WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

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AND THE
HENDRY PARTITIONING APPROACH*

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WP 1072-79 Revised August 1979

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* The author is grateful to Don Morrison of Columbia University, Jerry Herniter of Boston University, and David and Ben Butler of the Hendry Corporation for their comments on the previous draft of this paper.

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ABSTRACT

The entropy concept plays an important role in the derivation of the theoretical switching constant in the Hendry partitioning approach. This paper discusses the application of the entropy concept to the marketing analyses. It questions two of the assumptions in the Hendry derivation of the theoretical switching constant. A brief explanatory comment on mixed-mode partitioning structures is also made.

KEY WORDS: Entropy concept, Hendry model, Market partitioning, Product class segmentation.

1. INTRODUCTION

The market partitioning approach of the Hendry System continues to attract the attention of marketing practitioners and academics. Rubinson and Bass (1978) recently added a note to the growing collection of articles which have been written to provide insight into the Hendry System (e.g., Herniter 1974, Kalwani and Morrison 1977). The Hendry Corporation itself has published a book that describes different aspects of the Hendry System (The Hendry Corporation 1976). Major corporations, such as Coca Cola, General Foods, Lever Brothers, and Ogilvy and Mather, continue to use the Hendry approach for market partitioning and other marketing applications.

The theoretical switching constant is a central component of the Hendry partitioning approach. Its derivation is based on the entropy concept. This paper questions the derivation of the theoretical switching constant by examining two of its major assumptions. One of these assumptions is inconsistent with the very definition of entropy; the other assumption is arbitrary. The Hendry expression for the theoretical switching constant, as will be shown later, is a function only of the brand shares within a product category.[1]

This article is organized as follows. First, the Hendry partitioning approach is briefly described. This description includes a short explanatory comment on mixed-mode partitioning structures. A physics-based view of the entropy concept is then presented and applied in two marketing cases: (i) when all that is known about the system is that there are g brands, and (ii) when the shares of these g brands are known. The two major assumptions in the derivation of the theoretical switching constant are examined in light of the entropy concept. The appendices provide derivations of maximum entropy solutions for the two marketing cases.

2. HENDRY PARTITIONING APPROACH

The Key Relationship:

The Hendry partitioning approach provides an understanding of the direct versus indirect competition a product faces. In the Hendry model, two alternatives are assumed to be in direct competition if the switching to (and between) them from any other alternative is in direct proportion to their shares. Let X and Y be dry bleach brands, with brand X having twice the share of brand Y. Furthermore, suppose that switchers from brand Z, whether liquid or dry bleach purchasers, are twice as likely to choose X as to choose Y. Then, brands X and Y are said to be in direct competition. If, on the other hand, X is a dry bleach and Y is a liquid bleach and they are not in direct competition, the switchers are no longer apt to switch to them in proportion to their shares. Paraphrasing Butler (1975):

> Product alternatives are in direct competition if the switching to (and between) them is in proportion to their shares.

Kalwani and Morrison (1977) show that with two assumptions--(1) zero order choice process and (2) switching is proportional to share--the switching between alternative brands i and j in direct competition on two consecutive purchase occasions is given by

$$P_{(i,j)} = K_w S_i S_j .$$

K_w is the switching constant for the set of alternatives in direct competition and S_i and S_j are the shares of brands i and j, respectively.

The switching constant, K_w, can be shown to be a ratio of the actual switching under homogeneity in consumer purchase probabilities when each buyer has probability S_i of buying brand i (i = 1,2, ..., g). It takes the value zero when the buyers are completely loyal and always buy their favorite brand. It is equal to one, when the consumers are homogeneous and each buys brand i with probability S_i (i = 1,2, ..., g). Thus, K_w is a measure of the degree to which consumers are <u>not</u> precommitted to one brand or another and is a property of the choice category as a whole.

It should be noted that if the probability density function of consumers in a choice category with g alternatives is given by the Dirichlet distribution, then the interswitching between items i and j is given by equation (1). The Dirichlet probability density function is given by

$$f(p_1, p_2, \ldots, p_g) = \frac{\Gamma(\alpha_1 + \alpha_2 + \ldots + \alpha_g)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_g)} \prod_{i=1}^{g} p_i^{\alpha_i - 1}, \qquad (2)$$

$$0 < p_i < 1; \ \alpha_i > 0,$$

where $\sum_{i=1}^{g} p_i = 1$ and $\Gamma(\cdot)$ denotes the gamma function. The density function in equation (2) has (g-1) independent variates and can be rewritten as

$$f(p_1, p_2, \ldots p_g) = \frac{\Gamma(\psi)}{\prod_{i=1}^{g} \Gamma(\psi S_i)} \prod_{i=1}^{g} p_i^{\psi S_i - 1}, \qquad (3)$$

where
$$\Psi = K_W / (1-K_W) = \sum_{i=1}^{g} \alpha_i,$$

$$S_i = \alpha_i / \Psi, \qquad i = 1,2, \ldots, g.$$

It is easy to see from this alternative form of the Dirichlet distribution that $K_W = \Sigma\alpha_i/(\Sigma\alpha_i+1)$ is a property of the choice category and is a measure of the extent to which consumers in the choice category are not precommitted to one alternative or another.

