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Studies in Projective Combinatorics

by

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Abstract

The original goal of this work was to establish links between two mathematical fields which are theoretically quite distinct but practically closely related: the Grassmann-Cayley (GC) algebra and the theory of linear lattices (LL).

A GC algebra is essentially the exterior algebra of a vector space, endowed with the natural dual of wedge product, an operation which is called the meet. Identities between suitable polynomials in a GC algebra express geometric theorems in a very elegant language, as was first shown by Barnabei, Brini and Rota.

The leading example of a linear lattice is the lattice of subspaces of a vector space. In general, a linear lattice is a lattice of commuting equivalence relations, for which an elegant proof theory was first developed by Haiman. Inequalities between LL polynomials express geometric theorems which are characteristic free and dimension independent.

The main link between GC algebras and linear lattices developed in the present dissertation is the proof that a large class of GC identities introduced by Hawrylycz with the name *Arguesian identities of order 2,* has a lattical counterpart, and therefore the correspondent geometric theorems are lattical.

Since the proof theory for linear lattice is based on a graphical procedure, exploiting a class of graphs larger than the class of series-parallel graphs yielded two new ideas, as described next, the first of which is closely related to a study developed by Crapo.

The main result in the GC setting is the introduction of invariant operations associated to every suitable graph, which generalize the sixth harmonic on the projective line, and the correspondent expansion of the operation as a polynomial in the GC algebra. Issues concerning reducibility and composition of such graph operations are also investigated.

In the LL setting, the notion of a LL can be refined to isolate lattices in which all graphical operations defined in a GC algebra can be defined as well. We introduce this new class of lattices, for which we choose the name *strong* linear lattices, which more closely capture the geometric properties of lattices of projective spaces. It turns out that with little extra assumption, the algebra of the underlying field of a strong linear lattice –if any– can be almost completely recovered with little effort and elegant graphical proofs.

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Contents

Introduction

Classical projective geometry had been set aside for a long time, mostly for two reasons. Firstly, it is a widely shared opinion among mathematicians that most of the important results in this discipline have already been discovered. Secondly, synthetic methods for projective geometry are considered to be too difficult and are no longer used. In recent years, however, the development of two fields, the Grassmann-Cayley algebra and the theory of linear lattices, seem to converge to a common denominator: both can be useful settings to define, study and prove geometric theorems.

The Grassmann-Cayley algebra is the exterior algebra over a vector space endowed with the dual notion of the exterior product. If we call *join* the exterior product, the idea of introducing the natural dual operation, called *meet*, dates back to Grassmann, although much of his work in this area has been neglected for a long time. Only in the last few years has his theory been recognized, developed and expanded, first by Doubilet, Rota and Stein [**6**], then followed by other researches including Barnabei, Brini, Crapo, Hawrylycz, Kung, Huang, Rota, Stein, Sturmfels, White, Whitely and many others. Their research ranged from studying the bracket ring of the Grassmann-Cayley algebra, to the generalization to the supersymmetric algebra and application to many disciplines, including projective geometry, physics, robotics, and computer vision.

Linear lattice theory emerged only in recent years, after Haiman's main work on the subject [**8**]. Many mathematicians in lattice theory had been focusing on modular lattices, which are lattices that satisfy the following identity, discovered by Dedekind:

$$(c\wedge(b\vee a))\vee b = (c\vee b)\wedge(a\vee b).$$

Examples of modular lattices are lattices of subspaces of a vector space, lattices of ideal of a ring, lattices of submodules of a module over a ring, and lattices of normal subgroups of a group. For example, in the lattice of subspaces of a vector space the meet of two subspaces is their set theoretic intersection, and the join of two subspaces is the subspaces generated by them. It turns out that all modular lattices that occur in algebra are endowed with a richer structure. They are lattices of commuting equivalence relations. Two equivalence relations on the same set commute whenever they commute in the sense of composition of relations. Haiman chose the name *linear lattices* for such lattices of commuting equivalence relations. It is not known at the moment whether linear lattices may be characterized by identities. Nevertheless, they are tamer than modular lattices because of the existence of a proof theory, as developed by Haiman. What does such proof theory consist of? It is an iterative algorithm performed on a lattice inequality that splits the inequality into sub-inequalities by a tree-like procedure and eventually establishes that the inequality is true in all linear lattices, or else it automatically provides a counterexample. Thanks to Haiman's algorithm, many linear lattice inequalities can be easily verified. Some of them can be shown to interpret geometric theorem in a projective space.

The operations join and meet in a linear lattice are indicated by the same notation as the join and meet in a Grassmann-Cayley algebra. Moreover, geometric theorems in the projective space can often be derived from identities both in Grassmann-Cayley algebra and in a linear lattice, which look pretty much alike. Nevertheless, the meaning underneath the common symbols is completely different. One of the goals of the present dissertation is to investigate when and how the two identities are related, and whether one may be derived from the other.

The thesis is organized as follows: the first chapter is introductory, and is divided into two sections. The first section develops the basic notions of the Grassmann-Cayley algebra within the context of the exterior algebra of a Peano space, following the presentation of Barnabei, Brini and Rota [**3**]. We define the notion of an exterior polynomial as an expression in extensors, join and meet and recall some elementary properties about extensor polynomial which will be useful in the sequel. Examples of how these methods can be used to prove geometric propositions are also provided. The second section of Chapter 1 introduces the notion of a linear lattice, the translation of lattice polynomials into series-parallel graphs, and presents the proof theory for linear lattices as was developed by Haiman in [**8**].

The main idea of Chapter 2 is the following: the operations meet and join alone are not sufficient to describe all invariant operations on points or subspaces of a projective space. The simplest example of such an operation is the classical sixth harmonic, which constructs a sixth point given five points on the projective line. This operation cannot be described in terms of joins and meets alone within the projective line, since they are all trivial. The standard definition for this construction relies on the embedding of the projective line into a larger space and on some arbitrary choice of elements in this space, on which classical join and meet are then performed. It is natural to ask whether similar invariant operations can be defined on a larger set of points, or in a higher dimensional projective space. In Chapter 2 we provide a machinery to associate an invariant operation on projective subspaces of a projective space to every suitable graph, which we call *graphical operation.* The admissible graphs can be seen as a generalization of series-parallel graphs. Indeed, if the graph is a series-parallel graph, then the operation we obtain is the join-meet operation we would obtain by "reading" the graph as described in Section 2 of Chapter 1. That means that whenever the graph is a series connection of two subgraphs, the operation associated to the graph is the join of the resulting operations associated to the composing subgraphs, and whenever the graph is made of a parallel connection of two subgraphs, the operation associated to the graph is the meet of the operations associated to the composing subgraphs. Next, the graphical operations are expanded in terms of bracket polynomials, which means that we can explicitly write the operation associated to any graph in terms of an invariant polynomial in the Grassmann-Cayley algebra. For the projective line we also study the problem of reducing complex graphs operations in terms of compositions of simpler ones.

Since graphical operations, which are performed on the Grassmann-Cayley algebra, reflect operations on the projective space, we may wonder if we can express the same operations in a linear lattice, which is also a powerful setting for studying geometric propositions and operations. It is easily seen that the mere notion of linear lattice is not strong enough to express graphical operations. Chapter 3 is therefore dedicated to enrich a linear lattice with the same graphical operations we have in a Grassmann-Cayley algebra. We call *strong* such linear lattices. Particular attention is then dedicated to linear lattices in which the simplest graphical operation, the sixth harmonic, can be defined. We chose the name *Wheatstone lattices* for such lattices, since the graph representing the sixth harmonic looks like the Wheatstone bridge of circuit theory. The examples of linear lattices which are relevant to geometry are lattices of submodules of modules over a ring, and therefore they are lattices of subgroups of an Abelian group. Since in Chapter 3 we also show that every lattice of subgroups of an Abelian group is a Wheatstone lattice, that suggests that Wheatstone lattices are more appropriate a setting for studying geometry than linear lattices. The last section of Chapter 3 shows that the process of coordinatization developed by Von Staudt and Von Neumann can be carried on almost completely for a Wheatstone lattice, with simple graph-like proofs. The only missing property for the coordinate ring is the distributivity of "·" with respect to "+", but that is as far as we can get, since counterexamples of Wheatstone lattices for which the coordinate ring does not exist can be easily provided.

In Chapter 4 we go back to the initial problem that motivated this thesis. We look for connections between identities in a Grassmann-Cayley algebra and identities in a linear lattice. GC identities are fairly easy to check, but not easy to generate. The main tool for generating identities is described in Hawrylycz's thesis [**11**]. He provides a machinery which produces a class of identities that he calls *Arguesian*, since they generalize Desargues' theorem on projective plane. Most of classical results in projective geometry can be produced in this way. Mike Hawrylycz also conjectures that these identities are consequences of underlying lattical identities. In Chapter 4 we give a partial positive answer to this conjecture. We provide the lattical identities corresponding to a subclass of all Arguesian identities, more precisely the Arguesian identity of order 2. The simplest Arguesian identity of higher order corresponds to a theorem in the projective plane discovered by Bricard. Every attempt to prove the corresponding lattical identity failed, despite the powerful proof algorithm provided by Haiman. This suggests that Bricard theorem, and more in general Arguesian identities of order higher than 3, may not hold in a linear lattice, though they may hold for Wheatstone or strong lattices, which we conjecture. Chapter 4 contains many examples of lattical identities and closes with the classification of all Arguesian lattical identities for the 3-dimensional spaces.

CHAPTER 1

Background

1. The Grassmann-Cayley Algebra

We start by recalling some of the basic definitions and results of Grassmann-Cayley algebras.

DEFINITION 1.1. A **Peano Space** of **step n** is a pair $(V, [\cdot])$, where V is a vector space of finite dimension n over a field K, and the bracket $[\cdot]$ is an alternating non-degenerate n-linear form over V. In other words it is map

$$x_1, \ldots, x_n \mapsto [x_1, \ldots, x_n] \in K$$

where the vectors x_i belong to V, and satisfy the following properties:

(i) $[x_1, \ldots, x_n] = 0$ if $x_i = x_j$ for some $i \neq j$;
(ii) there exist elements $x_1, \ldots, x_n$ in V such that $[x_1, \ldots, x_n] \neq 0$;
(iii) for every α, β in K and x, y in V,

$$[x_1, \ldots, x_{i-1}, \alpha x + \beta y, x_{i+1}, \ldots, x_n] = \alpha[x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n] + \\ + \beta[x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n]. \quad (1.1)$$

The Peano space $(V, [\cdot])$ will often be referred to by V alone, to alleviate the notation. There are many ways to exhibit a bracket on a vector space V, the most common of which relies on the choice of a basis $e_1, \ldots, e_n$ for V, and the choice of a non-zero value for $[e_1, \ldots, e_n]$. In this way, given vectors $v_i = \sum_j a_{ij} e_j$, $i = 1, \ldots, n$, we can expand the bracket by linearity and obtain

$$[v_1, \ldots, v_n] = \det(a_{ij})[e_1, \ldots, e_n],$$

where $\det(a_{ij})$ is the usual determinant of a matrix. This presentation, in spite of being the most popular, is deceiving in our context, since we want to think of V as a geometric entity that goes beyond the mere choice of a basis.

Let us fix the notation for the exterior algebra of V, since it may differ from the standard one.

DEFINITION 1.2. The **exterior algebra** $\bigwedge(V)$ of the vector space V is obtained as the quotient of the free associative algebra on V by the ideal generated by v^2, for all v in V.

The exterior algebra of V is graded and decomposes as

$$\bigwedge(V) = \bigoplus_{i=0}^{n} \bigwedge^i(V), \text{ where } \dim_K(\bigwedge^i(V)) = \tbinom{n}{i}.$$

The product in $\bigwedge(V)$ will be more often denoted by "$\vee$" and will be called **join.** The elements in $\bigwedge^i(V)$ will be called **tensors** of **step i.** A tensor x of step i will be called **decomposable,** or an **extensor** if there exist vectors $v_1, \ldots, v_i$ so that $x = v_1 \vee \ldots \vee v_i$.

PROPOSITION 1.1. *Every tensor of step* 1 *is decomposable.*

Notice that $\bigwedge^n$ has dimension 1 over K, and the choice of a bracket for V corresponds to the choice of an isomorphism from $\bigwedge^n(V)$ to K.

The join is a powerful tool to express geometric operations, as the following propositions will show.

PROPOSITION 1.2. *Given a subspace W of V of dimension k, and two bases, $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ of W, then for some non-zero constant c,*

$$v_1 \vee \ldots \vee v_k = c\, w_1 \vee \ldots \vee w_k.$$

PROPOSITION 1.3. *Given vectors $v_1, \ldots, v_k$ in V, then $v_1 \vee \ldots \vee v_k \neq 0$ if and only if $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors.*

Thanks to these propositions, we can associate to each extensor x of step k a k-dimensional subspace W of V, and vice versa, though in a non-unique manner. We say that x **supports** W.

PROPOSITION 1.4. *Let x, y be two extensors that support vector spaces X and Y, respectively. Then*

(i) $x \vee y = 0$ *if and only if* $X \cap Y \neq \{0\}$.
(ii) *If* $X \cap Y = \{0\}$, *then the extensor* $x \vee y$ *is the extensor associated to the space spanned by* $X \cup Y$.

Multilinearity and skewness, moreover, imply the following results.

PROPOSITION 1.5. *Given x, y, z in $\bigwedge(V)$ and scalars α, β in K,*

$$(\alpha x + \beta y) \vee z = \alpha(x \vee z) + \beta(y \vee z)$$

PROPOSITION 1.6. *In $\bigwedge(V)$ the following hold:*

(i) *Given x, y of step h, k, then*

$$x \vee y = (-1)^{hk} y \vee x.$$

(ii) *Given vectors $v_1, \ldots, v_i$ and a permutation σ of $\{1, \ldots, i\}$;*

$$v_1 \vee \ldots \vee v_i = \operatorname{sgn}(\sigma) v_{\sigma_1} \vee \ldots \vee v_{\sigma_i}.$$

Given tensors $x_1, \ldots, x_i$, and a permutation σ of $\{1, \ldots, i\}$ we can form tensors $x = x_1 \vee \ldots \vee x_i$ and $y = x_{\sigma_1} \vee \ldots \vee x_{\sigma_i}$. It follows from proposition 1.6 that either $y = x$ or $y = -x$, the sign depending on the permutation and the steps of x and y. When no explicit sign is needed, we will use the following notation.

DEFINITION 1.3. Define $\mathbf{sgn(x, y)}$ to be the number (± 1) such that $x = \operatorname{sgn}(x, y) y$.

DEFINITION 1.4. Given an extensor $x = v_1 \vee \cdots \vee v_i$,

$$[x] = [v_1 \vee \cdots \vee v_i] = \begin{cases} 0 & \text{if } i < n \\ [v_1, \ldots, v_i] & \text{if } i = n \end{cases}$$

This definition is independent of the choice of the representation, and extends the bracket from V to $\bigwedge(V)$, by linearity. Moreover, given tensors $x_1, \ldots, x_i$, we can similarly define

$$[x_1, \ldots, x_i] = [x_1 \vee \ldots \vee x_i].$$

PROPOSITION 1.7. *Let x and y be two extensors of step k and $n-k$. Then*

$$[y, x] = (-1)^{k(n-k)}[x, y].$$

We are now ready to provide $\bigwedge(V)$ with another operation, called the **meet** ($\wedge$), which will play the dual role to the join in the geometry of the space V. We define the meet on the extensors and extend it by linearity to $\bigwedge(V)$. Recall that the join is understood when there is no symbol between two tensors.

DEFINITION 1.5. Given extensors $x = v_1 \cdots v_i$ and $y = w_1 \cdots w_j$, define $x \wedge y = 0$ if $i + j < n$ and

$$x \wedge y = \sum_{\substack{\sigma \in S^i \\ \sigma_1 < \ldots < \sigma_{n-j} \\ \sigma_{n-j+1} < \ldots < \sigma_i}} \operatorname{sgn}(\sigma)[v_{\sigma_1} \cdots v_{\sigma_{n-j}} w_1 \cdots w_j] v_{\sigma_{n-j+1}} \cdots v_{\sigma_i} \tag{1.2}$$

if $i + j \geq n$.

PROPOSITION 1.8. *Expression 1.2 equals to*

$$x \wedge y = \sum_{\substack{\sigma \in S^j \\ \sigma_1 < \ldots < \sigma_{i+j-n} \\ \sigma_{i+j-n+1} < \ldots < \sigma_j}} \operatorname{sgn}(\sigma)[v_1 \cdots v_i w_{\sigma_{i+j-n+1}} \cdots w_{\sigma_j}] w_{\sigma_1} \cdots w_{\sigma_{i+j-n}} \tag{1.3}$$

PROPOSITION 1.9. *In $\bigwedge(V)$ the following hold:*

(i) *The definition can be extended to $\bigwedge(V)$.*
(ii) *The meet is associative.*
(iii) *Let x and y be tensors of step i and j. Then*

$$y \wedge x = (-1)^{(n-i)(n-j)} \, x \wedge y.$$

There are more popular notations to indicate the unpleasant formulas (1.2) and (1.3). The first one is the **dotted notation**, using which the above formulas are written, respectively:

$$\begin{aligned} x \wedge y &= [\dot{v}_1 \cdots \dot{v}_{n-j} w_1 \cdots w_j] \dot{v}_{n-j+1} \ldots \dot{v}_i \\ &= [v_1 \cdots \ v_i \dot{w}_{i+j-n+1} \cdots \dot{w}_j] \dot{w}_1 \cdots \dot{w}_{i+j-n} \end{aligned}$$

where the sign and the summation is implicit in the notation. The most compact notation is the **Sweedler notation**, which comes from the Hopf

Algebra structure of $\bigwedge(V)$, where the sign, unfortunately, is again implicit. Formulas (1.2) and (1.3) in this notation become, respectively:

$$\begin{aligned} x \wedge y &= \sum_{(x)} [x_{(1)} y] x_{(2)} \\ &= \sum_{(y)} [x y_{(2)}] y_{(1)} \end{aligned}$$

The meet has an interesting property for supported spaces. Compare the following with proposition 1.4.

PROPOSITION 1.10. *Let x, y be two extensors that support X and Y, respectively. Then*

(i) $x \wedge y = 0$ *if and only if X, Y do not span V.*
(ii) *If X, Y span V, then the extensor $x \wedge y$ supports $X \cap Y$.*

DEFINITION 1.6. A Peano space $\bigwedge(V)$ of step n equipped with the two operations of join and meet is called the **Grassmann-Cayley algebra of step n** and denoted by $G(V)$, or $GC(n)$. It is a graded double algebra, $G_k(V)$ being the subspace of degree (or step) k.

Since $G_{n-1}(V)$ is n-dimensional, we can establish an isomorphism of $G_{n-1}(V)$ with V. Unfortunately, there is no canonical way of doing it. We can, however, rely on a choice of a basis for V. In what follows, we will use the shorter U to indicate $G_{n-1}(V)$.

DEFINITION 1.7. A basis $e_1, \ldots, e_n$ for the space V is **unimodular** if $[e_1, \ldots, e_n] = 1$.

DEFINITION 1.8. The extensor

$$E = e_1 \vee \ldots \vee e_n$$

of U will be called the **integral.** The integral is well defined and does not depend on the choice of a unimodular basis.

The integral behaves like an identity in the double algebra. More precisely,

PROPOSITION 1.11.

(i) *For every tensor x with step$(x) > 0$, we have*

$$x \vee E = 0, \qquad x \wedge E = x,$$

while, for every scalar k, we have

$$k \vee E = kE, \qquad k \wedge E = k.$$

(ii) *For every n-tuple $(v_1, \ldots, v_n)$ of vectors in v, we have the identity*

$$v_1 \vee \ldots \vee v_n = [v_1, \ldots, v_n] E.$$

Many identities between polynomials in $G(V)$ can be easily derived from the definitions. For a gallery of such identities the reader is referred to [**11**], [**3**]. I will just report the most significant ones in what follows.

