present an approximate model for a two-stage production/inventory system. In Section 4 we show how this analysis extends to an M-stage serial system. In Section 5 we discuss how the model might be generalized to more realistic production/inventory assembly systems. The basic model is a direct extension of the classical continuous-review model for a single item with stationary, stochastic demand, i.e. a reorder point, order quantity model (Hadley and Whitin [6], pp. 159-169). Below we state the key assumptions describing the operating mechanics of the system that we model.

- (i) We assume primarily a pure serial production/inventory system. In a serial system each stage has at most one succeeding stage and at most one preceding stage. Each stage is both an inventory control point and a production or processing center. In Section 5 we talk briefly about applying the model to more complex assembly systems.
- (ii) Independent demand occurs only for the end item at the final stage (item 1). The demand process for the end item is a stationary, stochastic process. For convenience, we might assume that the demand over a fixed time interval is a random variable with a normal distribution. All other stages experience demand that is dependent upon the replenishment policy for the end item.

- (iii) Associated with each stage is a deterministic production lead time. Once the production of a lot is initiated at a stage, that lot becomes available for inventory only after the fixed lead time. Furthermore, the entire lot is available after the fixed lead time.
- (iv) The production of a lot at a stage requires that all components (immediately preceding stages) be available at the start of production. Hence, the start of production results in an immediate draw-down of the component inventories.
- (v) The relevant costs at each stage are a fixed setup cost for the production of a lot and a linear inventory holding cost. We assume any variable production costs to be linear in the production lot size and can be ignored.
- (vi) End item demand that cannot be satisfied immediately from inventory is backordered at a cost proportional to the number of items backordered.
- (vii) We desire inventory control policies that minimize expected cost per unit time. We restrict attention to continuous-review stationary policies where each stage has a reorder point and an order quantity. Whenever the echelon inventory at a stage drops below the stage's reorder point, the stage initiates production of an order quantity provided that component stock is available. We require the order quantity at a stage to be an integral multiple of the order quantity at the succeeding stage. This latter assumption is consistent with the policies examined by Crowston et al. [5] and Schwarz and Schrage [14] for an analogous deterministic lot-sizing problem.

This set of assumptions clearly describes an idealized setting. Yet, the assumptions are similar in spirit to those required in the continuous-review, single-item model with stochastic demand. Alternatively, we can view these assumptions as describing a continuous-review analog to the system studied by Clark and Scarf [3], [4]. The most limiting assumption would seem to be that the production lead times are fixed; this removes any problem complexity arising from having to plan or allocate production capacity across the production stages. In addition, by assuming the form of the control policies, we greatly simplify the analysis. Yet, we assert that for the operating system described, the proposed control policies are most reasonable. Indeed we could not conceive of any other form of control policy. To illustrate the control policy, we discuss next a simple, two-stage example.

3. Two Stage Example

Before illustrating the continuous-review policy with a small example, we need first review the traditional reorder-point, reorder-quantity model for a continuous review, single stage system (e.g. Hadley and Whitin [6]). To do this we introduce the following notation:

Q - reorder quantity

R - reorder point

$C(Q,R)$ - expected cost per year as a function of the parameters Q and R

d - expected annual demand

h - holding cost per unit per year

b - backorder cost per unit

a - setup or ordering cost

$f(x|\ell)$ - probability density function for lead time demand, where ℓ denotes the length of the lead time

Now the expected cost function is the sum of the expected setup costs, holding costs and backorder costs. It is customary to approximate this cost expression as follows:

$$(1) \quad C(Q,R) \cong \frac{ad}{Q} + h\left(\frac{Q}{2} + \int_0^{\infty} (R-x)f(x|\ell)dx\right) + \frac{bd}{Q} \int_R^{\infty} (x-R)f(x|\ell)dx$$

In this expression the setup cost and backorder cost terms are exact, while the holding cost term is approximate. The holding cost term consists of the average cycle stock, $hQ/2$, and the expected safety stock,