Procedure:

The Hendry partitioning method is an iterative trial-and-error procedure. A hypothetical market-structure based on "expert-judgement" is set up and theoretical switching levels within and between product categories are computed. Empirical switching levels are then compared with the theoretical ones to determine the goodness of fit. Revisions in the hypothetical structure are surmised by noting where the theoretical switching levels exceed the empirical levels and vice versa. After one or more iterative attempts, a partitioning structure is identified which provides a reasonably good fit to the empirical data.

The empirical switching levels are obtained either from panel (or survey) data by comparing purchases on the previous choice occasion with those on the occasion prior to that. The theoretical switching levels require knowledge of the theoretical switching constant and market shares. In the Hendry model, the entropy concept is used to derive an expression for the theoretical switching constant. Its value is a function of only the brand shares within a product category

$$K_W = \frac{\sum_{i=1}^{g} \frac{S_i^2 \ln(1/S_i)}{1 + S_i \ln(1/S_i)}}{\sum_{i=1}^{g} S_i(1 - S_i)} , \qquad (4)$$

where S_i is the share of brand i.

To illustrate, consider the hypothetical partitioning of the bleach market displayed in Figure 1. Note that it is type-primary rather than brand-primary (or mixed-mode). There are five separate product categories and the Hendry partitioning approach requires comparison of the empirical and theoretical switching levels in each of these categories. Consider, for example, the dry bleach category. Assume that there are only three dry bleach brands: A, B, and C, whose shares, based on the purchases on the last choice occasion, are 0.5, 0.3, and 0.2, respectively. Table 1 displays the computation of the value of the theoretical switching constant. This value of K_W (=0.4145) is used to obtain the theoretical switching levels, using equation (1). The theoretical repeat purchase proportions are easily calculated by subtracting the switching levels from the share data. Figure 2 displays the theoretical switching and repeat purchase levels for brands A, B and C. If the observed switching levels - SI(·) and SO(·) - are close to the theoretical ones, then brands A, B and C are in direct competition as assumed.

INSERT TABLE 1 and FIGURES 1 AND 2 HERE

The use of Hendry partitioning approach leads to the identification of two forms of partitioning structures: (i) nested, and (ii) mixed-mode (Rubinson and Bass 1978). In mixed-mode partitioning structures, two product characteristics--say, brand label and type--simultaneously form the primary partitioning level. The theoretical switching constants for switching between brand labels and types are obtained by finding their respective shares within the total market.

Figure 3 is an illustration of a mixed-mode partitioning structure where both brand label and type simultaneously form the primary partitioning level. In Figure 3, this implies that the theoretical switching constant, K_{W_B}, for switching between brands A, B, and C is obtained by substituting A = 50%, B = 20%, and C = 30%, which are their shares within the total market. Similarly, the theoretical switching constant, across types I and II, K_{W_T}, is calculated by substituting type shares--I = 70% and II = 30%--within the total market.

INSERT FIGURE 3 HERE

In the nested structures, type-primary or brand-primary, partitioning is sequential. For instance, in a type-primary structure, the theoretical switching constants for switching between types is obtained by finding the type shares within the total market. Then, the theoretical switching constants for switching between brands are calculated separately for each product type. In other words, at the secondary partitioning level, product types act as separate markets.

3. THE ENTROPY CONCEPT

Entropy is a physics-based concept that can be applied to marketing problems. It is used in the derivation of the theoretical switching constant.

In the physical sciences entropy is defined as the logarithm of the number of ways a system can be arranged, subject to the constraints that are acting on it; that is, the number of microscopic states that are consistent with the macroscopic information about the system's parameters. In the following presentation the system refers to a population of N buyers of a given product class. Brand share information is an example of a macroscopic system parameter.

Before formally deriving the results, an intuitive evaluation of this definition may be helpful. First, note that additional constraints on a system decrease the entropy of the system; that is, the number of different arrangements of buyers that is consistent with the restrictions on the system will be less.

Consider the case where the only information that is available about the system is that there are g brands. Intuition would suggest that the maximum entropy solution for this case is when each consumer has a probability of 1/g of buying each of the g brands in the product class. This solution has the greatest uncertainty in the consumer's choice of product alternatives.

Next, consider the case where the market shares of each of the g brands are known. Let S_i represent the market share of brand i (i = 1,2, ..., g). An intuitive maximum entropy solution for this case is that each consumer has a probability S_i of buying brand i. Note that entropy is minimized when each consumer is completely loyal to a brand, and buys only that brand on every choice occasion. For this case, there is only one possible arrangement of consumers that is consistent with the brand share information.

