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A DISTRIBUTIONAL FORM OF LITTLE'S LAW

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ABSTRACT

For many system contexts for which Little's Law is valid a distributional form of the law is also valid. This paper establishes the prevalence of such system contexts and makes clear the value of the distributional form.

Keywords: Queues, Little's Law, priority, limit theorems

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1. Introduction

Over the twenty-five years since Little's Law first appeared (Little [11]), its simplicity and importance have established it as a basic tool of queueing theory (cf e.g. the survey paper by Ramalhoto, Amaral and Cochito [12]). Little's Law equates the expectations of two variates in a system. For many of the systems to which Little's Law is applicable, a stronger relation between the distributions of the two variates is available. The setting required is described in the theorem in Section 2. The distributional form of Little's law has been observed previously in special contexts, (Fuhrmann and Cooper [2], Servi [13], Svoronos and Zipkin [14]) but the generality of the setting and its importance for queueing theory has not been set forth.

Suppose that, for some ergodic queuing system, there is a class **C** of customers in the system with Poisson arrival rate λ. Let N be the ergodic number of customers in the system in that class and let T be the ergodic time in system spent by a customer in that class. Let $\pi_{NS}(u) = E[u^N]$ be the p.g.f. of N and let $\alpha_{TS}(s) = E(e^{-sT})$ be the Laplace-Stieltjes transform of T. Suppose it is known for class **C** that

$$\pi_{NS}(u) = \alpha_{TS}(\lambda - \lambda u) \tag{1}$$

i.e. that

$$N =^{d} K_{\lambda T} \tag{2}$$

Here K_θ is a Poisson variate with parameter θ . In words, the ergodic number of customers in that class in the system is equal in distribution to the number of Poisson customers arriving during an ergodic time in system for members of that class. A customer class for which the equality in distribution holds may be said to satisfy the **distributional form of Little's Law** or to be an LLD class. If one differentiates (1) at u = 1, one sees that LLD implies Little's Law, i.e. $E[N] = \lambda E[T]$ for that class. The converse is not true as will be seen.

For many of the system contexts for which Little's Law is valid , the distributional form of the law is also valid. The object of the paper is to demonstrate the prevalence of such system contexts, and to make clear the value of the distributional form.

2. The prevalence of LLD Classes

It is known (c.f.Kleinrock [10]) that the customers of an M/G/1 system with infinite queue capacity and FIFO discipline satisfy (1). A broader prevalence of the distributional form of Little's Law is suggested by the following two theorems.

Prop. 1 (Keilson and Servi [7])

Consider an M/G/1 type system with two classes of customers, infinite queue capacity and FIFO preempt-resume discipline for the low priority class. The distributional form of Little's Law is valid for the low priority customers.

Prop. 2.

Every class of customers in an M/G/1 type priority system for which all classes have Poisson arrivals and FIFO preempt-resume discipline over lower priority classes is an LLD class.

Proof. Each class of customers sees only the customers with higher priority. The totality of all customers with higher priority may be regarded as a single class having a Poisson arrival rate equal to the sum of all the arrival rates with higher priority and an effective service time distribution which is a weighted mixture of the service times with higher priority. The general result then follows from that for two classes.

These examples suggest an even broader validity. A more general theorem is given next.

Theorem.

Let an ergodic queueing system be such that for a given class C of customers,

a) arrivals are Poisson of rate λ;

b) all arriving customers enter the system, and remain in the system until served i.e there is no blocking, balking or reneging;

c) the customers remain in the system until served and leave the system one at a time in order of arrival (FIFO).

Then the distributional form of Little's Law is valid for that class C of customers.

The proof is based on two lemmas.

Lemma A.(Cf. Cooper [1] for the result due to Burke and Takács) Let N(t) be a time homogeneous process in continuous time on the lattice of non-negative integers with changes of ± 1 at sequences of successive arrival and departure epochs (τ^A_j) and (τ^D_j) respectively. Then $\lim_{j \to \infty} P[N(\tau^D_j+) \leq n] = \lim_{j \to \infty} P[N(\tau^A_j-) \leq n]$ when these limits exist.

Lemma B. (Cf. Wolff [17], [1]) Let N(t) be an ergodic population counting process in continuous time with Poisson arrivals at successive epochs (τ^A_j). Then

$$\lim_{t \to \infty} P[N(t) \leq n] = \lim_{j \to \infty} P[N(\tau^A_j+) \leq n]$$

Proof. Suppose that at t=0, the system is empty. Let τ_{Ak} be the arrival epoch of the k'th customer in class C, and let τ_{Dk} be the departure epoch for that customer. Let N(t) be the number of customers of class C in the system at time t. Let T_k be the time spent in the system by the k'th customer, i.e. let $T_k = \tau_{Dk} - \tau_{Ak}$. Then $N(\tau_{Dk}+) =^d K_{\lambda T_k}$. This is true, since all customers found at arrival by the k'th customer have left the system before that customer entered service, all customers arriving during his time in system are still

of such systems (cf. Keilson and Servi [8], Harris and Marchal [3]) , the server may take vacations after servicing a customer with a probability dependent on queue length. The server might not end a vacation duration until exactly N customers are waiting (where N is a prespecified positive integer) (cf. Yadin and Naor [16], Heyman [4]) Little's law holds in distribution for the customers in such systems provided they are served in order of arrival.