PROPOSITION 1.12. *Let x, y be extensors whose steps add up to n. Then*

$$x \vee y = (x \wedge y)E$$

PROPOSITION 1.13. *Let x, y, z be extensors whose steps add up to n. Then*

$$x \wedge (y \vee z) = [x, y, z] = (x \vee y) \wedge z$$

PROPOSITION 1.14. *Let $P(a_i, \vee, \wedge)$ be a non-zero extensor polynomial in $GC(n)$. Then $step(P) = k$ if and only if $\sum_i \mathrm{step}(a_i) \equiv k (\mathrm{mod}\ n)$.*

PROPOSITION 1.15. *Let $\{e_i\}$ be a basis for V. Define*

$$u_i = (-1)^{(i-1)} e_1 \vee \ldots \vee e_{i-1} \vee e_{i+1} \vee \ldots \vee e_n.$$

Then $\{u_i\}$ is a basis of U and $[e_i, u_j] = \delta_{ij}[e_1, \ldots, e_n]$.

DEFINITION 1.9. The set $\{u_i\}$ defined in proposition 1.15 is said to be the **cobasis**, or **dual basis**, of the basis $\{e_i\}$. The elements of U are called **covectors.**

The vector space U can be given a natural Peano structure $[[\cdot]]$ by defining, for $w_1, \ldots, w_n$ in U,

$$[[w_1, \ldots, w_n]] = w_1 \wedge \ldots \wedge w_n. \tag{1.4}$$

THEOREM 1.16 (Cauchy). *Let $v_1, \ldots, v_n$ be a basis of V, and $w_1, \ldots w_n$ its dual basis. Then*

$$[[w_1, \ldots, w_n]] = [v_1, \ldots, v_n]^{n-1}.$$

In particular, unimodularity of $\{v_i\}$ implies unimodularity of $\{w_i\}$. If we call $\bar{\vee}$ the product (or join) in $\bigwedge(U)$, we can then define the meet $\bar{\wedge}$ in $\bigwedge(U)$ to be the following:

$$x \bar{\wedge} y = \sum_{(x)} [[x^{(1)} y]] x^{(2)} = \sum_{(y)} [[x y^{(2)}]] y^{(1)},$$

where the superscript is used in the place of the subscript for the coproduct decomposition of x and y in $\bigwedge(U)$.

At this point we should make more explicit the duality between the exterior algebra of the join and the exterior algebra of the meet. This is achieved by introducing a **Hodge star operator.** The Hodge star operator relative to a given (unimodular) basis of V is the unique algebra isomorphism from $(G(V), \vee)$ to $(G(V), \wedge)$ that maps every element of the basis into its correspondent element of the cobasis, and the unit of K to the integral E. In symbols, the Hodge star operator is defined on a basis of $G(V)$ as follows, and extended by linearity to the whole $G(V)$.

DEFINITION 1.10. Let $e_1, \dots, e_n$ be any unimodular basis of V, and $u_1, \dots, u_n$ its correspondent cobasis. Then the star operator $*$ is defined on a basis of $G(V)$ as

$$\begin{aligned} *1 &= E \\ *e_i &= u_i \\ *(e_{i_1} \vee \dots \vee e_{i_k}) &= (*e_{i_1}) \wedge \dots \wedge (*e_{i_k}). \end{aligned}$$

PROPOSITION 1.17. *The Hodge star operator has the following properties:*

(i) $*$ *maps* $G_i(V)$ *isomorphically onto* G_{n-i};
(ii) $*(x \vee y) = (*x) \wedge (*y)$,
$*(x \wedge y) = (*x) \vee (*y)$, *for every* $x, y \in G(V)$;
(iii) $*1 = E$, $*E = 1$;
(iv) $*(*x) = (-1)^{k(n-k)}\, x$ *for every* $x \in G_k(V)$.

The duality established by the Hodge operator allows us to restate every result valid for the vectors of V in its dual form, namely as results for the covectors of V, as:

PROPOSITION 1.18. *Every covector is decomposable.*

PROPOSITION 1.19. *Every extensor of step* $k < n$ *can be written as the meet of* $n - k$ *covectors.*

We are going to consider polynomials that involve extensors of $G(V)$, join, meet and bracket. Since the extensors are the elements of $G(V)$ that bear the most significant geometric meaning, we refrain from stating the results in the general setting, although most of them would remain basically unchanged.

The first systematic study on identities involving polynomials in joins, meets and brackets was carried out in **[3]**, where the name *alternative laws* was first introduced. Alternative laws are nested in increasing order of generalization and complexity. Instead of providing a comprehensive list of such identities that might bore the reader, or reporting just the "most general one," assuming it existed, I will rather describe the alternative laws that I will use in the next chapters. The reader is referred to **[3]** for a deeper and more detailed study on the subject.

PROPOSITION 1.20. *Let* $x, y_1, \dots, y_k$ *be extensors of* $G(V)$. *Then*

$$x \wedge (y_1 \wedge \dots \wedge y_k) = \sum_{(x)} [x_{(1)}, y_1] \cdots [x_{(k)}, y_k] x_{(k+1)}$$

In case x is explicitly given as join of vectors, $x = v_1 \vee \dots \vee v_i$, the sum over the index (x) means the sum over all the possible choices

$$x_{(i)} = v_{l_1} \vee \dots \vee v_{l_j} \tag{1.5}$$

which are essentially distinct, that is not just permutations of composing vectors. The sign $\operatorname{sgn}(x, x_{(1)} \dots x_{(k+1)})$ is implicit in the decomposition.

Recall that every incomplete bracket evaluates to 0, thus both sides of (1.5) evaluate to zero unless

$$nk \leq \text{step}(x) + \text{step}(y_1) + \cdots + \text{step}(y_k) \leq n(k+1).$$

In general the step of a polynomial is the sum of the steps of the extensors constituting the polynomial minus n times the number of meets. Dualizing proposition 1.20 by a Hodge star operator, we obtain

PROPOSITION 1.21. *Let $x, y_1, \ldots, y_k$ extensors of $G(V)$. Then*

$$x \vee (y_1 \vee \ldots \vee y_k) = \sum_{(x)} [x_{(1)}, y_1] \cdots [x_{(k)}, y_k] x_{(k+1)}$$

COROLLARY 1.22. *Let $a_1, \ldots, a_n$ be vectors, and $\beta_1, \ldots, \beta_n$ be covectors. Then*

$$(a_1 \vee \ldots \vee a_n) \wedge (\beta_1 \wedge \ldots \wedge \beta_n) = \det([a_i, \beta_j]).$$

The next corollary is a simple version of what Brini, Barnabei and Rota call the straightening algorithm in [**3**], and will play a crucial role in the sequel.

COROLLARY 1.23. *Let x, y, z be extensors. Then*

$$[x][y, z] = \sum_{(x)} [x_{(1)} y][x_{(2)} z].$$

Here we work out some specific results for $GC(3)$ to motivate and clarify the notation.

THEOREM 1.24 (Desargues). *Let a, b, c, a', b', c' be six distinct points in the projective plane. Then the lines $L = aa'$, $M = bb'$, $N = cc'$ are concurrent (have a common point) if and only if the points $x = bc \cap b'c'$, $y = ac \cap a'c'$, $z = ab \cap a'b'$ are collinear (lie on a common line).*

In order to prove the theorem we need the following lemma.

LEMMA 1.25. *Let a, b, c, a', b', c' be vectors of $GC(3)$. Then the following identity holds:*

$$[abc][a'b'c'] aa' \wedge bb' \wedge cc' \cdot E = (bc \wedge b'c') \vee (ac \wedge a'c') \vee (ab \wedge a'b'). \tag{1.6}$$

PROOF. Let A, B, C be covectors such that $a' = B \wedge C$, $b' = A \wedge C$, and $c' = A \wedge B$. Then

$$b'c' = (A \wedge C) \vee (A \wedge B) = -((A \wedge C) \vee B) A = [a'b'c'] A,$$

and similarly for $a'c', a'b'$. With this substitution, $L \wedge M \wedge N$ becomes:

$$(a \vee (B \wedge C)) \wedge (b \vee (A \wedge C)) \wedge (c \vee (A \wedge B)) =$$
$$= ([aB]C - [aC]B) \wedge ([bA]C - [bC]A) \wedge ([cA]B - [cB]A)$$

which, by associativity and anticommutativity of the meet, expands to

$$([aB][bC][cA] + [aC][bA][cB]) A \wedge B \wedge C. \tag{1.7}$$

Similarly, $x \vee y \vee z$ expands to

$$([aB][bC][cA] + [aC][bA][cB])(a \vee b \vee c)[a'b'c']^3. \tag{1.8}$$

Recalling that $a \vee b \vee c = [abc]E$ and that, by Cauchy's theorem (1.16), $A \wedge B \wedge C = [a'b'c']^2$, polynomials (1.7) and (1.8) are equal up to a constant, which proves the lemma. □

REMARK . If our concern is the theorem, we don't need to keep track of the constants, since they don't play any role in the geometry, provided they are not zero.

PROOF (THEOREM). We identify a geometric object (point or line) with an extensor in $GC(3)$ supporting it, which bear the same name, as in the lemma. By the lemma, if the points $a, \dots, c'$ are distinct, then

$$\begin{array}{ccc} LHS\ of\ (1.6) = 0 & \Longleftrightarrow & RHS\ of\ (1.6) = 0 \\ \Updownarrow & & \Updownarrow \\ L, M, N\ concurrent & & x, y, z\ collinear \end{array}$$

which proves the theorem. □

In the same way most of the classical results of projective geometry can be proved. We recall here Pappus' theorem and a theorem due to Bricard, since we will mention them again in next chapters. The following identity appears in [**10**].

THEOREM 1.26 (Pappus). *For a, b, c, a', b', c' in GC(3), the following identity holds:*

$$(bc' \wedge b'c) \vee (ca' \wedge c'a) \vee (ab' \wedge a'b) = (c'b \wedge b'c) \vee (ca' \wedge ab) \vee (ab' \wedge a'c')$$

□

THEOREM 1.27 (Pappus). *If points a, b, c are collinear and a', b', c' are collinear and all distinct in the projective plane, then the intersections $ab' \cap a'b$, $bc' \cap b'c$ and $ca' \cap c'a$ are collinear.*

PROOF. Since a, b, c and a', b', c' are two sets of three collinear points, $ca' \wedge ab = c[a'ab]$ and $ab' \wedge a'c' = -b'[aa'c']$ and therefore the right hand side of 1.26 reduces to $[a'ab][a'ac'](c'b \wedge b'c) \vee c \vee b'$, which equals zero. Hence the left hand side vanishes as well, which implies that $ab' \cap a'b$, $bc' \cap b'c$, and $ca' \cap c'a$ are collinear. □

THEOREM 1.28 (Bricard). *Let a, b, c and a', b', c' be two triangles in the projective plane. Form the lines aa', bb', and cc' joining respective vertices. Then these lines intersect the opposite edges bc, ac, and ab in collinear points if and only if the join of the points $bc \cap b'c'$, $ac \cap a'c'$ and $ab \cap a'b'$ to the opposite vertices a', b' and c' form three concurrent lines.*

PROOF. In GC(3) the following identity holds

$$[abc]^2(aa' \wedge b'c') \vee (bb' \wedge a'c') \vee (cc' \wedge a'b') =$$
$$= [a'b'c']((bc \wedge b'c') \vee a) \wedge ((ac \wedge a'c') \vee b) \wedge ((ab \wedge a'b') \vee c)E. \tag{1.9}$$

The left hand side vanishes if and only if the right hand side vanishes, which directly implies the theorem. □

2. Linear Lattices

The set of subspaces of a vector space, endowed with two binary operations, intersection and subspace generated by, has a lattice structure in the usual sense [**4**]. For long time this lattice was considered as a modular lattice. Modular lattices are lattices that satisfy the following identity, discovered by Dedekind:

$$(c\wedge(a\vee b))\vee b=(c\vee b)\wedge(a\vee b).$$

The notion of modular lattice misled mathematicians for decades. In fact all modular lattices which occur in algebra are endowed with a richer structure. They are lattices of commuting equivalence relations. This property is fundamental in constructing a proof theory for linear lattices. In what follows, we are going to recall the basic notation we will need and the main results on linear lattices.

Given a set S, a **relation** on S is a subset R of $S\times S$. A relation is sometimes identified with a bipartite graph, namely, with a graph on the set S of vertices whose edges are the pairs (α,β) which belong to R.

On the set $\mathcal{A}$ of relations on S, all Boolean operations among sets are defined. For example, $\cup$ and $\cap$ are the usual union and intersection; similarly, one defines the complement of a relation. The identity relation is the relation $I=\{(\alpha,\alpha):\alpha\in S\}$. In addition, composition of relations is defined, which is the analog for relations of functional composition. If R and T are relations, define:

$$R\circ T=\{(\alpha,\beta)\in S\times S\ \mid\ \exists\,\gamma\in S\ \ s.t.\ (\alpha,\gamma)\in R,\ (\gamma,\beta)\in T\}.$$

Given a relation R, one defines the **inverse relation** as follows:

$$R^{-1}=\{(\alpha,\beta)|(\beta,\alpha)\in R\}.$$

It has been shown that this wealth of operations defined on the set of all relations on S does not suffice to give an algebraic characterization of the set of relations, and such an algebraic description remains to this day an open problem.

There is, however, a class of relations for which an extensive algebraic theory can be developed, and that is the class of commuting equivalence relations on a set S. We say that two relations R and T **commute** when $R\circ T=T\circ R$.

Recall that a relation R is an **equivalence relation** if it is reflexive, symmetric and transitive. In the notation introduced above, these three properties are expressed by $R\supseteq I$, $R^{-1}=R$, and $R\circ R\subseteq R$, where I is the identity relation. The equivalence classes of an equivalence relation form a **partition** of S.

Recall that a partition a of a set S is a family of nonempty subsets of S, called blocks, which are pairwise disjoint and whose union is the set S. Clearly, every partition a of S defines a unique equivalence relation whose equivalence classes are the blocks of a. In other words, the notions of equivalence relation and partition are mathematically identical, though psychologically different.

We denote by R_a or $R(a)$ the equivalence relation associated to the partition a. We will often write $\alpha R_a \beta$ in place of $(\alpha, \beta) \in R_a$. The set of partitions of a set S, denoted by $\Pi(S)$, is endowed with the partial order of **refinement**: we say that $a \leq b$ when every block of a is contained in a block of b. The refinement partial order has a unique maximal element $\hat{1}$, namely, the partition having only one block, and a unique minimal element $\hat{0}$, namely, the partition for which every block has exactly one element.

One verifies that the partially ordered set $\Pi(S)$ is a lattice. Lattice meets and joins, denoted by $a \vee b$ and $a \wedge b$, can be described by using the equivalence relations $R(a)$ and $R(b)$ as follows:

PROPOSITION 1.29. *For a, b partitions of S,*

$$R(a \wedge b) = R(a) \cap R(b) \tag{1.10}$$

$$R(a \vee b) = (R_a \circ R_b) \cup (R_a \circ R_b \circ R_a) \cup \cdots \cup (R_b \circ R_a) \cup (R_b \circ R_a \circ R_b) \cup \cdots \tag{1.11}$$

PROOF. (1.10) is immediate, while to prove (1.11) recall that $a \vee b$ is the smallest partition containing a and b. By imposing the transitivity property of $R(a \vee b)$ one obtains the long expression (1.11). □

It can be shown that the number of iterations of compositions of relations as well as of unions in (1.11) cannot be bounded a priori. As a matter of fact, pairs of relations can be classified according to the minimum number of terms in (1.11) required to express the join of the corresponding partitions. In what follows, we will be interested only in pairs of equivalence relations for which the expression for the join has the minimum allowable number of terms. It turns out that pairs of equivalence relations for which the right side of (1.11) reduces to two terms are precisely commuting equivalence relations. In symbols,

PROPOSITION 1.30. $R(a \vee b) = R_a \circ R_b$ *iff* $R_a \circ R_b = R_b \circ R_a$.

□

DEFINITION 1.11. A **linear lattice** is a sublattice of $\Pi(S)$ consisting of commuting equivalence relations.

The main example of linear lattice is given by the lattice of normal subgroups of a group. Let H be a normal subgroup of a group G. Let a, b be elements of G. Define the equivalence relation R_H by aR_Hb if and only if $a - b \in H$. In other words the equivalence classes are given by the cosets of H. It is immediate to verify that two such equivalence relations commute.

In particular, the set of submodules of a module has a linear lattice structure, and the same holds for linear subspaces of a vector space over a field. The join of two subspaces corresponds to the space generated by them, and the meet corresponds to the intersection. This motivates the

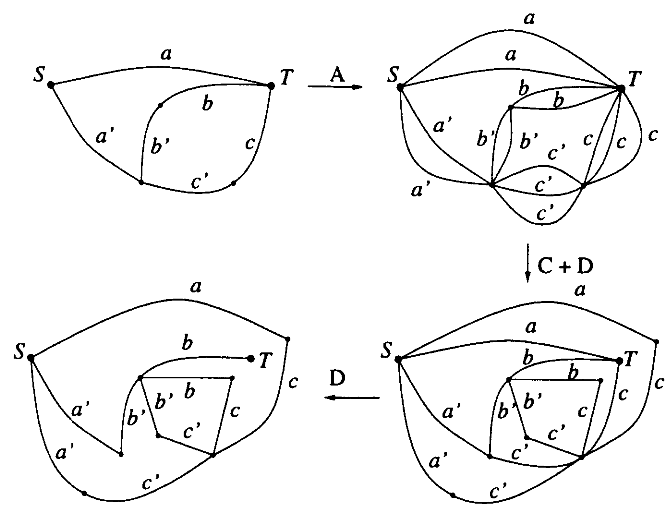


FIGURE 1. Proof of Arguesian inequality 1.32

REMARK . Theorem 1.32 is not just any inequality, it is the lattice version of Desargues theorem 1.24. To see this, suppose the lines aa', bb' and cc' are concurrent. Then a lies on the join of a' with point $bb' \cap cc'$, therefore the left hand side of (1.12) equals a. Let $x = bc \cap b'c'$, $y = ac \cap a'c'$, $z = ab \cap a'b'$. Inequality (1.12) implies that a lies on the join of b with point $a'b' \cap xy$, that implies that z belongs to xy, as wanted. Since Desargues theorem is self-dual, one implication will suffice.

CHAPTER 2

Invariant Operations

Operations on a projective space which do not depend on the choice of a coordinate system has not been studied much to date. We know that the GC algebra can be used to express an invariant operation in terms of a bracket polynomial. The geometric meaning of an invariant operation associated to a generic polynomial, however, is hard to understand in general. This is the first reason that led us to single out a large class of operations associated to precise geometric constructions from all possible invariant operations.

One might think that all geometric constructions that can be obtained from a set of projective subspaces of a projective space come from the classical join and meet operations. If that were true, than no geometric constructions could be realized on the projective line, since join and meet are trivial here. On the other hand, we know of examples of invariant operations, for instance the sixth harmonic, which can be geometrically defined on the projective line. The key point is the embedding of the projective line into a higher dimensional space and the choice of some external points. In this larger space the sixth harmonic is realized by join and meet operations, and independence from the choice of the external points must be shown.

The present work introduces a class of invariant operations which generalize the sixth harmonic construction. At the same time, the new operations can be visualized by graphs which are a generalization of the series-parallel graphs associated to the classical join and meet operations.

1. Graphical Operations

In what follows we are going to associate to every bipointed graph an invariant operation on the n-dimensional real projective space.

Let $G = (V, E)$ be a bipointed connected finite graph, where

$$V = \{v_1, \ldots, v_v\}$$

is the set of **vertices** and

$$E = \{e_1, \ldots, e_e\}, \qquad e_i = \{v_{s_i}, v_{t_i}\},\ s_i \neq t_i.$$

is the set of **edges**, and v_1, v_v, are the terminal vertices.

In addition, to each edge e_i it is associated a **weight** w_i such that $1 \leq w_i \leq n - 1$, for some n. For the rest of this work, we will refer to such a bipointed weighted graph by the simple word **graph**.

We are going to associate to each graph a e-ary operation on the lattice of linear subspaces of $\mathbb{R}^n$. Most of the results can be generalized to any field, although, for the sake of simplicity, we prefer to carry on the details only for reals numbers. Let

$$L(V) = L_0 \cup \cdots \cup L_n$$

be the lattice of subspaces of $\mathbb{R}^n$, where L_i is the set of i-dimensional subspaces.