Intuition aside, formal derivations of the maximum entropy solution in these two cases are considered next; (i) when all that is known about the system is that there are g brands, and (ii) when the shares of these g brands are known.

Formal Derivations

Case (i): Product class with g brands

Suppose that N_i of the N buyers of a product class chose brand i on a given purchase occasion, with the obvious restriction that

$$\sum_{i=1}^{g} N_i = N.$$

Furthermore, suppose that the process of consumers' selection of choice alternatives is probabilistic and that there are no restrictions on the number N_i that can end up choosing an item i (except that $0 \leq N_i \leq N$). In that case, the first one of the N consumers can choose any one of the g alternatives. The second buyer, once again, can choose any one of the g brands, and so on. Thus, there are altogether g^N possible ways in which the N consumers can choose among the g alternatives. Consider this set of g^N ways as a sample space S containing $W(S) = g^N$ sample points, each having the same probability, g^{-N}.

An event $E = \{N_i, i = 1, 2, \ldots, g\}$ is defined as a subset of sample points corresponding to a particular set of values N_i. The number of sample points in E (the number of ways in which the event can occur) is

$$W(E) = \frac{N!}{\prod_{i=1}^{g} N_i!} \quad \text{with} \quad \sum_{i=1}^{g} N_i = N. \tag{5}$$

The term W(E) represents the number of microscopic states that are consistent with the macroscopic information about the set of values N_i. Assuming equal probabilities for all sample points (since all of them are equally likely), the probability of the event, E, P(E), is given as

$$P(E) = W(E)g^{-N}. \tag{6}$$

The most probable event is determined by finding the event that can occur in the maximum number of ways, i.e., by maximizing W(E) or its monotonic transform, lnW(E). Taking natural logarithms in equation (5) gives

$$\ln W(E) = \ln N! = \sum_{i=1}^{g} N_i!. \qquad (7)$$

Equation (7) can be simplified by the use of Stirling's approximation which states that if an integer X is large enough, say X > 30, X! is given by

$$X! = (2 \pi X) X^X e^{-X},$$

from which

$$\ln X! \simeq 1/2 \ln (2 \pi X) + X \ln^X - X.$$

The first term in the expression above is of negligible magnitude since it is the logarithm of a large number whereas the other two terms are numbers themselves, that in marketing applications are likely to be of large magnitude. Hence,

$$\ln X! = X \ln X - X.$$

Therefore, equation (7) simplifies to

$$\ln W(E) = N \ln N - \sum_{i=1}^{g} N_i \ln N_i. \qquad (8)$$

The first term on the right hand side of equation (8) is a constant, so finding the most probable event is equivalent to maximizing the second term (with the minus sign). The values of N_i's which maximize entropy are found by constructing the Lagrangian:

$$L = -\sum_{i=1}^{g} N_i \ln N_i - \lambda_0 \left(\sum_{i=1}^{g} N_i - N \right). \qquad (9)$$

The N_i's which maximize the above Lagrangian can be easily shown to each equal to $\frac{N}{g}$. Substituting $N_i^* = \frac{N}{g}$ in equation (8) gives

$$\ln W(E) = N \ln g;$$

therefore,

$$W(E) = g^N. \qquad (10)$$

To summarize, when there are no constraints except that there are g brands, the maximum entropy solution (the most probable event) is obtained when each brand is purchased by the same number of consumers, $\frac{N}{g}$. This event can occur in g^N different number of ways.[2]

Let the purchase probabilities of consumer k for the various choice alternatives be given by

$$\underset{\sim}{P}_k = (P_{1k}, P_{2k}, \ldots, P_{gk}),$$

where $\sum_{i=1}^{g} P_{ik} = 1$. Furthermore, let F_k buyers out of N (where $\sum_k F_k = N$) make their purchase decisions according to the purchase probability vector $\underset{\sim}{P}_k$. The entropy, H_k, due to each consumer with probability vector, P_k, is given by

$$H_k = -\sum_{i=1}^{g} P_{ik} \ln P_{ik},$$

and that due to the population of N buyers is given by[3]

$$H = -\sum_k F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik}. \tag{11}$$

The values of P_{ik}'s which maximize entropy (assuming F_k's are fixed) are found by constructing the Lagrangian

$$L = -\sum_i F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik} - \sum_k \lambda_k \left(\sum_{i=1}^{g} P_{ik} - 1 \right).$$

The P_{ik}'s which maximize the above Lagrangian can be easily shown to be (see Appendix 1).

$$P_{ik} = \frac{1}{g}, \qquad i = 1, 2, \ldots, g, \quad \text{for all } k. \tag{12}$$

$$H = -N \sum_{i=1}^{g} \frac{1}{g} \ln \frac{1}{g} = N \ln g. \tag{13}$$

Taking the antilog of the entropy S gives the number of arrangements[4] of the system, g^N. This result matches the earlier derivation of the maximum entropy solution, in equation (10).