$h \int_0^{\infty} (R-x)f(x|\ell)dx$. This latter term is an approximation and represents

the expected net inventory level just prior to replenishment. The reason for this approximation is its analytic tractability relative to an exact expression. From the approximate cost expression, the best choice of policy parameters is given by the simultaneous solution to the following two expressions:

$$(2) \quad Q = \sqrt{\frac{2d(a + bz(R))}{h}}$$

and

$$(3) \quad \int_R^{\infty} f(x|\ell) dx = \frac{hQ}{bd}$$

where

$$z(R) = \int_R^{\infty} (x-R)f(x|\ell) dx .$$

For most common settings in which the backorder cost is significantly larger than the inventory holding cost, this solution from the approximate cost expression is nearly optimal. In what follows for a multi-stage system, we will use the analogous approximation for the expected holding cost as used in (1).

Now consider a two-stage system. The notation is similar to before, with the subscript denoting the stage: stage one is the downstream stage and represents the end item; stage two is the upstream stage and represents the component. We assume that the order quantities are chosen such that $Q_2 = nQ_1$ for n being a positive integer. The reorder point for the component, R_2 , applies to the echelon inventory, that being the inventory (on-hand and on-order) at either stage two (component) or stage one (end item). After placing an order, stage two receives its stock replenishment after a fixed lead time, call it ℓ_2 . We assume that whenever the components are replenished,

Q_1 units of the component are immediately forwarded for processing to end items, regardless of the inventory status of end items. This order quantity will arrive in the inventory for end items after ℓ_1 time units. Hence, a replenishment for components leads to a replenishment for end items, with the total lead time for the end-item order quantity being $\ell_1 + \ell_2$ time units after the initiation of the component replenishment. We call this a joint replenishment. Normal replenishment for end items occurs whenever the end-item inventory hits its reorder point and sufficient component inventory is on-hand. A joint replenishment is followed by exactly $n-1$ normal replenishments by stage one. This cycle then repeats, starting with the next joint replenishment. We depict this behavior for $n=2$ in Figure 1.

To derive a cost expression for this system, we consider separately each of the cost elements. For a given set of policy parameters, the expected annual ordering cost is easily found to be

$$(4) \quad \frac{a_1 d}{Q_1} + \frac{a_2 d}{nQ_1}$$

To derive the inventory holding cost, we first consider the echelon inventory for components over a complete order cycle. Just prior to the arrival of an order quantity of components, the on-hand echelon inventory is approximated by

$$\int_0^{\infty} (R_2 - x) f(x | \ell_2) dx$$

where $f(x | \ell_2)$ is the probability density function for demand over ℓ_2 time units, equal to the lead time at stage two. This expression is approximate since shortages are treated as negative inventory. Just after the order arrival, the on-hand echelon inventory is the above quantity plus $Q_2 = nQ_1$.