DEFINITION 2.1. A **specialization in step n** of the graph $G = (V, E)$ is a map that assigns to every edge e_i of G a w_i-dimensional subspace of $\mathbb{R}^n$. In symbols,

$$\begin{array}{rcl} s : E & \longrightarrow & L \\ e_i & \mapsto & s(e_i) = l_i, \quad \text{so that} \quad \dim(l_i) = w_i \end{array}$$

In order to define an invariant operation on the elements l_i, we need to introduce another notion. In what follows we fix an orientation of the graph, so that every edge e_i consists of a pair (v_{s_i}, v_{t_i}).

DEFINITION 2.2. A **geometric realization** of a specialized graph (G, s) is a map r that assigns to every vertex of G a point in $\mathbb{R}^n$ in a way compatible with the edges, in symbols,

$$\begin{array}{rcl} r : V & \longrightarrow & \mathbb{R}^n \\ v_j & \mapsto & r(v_j) = p_j, \quad \text{such that for every edge} \\ & & e_i = (v_{s_i}, v_{t_i}), \quad \text{we have} \quad p_{t_i} - p_{s_i} \in l_i. \end{array}$$

In this case we say that p_{s_i} and p_{t_i} are **l_i-equivalent**.

Let $\mathcal{R} \subset (\mathbb{R}^n)^v$ be the set of all geometric realizations of (G, s). Define

$$W = \{r(v_v) - r(v_1) \mid r \in \mathcal{R}\}.$$

PROPOSITION 2.1. *W is a linear subspace of $\mathbb{R}^n$.*

PROOF. For every subspace l_i we can choose w_i vectors $z_{i1}, \ldots, z_{iw_i}$ generating it. Let a geometric realization $r \in \mathcal{R}$ be fixed, and call $r(v_j) = p_j$. Every condition

$$p_{t_i} - p_{s_i} \in l_i$$

is equivalent to saying that there exist scalars $\lambda_{i1}, \ldots, \lambda_{iw_i}$ such that

$$p_{t_i} - p_{s_i} = \lambda_{i1} z_{i1} + \cdots + \lambda_{iw_i} z_{iw_i}.$$

If we put all the conditions together, we get the linear system of equations

$$\left\{ \begin{array}{l} \lambda_{11} z_{11} + \cdots + \lambda_{1w_1} z_{1w_1} + p_{s_1} - p_{t_1} = 0 \\ \qquad \vdots \qquad \vdots \qquad \vdots \\ \lambda_{e1} z_{e1} + \cdots + \lambda_{ew_e} z_{ew_e} + p_{s_e} - p_{t_e} = 0 \end{array} \right. \tag{2.1}$$

where the z_{km} are given and the p's and λ's are to be found. Since we are looking for the value of $p_v - p_1$, we might as well assume $p_1 = 0$ and look for the values of p_v that satisfy (2.1). Since the conditions imposed by system (2.1) are all linear and homogeneous, the set W is a linear subspace of $\mathbb{R}^n$, as wished. □

REMARK . The space W is independent of the orientation of G chosen.

REMARK . The space W is independent of the choice of z_{km}'s.

PROOF. Suppose we choose instead $\tilde{z}_{i1}, \dots, \tilde{z}_{iw_i}$, and are looking for the solutions of a system $\widetilde{(2.1)}$ (which is (2.1) with each z_{km} replaced by $\tilde{z}_{km}$). Since the z_{ij}'s generate l_i, then for every solution $\lambda_{i1}, \dots, \lambda_{iw_i}$ of (2.1), there exist $\tilde{\lambda}_{i1}, \dots, \tilde{\lambda}_{iw_i}$ such that

$$\lambda_{i1} z_{i1} + \cdots + \lambda_{iw_i} z_{iw_i} = \tilde{\lambda}_{i1} \tilde{z}_{i1} + \cdots + \tilde{\lambda}_{iw_i} \tilde{z}_{iw_i}.$$

Therefore, for every solution of (2.1) there is a solution to $\widetilde{(2.1)}$ that coincides on the p_i's, and vice versa. □

DEFINITION 2.3. For a fixed graph G, the operation that assigns the space W to linear spaces $l_1, \dots, l_e$ with the above construction will be denoted by $G(l_1, \dots, l_e)$, and called *graphical operation associated to G*.

Clearly this operation is invariant, since we made no appeal to a choice of coordinate system.

We can consider the operation just defined from a different point of view. We first state the projective version, that is:

FACT (projective construction). Choose a projection from $\mathbb{R}^n \setminus \{0\}$ to a $(n-1)$-dimensional real projective space P. Call $\tilde{l}_1, \dots, \tilde{l}_e$ the projective subspaces in P correspondent to the linear subspaces $l_1, \dots l_e$ of $\mathbb{R}^n$ under this projection. Embed $P \hookrightarrow \mathbb{P}^n$ in some n-dimensional projective space. A geometric realization is then a map from the vertices of the graph $\{v_i\}$ to points $\{p_i\}$ of $\mathbb{R}^n = \mathbb{P}^n \setminus P$ such that

$$e_k = \{v_i, v_j\} \implies (p_i \vee p_j) \wedge P \in \tilde{l}_k.$$

The result of the graph operation will be

$$\bigvee_{r \in \mathcal{R}} (p_1 \vee p_n) \wedge P$$

where the join is performed over all possible geometric realizations r of G.

Sometimes it is more convenient to perform the dual construction, that is obtained by translating verbatim the previous construction with the dual principle.

FACT (dual projective construction). Choose a projection π from $\mathbb{R}^n \setminus \{0\}$ to a $(n-1)$-dimensional projective space P. Call $\tilde{l}_1, \dots, \tilde{l}_e$ the projective subspaces in P correspondent to the linear subspaces $l_1, \dots l_e$ of $\mathbb{R}^n$ under this projection, as before.

A geometric realization is then a map from the vertices of the graph $\{v_i\}$ to hyperplanes $\{P_i\}$ of $\mathbb{R}^n$ such that

$$e_k = \{v_i, v_j\} \implies \pi(P_i \wedge P_j) \ni \tilde{l}_k.$$

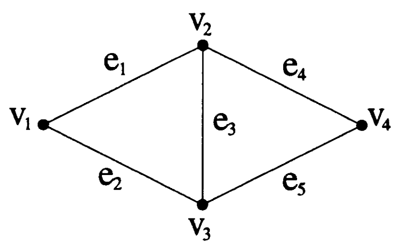
The result of the operation will be

$$\bigwedge_{r \in \mathcal{R}} \pi(P_1 \wedge P_n),$$

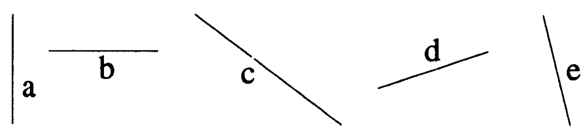
where the meet is performed over all possible realizations r of G.

The following example may clarify the discussion.

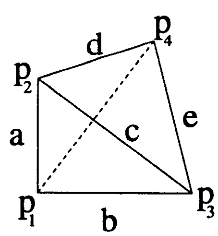
EXAMPLE 2.1. Let G be the following graph in step 2, called the **Wheatstone bridge**. The edges $e_1, \ldots, e_5$ have weight 1, and will be specialized to subspaces of dimension 1 of $\mathbb{R}^2$



Let a specialization of the graph be given, that is, to each edge $e_1, \ldots, e_5$ assign some 1-dimensional subspaces of $\mathbb{R}^2$. For instance we can choose the subspaces correspondent to the directions $a, \ldots, e$, as follows:



Then a geometric realization is given by the following picture, where the dotted line stands for the result of the graphical operation.



It is clear from the above picture why the graph is the natural way to represent such operation. The independence of the result from the choice of (compatible) p_2, p_3, p_4 was proved earlier in this section. A different choice corresponds to a rescaling of the picture.

Alternatively, we can visualize the same operation with the dual projective construction in the following way, as shown in figure 1: edges $e_1, \ldots e_5$ get specialized to points $a, \ldots, e$ of line P. The origin of $\mathbb{R}^2$ is at infinity, along the vertical direction, so that the projection from $\mathbb{R}^2 \setminus \{0\}$ to P is the vertical projection in the picture. The lines P_1, P_2 intersect in the point a', whose projection contains (it actually equals) a, and similarly for the other lines and points of the picture. The point f' is the intersection of line P_1 and P_4; its projection on P, f, gives the result of the operation. Notice that f is independent of the choices we made, although that is not so immediate from this point of view.

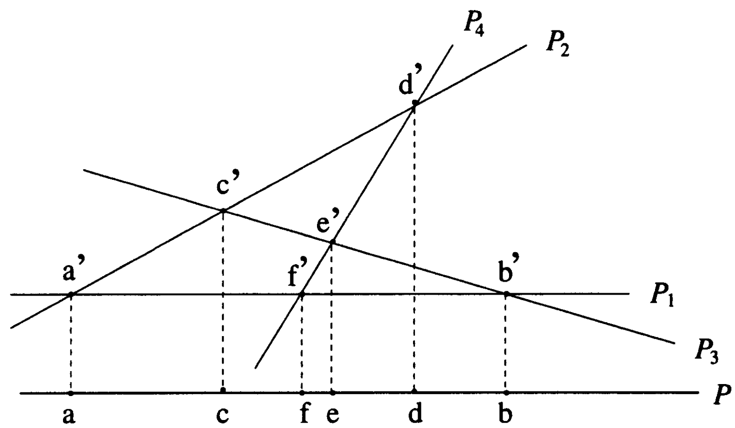


FIGURE 1. Dual projective construction for the Wheatstone bridge operation

The operation defined by the Wheatstone bridge is the well known sixth harmonic of 5 points on the projective line. The same construction can be found for instance in [**2**] and [**5**]. The construction defined in the present work differs substantially from the construction introduced in [**5**], although the simplest example for both constructions is the sixth harmonic. From a philosophical point of view, in the present dissertation we study invariant operations, while in [**5**] the authors study conditions for lifting. From a mathematical point of view, the work [**5**] only deals with what Crapo and Rota call *first order syzygies*, while the present work does not have such a restriction. For instance, a 4-ple vertex for a graph in step 2 corresponds to a second order syzygy, in Crapo and Rota language. Moreover, all edges of a graph can be specialized to distinct points (or higher dimensional spaces), and we will see in the next sections that the correspondent operation expands to a *multilinear* bracket polynomial in the GC algebra.

We have just shown how a weighted graph defines an operation

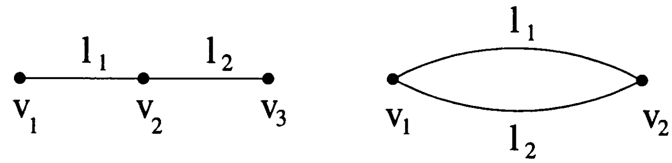
$$\begin{array}{rcl} L_{w_1} \times \cdots \times L_{w_e} & \longrightarrow & L \\ (l_1, \ldots, l_e) & \mapsto & l. \end{array}$$

What is the dimension of the resulting subspace l? Let $w = \sum w_i$, and define

$$d_G \doteq n(v - e - 1) + w. \tag{2.2}$$

It will turn out that d_G is the dimension of the resulting subspace l, for generic $l_1, \dots l_e$, provided the graph satisfies some suitable rigidity and irreducibility properties. Indeed, the number d_G is obtained as the difference between the number of free variables for the system (2.1) and the number of linear conditions on them.

EXAMPLE 2.2. The graphs



correspond to the usual join and meet operations, respectively. The join yields dimension w_1+w_2 and the meet yields w_1+w_2-n, for generic elements l_1, l_2, as expected. In what follows, when we talk of the dimension, or step, of a graph G, we mean the dimension of the resulting $G(l_1, \dots, l_e)$ for generic elements $l_1, \dots, l_e$, that is, belonging to a Zariski open set. The assumption that $l_1, \dots, l_e$ are generic elements will be understood for the rest of the chapter.

DEFINITION 2.4. Two specialized graphs G_1, G_2 are equivalent ($\simeq$) if the result of the associated operation is the same.

In what follows the term *subgraph* will stand for a pair $H = (V_H, E_H)$ where V_H and E_H are subsets of V and E, respectively. Note that a subgraph need not be a graph. Given a subgraph H of G with v_H vertices and e_H edges, we can similarly define

$$d_H \doteq n(v_H - e_H - 1) + \sum_H w_i.$$

DEFINITION 2.5. A subgraph $H \subset G$ is said to be **vertex-closed** if it is connected and contains all vertices of its edges, in symbols,

$$e_i = (v_j, v_k) \in E_H \Longrightarrow v_j, v_k \in V_H;$$

it is **edge-closed** if it is connected and contains all edges of its vertices, in symbols,

$$v_j \in V_H \Longrightarrow e_i \in E_H \text{ for every } e_i \ni v_j.$$

PROPOSITION 2.2 (Quotient). *Let $H \subset G$ be a vertex-closed subgraph such that $d_H \leq 0$. Then the graph G/H obtained from G by collapsing H to a point is equivalent to G, for any (generic) specialization. We write*

$$G \simeq G/H.$$

PROOF. Consider the lines of the system (2.1) corresponding to the edges of H. Fix the value of one of the points of H, say 0, without loss of generality. The inequality $d_H \leq 0$ implies that the homogeneous linear system corresponding to the lines of H has more conditions than variables.

As a consequence, zero is the unique solution, for generic values of the l_i in H. Therefore, we conclude that all points of H must be the same, and can be treated as one, without effecting the remaining points of G. □

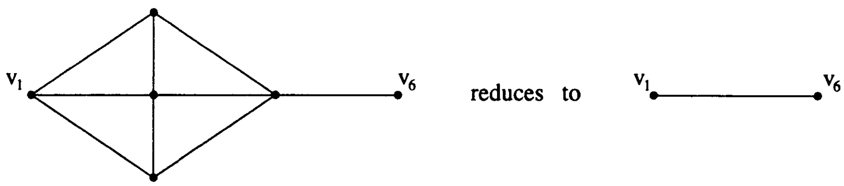


FIGURE 2. Example of quotient in step 2.

PROPOSITION 2.3 (Excision). *Let $H \subset G$ an edge-closed subgraph not containing v_1, v_v and such that $d_H \geq -n$. Then the graph $G\backslash H$ obtained from G by removing H is equivalent to G. We write*

$$G \simeq G\backslash H.$$

PROOF. Any geometric realization of G can be restricted to become a geometric realization of $G\backslash H$. Conversely, a geometric realization of $G\backslash H$ extends to a realization of G only if the restriction of system (2.1) to the lines of H has a solution. Since

$$w_H + n(v_H - e_H) \geq 0,$$

the system has at least a solution, therefore every geometric realization of $G\backslash H$ is one of G, and vice versa. □

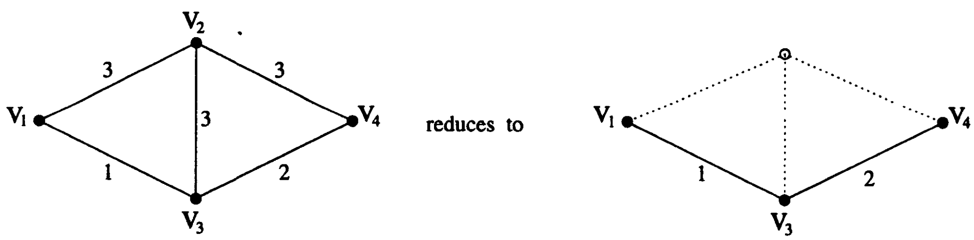


FIGURE 3. Example of excision in step 4.

DEFINITION 2.6. A graph G is said to be **admissible** if $1 \leq d_G \leq n-1$ and **irreducible** if

$$G \simeq G/H \quad \text{implies} \quad H = \text{a single vertex}, \quad \text{and}$$
$$G \simeq G\backslash H \quad \text{implies} \quad H = \phi$$

A graph is not irreducible if and only if the corresponding system (2.1) has redundant rows. Therefore we conclude the following:

PROPOSITION 2.4. *An admissible irreducible graph G defines an operation of step d_G.*

□

From now on we will only consider admissible irreducible graphs. We conclude this section with a couple of words on the **composition** of two graph operations. Given a graph H of step k and a graph G with an edge e_i of weight k, we can perform the composition

$$G(l_1, \ldots, l_{i-1}, H(m_1, \ldots, m_{e_H}), l_{i+1}, \ldots, l_{e_G}).$$

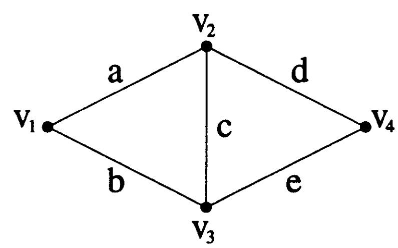
This is obtained, graphically, by removing the edge $l_i = \{v_{s_i}, v_{t_i}\}$ from G and attaching the graph H at its place, where the terminal vertices of H are identified with v_{s_i}, v_{t_i}.

2. Graphs in Step 2

In this section we are going to focus on operations defined on subspaces of $\mathbb{R}^2$. In this setting the edges can only have weight 1, and the graph can only be of step 1. These two conditions together imply that the graph G satisfy $2v - e = 3$. In particular, a graph must have an odd number of edges. For $e=1$ we get a trivial example, and for $e=3$ a reducible one is obtained. The first interesting instance, $e = 5$, was shown in example 2.1, the so called Wheatstone bridge (or sixth harmonic). Notice that this operation could not be defined by meets and joins alone, since they are trivial in $\mathbb{P}^1$. Nevertheless, it could be shown to be definable by meets and joins alone in $\mathbb{P}^2$ provided we embed $\mathbb{P}^1$ in $\mathbb{P}^2$ and make some random choice of external points. We have to show, of course, the independence from the choice of these points.

Now that we have the first non-trivial operation at hand, it is natural to ask when and how complex graphs can be reduced to compositions of Wheatstone bridge operations. A deeper and more complete study would focus on the algebra of graphs subject to equivalence relations and compositions. Some partial results along these directions we will be shown shortly.

PROPOSITION 2.5 (Symmetry). *Let's call $W(a,b|c|d,e)$ the operation associated to the Wheatstone bridge:*

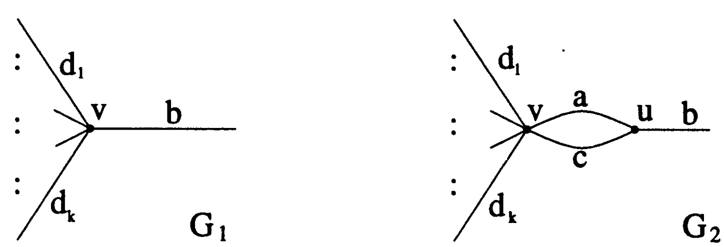


Then

$$W(a,b|c|d,e) = W(b,a|c|e,d) = W(d,e|c|a,b) = W(e,d|c|b,a).$$

□

PROPOSITION 2.6 (Expansion). *Let G_1, G_2 be two graphs that are congruent except in a region where they look as follows.*



Then if $a \neq c$, the graphs are equivalent.

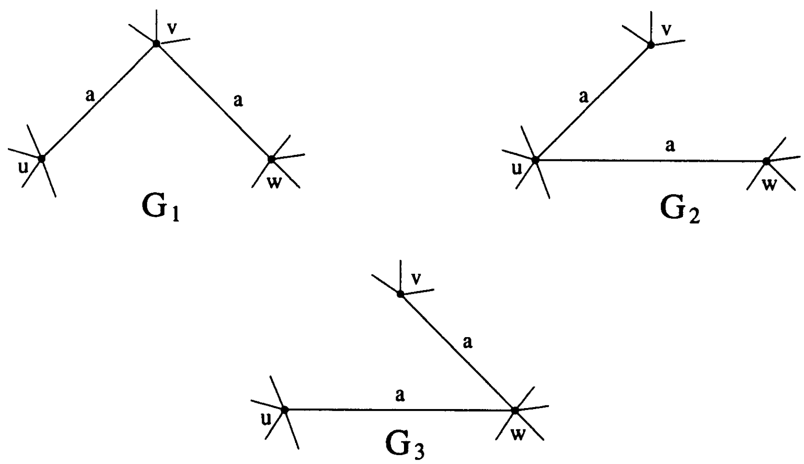
PROOF. If we indicate by $a \wedge c$ the vertex-closed subgraph of G_2 containing the edges a and c, then we can take the quotient and obtain

$$G_1 \equiv G_2/(a \wedge c).$$

□

DEFINITION 2.7. If α, β are two points in $\mathbb{R}^n$ whose difference lies on a subspace a, we say that α and β are **a-equivalent**.