Case (ii): Product class with g brands when brand shares are known

The values of P_{ik}'s that maximize entropy of the situation are found by constructing the Lagrangian

$$L = -\sum_k F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik} - \sum_k \lambda_k \left(\sum_{i=1}^{g} P_{ik} - 1 \right)$$

$$- \sum_{i=1}^{g} \alpha_i \left(\sum_k F_k P_{ik} - S_i \right), \tag{14}$$

where S_i is the share of item i. The third term in equation (14) accounts for the constraints on brand shares. It is shown in Appendix 2 that the P_{ik}'s which maximize the above Lagrangian are

$$P_{ik} = S_i, \qquad \text{for all } k. \tag{15}$$

Thus, the maximum entropy solution is obtained when the population of N buyers is homogeneous and each consumer has a probability S_i of choosing brand i. The entropy of the system can be obtained by substituting $P^*_{ik} = S_i$ in equation (11).

$$H = -N \sum_{i=1}^{g} S_i \ln S_i = \sum_{i=1}^{g} \ln S_i^{-NS_i}. \tag{16}$$

The number of arrangements, W(E; Si's), is given by taking the antilog of H

$$W(E; S_i\text{'s}) = \prod_{i=1}^{g} S_i^{-NS_i}. \tag{17}$$

As expected, the number of arrangements W(E; S_i's) when brand shares are known is smaller than W(E) where there are no constraints on market shares. The system entropy is reduced by the additional constraint on brand shares.

4. DERIVATION OF THE THEORETICAL SWITCHING CONSTANT

Actual consumers exhibit brand loyalties in their purchase behavior. They tend to buy their favorite brands repeatedly. They have a higher probability of buying one of the brands (their favorite one) and consequently, a smaller probability of purchasing any of the other brands in the product category. In other words, consumers' purchase behavior reveals self-imposed restrictions through their tendency to buy their favorite brands repeatedly. These constraints imply a reduction in the entropy of the system from case (ii) where brand loyalties were not known.

In the Hendry model, V_i is the expected value of the unconditional probability of repeat buying of brand i. It is the proportion of the total population of N consumers who are expected to repeat buy brand i and represents loyalty toward brand i at the aggregate market level. The term NV_i is the expected number of repeat buyers of brand i. Similarly, U_i denotes the expected value of the unconditional probability of switching into brand i from one of the other items. These definitions of V_i and U_i give

$$NS_i = NV_i + NU_i, \tag{18}$$

where the term NS_i is the expected number of buyers of brand i, and the terms NV_i and NU_i denote the expected repeat buyers of brand i and switchers into brand i, respectively. Now, let $W(E;\ S_i\text{'s},\ V_i\text{'s})$ denote the number of system arrangements when there is knowledge of the g brand shares, S_i's, and consumer loyalties towards the g brands, V_i's. Then, it follows from the definition of the entropy concept that

$$W(E;\ S_i\text{'s},\ V_i\text{'s}) \leq W(E;\ S_i\text{'s}) \leq W(E),$$

or substituting from equations (10) and (17)

$$W(E;\ S_i\text{'s},\ V_i\text{'s}) \leq \prod_{i=1}^{g} S_i^{-NS_i} \leq g^N = \prod_{i=1}^{g} g^{N/g} \tag{19}$$

Note that the expressions for the number of arrangements $W(E)$ and $W(E;\ S_i\text{'s})$ are orthogonal in terms of the contribution of the number of arrangements due to each item i. That is, the total number of arrangements is a product of the number of arrangements of the buyers of each of the g brands. Assuming that $\Omega = W(E;\ S_i\text{'s},\ V_i\text{'s})$ is also orthogonal, gives

$$\Omega = \prod_{i=1}^{g} \Omega_i = W(E;\ S_i\text{'s},\ V_i\text{'s}).$$

From the definition of the entropy concept, the additional knowledge of consumers' brand loyalties requires that for each brand i,

$$\Omega_i \leq S_i^{-NS_i}, \qquad (20)$$

since the market share as well as consumers' loyalty for each brand are known.

The Hendry model, at this point, with no prior justification, introduces a strong assumption to transform the above inequality into the equality:

$$\Omega_i = (S_i^{-NS_i})^{V_i}, \qquad (21)$$

where the exponent V_i has the same definition as in equation (18) and denotes the expectation of the probability of repeat buying brand i. Algebraically, it is clear that the value of the exponent used to transform the above inequality into an equality should be in the range 0 to 1. But, that it should be V_i is strictly an assumption. The exponent could be U_i or some other function of U_i or V_i. If anything, using the exponent U_i would at least be consistent with the definition of entropy since in that case larger values of U_i would imply greater values of Ω_i and therefore, more "randomness" or "disorder." In the extreme case of all V_i's equal to zero (implying maximum switching or no brand loyalty), the above assumption implies that the system could be arranged in a unique way. This contradicts the definition of entropy which states that the number of arrangements of the system increases as the restrictions on the system are relaxed.