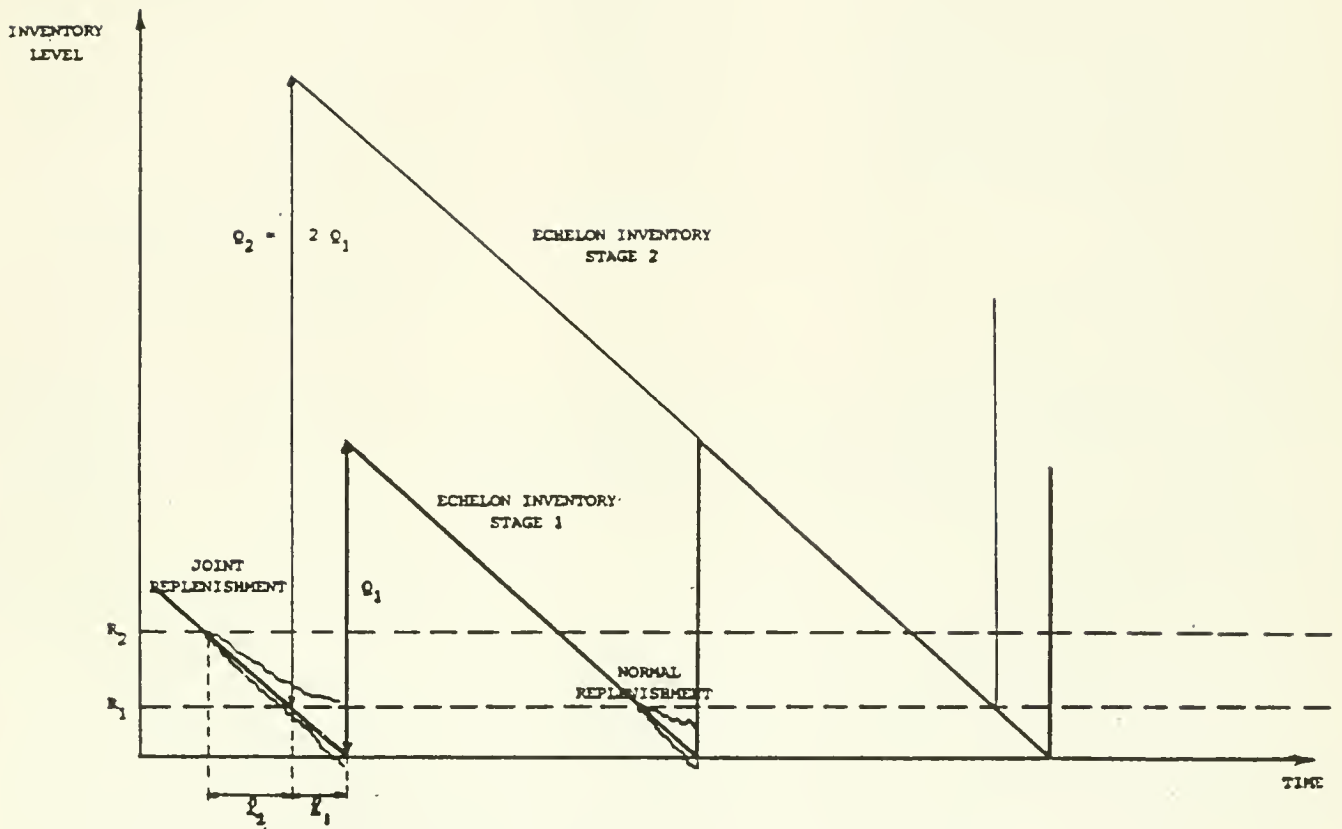


Figure 1: Two Stage Example

Since we assume demand to be stationary, the expected echelon inventory holding cost over a complete order cycle is

$$(5) \quad h_2 \left(\frac{nQ_1}{2} + \int_0^{\infty} (R_2 - x) f(x | l_2) dx \right),$$

where h_2 is the per unit echelon holding cost.

The derivation for stage one is quite similar, but now we must distinguish between the types of replenishment. Consider an order cycle at stage one. In $n-1$ out of n times we have a normal replenishment, and the on-hand inventory just prior to the order arrival is

$$\int_0^{\infty} (R_1 - x) f(x | l_1) dx.$$

In one out of n times we have a joint replenishment, and the on-hand inventory just prior to the order arrival is approximated by

$$\int_0^{\infty} (R_2 - x) f(x | l_1 + l_2) dx,$$

since effectively the lead time is $l_1 + l_2$ with a reorder point of R_2 . We assume here that even though the reorder point R_2 applies to the echelon stock for the component, that all of this stock will have been processed to end items by the time the replenishment of end items arrives. This will almost certainly be true for most cases of interest. Again, the on-hand inventory after an order arrival is the above quantity plus Q_1 . Hence, the expected echelon inventory holding cost associated with stage one is

$$(6) \quad h_1 \cdot \left(\frac{Q_1}{2} + \frac{n-1}{n} \int_0^{\infty} (R_1 - x) f(x | l_1) dx \right. \\ \left. + \frac{1}{n} \int_0^{\infty} (R_2 - x) f(x | l_1 + l_2) dx \right)$$

where now h_1 denotes the echelon holding cost for the end item. A somewhat-more rigorous justification for this approximation is given in the appendix.