PROPOSITION 2.7 (Transitivity). *Let three graphs be congruent except in a region where they look as follows.*



Then they are equivalent.

PROOF. In G_1, u and v are a-equivalent, and the same for v and w. Hence, by transitivity, u and w are a-equivalent, and we can connect them by an a-edge. At this point we can delete either one of the edges $\{u, v\}$, $\{v, w\}$, since it is redundant, and obtain either the graph G_2 or G_3. Alternatively, we can consider the two lines of the system (2.1) correspondent to the two a-edges. They look like

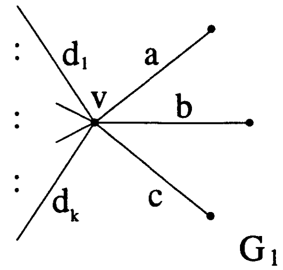
$$\begin{aligned} u - v &= \lambda_1 a \\ v - w &= \lambda_2 a \end{aligned}$$

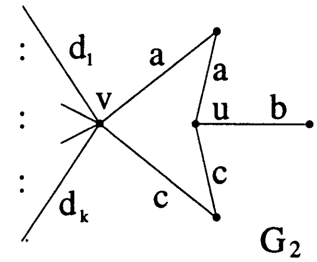
By adding these equations we get

$$u - w = \lambda_3 a, \qquad \lambda_3 = \lambda_1 + \lambda_2$$

which can replace either one of them. □

LEMMA 2.8 (Detachment). *Let G_1, G_2 be two graphs that are congruent except in a region where they look as follows:*





Then they are equivalent.

PROOF. First perform an expansion operation at the vertex v over the edge b of G_1 (with edges a and c) and then by transitivity on the a's and c's, we obtain the graph G_2. □

DEFINITION 2.8. We call the operation from G_1 to G_2 **detachment over v along b with a, c support.**

DEFINITION 2.9. A graph $G = (V, E)$ is **triangulable** if every edge of G belongs to a triangle, in symbols

$$\forall h=\{u,v\} \in E \qquad \exists\, z \in V \qquad \text{such that} \quad \{z,v\},\{z,u\} \in E.$$

THEOREM 2.9. *Every triangulable graph G is equivalent to an iteration of bridge operations.*

PROOF. A vertex will be called n-tuple if it belongs to n distinct edges. We will proceed by induction on the number of vertices of G, two vertices connected by an edge being the base of the induction.

CLAIM . There exist at least one non-terminal triple vertex.

Note that a non-terminal vertex v cannot be single nor double, otherwise the subgraph H consisting of v and all its l edges would have $d_H = -l \geq -2$ and could be excised. Similarly, a terminal vertex v cannot be single: call H the subgraph of G consisting of all vertices of G but v and all edges of G but the one containing v. This is a vertex-closed subgraph of G for which $d_H = 0$, therefore by quotient operation G could be reduced to G/H.

At this point, if every non-terminal vertex were at least quadruple, then we would have at least

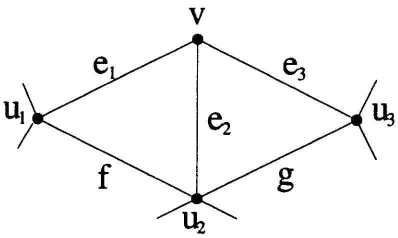
$$\frac{4(v-2)+2\cdot 2}{2}$$

edges, which implies

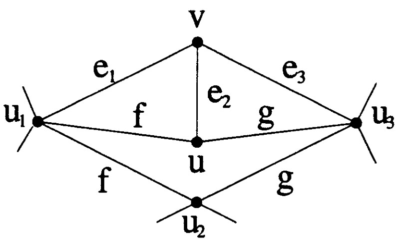
$$0 = 2v - e - 3 \;\leq\; 2v - 2v + 4 - 2 - 3 = -1.$$

The contradiction proves our claim.

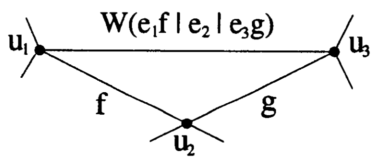
Let v be a triple vertex of G, and let $e_1 = \{v, u_1\}$, $e_2 = \{v, u_2\}$ and $e_3 = \{v, u_3\}$ be its edges. Since G is triangulable, e_1 makes a triangle either with e_2 or e_3; without loss of generality, suppose it does with e_2. Similarly, e_3 makes a triangle with either e_1 or e_2, and we can assume, without loss of generality, that it makes the triangle with e_2. The portion of G near v now looks like



for some f, g, where $\{u_1, u_2, u_3\}$ may contain terminal vertices. Now by detachment operation performed on u_2 along e_2 with f, g support, the above graph is equivalent to



Since neither u nor v are terminal vertices, we can replace the Wheatstone bridge by a single edge representing it, namely



Using the bridge operation we got rid of 3 edges and one vertex and have replaced them with one new edge, h. Since fgh is a triangle and the rest of the graph has not changed, we still have a triangulable graph with one vertex less, which completes the inductive step. □

Sometimes even non-triangulable graphs can be reduced by detachment, to compositions of bridge operations, as in figure 4. Sometimes they cannot, as in figure 5. The graph G_2 of figure 5 is the simplest graph that contains no triangles, and we call it the diamond graph.

Another interesting operation is the following:

PROPOSITION 2.10 (Diagonal exchange). *The following graphs are equivalent*

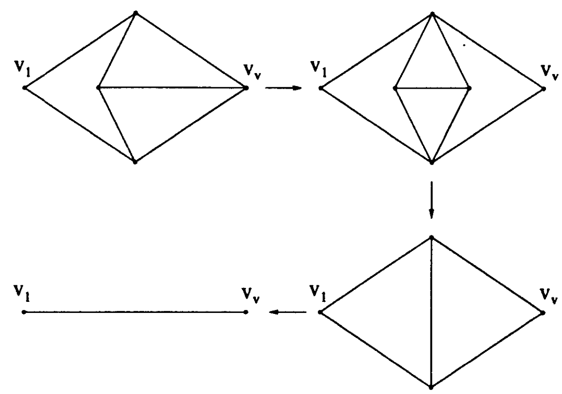


FIGURE 4. Example of reducible graph, with reduction.

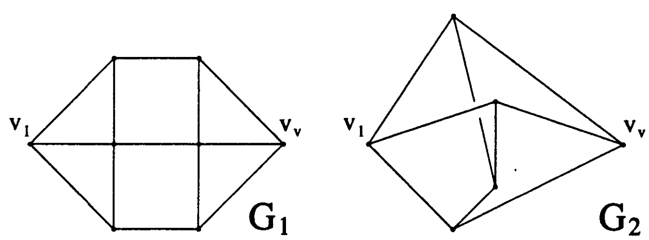
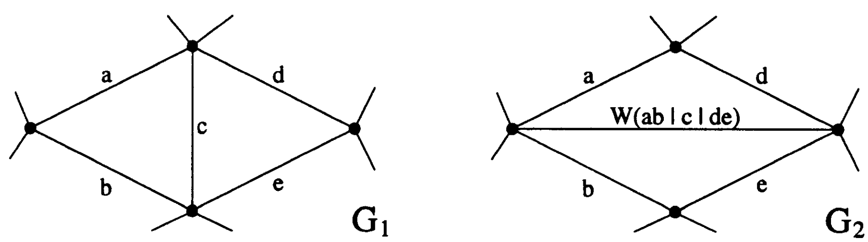


FIGURE 5. Examples of non reducible graphs.



PROOF. We can go from G_1 to G_2 by two detachment operations along c and a bridge substitution. □

By performing the diagonal exchange twice, we obtain the same graph as G_1, with c replaced by $W(ad\,|\,W(ab|c|de)\,|\,be)$. This can be shown to imply

COROLLARY 2.11.

$$W(ad\,|\,W(ab|c|de)\,|\,be) = c.$$

3. Bracket Expansion in Step 2

The purpose of this section is to provide a purely combinatoric description of the graph operations in terms of bracket polynomials.

The expansion in step 2 we are going to consider could be treated as a special case of the more general step n, which will be explored in next section. We present it separately for two reasons. First, in step 2 the expansion can be reduced to a very nice formula that deserves special attention. Second, the reader can get acquainted with the techniques in a special case where the notation is much simpler.

In what follows, an admissible irreducible graph $G = (V, E)$ in step 2 will be chosen. We have shown in section 1 that the result of the graph operation, $G(l_1, \ldots, l_e)$, is given by the set of p_v that solves system (2.1), provided that $p_1 = 0$.

By abuse of notation, we can choose a coordinate system for $\mathbb{R}^2$ so that $l_i = <(l_i, 1)>$, and the context will make it clear when we are referring to the point in the projective space and when to the x-coordinate of the point. Each point p_j of a geometric realization will have coordinates (p_j^x, p_j^y). For p_v^x, p_v^y we will use the shorter names x, y. While we show the general reduction of the system (2.1) that leads to an expression in bracket polynomials, we apply the procedure to an easy example, the Wheatstone bridge, to better visualize it.

The author believes that a more direct proof exists. At the moment, however, we have to cope with this technical proof, which is hard to follow if the reader does not work out each single step along the way.

STEP 1: Choose any orientation of the edges of G, for instance the one shown in figure 6. We can rewrite the system (2.1) as follows, where we have

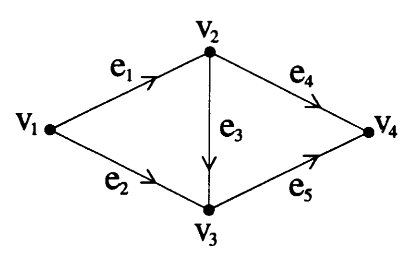


FIGURE 6. An orientation for the Wheatstone bridge

deleted the columns correspondent to p_1, since p_1 equals zero.

$$
\begin{array}{c}
\begin{array}{cc}
\qquad\overbrace{\hphantom{l_1\ 0\ \dots\ \dots\ 0}}^{e \text{ columns}} & \qquad\overbrace{\hphantom{e{+}2j{-}3 \qquad\qquad e{+}2k{-}3}}^{2(v-1)\text{ columns}}
\end{array}\\
2e\left\{\left|\begin{array}{cccccccc}
l_1 & 0 & \dots & \dots & 0 & & & \\
1 & 0 & \dots & \dots & 0 & & & \\
0 & l_2 & 0 & \dots & 0 & e{+}2j{-}3 & & e{+}2k{-}3\\
0 & 1 & 0 & \dots & 0 & \downarrow & & \downarrow\\
\dots & \dots & \dots & \dots & \dots & -1 & \leftarrow 2i{-}1 \rightarrow & 1\\
\dots & \dots & \dots & \dots & \dots & -1 & \leftarrow 2i \rightarrow & 1\\
0 & \dots & \dots & \dots & l_e & \uparrow & & \uparrow\\
0 & \dots & \dots & \dots & 1 & e{+}2j{-}2 & & e{+}2k{-}2
\end{array}\right|\right.
\end{array}
\left|\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_e\\ p_2^x\\ p_2^y\\ \vdots\\ x\\ y\end{array}\right|
=
\left|\begin{array}{c}0\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ 0\end{array}\right|
\qquad (2.3)
$$

For every edge $e_i = (v_j, v_k)$ the $+1$ and -1 in the right part of the matrix are located in the position indicated, and elsewhere there are zeros. In other words, the right part of the matrix is an incidence matrix for the vertices and the edges of G. In the Wheatstone bridge example, this looks like:

$$
\begin{array}{r}
\\
e_1\left\{\begin{array}{c}\\ \\ \end{array}\right.\\
e_2\left\{\begin{array}{c}\\ \\ \end{array}\right.\\
e_3\left\{\begin{array}{c}\\ \\ \end{array}\right.\\
e_4\left\{\begin{array}{c}\\ \\ \end{array}\right.\\
e_5\left\{\begin{array}{c}\\ \\ \end{array}\right.
\end{array}
\left|\begin{array}{ccccc|cc|cc|cc}
 & & & & & \multicolumn{2}{c}{\overbrace{}^{v_2}} & \multicolumn{2}{c}{\overbrace{}^{v_3}} & \multicolumn{2}{c}{\overbrace{}^{v_4}}\\
l_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & l_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & l_3 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & l_4 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & l_5 & 0 & 0 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1
\end{array}\right|
\left|\begin{array}{c}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ p_2^x\\ p_2^y\\ p_3^x\\ p_3^y\\ x\\ y\end{array}\right|
=
\left|\begin{array}{c}0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{array}\right|
\qquad (2.4)
$$

It is enough to find one pair (x, y) that solves the system (2.3), since any other will be a multiple of it. We can assume that the choice of coordinates allows $y = 1$ to be a solution, for some x. At this point we can move the column correspondent to y on the right hand side of the equal sign, and solve the square system by Cramer's rule. Before doing that, we will reduce the system (2.3) by row operations, as described in next step.

STEP 2: Call A the matrix of the system (2.3). By subtracting l_i times the $(2i)^{\text{th}}$ row from the $(2i-1)^{\text{th}}$ row, for $i = 1, \dots, l$, and then moving all even rows at the top, the matrix A will be

$$
\begin{bmatrix} I & * \\ 0 & B \end{bmatrix}
$$

where I is the identity matrix and B has the following entries in the positions indicated, and zero elsewhere.

$$\begin{bmatrix} \overset{2j-3}{\downarrow} & & & & & \overset{2k-3}{\downarrow} & \\ \mathbf{-1} & l_i & \leftarrow & i & \rightarrow & \mathbf{1} & -l_i \\ & \underset{2j-2}{\uparrow} & & & & & \underset{2k-2}{\uparrow} \end{bmatrix}$$

In other words, the i^{th} row, that corresponds to edge $e_i = (v_j, v_k)$, has the $-1, l_i$ in the columns corresponding to vertex j, and $1, -l_i$ in the column corresponding to vertex k. For our example (2.4), B becomes

$$\begin{bmatrix} 1 & -l_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -l_2 & 0 & 0 \\ -1 & l_3 & 1 & -l_3 & 0 & 0 \\ -1 & l_4 & 0 & 0 & 1 & -l_4 \\ 0 & 0 & -1 & l_5 & 1 & -l_5 \end{bmatrix} \tag{2.5}$$

With this reduction, if we call $\widetilde{B}$ the part of B made of the first $2v - 4$ columns (that is, all but the last two) and call c_x, c_y the last two columns, by Cramer's rule the result of the graph operation becomes

$$\begin{aligned} x &= \frac{-|\widetilde{B}c_y|}{|\widetilde{B}c_x|} \\ y &= 1 \qquad \text{or, equivalently,} \\ x &= -|\widetilde{B}c_y| \\ y &= |\widetilde{B}c_x| \end{aligned}$$

In order to reduce the matrix $\widetilde{B}$ even farther, we will use the following well-known lemma in the next step.

LEMMA 2.12. *For every connected graph G, there exists a connected, simply connected subgraph S containing all vertices of G. S is called a* **spanning tree** *for G.*

STEP 3: Call F the set of terminal edges of G, that is edges that contain the vertex v_v. Choose a spanning tree S for $G \backslash \{v_v \cup F\}$ The edges of S will be called **skeleton edges** and the remaining edges of G will be called **simple edges.** We can assume the orientation we chose in step 1 was such that the skeleton edges are oriented outward the vertex v_1, which we take as the root of the spanning tree. This defines a partial order on the skeleton edges, smaller edges pointing toward bigger ones. We can distinguish the skeleton edges by a double arrow, as in figure 7. By the defining property of S, for each non-terminal vertex there is exactly one skeleton edge pointing to it. Since $2v - e = 3$, we have $v - 2$ skeleton edges and $v - 1$ simple edges.

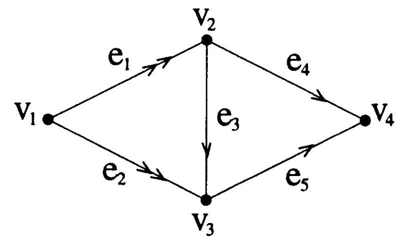


FIGURE 7. An oriented spanning tree for the Wheatstone bridge

PROPOSITION 2.13. *The result of the bracket operation is*

$$\begin{aligned} x &= -|C\bar{c}_y| \\ y &= |C\bar{c}_x| \end{aligned} \tag{2.6}$$

where C is a $(v-1)\times(v-1)$ matrix whose columns are labeled by skeleton edges $d_1,\ldots,d_{v-2}$ and rows by simple edges $g_1,\ldots,g_{v-1}$, $\bar{c}_x$ and $\bar{c}_y$ are the restrictions of c_x and c_y to the row correspondent to the simple edges, and the (i,j) entry for C is $(\lambda_{ij}-\mu_{ij})[d_jg_i]$, where

$$\lambda_{ij} = \begin{cases} 1 & \text{if the tip of } g_i \text{ lies on a skeleton path from } d_j \\ 0 & \text{else} \end{cases}$$

$$\mu_{ij} = \begin{cases} 1 & \text{if the toe of } g_i \text{ lies on a skeleton path from } d_j \\ 0 & \text{else.} \end{cases}$$

If the (d_j, g_i) entry is not zero we say that skeleton edge d_j **sees** the edge g_i. In our example, we have:

$$C = \begin{bmatrix} -[l_1l_3] & [l_2l_3] \\ -[l_1l_4] & 0 \\ 0 & -[l_2l_5] \end{bmatrix} \quad \bar{c}_x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \bar{c}_y = \begin{pmatrix} 0 \\ -l_4 \\ -l_5 \end{pmatrix}$$

Before showing the proof of the proposition, we would like to point out the importance of this result. Note that $\bar{c}_x$ will only have 0 and 1 as entries, and $-\bar{c}_y$ will have l_i's in the same position where $\bar{c}_x$ has the 1's, and zeros elsewhere. When we expand $y = |C\bar{c}_x|$, we obtain a polynomial in the brackets of the points $l_1,\ldots,l_e$, or, equivalently, a linear combination of the y-coordinates of the points l_i, which are all 1, with products of brackets as coefficients. When we expand $x = -|C\bar{c}_y|$ we obtain the same linear combination, but with the x-coordinates of the points in place of the y-coordinates. This corresponds to saying that the result of the graph operation *is* that linear combination, where we replace the coordinates by the actual points, in the Grassmann-Cayley algebra. We get, in this way, an expansion in term of brackets of the operation associated to a graph.

EXAMPLE 2.3. The Wheatstone bridge operation expands to

$$W(l_1,l_2|l_3|l_4,l_5) = -l_4[l_1l_3][l_2l_5] + l_5[l_1l_4][l_2l_3]. \tag{2.7}$$

PROOF OF PROPOSITION. Every row of C corresponds to an edge. The columns of C in odd position correspond to the x-coordinate of non-terminal vertices, and the columns in even position correspond to the y-coordinate of non-terminal vertices. Since non-terminal vertices correspond to skeleton edges, we can label each column with its correspondent skeleton edge. The skeleton edges are partially ordered, and we can extend this partial order to the (pairs of) columns of C. Perform now the following operation: to each column add all the columns bigger than it with respect to this partial order. During such operation, even columns will be added to even columns and odd columns to odd columns. Take a closer look at the rows of the matrix we obtain with this operation: the j^{th} skeleton row will have $1, -d_j$ in positions $2j-1$ and $2j$, respectively, and zero elsewhere. The simple row g_i will have $1, -d_j$ in positions $2j-1, 2j$ if the tip of g_i lies on a skeleton path from d_j, and will have $-1, g_j$ if the toe of g_i lies on a skeleton path from d_j, and zero in all other cases (including when both tip and toe lie on some paths from d_j. Now we perform rows operations to $(C\bar{c}_x\bar{c}_y)$, according to the following scheme. Each skeleton row must be added to, or subtracted from, every simple row in such a way to cancel all the 1's of the odd columns of simple rows. After this operation each odd column, correspondent to skeleton edge d_i, will have all zero entries except for the position correspondent to d_i, where there will be a 1. Note that the previous operation does not change the columns $\bar{c}_x, \bar{c}_y$. We can now simplify the determinant with a Laplace expansion on odd columns. We need not keep track of the sign, since if it changes for x, it will change for y as well, leaving the result unchanged. By noticing that, for vectors $\mathbf{a} = (a, 1)$, $\mathbf{b} = (b, 1)$ we have $[\mathbf{a}, \mathbf{b}] = b - a$, the proof is complete.