The second major assumption introduced in the Hendry model is that

$$\ln\Omega_i = NU_i, \tag{22}$$

which can also be expressed as

$$\ln\Omega = \sum_{i=1}^{g} \ln\Omega_i = \sum_{i=1}^{g} NU_i = NU. \tag{23}$$

Recall that the logarithm of the number of possible arrangements of a system is equal to the entropy of the system. Hence, the left hand side term of equation (22) or (23) represents the system entropy. The term NU represents the expected number of consumers who do <u>not</u> repeat buy their previous choice. But should the system entropy be equal to the expected number of switchers? Examining the two cases from the previous section may be helpful.

In the first case where the only information available is that there are g brands, the logarithm of the number of system arrangements equals $N\ln g$. The expected proportion of switchers in this case is $\sum_{i=1}^{g} N(\frac{1}{g})\ (1 - \frac{1}{g})$.

These two quantities are <u>not</u> equal:

$$N \ln g \neq \sum_{i=1}^{g} N\frac{1}{g}\left(1 - \frac{1}{g}\right) = N\left(1 - \frac{1}{g}\right). \tag{24}$$

Similarly, in the case where brand shares are known, these quantities, again, are not equal:

$$N \sum_{i=1}^{g} S_i \ln\left(\frac{1}{S_i}\right) \neq N \sum_{i=1}^{g} S_i(1 - S_i). \tag{25}$$

In the second chapter on consumer dynamics published by the Hendry Corporation (1971), this assumption is moved downstream, but its arbitrariness still remains.[5]

Kalwani and Morrison (1977) have shown that the switching constant is the ratio of the expected value of actual total switching to the expected value of total switching for homogeneous consumer purchase probabilities when each consumer buys brand i with probability S_i $(i = 1, 2, \ldots, g)$. Hence,

$$K_W = \frac{\sum_{i=1}^{g} U_i}{\sum_{i=1}^{g} S_i(1 - S_i)}. \tag{26}$$

From equations (21) and (22),

$$U_i = -S_i V_i \ln S_i. \tag{27}$$

Substituting from $V_i = S_i - U_i$ from equation (18) into equation (27) gives

$$U_i = \frac{-S_i^2 \ln S_i}{1 - S_i \ln s_i}. \tag{28}$$

Finally, substituting equation (28) in equation (26) gives the Hendry expression for the theoretical switching constant

$$K_W = \frac{\sum_{i=1}^{g} \frac{-S_i^2 \ln S_i}{1-S_i \ln S_i}}{\sum_{i=1}^{g} S_i(1-S_i)} = \frac{\sum_{i=1}^{g} \frac{S_i^2 \ln(1/S_i)}{1+S_i \ln(1/S_i)}}{\sum_{i=1}^{g} S_i\,(1 - S_i)} \qquad (29)$$

As stated earlier, the theoretical switching constant depends only on the shares of brands within a product class.

5. DISCUSSION AND CONCLUSIONS

In the previous section, questions were raised concerning two of the assumptions in the Hendry derivation of the theoretical switching constant. It must be stressed that this criticism was directed solely at the derivation and the resulting Hendry expression (equation (29)) for the value of the theoretical switching constant. No fault was found with the relationship that switching is proportional to share that is used to define items in direct competition. This, however, is not a theoretical, but an empirical relationship which may or may not hold in a given situation. The Hendry people based on empirical analyses of consumer panel data in many different product categories contend that it most often does and, as discussed below, use it extensively in identifying partitioning structures.

In the Hendry model, this empirical switching relationship is used to define direct competition within a product category. It is also used to analyze a product category in disequilibrium. For illustration, consider the dry-bleach example in Figure 2. Suppose that brand C is "under disturbance" as a result of an aggressive promotion of product design improvements, and that the campaign is successful. The Hendry model posits that in this situation the switching levels between brands A and B will still be balanced. The gain in share of brand C will be drawn from brands A and B in proportion to their shares. Thus, five-eights of the gain in the share of brand C will come from brand A.

Recall that the switching is proportional to share relationship, when combined with the zero-order assumption, leads to equation (1) for switching between items i and j. There is considerable empirical and theoretical support for equation (1) which states that the switching between two choice alternatives is equal to a constant times their shares within the choice category.[6] Under these assumptions, Kalwani and Morrison (1977) show that the empirical switching constant for a given product category is equal to the ratio of actual switching to the expected switching when consumer choice probabilities are homogeneous and each individual's probability of choosing alternative i is equal to its share within the category S_i. Further, they show that the empirical switching constant is also obtained by dividing the interswitching between any pair of items in the product category by their shares within the category.