Since backorders occur only for the end items, we must distinguish between the types of replenishment for the end item to compute the expected backorder cost. As a consequence of a normal replenishment, the expected backorder cost is just

$$b \int_{R_1}^{\infty} (x-R_1)f(x|\ell_1)dx.$$

Similarly for a joint replenishment we have

$$b \int_{R_2}^{\infty} (x-R_2)f(x|\ell_1+\ell_2)dx$$

since the replenishment is triggered at R_2 and requires $\ell_1+\ell_2$ time units to complete. We again make the assumption that the entire component inventory of R_2 is available to satisfy end item demand within the joint lead time.

By combining these terms in the proper proportions we get the expected backorders per replenishment of end items. By weighting this by the annual number of replenishments, we find the annual expected backorder cost:

$$(7) \quad \frac{bd}{Q_1} \cdot \left(\frac{n-1}{n} \int_{R_1}^{\infty} (x-R_1)f(x|\ell_1)dx \right. \\ \left. + \frac{1}{n} \int_{R_2}^{\infty} (x-R_2)f(x|\ell_1+\ell_2)dx \right).$$

By summing (4-7) we obtain the expected cost as a function of the parameters, Q_1 , R_1 , R_2 and n , call it $C(Q_1, R_1, R_2, n)$. This expression is not exact due to the approximations noted above. If we assume that a value

of $n > 1$ is given, then to minimize the expected cost we choose the remaining parameters as the simultaneous solution to the following equations:

$$(8) \quad Q_1 = \sqrt{\frac{2d(a_1 + \frac{a_2}{n} + bz(R_1, R_2, n))}{h_1 + nh_2}}$$

$$(9) \quad \int_{R_1}^{\infty} f(x|\ell_1)dx = \frac{h_1 Q_1}{bd}$$

$$(10) \quad \int_{R_2}^{\infty} f(x|\ell_1 + \ell_2)dx = \frac{(h_1 + nh_2)Q_1}{bd}$$

$$\text{where } z(R_1, R_2, n) = \frac{n-1}{n} \int_{R_1}^{\infty} (x-R_1)f(x|\ell_1)dx \\ + \frac{1}{n} \int_{R_2}^{\infty} (x-R_2)f(x|\ell_1 + \ell_2)dx.$$

For the case with $n=1$, the problem collapses to a one-stage problem for which the solution given by (2)-(3) is applicable (i.e. $a = a_1 + a_2$, $h = h_1 + h_2$, $\ell = \ell_1 + \ell_2$). The solution to the equations (8)-(10) can be found via an iterative procedure that alternates between solving (8) for Q_1 given R_1 and R_2 , and solving (9)-(10) for R_1 and R_2 given Q_1 .

We note that the above solution is very similar to that for the single-stage system. We can interpret the order quantity Q_1 as an economic order quantity. The ordering cost has been augmented to reflect the coordinated replenishments and the expected backorder cost per cycle. The holding cost also reflects the coordinated replenishments. The reorder points are found by setting their respective stockout probabilities equal to a critical ratio. Interestingly, these critical ratios differ, with the critical ratio (stockout probability) being larger for the component

than for the end item. Although it is possible for R_2 to be less than R_1 , we will typically have R_2 greater than R_1 , as one might expect.

Determining the best choice for n is not as simple. For given values of Q_1 , R_1 and R_2 , the cost expression $C(Q_1, R_1, R_2, n)$ is unimodal in n . Hence, for this case we can easily find the best choice for n by a simple line search. Conceivably then, one could iterate between finding the best choice for Q_1 , R_1 and R_2 for a given value of n and finding the best choice for n for given values of Q_1 , R_1 and R_2 . This iterative procedure will obtain improved solutions at each step and is guaranteed to converge since we can bound the largest possible choice for n . However, there is no guarantee that this procedure finds the global optimal solution; rather, this procedure would only ensure a local optimum. An alternate procedure might be to search over possible values for n , where for each candidate value of n the best choice for Q_1 , R_1 and R_2 are determined from (8)-(10). We conjecture that the objective value as a function of n , assuming that the conditionally-optimal choices for Q_1 , R_1 and R_2 are used, is unimodal in n ; hence we could perform the line search quite efficiently. We suggest as a starting value for n to use the optimal choice for n for the deterministic problem (Schwarz [13]).