□

EXAMPLE 2.4. We will illustrate the steps of the previous procedure on the Wheatstone bridge, with a different choice of spanning tree. For instance, consider e_1 and e_3 as skeleton edges, with the same orientation as in figure 7. The starting matrix $\tilde{B}$ is given once again by the first 4 columns of (2.5). The column operation in this case consists in adding the third column to the first and the fourth to the second, and transforms $\tilde{B}$ into the matrix displayed below, on the left. The row operation then consists in subtracting the first row from the second one, and adding it to the fourth and fifth. Similarly, subtract the third row from the second and add it to the fifth. The result is shown in matrix on the right:

$$\begin{bmatrix} 1 & -l_1 & 0 & 0 \\ 1 & -l_2 & 1 & -l_2 \\ 0 & 0 & 1 & -l_3 \\ -1 & l_4 & 0 & 0 \\ -1 & l_5 & -1 & l_5 \end{bmatrix} \quad \begin{bmatrix} 1 & -l_1 & 0 & 0 \\ 0 & l_1 - l_2 & 0 & l_3 - l_2 \\ 0 & 0 & 1 & -l_3 \\ 0 & -l_1 + l_4 & 0 & 0 \\ 0 & -l_1 + l_5 & 0 & -l_3 + l_5 \end{bmatrix}$$

The right matrix, after eliminating first and third rows and columns, and rewriting $l_i - l_j$ as $[l_i l_j]$, becomes

$$C = \begin{bmatrix} [l_1 l_2] & [l_3 l_2] \\ -[l_1 l_4] & 0 \\ -[l_1 l_5] & -[l_3 l_5] \end{bmatrix}$$

This is the matrix we would expect from proposition 2.13. Note that we can solve and find the result of the operation, as we did before, and obtain

$$l = W(l_1, l_2 | l_3 | l_4, l_5) = l_4([l_1 l_2][l_3 l_5] - [l_3 l_2][l_1 l_5]) + l_5[l_1 l_4][l_3 l_2],$$

which, by straightening (corollary 1.23), and up to a sign, equals 2.7.

COROLLARY 2.14. *Each graph operation in step 2 expands to a bracket polynomial.*

□

The fact that the graph operation is invariant already implied corollary 2.14, but proposition 2.13 also provides an *explicit* expansion in brackets. In the next theorem we are going to provide a purely combinatoric description of such expansion. The bracket polynomial obtained will be multilinear in the elements l_i.

DEFINITION 2.10. Let $G = (V, E)$ a graph and $f \in F$ a terminal edge. A **basic monomial for** $(\mathbf{G}, \mathbf{f})$ is a product

$$[l_{i_1} l_{i_2}] \cdots [l_{i_{e-2}} l_{i_{e-1}}] \tag{2.8}$$

such that

(i) l_{i_j} are all distinct;
(ii) $f \notin \{l_{i_j}\}$;
(iii) $i_{2j-1} < i_{2j}$;
(iv) each bracket contains a pair of edges that meet at a non-terminal vertex v. We will say that the bracket **belongs to** the vertex v.
(v) different brackets belong to different vertices.

We call each of the brackets of (2.8) an **admissible bracket**.

EXAMPLE 2.5. For figure 8, the only basic monomial for (G, f) is

$$[de][ac][bg],$$

and the basic monomials for (G, g) are

$$[df][ac][be], \qquad [ef][bc][ad].$$

The operation associated to a graph will turn out to be linear combination of basic monomials with coefficients ± 1. In order to understand the sign of each basic monomial, we need to introduce two additional definitions. In what follows we assume that E is an ordered set with all skeleton edges smaller than simple edges.

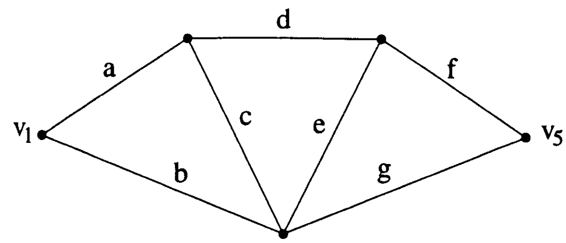


FIGURE 8

DEFINITION 2.11. If σ is a permutation of E, define

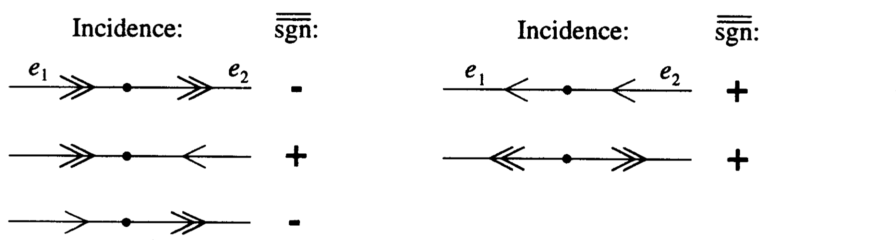
$$\overline{\operatorname{sgn}}(\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\rho),$$

where τ is the permutation of skeleton edges inside σ, and ρ is the permutation of simple edges inside σ.

EXAMPLE 2.6. Suppose the skeleton edges are A, B and the simple edges are a, b, c, ordered lexicographically, then

$$\overline{\operatorname{sgn}}(AbBac) = \operatorname{sgn}(AB)\operatorname{sgn}(bac) = 1 \cdot -1 = -1.$$

DEFINITION 2.12. Given a pair (e_1, e_2) of edges identifying a vertex, define $\overline{\overline{\operatorname{sgn}}}(e_1, e_2)$ as follows, where the sign changes if we change the orientation of a simple vertex.



THEOREM 2.15. *Let $\mathcal{M}(G, f)$ be the set of basic monomials for (G, f), and F be the set of terminal edges. Then*

$$l = G(l_1, \ldots, l_e) = \sum_{f \in F} f \cdot \left(\sum_{m_f \in \mathcal{M}(G,f)} \operatorname{sgn}(m_f)\, m_f \right)$$

where

$$\operatorname{sgn}(m_f) = \overline{\overline{\operatorname{sgn}}}(l_{i_1} l_{i_2}) \cdots \overline{\overline{\operatorname{sgn}}}(l_{i_{e-2}} l_{i_{e-1}}) \cdot \overline{\operatorname{sgn}}(f l_{i_1} \ldots l_{i_e - 1}). \tag{2.9}$$

COROLLARY 2.16 (multilinearity). *Let x,y be points in $\mathbb{P}^2$, and a, b be scalars. Then*

$$\begin{aligned} G(l_1, \ldots, l_{i-1}, ax + by, l_{i+1}, \ldots, l_e) = aG(l_1, \ldots, l_{i-1}, x, l_{i+1}, \ldots, l_e) \\ + bG(l_1, \ldots, l_{i-1}, y, l_{i+1}, \ldots, l_e). \end{aligned} \tag{2.10}$$

□

EXAMPLE 2.7. The expansion of the diamond graph (figure 9) is the following:

$$g[af][de][ci][bh] - h[be][df][ci][ag] + i[cd][ef][bh][ag].$$

The basic monomials are easily verified. For the signs, we have:

$$\begin{aligned}
\overline{\mathrm{sgn}}(gafdecibh) &= \mathrm{sgn}(adcb)\,\mathrm{sgn}(gfeih) = -1 \cdot 1 = -1 \\
\overline{\overline{\mathrm{sgn}}}(af) &= 1 \\
\overline{\overline{\mathrm{sgn}}}(de) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(ci) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(bh) &= -1,
\end{aligned}$$

which, multiplied, give the sign of the first monomial,

$$\begin{aligned}
\overline{\mathrm{sgn}}(hbedfciag) &= \mathrm{sgn}(bdca)\,\mathrm{sgn}(hefig) = 1 \cdot 1 = 1 \\
\overline{\overline{\mathrm{sgn}}}(be) &= 1 \\
\overline{\overline{\mathrm{sgn}}}(df) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(ci) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(ag) &= -1,
\end{aligned}$$

which, multiplied, give the sign of the second monomial, and

$$\begin{aligned}
\overline{\mathrm{sgn}}(icdefbhag) &= \mathrm{sgn}(cdba)\,\mathrm{sgn}(iefhg) = -1 \cdot -1 = 1 \\
\overline{\overline{\mathrm{sgn}}}(cd) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(ef) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(bh) &= -1 \\
\overline{\overline{\mathrm{sgn}}}(ag) &= -1,
\end{aligned}$$

which, multiplied, give the sign of third monomial.

PROOF OF THEOREM. We just sketch the main steps of the proof, since the best way to understand it is to work directly on an example. The starting point for the proof is proposition 2.13. As remarked at the end of proposition 2.13, we just need to expand the determinant of (2.6), where the last column contains the terminal edges. We compute the determinant by Laplace expansion, starting from last column. By this first step we eliminate, in turns, one of the terminal edges and its correspondent row.

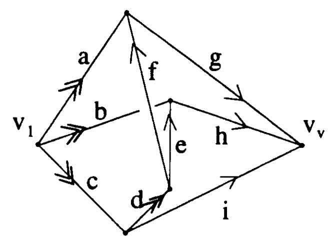
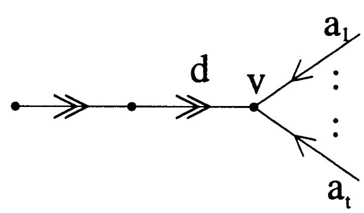


FIGURE 9. Diamond graph

We proceed by Laplace expansion on selected columns, as follows: recall that skeleton edges are partially ordered ($\preceq$), and select any column correspondent to a maximal skeleton edge, say d. The portion of the graph close to d looks like:



The only non zero entries for such column are in correspondence with the simple edges $a_1, \ldots, a_t$, and are $\pm[da_i]$, the sign being positive (negative) if the simple edge points towards (from) v. If $t = 1$ then the Laplace expansion is very simple: we can record the factor $\pm[da_i]$, eliminate the correspondent row and column and move on to another column. This is exactly what we expected, since $[da_i]$ is the only admissible bracket for v.

Now suppose $t > 1$, so that the expansion will be a polynomial with t terms. In what follows we are going to use the following lemma.

LEMMA 2.17. *Let a, b, c, d be* 4 *points in $GC(2)$. Then*

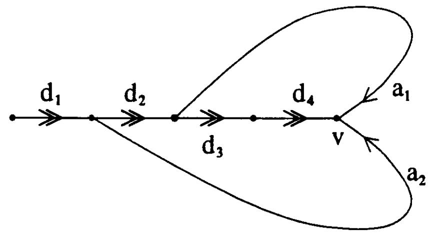
$$[ab][cd] - [ac][bd] + [ad][bc] = 0.$$

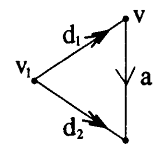
PROOF. This is a particular case of corollary 1.23. □

DEFINITION 2.13. The **backpath** of a skeleton edge d is the set of skeleton edges less than d.

DEFINITION 2.14. Let a be a simple edge seen by a maximal skeleton edge d. We say that a **dies** at the skeleton edge h, or $D(a) = h$, where h belongs to the backpath of d, if h is the largest skeleton edge that does not see a. In case that such h does not exist, we say that a never dies, since d_1 and d_2 are not commensurable.

EXAMPLE 2.8. In the following example on the left, $D(a_1) = d_2$ and $D(a_2) = d_1$, and on the right we see a case where a never dies.





At this point we start expanding the column correspondent to d. We have an expansion that contains $[da_1]$. Every other a_i, $i \neq 1$, will be used either with some $h \not< g$ or with some $h < g$, h larger than $D(a_i)$. In the first case we are all set, since $[da_1]$ is an admissible bracket, and as for $[ha_i]$ we can assume that it can be transformed into an admissible one as well, by inductive reasoning.

In the second case we have a term $[ha_i]$ with $h < d$, and this is not an admissible bracket. There are two cases:

(i) $h \not\leq D(a_1)$ or a_i never dies
(ii) $h \leq D(a_1)$

CASE (i): h sees a_1, therefore among all the possible expansions we have

$$[da_1][ha_i]P, \tag{2.11}$$

for some P, and we also have

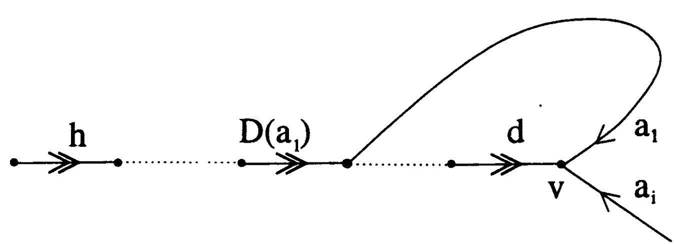
$$-[da_i][ha_1]P, \tag{2.12}$$

where P is the same as before. By combining (2.11) and (2.12) and using lemma 2.17, we get

$$-[hd][a_1a_i]P. \tag{2.13}$$

In (2.13), $[a_1a_i]$ is an admissible bracket for the vertex v, and $[hd]$ can be seen as if we made d become a simple edge. Admissibility of $[hd]$ is taken care of by inductive reasoning again.

CASE (ii): We have something that looks like:



Among all the possible expansions, we have

$$[da_1][ha_i]P, \tag{2.14}$$

and this time we do not have

$$- [da_i][ha_1]P. \tag{2.15}$$

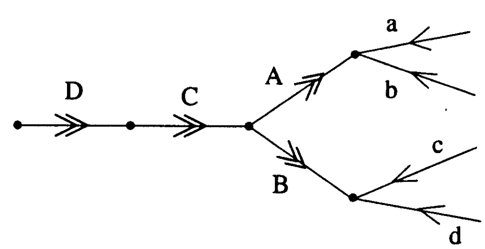
Nevertheless, we can add and subtract (2.15), and combine one of them with (2.14), and obtain

$$- [dh][a_1 a_i]P + [da_i][ha_1]P. \tag{2.16}$$

The first term of (2.16) has $[a_1 a_i]$, which is admissible, and $[dh]$, which can be shown to be transformed into an admissible one by induction. The second term of (2.16) has $[da_i]$, that is admissible, and $[ha_1]$, that is treated again by induction. In this way we expand the column correspondent to vertex v, and can move on to another column. Keeping track of the sign is a tedious process that yields formula (2.9). □

We would like to show an example that illustrates the idea of previous proof. In what follows we will use lemma 2.17 many times.

EXAMPLE 2.9. If a portion of a graph looks like



then the corresponding portion of the matrix associated to it is:

$$\begin{array}{c|cccc|} & A & B & C & B \\ a & [Aa] & & [Ca] & [Da] \\ b & [Ab] & & [Cb] & [Db] \\ c & & [Bc] & [Cc] & [Dc] \\ d & & [Bd] & [Cd] & [Dd] \end{array}$$

By expanding the first column, we get:

$$[Aa]\begin{vmatrix} & [Cb] & [Db] \\ [Bc] & [Cc] & [Dc] \\ [Bd] & [Cd] & [Dd] \end{vmatrix} - [Ab]\begin{vmatrix} & [Ca] & [Da] \\ [Bc] & [Cc] & [Dc] \\ [Bd] & [Cd] & [Dd] \end{vmatrix}$$

By expanding again the reduced matrices by their first columns, we get:

$$- [Aa][Bc]\begin{vmatrix} [Cb] & [Db] \\ [Cd] & [Dd] \end{vmatrix} + [Aa][Bd]\begin{vmatrix} [Cb] & [Db] \\ [Cc] & [Dc] \end{vmatrix} +$$

$$+ [Ab][Bc]\begin{vmatrix} [Ca] & [Da] \\ [Cd] & [Dd] \end{vmatrix} - [Ab][Bd]\begin{vmatrix} [Ca] & [Da] \\ [Cc] & [Dc] \end{vmatrix} \tag{2.17}$$

Now,

$$\begin{vmatrix} [Cb] & [Db] \\ [Cd] & [Dd] \end{vmatrix} = [Cb][Dd] - [Cd][Db] = [CD][bd],$$

and similarly for the other 2×2 matrices. With this reduction, (2.17) becomes:

$$\begin{aligned}
& -[Aa][Bc][CD][bd] + [Aa][Bd][CD][bc] + \\
& +[Ab][Bc][CD][ad] - [Ab][Bd][CD][ac] \\
= \; & [CD]([Aa]([Bd][bc] - [Bc][bd]) + [Ab]([Bc][ad] - [Bd][ac])) \\
= \; & [CD]([Aa][Bb][dc] + [Ab][Ba][cd]) \\
= \; & [CD][cd](-[Aa][Bb] + [Ab][Ba]) \\
= \; & -[CD][cd][AB][ab]
\end{aligned}$$

Notice that this is the only basic monomial for this graph, and the sign matches with (1.6)

4. Bracket Expansion in Step n

In this section we are going to find the bracket expansion of the graph operation when the step is bigger than 2. The expression we obtain is not as compact as the previous case. Nevertheless, we obtain a multilinear bracket expansion. Multilinearity is a key property, since N. White [**15**] discovered an algorithm which reads a multilinear bracket polynomial and returns its expression in joins and meets, if such an expression exists.

Let $a, \dots, z$ be linear subspaces of $\mathbb{R}^{n+1}$ associated to the edges of the graph G. If a is w_a-dimensional, we can fix vectors $a_1, \dots, a_{w_a}$ such that

$$a = a_1 \vee \cdots \vee a_{w_a},$$

and similarly for the remaining edges.

LEMMA 2.18. *The graph G is equivalent to a graph $\widehat{G}$ where an edge of G with label a and weight w_a is replaced by a chain of w_a edges of weight 1 labeled by $a_1, \dots, a_{w_a}$.*

PROOF. This comes at once from the composition property mentioned at the end of section 1, since we express a as a join of vectors, which corresponds to a graph made of a linear chain. □

We can apply lemma 2.18 for every edge of G and obtain an equivalent graph H whose edges have weight 1. As we did in the previous section, we will solve system (2.1) by Cramer's rule. Before doing that we reduce the system by a sequence of row and column operations, as described in the following steps.

STEP 1: Choose a coordinate system such that every vector a_i is of the form

$$a_i = \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^n \\ 1 \end{pmatrix}$$

Let F be the set of terminal edges. Choose a spanning tree for $H \backslash F$. Call an edge **simple** if it is a non-terminal edge and does not belong to the spanning tree. Orient every simple edge at wish, and every terminal edge toward v_v. With these choices, and assuming that $p_1 = 0$, as we did in step 2, the system 2.1 of section 1 becomes:

$$\begin{array}{c|cccccccc|} & a_1 & \cdots\cdots & & z_{w_z} & v_2 & \cdots & v_v \\ a_1 & a_1 & 0 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & & \vdots & & I_{c_j}^{v_i} & \\ z_{w_z} & 0 & \cdots & 0 & z_{w_z} & & & \end{array} \left| \begin{array}{c} \lambda_{a_1} \\ \vdots \\ \lambda_{z_{w_z}} \\ p_2 \\ \vdots \\ p_v \end{array} \right| = 0 \qquad (2.18)$$

where the following notation holds: every line, labeled by $a_1, \dots, z_{w_z}$ is in fact a multiline consisting of $n+1$ lines. Every column labeled by $a_1, \dots, z_{w_z}$ is an actual column, and every column labeled by $v_2, \dots, v_v$ is a multicolumn consisting of $n+1$ columns. For each i, p_i is the vector $(p_i^1, \dots, p_i^{n+1})$ of the coordinates of point p_i. The intersection of a multiline with a multicolumn determines a multiposition. The entry $I_{c_j}^{v_i}$ in multiposition (c_j, v_i) is a square $n+1$-dimensional matrix, for which,

$$I_{c_j}^{v_i} = \begin{cases} I & \text{if } v_i \text{ is the terminal vertex of the edge } c_j \\ -I & \text{if } v_i \text{ is the initial vertex of the edge } c_j \\ 0 & \text{else,} \end{cases}$$

where I is the identity matrix.