C. **M/G/1 systems with preemptive interruptions at clock ticks** (Cf. Keilson and Servi [6]). If a Poisson stream of ordinary customers with iid service times is preempted by other iid tasks arriving at clock ticks, then Little's law holds in distribution for the time to service completion of the ordinary customers.

D. **Cyclic service systems** (Cf. Takagi [15]) Suppose Poisson customers queue at service sites and a single server moves cyclicallly between such sites, either serving the customers at the sites to exhaustion, employing a Bernoulli schedule (Keilson and Servi [5]), or serving at most K customers at a site before moving on. Then the customers at any site are an LLD class provided they are served in order of arrival. The changeover time of the server between sites does not disturb the LLD property.

E. **Tandem server systems (M/G/G/ . . /G).** Consider K servers in tandem each with infinite queue capacity and each having iid service times for successive arrivals. If arrivals to the system are Poisson, and FIFO discipline is maintained throughout the system, then Little's law in distribution relates the time spent in the system to the number of customers in the system at ergodicity.

4. The value of the distributional form of Little's Law.

Apart from the relations between means and variances given in (3), the distribution information has other benefits, some of which are listed below.

Stochastic bounds

It is well known that the Poisson variate K_θ increases stochastically with θ. It then follows from (2) that **N** increases stochastically when **T** increases. It is sometimes easy to obtain a distribution bound for a time in system. When one can do so one automatically has a distribution bound for the number in the system.

Heavily loaded systems.

Suppose it is known that **T** is exponentially distributed to good approximation. It is then a direct consequence of the distributional form of Little's Law that for an LLD system **N** is geometrically distributed. The single parameter needed for the distribution is then available from the first moment. For heavily loaded systems, knowledge that **T** is asymptotically exponential in distribution implies that **N** is asymptotically geometric in distribution . An easy example is that of the number in the system for M/G/1 with FIFO discipline.

Decomposition Results.

For a broad class of customers, Fuhrmann and Cooper [2] proved that the ergodic number of customers in the system is distributed as the sum of two random variables, one of which is the ergodic number in the system of an ordinary M/G/1 queue. This class is a proper subset of the LLD class. Every LLD class for which **N** decomposes has a decomposition for **T**. This decomposition has also been observed in [2] via separate reasoning.

5. The LLD property for the number in queue.

The ordinary form of Little's Law is applicable both to the time in system and to the time in queue. It is natural to try to find an LLD result for the time in queue and the number in queue by transferring the ingredients of the proof of the theorem to a subsystem consisting of a queue only. Direct application of the theorem, however does not work. The difficulty arises from the requirement in the theorem that the arrival process be Poisson . For M/G/1, say, arrivals bypass the queue when the server is idle . The arrival

process to the queue (as against to the system) is then a Poisson point process with censored epochs i.e. is not Poisson. Nevertheless , for an M/G/1 system, the time in queue and the number in queue do obey Little's Law in distribution.

This may be first verified from the familiar M/G/1 results . Let $\pi_{NQ}(u)$ be the pgf of the ergodic number in the queue and let $\alpha_{TQ}(s)$ be the transform of the ergodic time in queue. Let $\pi_{NS}(u)$ be the pgf of the ergodic number in the system and let $\alpha_{TS}(s)$ be the transform of the ergodic time in system. Let $\alpha_T(s)$ be the transform of the service time. We first note that $\pi_{NQ}(u)$ has contributions from the idle state and from the states where the server is busy. One then has

$$\pi_{NQ}(u) = \frac{\pi_{NS}(u) - (1-\rho)}{u} + (1-\rho) .$$

From our theorem, $\pi_{NS}(u) = \alpha_{TS}(\lambda-\lambda u) = \alpha_{TQ}(\lambda-\lambda u)\alpha_T(\lambda-\lambda u)$. One has

$$\pi_{NQ}(u) = \frac{\alpha_{TQ}(\lambda-\lambda u)\alpha_T(\lambda-\lambda u) - (1-\rho)(1-u)}{u} .$$

Since the time in queue at ergodicity coincides with the ergodic waiting time , we may employ the Pollaczek-Khintchine Law for $\alpha_{TQ}(s)$ to see that

$$\pi_{NQ}(u) = \frac{\dfrac{(1-\rho)(1-u)\alpha_T(\lambda-\lambda u)}{\alpha_T(\lambda-\lambda u)-u} - (1-\rho)(1-u)}{u}$$

$$= \frac{(1-\rho)(1-u)\left\{ \dfrac{\alpha_T(\lambda-\lambda u)}{\alpha_T(\lambda-\lambda u)-u} - 1 \right\}}{u} = \frac{(1-\rho)(1-u)}{\alpha_T(\lambda-\lambda u)-u} = \alpha_{TQ}(\lambda-\lambda u) ,$$

as required.

The validity of Little's Law in distribution could also have been obtained from our theorem with the help of the following artifice. One could replace the M/G/1 system by an M/G/1 vacation system (e.g., [1]) where the server takes a vacation of duration D whenever it finds the queue idle . In this way

the queue is treated as a subsystem with Poisson arrivals and the theorem gives the LLD result. When a sequence of such subsystems with vacation durations D_j is considered, and $D_j \rightarrow 0$, LLD holds for each j and hence in the limit for M/G/1.

The same artifice or a similar artifice could be applied to all of the customer classes described in Section 3 to conclude that the LLD property is available for the ergodic number in queue and time in queue.

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