4. M-Stage Serial System

The analysis of the two-stage system extends directly to M-stage serial systems for $M > 2$. We need define the integer variables n_i to be the relative order frequency between stage i and stage $i-1$, for $i = 2, 3, \dots, M$. Hence, the order quantity for stage i is $Q_i = n_i Q_{i-1}$, and $Q_i = n_i n_{i-1} \dots n_2 Q_1$. For convenience, we define $r_i = n_i n_{i-1} \dots n_2$ so that $Q_i = r_i Q_1$ and r_{i+1}/r_i is an integer. We assume, as in the two-stage example, that the reordering policy is a "nested" policy. That is, for any two successive stages, say k and $k+1$, the lower stage orders exactly n_k times for each order by the higher stage. Furthermore, whenever stage $k+1$ orders, there is a joint replenishment with stage k and all lower stages.

We now express the expected annual cost as a function of the policy parameters. To do this we will again consider separately the three types of costs: setup or ordering, inventory holding and backorder. The expected annual ordering cost is now given by

$$(11) \quad \frac{d}{Q_1} \cdot \left(a_1 + \frac{a_2}{r_2} + \frac{a_3}{r_3} + \dots + \frac{a_M}{r_M} \right).$$

To express the expected holding cost we consider each stage separately. For stage one (end item) the derivation is analogous to that for the two-stage example, but more complex. There are now M types of replenishment at stage one. A normal replenishment occurs whenever stage one reorders without any other stage having reordered; a "two-stage" joint replenishment occurs when stage two reorders followed by stage one reordering; a "k-stage" joint replenishment occurs when stage k reorders and triggers subsequent reorderings by stage $k-1, k-2, \dots, 1$. The initiation of an "M-stage" joint replenishment is a regeneration point and we define a cycle to be the time between successive

"M-stage" joint replenishments. A cycle entails $r_M = n_2 n_3 \dots n_M$ replenishments by stage one. Of these r_M replenishments by stage one, exactly one corresponds to an "M-stage" joint replenishment, n_{M-1} of the replenishments are from "M-1 stage" joint replenishments, $n_M(n_{M-1}-1)$ of the replenishments are from "M-2 stage" joint replenishments, and so on. In general, there are $n_M n_{M-1} \dots n_{k+2}(n_{k+1}-1) = r_M(\frac{1}{r_k} - \frac{1}{r_{k+1}})$ replenishments that correspond to "k-stage" joint replenishments. Finally, there are $n_M \dots n_3(n_2-1) = r_M(1 - \frac{1}{r_2})$ normal replenishments at stage one. Now by repeating the arguments used in the two-stage example and in the appendix, we approximate the inventory holding cost for stage one (end item) by the following:

$$\begin{aligned}
 (12) \quad h_1 \cdot & \left\{ \frac{Q_1}{2} + \left(1 - \frac{1}{r_2}\right) \int_0^\infty (R_1-x) f(x|\ell_1) dx \right. \\
 & + \left(\frac{1}{r_2} - \frac{1}{r_3}\right) \int_0^\infty (R_2-x) f(x|\ell_1+\ell_2) dx \\
 & + \dots \\
 & + \left(\frac{1}{r_{M-1}} - \frac{1}{r_M}\right) \int_0^\infty (R_{M-1}-x) f(x|\ell_1 + \dots + \ell_{M-1}) dx \\
 & \left. + \frac{1}{r_M} \int_0^\infty (R_M-x) f(x|\ell_1 + \dots + \ell_M) dx \right\}.
 \end{aligned}$$

The first term within the brackets is just the average cycle stock. The remaining terms comprise the expected safety stock level, which is modeled as a weighted average of the remaining inventory over all types of replenishments.