STEP 2: Consider the multiline a_1. Subtract a_1^j times the last row from the j^{th} one, for each $j = 1, \dots, n$. In this way the first column of the matrix A will have only one non-zero entry, and can be disposed of when we solve the system. By proceeding in this way for all multilines of A, we end up deleting all the columns on the left (the ones labeled by edges) and are left with

$$B = \begin{array}{c|ccc|} & v_2 & \dots & v_v \\ a_1 & & & \\ \vdots & & \pm J_{c_j}^{v_i} & \\ z_{w_z} & & & \end{array}$$

where the multientry $J_{c_j}^{v_i}$ is a $(n-1 \times n)$ matrix that is nonzero only if v_i is a vertex of c_j, in which case

$$J_{c_j}^{v_i} = \left| \begin{array}{cccccc} 1 & 0 & \dots\dots & & 0 & c_j^1 \\ 0 & 1 & 0 & \dots & 0 & c_j^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots\dots & & 0 & 1 & c_j^n \end{array} \right|$$

and the sign $+$ or $-$ is, again, in accordance with the vertex v_i being a terminal or an initial vertex for c_j.

In what follows, by the sum of two multicolumns (or multilines), we mean the sum of each correspondent column (or line). Let $d_1, \dots, d_d$ be the edges of the spanning tree of H. The labels v_i's for all multicolumns but the last one can be replaced by the labels d_i's, where each nonterminal vertex is replaced by the unique edge of the spanning tree that points to it.

The skeleton edges are partially ordered, and we can extend this partial order to the multicolumns of B. As we did in the previous section, perform the following operation: to each multicolumn add all the multicolumns bigger than it with respect to this partial order. As a result, every skeleton multiline c_j will have only one non-zero multientry, $J_{c_j}^{c_j}$, in the (c_j, c_j) position. This allows us to add (or subtract) the multiline correspondent to a skeleton edge to (from) all simple multiline with a non-zero multientry in position c_j. After this operation, every column, except the last one, of every skeleton edge will have only one non-zero entry, and therefore can be eliminated when we are to solve the system. Note that in this procedure the last multicolumn, associated to the terminal vertex v_v, is not changed. Thanks to this reduction procedure, we have just shown the following:

PROPOSITION 2.19. *Let $d_1, \ldots, d_d$ be the skeleton edges, $g_1, \ldots, g_g$ be the simple edges and $f_1, \ldots, f_f$ be the terminal edges. Then the result of the graph operation is given by the set of vectors $p = (p^1, \cdots, p^{n+1})$ which satisfy the following linear system:*

$$\begin{array}{c} \\ g_1 \\ \vdots \\ g_g \\ f_1 \\ \vdots \\ f_f \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & \ldots & d_d & v_v \end{array} \\ \left|\begin{array}{cc} & 0 \\ & \vdots \\ K_h^{d_i} & 0 \\ & F_1 \\ & \vdots \\ & F_f \end{array}\right| \end{array} \begin{array}{c} \\ \left|\begin{array}{c} \mu_1 \\ \vdots \\ \mu_d \\ p^1 \\ \vdots \\ p^{n+1} \end{array}\right| \end{array} = 0 \tag{2.19}$$

where, for the skeleton edge d_i and simple or terminal edge h, we have:

$$K_h^{d_i} = \begin{cases} \left|\begin{array}{c} d_i^1 - h^1 \\ \vdots \\ d_i^n - h^n \end{array}\right| = d_i - h & \text{if only the } \textbf{tip} \text{ of } h \text{ lies on a path from } d_i \\ -\left|\begin{array}{c} d_i^1 - h^1 \\ \vdots \\ d_i^n - h^n \end{array}\right| = h - d_i & \text{if only the } \textbf{toe} \text{ of } h \text{ lies on a path from } d_i \\ 0 & \text{else} \end{cases}$$

and, for every terminal edge f_i,

$$F_i = \left|\begin{array}{cccccc} 1 & 0 & \ldots\ldots & & 0 & f_i^1 \\ 0 & 1 & 0 & \ldots & 0 & f_i^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots. & & 0 & 1 & f_i^n \end{array}\right|$$

□

DEFINITION 2.15. Let g be either a simple or a terminal edge. Define the set of skeleton edges that **see** g to be

$$\mathcal{S}(g) = \{d_j \mid \text{there exists a skeleton path from } d_j \text{ to either the tip or the toe of } g\text{, but not both.}\}$$

The sign $\mathbf{sgn(d_j, g)}$ is positive if d_j sees the tip of the edge g and negative if it sees the toe of edge g.

DEFINITION 2.16. A bracket

$$[d_{i_1} \cdots d_{i_n} g]$$

is **compatible** for the graph G if

$$d_{i_1}, \ldots, d_{i_n} \in \mathcal{S}(g), \quad \text{and } d_{i_1} < \cdots < d_{i_n}$$

and is **negative** if the number of d_j's that see the toe of g is odd. Similarly, for a terminal edge f, a join

$$d_{i_1} \vee \ldots \vee d_{i_k} \vee f$$

is compatible if

$$d_{i_1}, \ldots, d_{i_k} \in \mathcal{S}(f), \quad \text{and } d_{i_1} < \cdots < d_{i_k}.$$

We are now ready to express the graph operation in term of brackets.

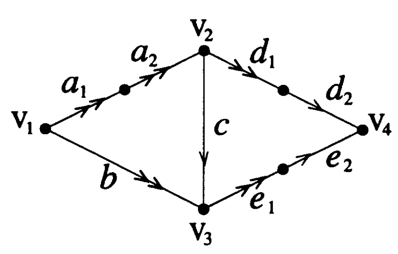
THEOREM 2.20. *Let a graph H with weight 1 edges be given. Let an orientation and a spanning tree be chosen, so that $\{d_1, \ldots, d_d\}$, are the skeleton edges, $\{g_1, \ldots, g_g\}$ are the simple edges and $\{f_1, \ldots, f_f\}$ are the terminal edges. Then the graph operation expands to*

$$\sum_{\sigma \text{ compatible}} \pm [d_{\sigma(1)} \cdots d_{\sigma(n)} g_1] \cdots [d_{\sigma(gn-n+1)} \cdots d_{\sigma(gn)} g_g] \cdot \\ \cdot (d_{\sigma(gn+1)} \cdots d_{\sigma(t)} f_f) \wedge \ldots \wedge (d_{\sigma(s)} \cdots d_{\sigma(p)} f_1), \quad (2.20)$$

where the sum is taken over all permutations for which the brackets and the joins are compatible, and where the sign is given by

$$\text{sgn}(\sigma) \cdot (-1)^{(\#\text{ of negative brackets})}.$$

EXAMPLE 2.10. Before proving the theorem, we want to illustrate an example that will prove this theorem to be simpler than it may appear at a first glance. Let H be the graph in step 3 shown below, for which we fix the orientation and the spanning tree as indicated.



Then the only simple edge is c, for which we have the compatible brackets

$$[a_1 a_2 c], \qquad [a_1 b c], \qquad [a_2 b c],$$

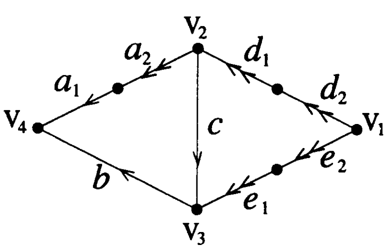
the first of which is positive and the others negative. Every remaining skeleton edge is then forced to choose the only terminal edge it sees, yielding:

$$\begin{aligned} G(a,b,c,d,e) &= +[a_1a_2c]\, be_1e_2 \wedge d_1d_2 \\ &\quad +[a_1bc]\, e_1e_2 \wedge a_2d_1d_2 + \qquad (2.21) \\ &\quad -[a_2bc]\, e_1e_2 \wedge a_1d_1d_2 \\ &= [ac][be]d + ((a\wedge bc)\vee d)e. \qquad (2.22) \end{aligned}$$

In (2.22), $a = a_1 \vee a_2$ and similarly for d, e. The compact expression 2.22 is readily seen to be equivalent to (2.21), by expanding the meet $(a\wedge bc)$ on the left.

REMARK . By choosing a different spanning tree, we obtain the same operation, but not the same expansion, in general. This provides us with a machinery for producing equalities, as the following example will illustrate.

EXAMPLE 2.11. We can reverse the initial and terminal vertices and choose the spanning tree as shown below.



The bracket expansion in that case becomes:

$$\begin{aligned} G(a,b,c,d,e) &= +[d_1d_2c]\, e_1e_2b \wedge a_2a_1 \\ &\quad - \sum_{\sigma,\tau\in\mathcal{S}^2} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)[ce_{\sigma(1)}d_{\tau(1)}]\, ad_{\sigma(2)} \wedge be_{\tau(2)} \\ &= -[cd][be]\, a + ((a\wedge d)\vee c)\wedge e)\vee b. \qquad (2.23) \end{aligned}$$

As a corollary, we have that expression (2.22) equals (2.23) in every Grassmann-Cayley algebra of step 3, up to a sign.

PROOF OF THEOREM. As in previous section, we are going to sketch the main steps of the proof. A satisfactory understanding can be achieved only by working out a specific example.

Let k be the step of the graph H. Let $p = (p^1, \cdots, p^{n+1})$ be solution to the system (2.19). We have enough degrees of freedom to fix the first $k-1$ coordinates of p to be 0, the $(n+1)^{\text{th}}$ to be 1, and look for the remaining coordinates, so that the linear system becomes a square system and we can use Cramer's rule to solve it. With this assumption, p_j, $(k \le j \le n)$ is given by

$$p_j = (-1)^j \frac{\begin{array}{c} \begin{array}{ccc} d_1 & \cdots & d_d \quad v_v \end{array} \\ \begin{array}{c} g_1 \\ \vdots \\ g_g \\ f_1 \\ \vdots \\ f_f \end{array} \left| \begin{array}{cc} & 0 \\ & \vdots \\ K_h^{d_i} & 0 \\ & \widehat{F}_1^j \\ & \vdots \\ & \widehat{F}_f^j \end{array} \right| \end{array}}{\left| \begin{array}{cc} & 0 \\ & \vdots \\ K_h^{d_i} & 0 \\ & \widehat{F}^{n+1} \\ & \vdots \\ & \widehat{F}^{n+1} \end{array} \right|} \quad \text{where} \tag{2.24}$$

$$\widehat{F}_i^j = \left| \begin{array}{ccccc} \multicolumn{4}{c}{\overbrace{\hphantom{0 \ \cdots\cdots\cdots \ 0}}^{n+1-k}} & \\ 0 & \cdots & \cdots & 0 & f_i^1 \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & f_i^{k-1} \\ 1 & 0 & \cdots & 0 & f_i^k \\ 0 & 1 & 0 \ \cdots & 0 & f_i^{k+1} \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & f_i^j \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & f_i^n \end{array} \right|$$

In other words $\widehat{F}_i^j$ consists of all the columns of F_i except columns $1, \ldots, k$, and j, and

$$\widehat{F}^{n+1} = \begin{array}{r} k-1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ n-k+1 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \end{array} \left| \begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\hphantom{0 \ \cdots\cdots\cdots\cdots \ 0}}^{n+1-k}} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ 0 & \cdots & 0 & 1 \end{array} \right|$$

In other words, $\widehat{F}^{n+1}$ consists of the k^{th} up to the n^{th} column of F_i (any i).

For expression (2.24), we can rescale everything and get rid of the denominator, which is the same for every j.

LEMMA 2.21. *Let $c_1, \dots c_n, b$ be $(n+1)$-dimensional vectors whose $(n+1)^{th}$ entry is* 1. *Then*

$$\begin{vmatrix} c_1^1 - b^1 & \dots & c_n^1 - b^1 \\ \vdots & & \vdots \\ c_1^n - b^n & \dots & c_n^n - b^n \end{vmatrix} = [c_1 \cdots c_n b] \tag{2.25}$$

PROOF. The left hand side of (2.25) equals

$$\begin{vmatrix} c_1^1 - b^1 & \dots & c_n^1 - b^1 & b^1 \\ \vdots & & \vdots & \vdots \\ c_1^n - b^n & \dots & c_n^n - b^n & b^n \\ 0 & \dots & 0 & 1 \end{vmatrix}$$

then by adding the last column to all the others we obtain the right hand side of (2.25). □

We now expand the determinant at the top of (2.24) by generalized Laplace expansion on each non-terminal multiline. Thanks to lemma 2.21, we see that every term in the expansion corresponds to a compatible bracket. It remains to show how we obtain the second line of (2.20). We work that out in 4 steps:

STEP 1: At this stage the matrix (2.24) is reduced to terminal multilines and some columns plus the terminal multicolumn. Subtract the first multiline from the others. After this operation, the terminal multicolumn will have all the columns, except the last one, with only one non-zero entry, and can therefore be eliminated, up to a sign.

STEP 2: Add the last column to all the columns with non-zero entries in the first multiline.

LEMMA 2.22. *Let $a_1, \dots, a_k$ be $(n+1)$-dimensional vectors. If $r = (r^1, \dots, r^{n+1})$ is the unique vector of $a_1 \vee \dots \vee a_k$ with $r^1 = \dots = r^{k-1} = 0$, $r^{n+1} = 1$, then*

$$r^j = \begin{vmatrix} a_1^1 & \dots & a_k^1 \\ \vdots & & \vdots \\ a_1^{k-1} & \dots & a_k^{k-1} \\ a_1^j & \dots & a_k^j \end{vmatrix}$$

□

STEP 3: Expand by Laplace along the first multiline. By lemma 2.22, all possible terms will be of the form

$$h_1 \vee \dots \vee h_k,$$

where the h_i's are to be chosen among f_1 and the remaining skeleton edges that see f_1.

STEP 4: We continue the Laplace expansion along the following multilines, noticing that these are multilines with n lines each, and when matched with n columns give an actual bracket.

The edges that see f_1 which were not used in step 3 are to be used to complete the brackets of other multilines. This is equivalent to expanding the meet of (2.20) $(f-1)$ times on the right. Keeping track of the sign for each term of the summation is not too hard, and completes the proof. □

CHAPTER 3

Linear Lattices

In this chapter we are going to define a subclass of the class of linear lattice, which we decided to call *Wheatstone lattices*. These lattices arise naturally in geometry. The lattice of a projective space, for instance, is a Wheatstone lattice. The main idea of the chapter is to consider a lattice in which we are able to define all the graphical operations of a GC algebra. The details, however, are carried on only for the simplest of non series-parallel graphs, the Wheatstone bridge, which is the most important since it allows us to define a quite strong algebraic structure on the lattice.

1. Wheatstone Lattices

Let $L = (A, \Gamma)$ be a lattice of equivalence relations, where $A = \{a, b, \dots\}$ is the set of equivalence relations over the set $\Gamma = \{\alpha, \beta, \dots\}$. It can be shown that every lattice has such a representation. Moreover we can assume that

$$\begin{aligned} \hat{0} &= \bigwedge_{a_i \in A} a_i = \phi \\ \hat{1} &= \bigvee_{a_i \in A} a_i = \Gamma. \end{aligned}$$

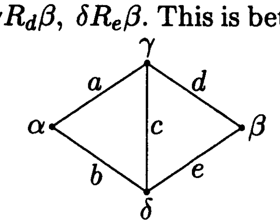
We write $\alpha R_a \beta$ to say that α and β are in the same equivalence class a.

Recall that we can represent L by a graph where the elements of Γ label the vertices and the elements of A label the edges. If an edge is labeled by a then its vertices are a-equivalent.

DEFINITION 3.1. Given a, b, c, d, e in A, define the relation

$$f = \left({a \atop b}\ c\ {d \atop e} \right) \tag{3.1}$$

as follows: α and β are f-equivalent if and only if there exist γ, δ in Γ such that $\alpha R_a \delta$, $\alpha R_b \delta$, $\gamma R_c \delta$, $\gamma R_d \beta$, $\delta R_e \beta$. This is better expressed by the graph:

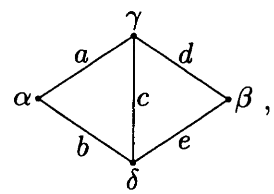


Notice that the relation just defined need not be an equivalence relation.

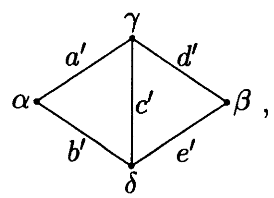
DEFINITION 3.2. A **Wheatstone Lattice** (WL for short) is a lattice of equivalence relations such that the operation defined in 3.1 is an equivalence relation. We call $\left(\begin{smallmatrix} \cdot & & \cdot \\ & \cdot & \\ \cdot & & \cdot \end{smallmatrix}\right)$ a **bridge operation** on L.

PROPOSITION 3.1. *The bridge operation is order preserving*

PROOF. Let $f = \left(\begin{smallmatrix} a & & d \\ & c & \\ b & & e \end{smallmatrix}\right)$, $f' = \left(\begin{smallmatrix} a' & & d' \\ & c' & \\ b' & & e' \end{smallmatrix}\right)$, where $a \le a', \ldots, e \le e'$. Then, if for some δ, γ



clearly



that shows that every block of f is contained in a block of f'. □

PROPOSITION 3.2. *The bridge operation satisfies the following identities:*

$$\left(\begin{matrix} a & & d \\ & c & \\ b & & e \end{matrix}\right) = \left(\begin{matrix} b & & e \\ & c & \\ a & & d \end{matrix}\right) = \left(\begin{matrix} d & & a \\ & c & \\ e & & b \end{matrix}\right) = \left(\begin{matrix} e & & b \\ & c & \\ d & & a \end{matrix}\right) .$$

PROOF. The proof is immediate from the fact that $f = \left(\begin{smallmatrix} a & & d \\ & c & \\ b & & e \end{smallmatrix}\right)$ is an equivalence relation. □

PROPOSITION 3.3. *A Wheatstone lattice is a linear lattice.*

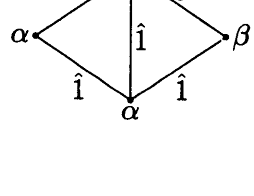
PROOF. We need to show that equivalence relations commute, namely that if

$$\alpha R_a \gamma, \quad \gamma R_b \beta,$$

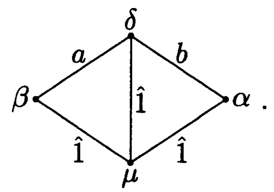
then there exists σ such that

$$\alpha R_b \sigma, \quad \sigma R_a \beta.$$

Define the relation $f = \left(\begin{smallmatrix} a & & b \\ & \hat{1} & \\ \hat{1} & & \hat{1} \end{smallmatrix}\right)$. Then $\alpha R_f \beta$, since we have



From the fact that L is a Wheatstone lattice, f is an equivalence relation, therefore we also have $\beta R_f \alpha$. This means that there exists δ, μ such that



In particular, $\alpha R_b \delta$, $\delta R_a \beta$, and the proof is therefore complete. □

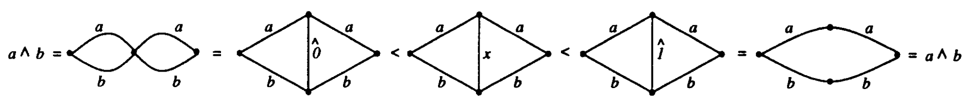
Notice that we have shown the join operation to be a particular case of the bridge operation. The same holds for the meet, as we will see shortly. This allows us to redefine a WL by means of the bridge operation alone.