Presumably, these switching relationships can be used to identify items in direct competition. The problem, however, is that while the above switching patterns always hold for items in direct competition, they may also, although rarely, hold for items that are not in direct competition.

Consider a grouping of the following three items: a brand of toothpaste, a brand of catsup, and a brand of coffee. These items are obviously not in direct competition. Assume that the zero-order process assumption holds. The switching between these items will be random with the empirical switching constant equal to one. All of the above switching relationships will then be satisfied despite the fact that the items are not in direct competition.

This suggests the need for theoretical criteria which can be used to obtain expected switching levels and thus to help verify a partitioning hypothesis. In the Hendry model this is accomplished by obtaining an expression for the theoretical switching constant which is a function of shares of choice alternatives within the category. The value of the empirical switching constant is compared with that of the theoretical switching constant by comparing the empirical and theoretical switching levels. Therefore, if the value of the theoretical switching constant as given by equation (29) is incorrect, true partitioning hypotheses may be rejected and false ones accepted. Theoretical switching constant is also important in analyzing switching relationships within product categories that are at disequilibrium. It helps to identify the item that is "under disturbance" and the gain (or loss) in its share that is being drawn proportionately from competing items. Improved assumptions are, therefore, needed in the derivation of the theoretical switching constant since it plays such a central role in the Hendry partitioning approach.

APPENDIX I

This appendix derives the set of consumers' purchase probabilities that maximizes the system entropy when the only information that is available is that there are g brands in the market. The entropy of the system, H, from the population of N buyers is given by

$$H = -\sum_k F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik}, \tag{A1}$$

where P_{ik} is consumer k's probability of purchasing brand i and F_k is the number of buyers out of N who make their purchase decisions according to the purchase probability vector $\underline{P_k}$. The values of P_{ik}'s that maximize the system entropy are found by constructing the Lagrangian

$$L = -\sum_k F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik} - \sum_k \lambda_k \left(\sum_{i=1}^{g} P_{ik} - 1 \right).$$

To maximize L, set the partial derivatives of L with respect to P_{ik} equal to zero, which gives

$$\frac{\partial L}{\partial P_{ik}} = -F_k \ln P_{ik} - F_k - \lambda_k = 0, \tag{A2}$$

for i - 1, 2, ..., g and all k. Equation (A2) implies

$$\ln P_{ik} = (\lambda_k + F_k)/F_k, \qquad i = 1, 2, \ldots, g, \text{ for all } k \tag{A3}$$

Since the expression for P_{ik} is independent of i, the P_{ik}'s are equal for any given consumer k. Therefore

$$P_{ik} = \frac{1}{g}, \qquad i = 1, 2, \ldots, g, \text{ for all } k \quad (A4)$$

which implies that the entire population of N consumers is homogeneous and that each consumer buys each of the g brands with probability $\frac{1}{g}$. Substituting $P^*_{ik} = \frac{1}{g}$ in equation (A1), the entropy of the system is given by

$$H = -N \sum_{i=1}^{g} \frac{1}{g} \ln \frac{1}{g} = N\ln g. \qquad (A5)$$

The number of arrangements is the antilog of H which is g^N as before (see equation (10)).

APPENDIX 2

This appendix derives the set of consumers' purchase probabilities that maximizes the system entropy when brand shares are known. The values of P_{ik}'s which maximize entropy for this situation are found by constructing the Lagrangian

$$L = -\Sigma F_k \sum_{i=1}^{g} P_{ik} \ln P_{ik} - \sum_k \lambda_k \left(\sum_{i=1}^{g} P_{ik} - 1\right)$$

$$- \sum_{i=1}^{g} \alpha_i \left(\sum_{i=1}^{g} F_i P_{ik} - 1\right), \qquad (A6)$$

where S_i is the market share of brand i. The third term in equation (A6) represents the brand shares constraints. To maximize the Lagrangian in equation (A6), set the partial derivatives of L with respect to P_{ik} equal to zero, which gives

$$\frac{\partial L}{\partial P_{ik}} = -F_k \ln P_{ik} - F_k - \lambda_k - \alpha_i F_k = 0 \qquad \text{(A7)}$$

for i = 1, 2, ..., g for all k. Equation (A7) implies

$$\ln P_{ik} + \alpha_i = \lambda_k / Fk, \qquad \text{for all i and k.} \qquad \text{(A8)}$$

The above equation can be rewritten for brand j as follows

$$\ln P_{jk} + \alpha_j = \lambda_k / F_k, \qquad \text{for all j and k.} \qquad \text{(A9)}$$

This implies for consumer k

$$\ln\left(\frac{P_{ik}}{P_{jk}}\right) = \alpha_j - \alpha_i, \qquad \text{(A10)}$$

where i and j are any two of the g brands. Similarly, for consumer 1,

$$\ln\left(\frac{P_{i1}}{P_{j1}}\right) = \alpha_j - \alpha_i. \qquad \text{(A11)}$$