For stage k , $k > 1$, the derivation of the expected holding cost is virtually identical. Due to the nested structure of the reordering decisions,

the echelon inventory at stage k depends only on the replenishments by stage k and by higher stages (i.e. stage $k+1$, ... stage M). Indeed we can view the stage- k echelon inventory as if it were the end-item inventory of an $(M-k+1)$ -stage serial system consisting of stages $k, k+1, \dots, M$. By repeating the arguments used for the end item, we obtain the following for the expected echelon inventory holding cost for stage $k, k = 2, 3, \dots, M$:

$$\begin{aligned}
 (13) \quad h_k \cdot & \left\{ \frac{r_k Q_1}{2} + \left(1 - \frac{r_k}{r_{k+1}}\right) \int_0^{\infty} (R_k - x) f(x | \ell_k) dx \right. \\
 & + r_k \left(\frac{1}{r_{k+1}} - \frac{1}{r_{k+2}} \right) \int_0^{\infty} (R_{k+1} - x) f(x | \ell_k + \ell_{k+1}) dx \\
 & + \dots \\
 & + r_k \left(\frac{1}{r_{M-1}} - \frac{1}{r_M} \right) \int_0^{\infty} (R_{M-1} - x) f(x | \ell_k + \dots + \ell_{M-1}) dx \\
 & \left. + \frac{r_k}{r_M} \int_0^{\infty} (R_M - x) f(x | \ell_k + \dots + \ell_M) dx \right\}
 \end{aligned}$$

The interpretation of this term is identical to that given for (12).

To represent the expected annual backorder cost we must account for the different types of replenishments for the end item. By extending the earlier arguments for the two-stage example, we obtain the following expression:

$$\begin{aligned}
 (14) \quad \frac{bd}{Q_1} \cdot & \left\{ \left(1 - \frac{1}{r_2}\right) \int_{R_1}^{\infty} (x - R_1) f(x | \ell_1) dx \right. \\
 & + \left(\frac{1}{r_2} - \frac{1}{r_3} \right) \int_{R_2}^{\infty} (x - R_2) f(x | \ell_1 + \ell_2) dx \\
 & + \dots + \frac{1}{r_M} \int_{R_M}^{\infty} (x - R_M) f(x | \ell_1 + \dots + \ell_M) dx \left. \right\}. \\
 \underline{\underline{\text{def}}} \quad & \frac{bd}{Q_1} z(\underline{R}; \underline{n})
 \end{aligned}$$

The term $z(\underline{R}; \underline{n})$ is defined above to be the expected backorder per replenishment at stage one.

The total expected cost as a function of the policy parameters is the sum of (11)-(14). If we assume that the $\{n_i\}$ values are specified, then we minimize the expected cost by choosing the remaining parameters by the simultaneous solution to the following equations:

$$(15) \quad Q_1 = \sqrt{\frac{2d(a_1 + \frac{a_2}{r_2} + \frac{a_3}{r_3} + \dots + \frac{a_M}{r_M} + bz(\underline{R}; \underline{n}))}{h_1 + r_2 h_2 + \dots + r_M h_M}}$$

$$(16) \quad \int_{R_1}^{\infty} f(x | \ell_1) dx = \frac{h_1 Q_1}{bd}$$

$$(17) \quad \int_{R_k}^{\infty} f(x | \ell_1 + \ell_2 + \dots + \ell_k) dx = \frac{(h_1 + r_2 h_2 + \dots + r_k h_k) Q_1}{bd}$$

for $k = 2, \dots, M$

Equation (17) is applicable only when $r_{k+1} > r_k$ or equivalently $n_{k+1} > 1$.

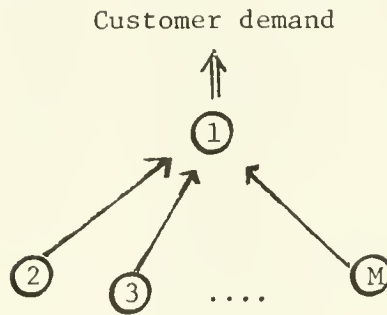
If $n_{k+1} = 1$, then the determination of R_k is not relevant since stage k reorders only whenever stage $k+1$ reorders. We note that the reorder points are such that the stockout probabilities associated with each stage's replenishments increases with the stages. Also, although typically $R_k < R_{k+1}$, it is possible for $R_k > R_{k+1}$ as we saw in the two-stage example.