PROPOSITION 3.4. *For every a, b, x in A,*

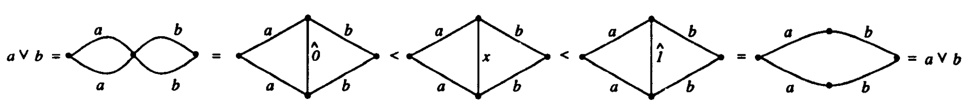
$$\left(\begin{matrix} a & & a \\ & x & \\ b & & b \end{matrix}\right) = a \wedge b$$

$$\left(\begin{matrix} a & & b \\ & x & \\ a & & b \end{matrix}\right) = a \vee b$$

PROOF. Recall that α, β are $\hat{0}$-equivalent only if $\alpha = \beta$ and that α is $\hat{1}$-equivalent to β for every α, β. Using $\hat{0} \leq x \leq \hat{1}$ and proposition 3.1, we have



and



□

We have seen in the previous chapter that the lattice of a projective space is a Wheatstone lattice. In step 2 we have a nice expression of the Wheatstone operation in terms of brackets:

$$\left(\begin{matrix} a & & d \\ & c & \\ b & & e \end{matrix}\right) = [ac][be]d - [bc][ad]e. \tag{3.2}$$

If we switch b with d, we obtain

$$\left(\begin{matrix} a & & b \\ & c & \\ d & & e \end{matrix}\right) = [ac][de]b - [dc][ab]e. \tag{3.3}$$

Equations (3.2) and (3.3) can be shown to be equivalent, thanks to lemma 2.17:

$$\begin{aligned}[ac][be]d - [bc][ad]e &= [ac][bd]e + [ac][de]b - [bc][ad]e \\ &= [ac][de]b + [ab][cd]e\end{aligned}$$

We want to find a suitable definition of commutativity for a Wheatstone lattice. One is tempted to define a WL commutative if, for any $a, \dots, e$,

$$\left({}^{a}_{b}\ c\ {}^{d}_{e} \right) = \left({}^{a}_{d}\ c\ {}^{b}_{e} \right). \tag{3.4}$$

This is too strong, however, since even for a Wheatstone lattice that comes from a GC algebra, (3.4) holds only for generic $a, \dots, e$. In fact, if $a \neq b$,

$$\left({}^{a}_{a}\ a\ {}^{b}_{b} \right) = a \quad \text{while} \quad \left({}^{a}_{b}\ a\ {}^{a}_{b} \right) = \hat{0}.$$

One solution could be to introduce the notion of generic elements in a linear lattice. Another one could be to define the lattice commutative if $\left({}^{a}_{b}\ c\ {}^{d}_{e} \right)$ is comparable with $\left({}^{a}_{d}\ c\ {}^{b}_{e} \right)$, that is, $\left({}^{a}_{b}\ c\ {}^{d}_{e} \right)$ is either bigger or smaller than $\left({}^{a}_{d}\ c\ {}^{b}_{e} \right)$. This is the case, at least, for a GC algebra. Neither of these solutions satisfies the author, therefore the definition remains vacant at the moment.

The need for a definition of a commutative WL comes from combining the following two classical results, which can be found in [**2**] and [**1**].

PROPOSITION 3.5. *For generic lines $a, \dots, e$ in a projective plane P,*

$$\left({}^{a}_{b}\ c\ {}^{d}_{e} \right) = \left({}^{a}_{d}\ c\ {}^{b}_{e} \right)$$

if and only if Pappus' theorem (1.27) holds in P.

PROPOSITION 3.6. *Let P be a projective plane over a division ring K. Then K is commutative if and only if Pappus' theorem holds.*

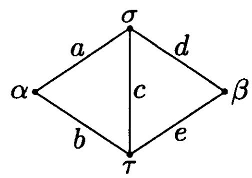
The notion of a Wheatstone lattice is not merely the restatement of projective geometry in a new language. A WL need not be a projective space, yet, it is a refinement of the notion of linear lattice which bears more geometrical meaning.

A large class of WL is given by the Abelian groups:

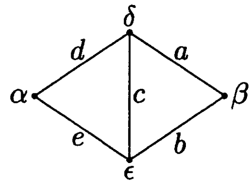
PROPOSITION 3.7. *The lattice of subgroups of an Abelian group is a Wheatstone lattice.*

PROOF. We need to show that the relation (3.1) is an equivalence relation and that the equivalence classes are the cosets of a subgroup.

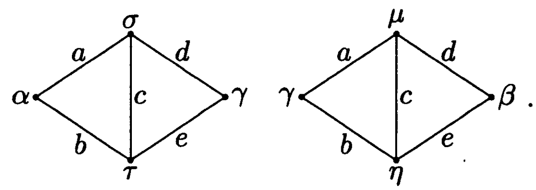
Reflexivity is straightforward. As for symmetry, let $a, b, c, d, e, \alpha, \beta$ be given and suppose there exist σ, τ such that



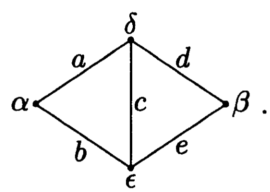
Define $\delta = \alpha + \beta - \sigma$ and $\epsilon = \alpha + \beta - \tau$. Then



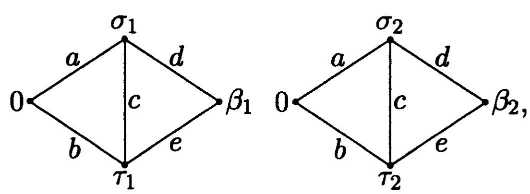
is easily verified, and proves symmetry. To show that transitivity holds, suppose



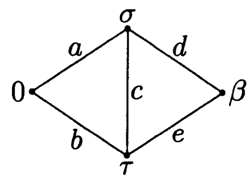
Define $\delta = \sigma + \mu - \gamma$, $\epsilon = \tau + \eta - \gamma$. Then it is immediate to see that



It remains to be shown that the equivalence classes are the cosets of a subgroup. It is enough to show that the elements equivalent to 0 form a subgroup. If



then it can be easily checked that for $\sigma = \sigma_1 - \sigma_2$, $\tau = \tau_1 - \tau_2$, $\beta = \beta_1 - \beta_2$, we have



which proves our claim. □

The question whether normal subgroups of a group and more general linear lattices have a WL structure is still open.

The Wheatstone graph is the simplest graph which is not series-parallel. Wheatstone lattices can be generalized to lattices in which every graphical operation is defined. We decided to call **strong lattices** such lattices. Lattices that come from GC algebra are strong. Many issues can be investigated at this point: which linear lattices are strong? Can a geometry be recovered from a strong lattice? A partial answer to this question is analyzed in the next section.

2. Algebraic Structures

A lattice is *complemented* if for any element a there exists an element b such that $a \vee b = \hat{1}$ and $a \wedge b = \hat{0}$. It is a well known result by Von Neumann that given a complemented modular lattice L of dimension bigger or equal than 4, we can construct a regular ring $\mathcal{R}$ associated to it, in such a way that L is isomorphic to the lattice of all principal right ideals of $\mathcal{R}$. In other words, we can construct a geometry on L. As a consequence, L is a linear lattice. No direct proof of the implication *modular complemented dimension bigger than 3* $\Rightarrow$ *linear complemented* has been found yet. The original proof by Von Neumann is pretty complex, and can be found in [**14**].

In this section we are going to explore to what extent the geometry can be derived from a Wheatstone lattice. A Wheatstone lattice is linear, as we have seen, but it is not complemented in general. We will show how to prove all the properties of a division ring, except distributivity, in a very elegant and simple way. We cannot prove distributivity, otherwise we would obtain a geometry from non-complemented lattices (as some Wheatstone lattices are), which is impossible.

Let $0, 1, \omega$ be elements of a Wheatstone lattice L, fixed once and for all. The reader should be careful not to confuse 0 and 1 with $\hat{0}$ and $\hat{1}$, the least and greatest elements of L. Moreover, in what follows, the symbol $\leq$ will indicate the order relation of L as a lattice.

Following Von Neumann footsteps, define two binary operations $(+, \cdot)$ on L as follows:

$$a + b = \left(\begin{smallmatrix} \omega & & b \\ & 0 & \\ a & & \omega \end{smallmatrix} \right) \tag{3.5}$$

$$a \cdot b = \left(\begin{smallmatrix} \omega & & b \\ & 1 & \\ a & & 0 \end{smallmatrix} \right). \tag{3.6}$$

Define also the following unary operations:

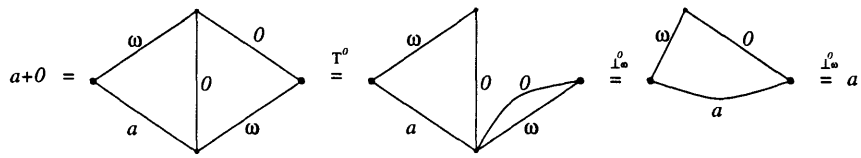
$$-a = \left(\begin{smallmatrix} \omega & & 0 \\ & a & \\ 0 & & \omega \end{smallmatrix} \right) \tag{3.7}$$

$$a^{-1} = \left(\begin{smallmatrix} \omega & & 1 \\ & a & \\ 1 & & 0 \end{smallmatrix} \right) \tag{3.8}$$

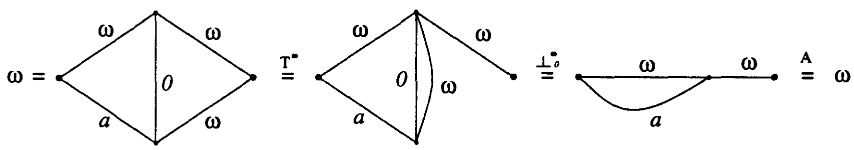
Recall that $a \perp b$ means $a \vee b = \hat{1}$ and $a \wedge b = \hat{0}$. The following properties hold under the condition $0 \perp 1 \perp \omega$, although some of them may hold under weaker assumptions. In the following pictures, symbol $\perp_b^a$ over an equal sing will indicate that we are using orthogonality of a and b. The symbol T^a will indicate that we are using transitivity on two consecutive edges labeled by a, and replace one of them by an appropriate a edge. Finally, the symbol "A" will indicate the absorption law $a \vee (a \wedge b) = a$. The following properties hold in every Wheatstone lattice:

(i) $a + b = b + a$.
This is immediate from the definition of sum.

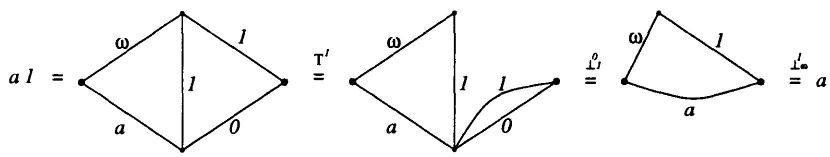
(ii) $a + 0 = a$.



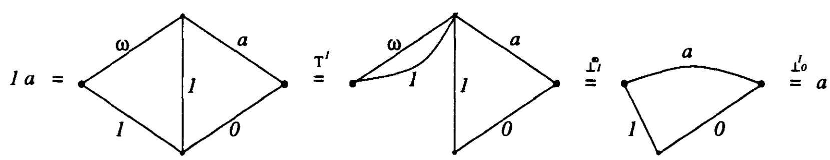
(iii) $a + \omega = \omega$.



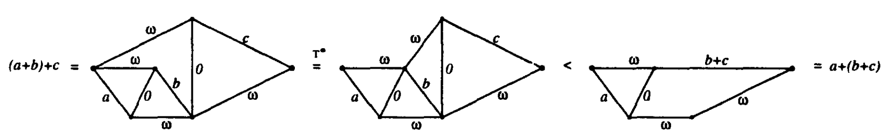
(iv) $a \cdot 1 = a$.



(v) $1 \cdot a = a$.

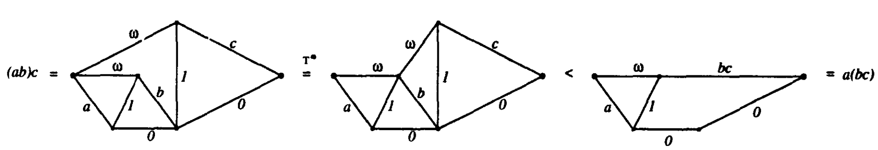


(vi) $(a + b) + c = a + (b + c)$.
The following shows $(a + b) + c \leq a + (b + c)$.



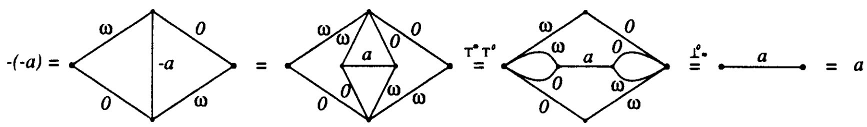
The opposite inequality is automatic by commutativity of the sum.

(vii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
The following shows $(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$.



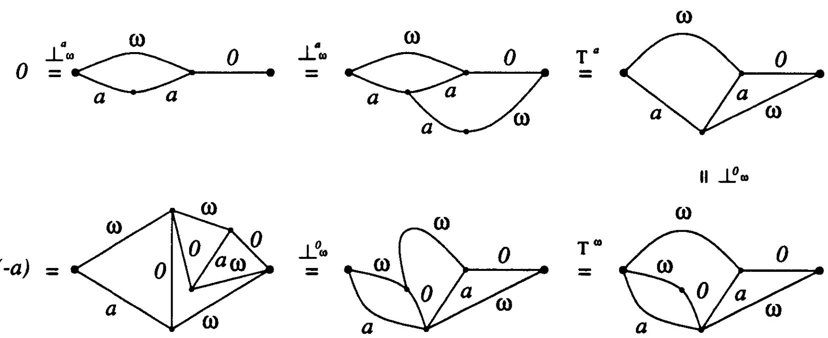
The opposite inequality is similarly proved.

(viii) $-(-a) = a$.



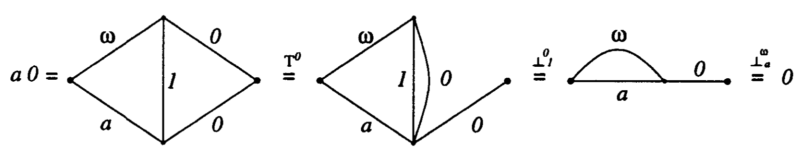
(ix) $(a^{-1})^{-1} = a$.
The proof is basically the same as the previous one.

(x) $\forall a \perp \omega,\ a + (-a) = 0$.

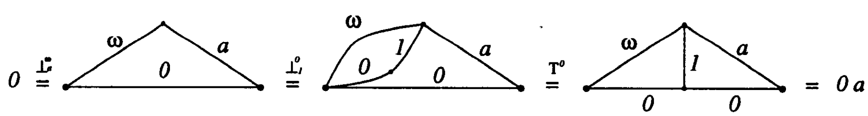


(xi) $\forall a \perp \omega,\ a \perp 0$, we have $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
The proof is similar to the previous one.

(xii) $\forall a \perp \omega,\ a \cdot 0 = 0$.



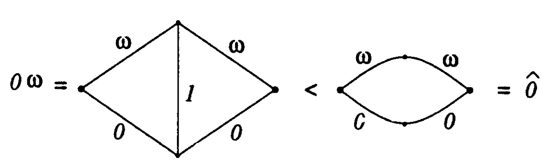
(xiii) $\forall a \perp \omega,\ 0 \cdot a = 0$.



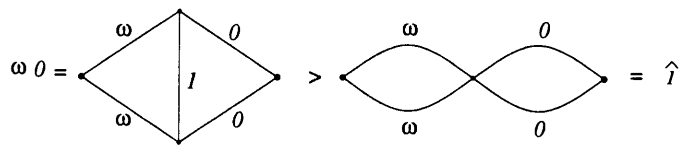
(xiv) $\forall a \perp 0,\ a \cdot \omega = \omega \cdot a = \omega$.
The proofs of these are similar to the previous two.

With this terminology we can give a meaning to expressions that usually do not make much sense:

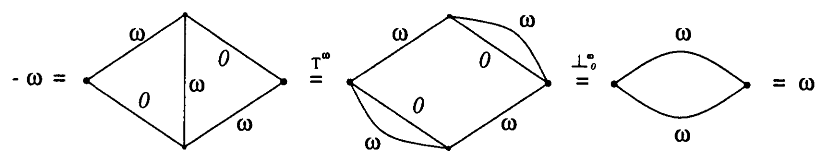
(xvi) $0 \cdot \omega = \hat{0}$.



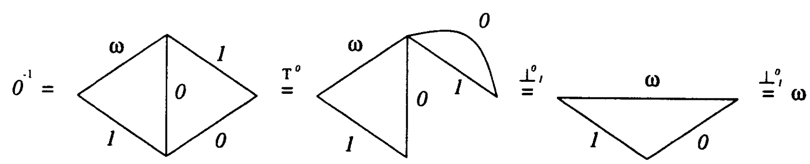
(xvii) $\omega \cdot 0 = \hat{1}$.



(xviii) $-\omega = \omega$.



(xix) $0^{-1} = \omega$



(xx) $\omega^{-1} = 0$.

The proof of this is similar to the previous one.

CHAPTER 4

Geometry

1. Lattical Theorems

The Grassmann-Cayley algebra has proven to be a useful setting for proving and verifying geometric propositions in a projective space P. The propositions they deal with are statements on incidence of projective subspaces of P. Classic theorems of projective geometry, such as the theorem of Desargues, Pappus, Bricard, and many generalizations of them, can be realized as identities in the Grassmann-Cayley algebra, as we have seen in previous chapters. Some of them can also be derived as inequalities in a linear lattice. In what follows we are going to study which theorems have such a lattical interpretation.

DEFINITION 4.1. Let $I : P = Q$ be a GC identity which implies a geometric theorem $T : R \Longleftrightarrow S$, where R and S are geometric statements. Let J be a LL inequality, and suppose that if J holds then it implies $(R \Longrightarrow S)$. We say that J is a (left) **lattical semi-analog** of I. Similarly define the right lattical semi-analog. In case I has left and right lattical semi-analogs which hold in every linear lattice, we say that I is a **lattical GC identity** and theorem T is a **lattical theorem.**

We will see that lattical semi-analogs are not hard to find, but they need not hold in general.

For instance, Pappus' theorem (1.27) cannot be lattical, since it only holds in projective spaces over commutative fields. On the other hand, Desargues' theorem (1.24) is lattical, as shown in theorem 1.32 and following remark. For other theorems, as Bricard's (1.28), the answer is not known yet, though the author believes that it is not lattical. The following inequality, for instance, is a lattical semi-analog of Bricard's identity 1.9, but it has not been proved to hold in every linear lattice, nor a counterexample has been found yet.

$$a\wedge(a'\vee(b'c'\wedge((bb'\wedge a'c')\vee(cc'\wedge a'b')))) \leq$$
$$\leq a\wedge((bc\wedge b'c')\vee(((ac\wedge a'c')\vee b)\wedge((ab\wedge a'b')\vee c))) \quad (4.1)$$

Let P be a polynomial in a Grassmann-Cayley algebra involving only join, meet, and extensors. By a **subexpression** $Q \subseteq P$ we mean that the expression Q in extensors, join and meet is a subexpression of the parenthesized expression P in the alphabets given by the extensors and binary operations $\vee$ and $\wedge$.

Let a be a subexpression of P. Then P can be written as

$$P = ((\cdots((a \vee M_1) \wedge M_2) \vee M_3) \cdots M_{k-1}) \wedge / \vee M_k, \tag{4.2}$$

for some polynomials $M_1, \ldots M_k$. Last operation is a meet if k is even and a join if k is odd.

A polynomial P in a GC algebra of step n is of **full step** if its step is either zero or n. Recall that if $\text{step}(A) + \text{step}(B) = n$, then $A \vee B = (A \wedge B)E$, where E is the integral. We write $R \equiv S$ to say that either $R = S \cdot E$ or $S = R \cdot E$. In what follows every polynomial will be of full step, unless otherwise specified.