Therefore

$$\frac{P_{ik}}{P_{jk}} = \frac{P_{i1}}{P_{j1}}$$

or,

$$P_{ik} = P_{i1} = S_i, \qquad i = 1, 2, \ldots, g. \tag{A12}$$

Thus, the maximum entropy solution is obtained when the entire population of N consumers is homogeneous and each has a purchase probability of S_i of buying brand i. The entropy of the system can be obtained by substituting for P^*_{ik} in equation (A1)

$$H = -N \prod_{i=1}^{g} S_i \ln S_i = \sum_{i=1}^{g} \ln S_i^{-NS_i} \tag{A13}$$

The number of arrangements are given by the antilog of H

$$W(E; S_i\text{'s}) = \prod_{i=1}^{g} S_i^{-NS_i} \tag{A14}$$

where S_i is the market share of brand i.

Footnotes

1. In general terms, the theoretical switching constant is a parameter which helps predict the theoretical switching between choice alternatives within a product category (e.g., the choice of instant versus ground coffees within the coffee market, the choice of brands within the ground coffee category). For the sake of simplicity, in deriving numerical results in the third and fourth sections of this paper, the term brand is used as a surrogate for the term choice alternative.

2. Substituting for W(E) from equation (10) in equation (6) leads to P(E) = 1. This suggests that the probability distribution described by equation (1) is sharply peaked and further, that the total number of system arrangements is equal to the maximum number of system arrangements. This is consistent with the findings in the physical sciences (Reif 1965, p. 111).

3. This is a new expression for system entropy where the term system refers to the population of N buyers. It can be extended to the continuous case by substituting the integral signs for the summation notation. As shown in this paper its maximization leads to intuitively appealing results.

4. It is easy to show that the logarithm of the number of ways a system can be arranged gives the entropy of the system (Reif 1965, p. 119).

5. During a personal conversation on the previous draft of this paper, David Butler of the Hendry Corporation defended this assumption in that both sides of equation (22) have the same units (number of buyers). This is correct, which, however, only implies that

$$\alpha \ln\Omega = NU, \qquad 0 \leq \alpha \leq 1.$$

That, in this case with knowledge of brand loyalty, the logarithm of the number of system arrangements should be less than the expected number of switchers is consistent with the results of the two cases analyzed in Section 3. This is clear from equations (24) and (25) which reveal that the left hand side terms representing the logarithm of the number of system arrangements are less than the right hand side terms representing the expected number of switchers.

6. The duplication-of-purchase law of Ehrenberg and his colleagues (Ehrenberg 1972) has the same mathematical form as equation (1), and has been shown to hold good for a wide variety of time-periods and product fields (Ehrenberg 1972, Ehrenberg and Goodhardt 1968 and 1969). Bass (1974) has shown that equation (1) can be obtained more fundamentally, starting with individual consumer preferences and combining them into purchase probabilities through Luce's choice axiom.

TABLE 1

Computation of the Theoretical Switching Constant

Brand	Number of Buyers	Brand Share, S_i	$\dfrac{S_i^2 \ln(1/S_i)}{1 + S_i \ln(1/S_i)}$	$S_i(1 - S_i)$
A	500	0.5	0.1287	0.25
B	300	0.3	0.0796	0.21
C	200	0.2	0.0487	0.16
	1,000	1.0	0.2570	0.62

$$K_w = \frac{0.2570}{0.6200} = 0.4145$$

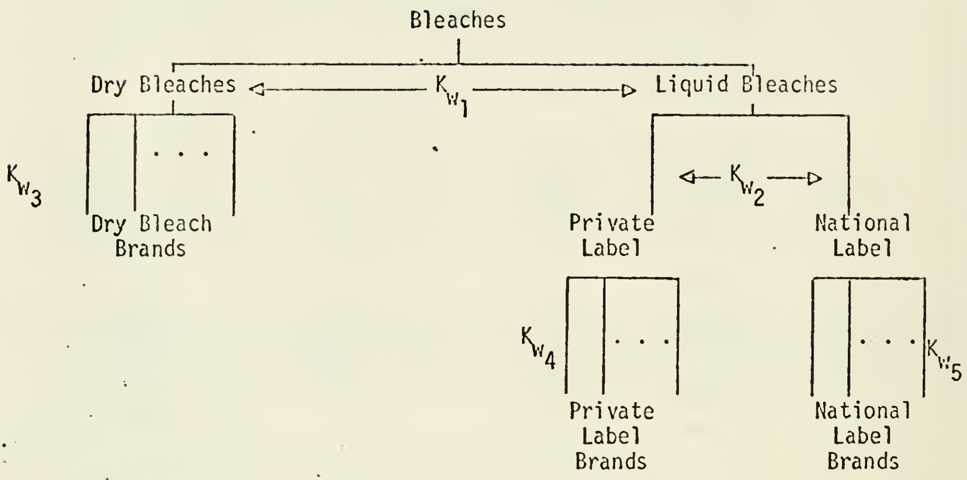


Figure 1 Legend:

K_{w_1} = Switching constant describing switching across dry versus liquid bleach categories.