We solve (15)-(17) by the iterative procedure that alternates between determining the order quantity and determining the reorder points. Determining the optimal choice for the $\{n_i\}$ values is much more difficult. It would seem to require an enumeration over all candidate solutions, most likely by a branch-and-bound procedure. We expect a very good starting solution for such a procedure would be to use the optimal $\{n_i\}$ values from the deterministic version of this problem that is easily solved (Schwarz [13]).

5. Extensions

In this section we first present one extension for a specific two-echelon assembly system, and then discuss the feasibility of more general extensions.

Consider a two-echelon, M-stage assembly system as depicted below.



We assume that the order quantities for the components are integral multiples of the order quantity for end item, i.e. $Q_i = n_i Q_1$ for n_i being a positive integer, $i = 2, \dots, M$. To develop a cost function for this system, we need to characterize the types of joint replenishments that can occur. We define a "stage-k" joint replenishment as a joint replenishment with the end item in which stage k is the component with the longest replenishment lead time amongst those components that will be simultaneously replenished. A "stage-k" replenishment is initiated when the echelon inventory of stage k reaches its reorder point. If component i is to be simultaneously replenished with stage k, then it is reordered exactly $\ell_k - \ell_i$ time units after the order for k. A normal replenishment occurs whenever the end item reorders without any component having jointly reordered. Without loss of generality, we assume that $\ell_2 \geq \ell_3 \geq \dots \geq \ell_M$.

Suppose that the order frequencies $\{n_i\}$ are relatively prime. With a little thought we can see that there is a regenerative cycle consisting of $n_2 n_3 \dots n_M$ replenishments by the end item. Of these replenishments, exactly $n_3 n_4 \dots n_M$ replenishments are "stage-two" joint replenishments with an effective lead time of $\ell_2 + \ell_1$; $(n_2 - 1) n_4 n_5 \dots n_M$ of the replenishments are "stage-three" joint replenishments with a lead time of $\ell_3 + \ell_1$; and, in

general, $(n_2-1)(n_3-1) \dots (n_{k-1}-1)n_{k+1} \dots n_M$ of the replenishments are "stage-k" joint replenishments with a lead time of $\ell_k + \ell_1$, for $k = 3, \dots, M-1$. Finally, there are $(n_2-1)(n_3-1) \dots (n_{M-1}-1)$ "stage-M" joint replenishments and $(n_2-1)(n_3-1) \dots (n_M-1)$ normal replenishments.

By knowing the types of joint replenishments in a regenerative cycle, we can now apply the logic previously used to characterize both the expected holding costs and the expected backorder cost. With this we can develop a cost function of the policy parameters. The minimization of this cost function results in a solution that would be quite analogous to that for a serial system, namely (15)-(17). Once again we must assume that the order frequencies are given, and furthermore, that they are relatively prime. If the order frequencies are not relatively prime, then this approach would overstate the number of "stage-k" joint replenishments for $k > 2$, and would understate the number of normal replenishments. As a consequence, the cost function will tend to be overstated for the range of policy parameters of interest. Nevertheless, optimization of this overstated cost function may lead to a reasonable choice of policy parameters.

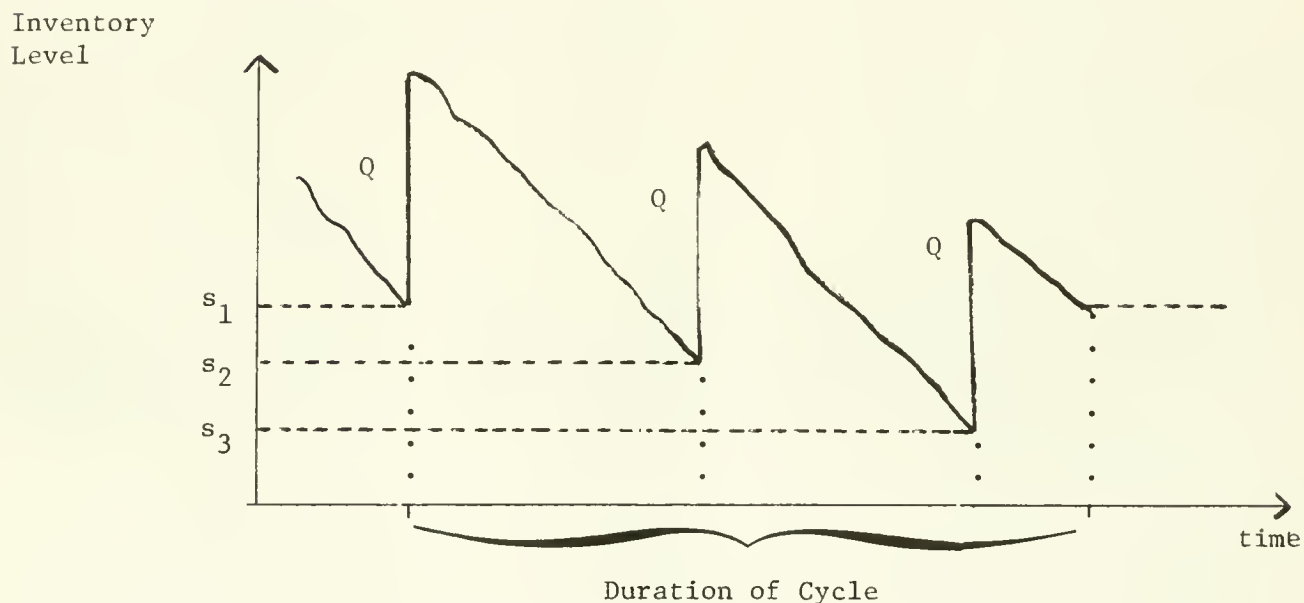
An alternative approach for the case when the order frequencies are not relatively prime would be to explicitly enumerate the number of joint replenishments of each type over a regenerative cycle. We could then form a cost function that we would minimize as in the earlier cases. The difficulty with this approach is the lack of a closed-form expression for the number of joint replenishments of a particular type.

For more complex assembly networks, the analysis soon becomes hopeless. One possible approach is to force the network into behaving like a serial system so that our earlier analysis could be applied. Otherwise, the key

issue is again to identify the types and to enumerate the occurrences of joint replenishments. In principle, we can always do this. But the task is quite overwhelming even for networks with modest complexity. Clearly some new ideas and insights are needed to address these problems.

APPENDIX

In this section we provide a general justification for the approximate expression that we use for the average echelon inventory level for any stage. Consider a regenerative cycle in which a particular inventory (or echelon inventory) is replenished exactly n times in each cycle. Each replenishment quantity is equal to Q , and we assume that the demand is stationary, albeit stochastic. Assume that just prior to the arrival of the i^{th} replenishment in the cycle, the expected net inventory level is s_i , $i = 1, 2, \dots, n$. The following illustration ($n=3$) may be of help:



Now the nature of the approximation is to use these expectations (s_i) to calculate the average (echelon) inventory level over the regenerative cycle. Given that the demand process is stationary and continuous, the average inventory level is just the area in the n trapezoids (one for each replenishment in the cycle) divided by the length of the regenerative cycle. In general, we state this as

$$(A1) \quad \left(\sum_{i=1}^n \frac{Q + s_i + s_{i+1}}{2} \cdot \frac{Q + s_i - s_{i+1}}{d} \right) \div \frac{nQ}{d}$$

where $s_{n+1} = s_1$ and d is the annual demand rate. After simplification, (A1) just reduces to

$$(A2) \quad \frac{Q}{2} + \sum_{i=1}^n \frac{s_i}{n}$$

For instance, in the two-stage system we have

$$s_1 = \int_0^{\infty} (R_2 - x) f(x | l_1 + l_2) dx$$

$$s_i = \int_0^{\infty} (R_1 - x) f(x | l_1) dx \quad \text{for } i=2, 3 \dots n.$$

Then, (A2) is just

$$\frac{Q}{2} + \frac{s_1}{n} + \frac{(n-1)s_2}{n}$$

as was given in Section 3. Similar derivations are possible for the average echelon inventory levels for the M-stage serial system in Section 4.

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