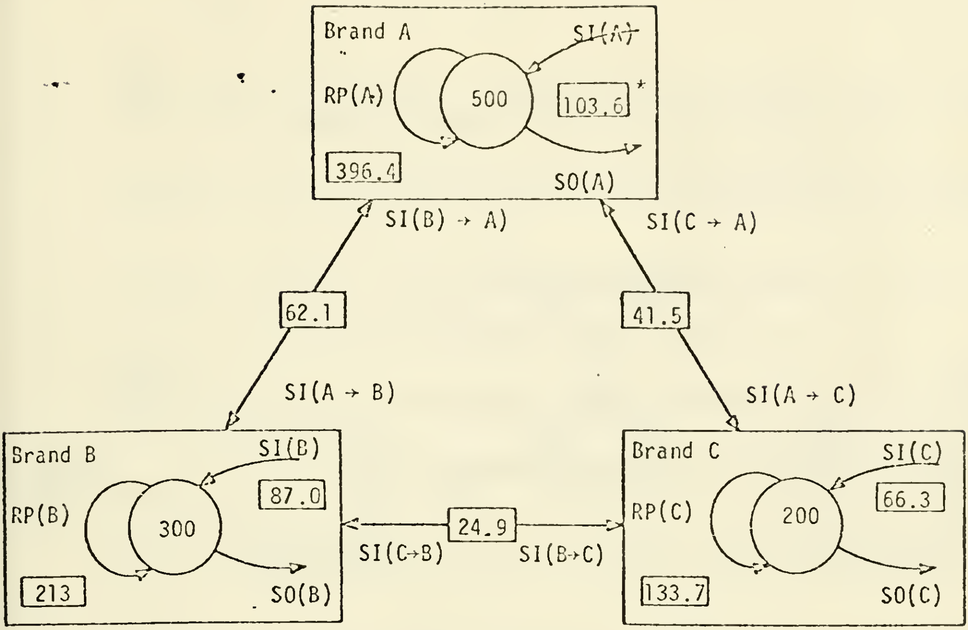
K_{w_2} = Switching constant describing switching between private label and national label liquid bleaches.

K_{w_3} = Switching constant describing switching between dry bleach brands.

K_{w_4} = Switching constant describing switching between private label liquid bleach brands.

K_{w_5} = Switching constant describing switching between national label liquid bleach brands.

FIGURE 1. A Hypothetical Type-Primary Partitioning of the Bleach Market.



<u>LEGEND:</u> SI(·) = Observed number of consumers switching to a brand.
SO(·) = Observed number of consumers switching out of a brand.
RP(·) = Observed number of consumers who repeat buy a brand.

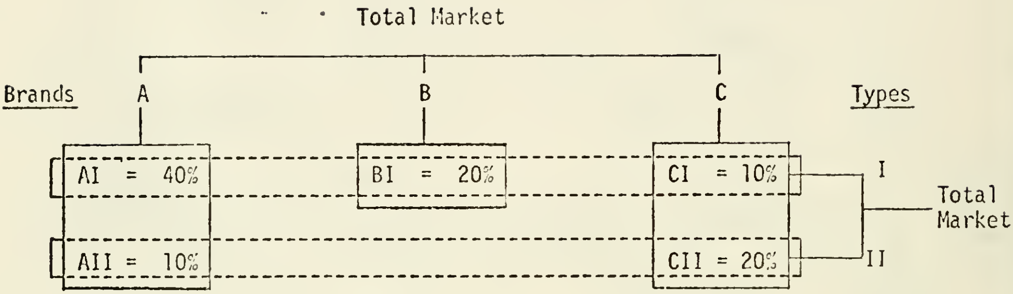
Theoretical switching levels are displayed in small rectangular boxes.

<u>NOTES:</u>

1. $103.6 = N\,K_W\,S_A(1 - S_A)$
   $= 1000\,(0.4145)(.5)(.5)$
2. $SI(A) = SI(B \rightarrow A) + SI(C \rightarrow A)$

*A total of 500 buyers chose brand A on the last choice occasion. Of them, 103.6 were expected to be switching into brand A and 396.4 (= 500 - 103.6) were expected to be repeat buying it.

FIGURE 2. Computation of the Theoretical Switching Levels



K_{W_B} (Across Brands) = 0.428*

K_{W_T} (Across Types) = 0.523**

* Computed by substituting $S_1 = 0.5$, $S_2 = 0.2$, and $S_3 = 0.3$ in equation (4).

** Computed by substituting $S_1 = 0.7$, and $S_2 = 0.3$ in equation (4).

FIGURE 3. A Hypothetical Mixed-Mode Partitioning Structure

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