LEMMA 4.1. *Let P be a polynomial as in* (4.2). *Then*

$$P \equiv a \wedge (M_1 \vee (M_2 \wedge (M_3 \vee (\cdots (M_{k-1} \vee / \wedge M_k)) \cdots))). \tag{4.3}$$

PROOF. The proof is by induction on k. For $k = 1$,

$$a \vee M_1 \equiv (a \wedge M_1)E.$$

For $k = 2$,

$$(a \vee M_1) \wedge M_2 \equiv (a \vee M_1) \vee M_2 = a \vee (M_1 \vee M_2) \equiv a \wedge (M_1 \vee M_2).$$

Suppose now $k > 2$ and the statement true up to $k - 1$. Call

$$\begin{aligned} M &= (a \vee M_1) \wedge M_2 \\ N &= M_3 \vee (M_4 \wedge \cdots)) \end{aligned}$$

Then

$$\begin{aligned} P &= ((\cdots((M \vee M_3) \wedge M_4) \cdots M_{k-1}) \wedge / \vee M_k \equiv, \text{ by inductive hypothesis} \\ &\equiv M \wedge (M_3 \vee (M_4 \wedge \cdots)) = \\ &= M \wedge N = \\ &= (a \vee M_1) \wedge M_2 \wedge N \equiv \\ &\equiv (a \vee M_1) \vee (M_2 \wedge N) = \\ &= a \vee (M_1 \vee (M_2 \wedge N)) \equiv \\ &\equiv a \wedge (M_1 \vee (M_2 \wedge N)). \end{aligned}$$

□

We call the right hand side of (4.3) the **a-unfolding** of P, and indicate it with $a \wedge \widetilde{P}_a$.

PROPOSITION 4.2. *Let $I : P = Q$ be an identity, and a be a subexpression of P and Q of step 1. Then*

$$a \wedge \widetilde{P}_a \leq a \wedge \widetilde{Q}_a \tag{4.4}$$

is a left lattical semi-analog of I.

PROOF. By lemma 4.1, I is equivalent to

$$a \wedge \widetilde{P}_a = a \wedge \widetilde{Q}_a, \tag{4.5}$$

from which we can derive the geometric theorem

$$a \in \widetilde{P}_a \Longrightarrow a \wedge \widetilde{P}_a = 0 \Longrightarrow a \wedge \widetilde{Q}_a = 0 \Longrightarrow \text{ either } a \in \widetilde{Q}_a \text{ or } \widetilde{Q}_a = 0. \tag{4.6}$$

The possibility $\widetilde{Q}_a = 0$ can be seen as a degenerate version of the geometric theorem.

On the other hand, if equation (4.4) holds in every linear lattice, it implies that

$$a \leq \widetilde{P}_a \Longrightarrow a \leq \widetilde{Q}_a \tag{4.7}$$

holds in every linear lattice, in particular in the lattice of subspaces of a projective space, where it means

$$a \in \widetilde{P}_a \Longrightarrow a \in \widetilde{Q}_a, \tag{4.8}$$

as wanted. □

A large class of identities in a Grassmann-Cayley algebra was found by Mike Hawrylycz in [**11, 12**], which may be viewed as a generalization of alternative laws in the sense of Barnabei, Brini and Rota [**3**]. Hawrylycz calls Arguesian this class of identities, since they may be considered as a generalization of Desargues theorem. In what follows we are going to show that a subclass of the Arguesian identities (Arguesian identities of order 2) is lattical. This means that geometric theorems implied by identities in a Grassmann-Cayley algebra can be proved to be a consequence of inequalities in a linear lattice.

Following Hawrylycz, we take two sets of alphabets, $\mathbf{a} = \{a_1, \dots, a_n\}$ is the set of vectors and $\mathbf{X} = \{X_1, \dots, X_n\}$ is the set of covectors, where n is the step of the GC algebra. We will use lowercase letters to indicate vectors and uppercase for covectors. We will also use the convention that the juxtaposition of vectors denotes their join while the juxtaposition of covectors denotes their meet.

By an **Arguesian polynomial** we mean an expression P in a GC algebra of step n involving only joins, meets and the two sets of variables, $\mathbf{a}$ and $\mathbf{X}$, such that either

(i) P has $k > 1$ occurrences for every covector and 1 occurrence for every vector, in which case P will be called **type I** of order k, or
(ii) P has $k > 1$ occurrences for every vector and 1 occurrence for every covector, in which case P will be called **type II** of order k.

An Arguesian polynomial P is proper if $0 < \text{step}(Q) < n$ for every subexpression $Q \subset P$. Unless otherwise stated, we shall assume that P is proper. Notice that it follows from proposition 1.14 that step(P) is either 0 or n.

Given Arguesian polynomials P and Q, define $P \stackrel{E}{\equiv} Q$, read P is **E-equivalent** to Q, if there exists a real-valued function r of $[a_1, \cdots, a_n]$

and $[[X_1, \cdots, X_n]]$, such that the identity $P = rQ$ is valid in a GC algebra, where we allow that either side may be multiplied by the integral extensor E. E-equivalence incorporates the fact that the scalar brackets $[a_1, \cdots, a_n]$, $[[X_1, \cdots, X_n]]$ and the overall sign difference of P and Q have no bearing on the geometry. Multiplication by the integral, E, merely formalizes the equivalence $P \vee Q = (P \wedge Q) \cdot E$ when $\text{step}(P) + \text{step}(Q) = n$. An **Arguesian identity** of order k is an identity $P \overset{E}{\equiv} Q$ where P is an Arguesian polynomial of type I, order k and Q is an Arguesian polynomial of type II, order k.

In his Ph.D. thesis, Mike Hawrylycz studies a class of Arguesian identities for every order k, as shown in theorem 3.1 of chapter 3 of [**11**]. However, it may be seen from the statement of theorem 3.1 that identities of order 2 are substantially different from identities of larger order. This difference is fundamental in proving that Arguesian identities of order 2 are lattical. For such a reason we decided to report only the portion of theorem 3.1 that deals with Arguesian identities of order 2. The reader is referred to [**11**] for the complete proof. From now on the order 2 will be understood unless otherwise specified. In what follows, the notation has slightly changed from the original theorem.

DEFINITION 4.2. Let $\mathbf{a}$ be a set of n vectors and $\mathbf{X}$ be a set of n covectors. By an **incidence matrix** of $(\mathbf{a}, \mathbf{X})$ we mean a $n \times n$ matrix $\{T(a, X)\}$ with entries in $\{0, 1\}$ such that every row and every column have at least 2 non-zero entries, and such that no two rows or columns are equal. For $a \in \mathbf{a}$, $T(a, \cdot)$ is the set of covectors X_j such that $T(a, X_j) = 1$. Similarly, for $X \in \mathbf{X}$, $T(\cdot, X)$ is the set of vectors a_i such that $T(a_i, X) = 1$.

THEOREM 4.3 (Hawrylycz). *Let an incidence matrix T be given. For a in $\mathbf{a}$, form the extensors*

$$e_a = \Big(\bigwedge_{X_j \in T(a,\cdot)} X_j\Big) \vee a.$$

Similarly, for X in $\mathbf{X}$, form the extensors

$$e_X = \Big(\bigvee_{a_i \in T(\cdot, X)} a_i\Big) \wedge X.$$

Let P be a type I Arguesian polynomial in a Grassmann-Cayley algebra of step n formed starting from the set $\{e_a\} \cup \mathbf{X}$ and applying repeatedly the following rules

(i) *Given polynomial R whose covectors $C(R)$ contain no repeated labels of X, and a basic extensor e_a with $C(R) \subset T(a, \cdot)$, set*

$$R' = (R \wedge (\bigwedge Y_i)) \vee a, \tag{4.9}$$

where Y_i range in $T(a, \cdot) \backslash C(R)$.

(ii) *Given polynomials R, S, form $R \wedge S$.*

Let Q be type II Arguesian formed using the set $\{f_X\} \cup \mathbf{a}$ and dual rules i) and ii). If P Q are type I, II Arguesian polynomial formed by these rules, then

$$P \overset{E}{\equiv} Q. \tag{4.10}$$

In order to show how to obtain the lattical version of identity (4.10), we need to provide a better description of Arguesian identities. The following procedure describes the series-parallel graph equivalent to a given Arguesian identity. The reader is encouraged to follow the procedure with the help of example 4.1.

PROCEDURE 4.1. Rule i) defines a partial order on the set of vectors $\mathbf{a}$ in the following way: whenever we use (4.9), set a greater ($\succ$) than every vector of R. Clearly if $a \succ b$ then $T(a,\cdot) \supset T(b,\cdot)$, but the converse does not hold in general. Moreover, whenever we use rule (4.9), $\{Y_i\}$ will be called the set of **covectors of** a. If, instead, a is a minimal vertex, then $T(a,\cdot)$ will be the set of covectors of a.

Similarly define, by duality, a partial order on covectors and the notion of vectors belonging to a given covector.

With this notation, a type I Arguesian polynomial P is the join-meet expansion of the series-parallel graph obtained in the following way: from a root vertex S draw a tree whose edges are labeled by the vectors according to their partial order, maximal vectors being connected to the root and leaves being at the end of minimal vectors. Take an external point T and connect T with the vertex of the tree, in such a way that the end of a tree edge labeled by a vector a is connected to T by one edge for each (and labeled by) covector of a. A this point connect T and S by one edge for each (and labeled by) covector that has labeled only one edge. If we read the graph so obtained with S and T being its terminal vertices, we obtain the polynomial P.

Similarly, we can translate a type II Arguesian polynomial Q in the following way: draw a circle and divide it into $2n$ edges. Each edge will be labeled by a vector. Then connect vertices of the circle by arcs, which will be labeled by covectors, in such a way that for any covector X, the arc labeled by X will enclose all its vectors and all covectors smaller than X. By placing the terminal vertices S and T at the ends of a maximal edge, we obtain the graph of a polynomial which is equivalent to Q (it is in fact equal to Q up to a $\vee/\wedge$ exchange).

EXAMPLE 4.1. The following 3×3 incidence matrix

	A	B	C
a		*	*
b	*	*	*
c	*	*	

produces, by theorem 4.3, the following Arguesian identity

$$(((BC\vee a)\wedge A)\vee b)\wedge(AB\vee c)\wedge C \stackrel{E}{\equiv} (((bc\wedge A)\vee a)\wedge B)\vee((ab\wedge C)\vee c), \tag{4.11}$$

whose correspondent graphical version is shown in figure 1. Equation (4.11)

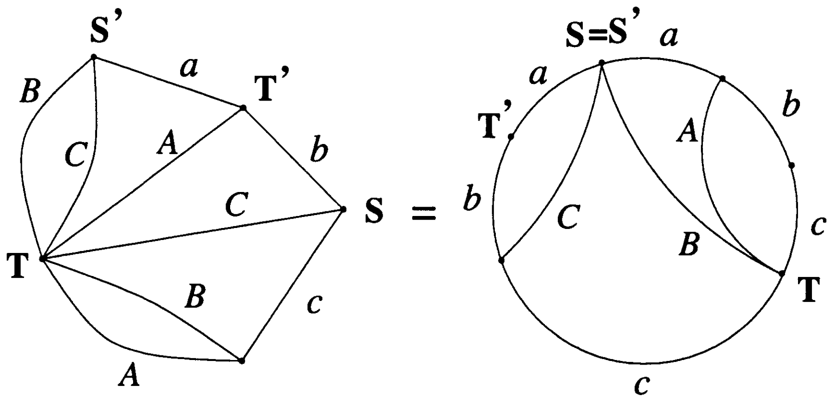


FIGURE 1. Series-parallel graph for equation (4.11)

is what Hawrylycz [**11**] calls the "third identity," since it completes the classification of planar identities, along with Desargues' (equation 1.6) and Bricard's (equation 1.9). The third identity and Desargues' are the only planar identities of order 2, and the geometric theorems they imply can be shown to be equivalent.

PROPOSITION 4.4. *A left semi-analog of* (4.11) *is given by*

$$a\wedge(BC\vee(A\wedge(b\vee((AB\vee c)\wedge C)))) \leq a\wedge(b\vee(C\wedge(c\vee(((bc\wedge A)\vee a)\wedge B)))) \tag{4.12}$$

and holds in every linear lattice.

PROOF. To see that (4.12) is a left semi-analog of (4.11), unfold the left hand side of (4.11) with respect to a and the right hand side with respect to the second occurrence of a. This is graphically achieved by placing the new terminal vertices S' and T' at the end of the edge labeled by a in figure 1. To prove inequality (4.12), starting from the graph at the left of figure 1, double the edges a, b and c and detach in T', S, to obtain the graph at the left of figure 2. This may be redrawn better, as seen on the right of figure 2. From here, by transitivity applied to covectors edges, we obtain the graph at the right of figure 1, as wanted. □

For completeness, we would like to report here thc geometric theorem equivalent to the third identity:

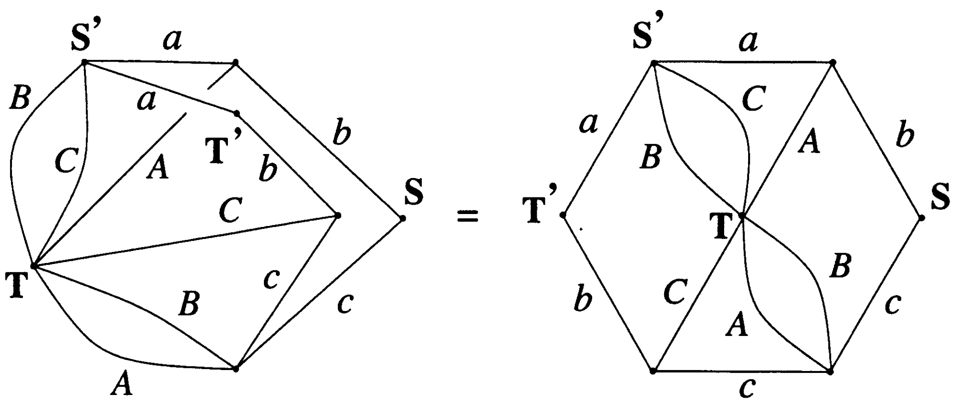


FIGURE 2. Proof of inequality 4.12

THEOREM 4.5. *Let a, b, c and a', b', c' be two triangles in a projective plane. The points $aa' \cap b'c'$, $cc' \cap a'b'$ and b are collinear if and only if the line $a'c'$ and the lines given by joining the points $ab \cap a'b'$ and $bc \cap b'c'$ with c and a, respectively, are concurrent.*

PROOF. By setting $A = b'c'$, $B = a'c'$, $C = a'b'$, the statement follows at once from identity (4.11). One direction of the theorem also follows from (4.12), and that suffices since the theorem is self-dual. □

We are now ready to generalize previous arguments to every Arguesian identity.

LEMMA 4.6. *Let $I : P \overset{E}{\equiv} Q$ be an Arguesian identity, where P is type II and Q is type I. Let a be any vector of P, then*

$$a \wedge \widetilde{P}_a \leq a \wedge \widetilde{Q}_a \tag{4.13}$$

holds in every linear lattice.

PROOF. The method is fairly intricate to be described by a purely theoretical proof. While we will outline the main steps, the reader is encouraged to follow on a real example, for instance the proof of inequality

$$a \wedge (ABC \vee ((c \vee BE) \wedge (e \vee DE) \wedge (b \vee (A \wedge (d \vee CD))))) \leq$$
$$\leq a \wedge (b \vee (A \wedge (d \vee (C \wedge ((B \wedge ca) \vee (E \wedge ce) \vee D \wedge ebd))))), \tag{4.14}$$

which comes from the incidence matrix

	A	B	C	D	E
a	*	*	*		
b	*		*	*	
c		*			*
d			*	*	
e				*	*

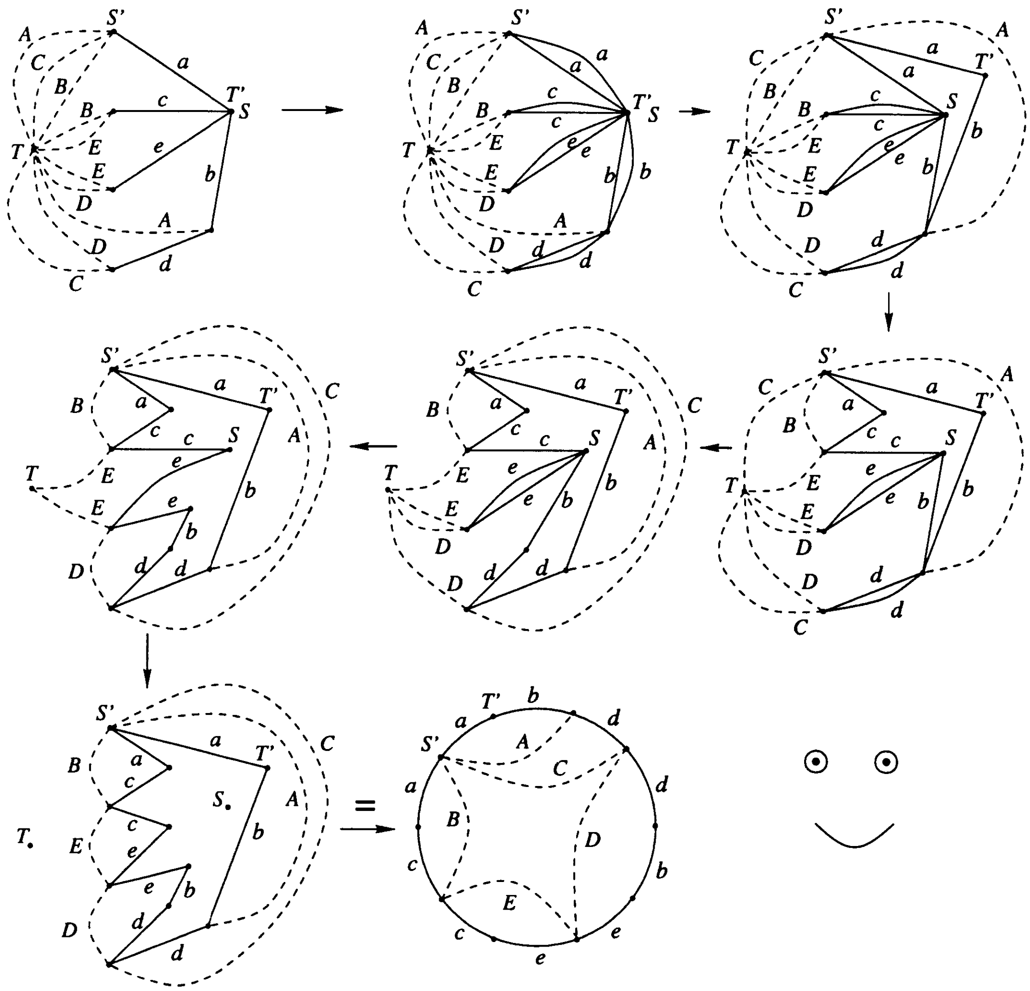


FIGURE 3. Proof of inequality 4.14

The steps of the proof are shown in figure 3.

Translate both sides of (4.13) into series-parallel graphs, as described in procedure 4.1, with the terminal vertices S', T' being the endpoints of an a-edge. For the left hand side, it may be more convenient to draw just the tree of the vector edges, and label each vertex by all covectors edges that should be attached to it, without actually drawing them, since it is understood that they all will end on the external point T. Double every edge of the tree, according to rule (A) of theorem 1.31. The leading idea is that by suitable detachments of tree vertices, we obtain the circle correspondent to the right hand side of (4.13), where the arcs are obtained by transitivity rule (B) on covectors, and subsequent deleting of vertex T and all its edges.

We will study this in more detail. Every covector A labels two tree vertices, name them v_a and w_a. Since every vertex may be labeled by more than one covector, vertices do have more than one name. Fix an order for the covectors, for instance the alphabetic order. Start from the first covector, A. From the way the type II polynomial is constructed from the incidence matrix, it can be seen that the shortest tree path connecting v_a with w_a covers exactly the vectors $T(\cdot, A)$ of the incidence matrix. Detach all the vertices along the path, except v_a and w_a, from the rest of the tree. Apply transitivity to A and connect directly v_a, w_a with an A arc, upon deleting the two A-edges from v_a, w_a to T. Move on to the next covector and perform the same procedure; keep doing the same for all covectors. Since we performed n detachments, we obtain a circle. The arcs connecting the vertices of the circle have the required property, by construction. □

REMARK . It is remarkable to notice that terminal vertices S', T' play no role in the preceding proof. We need them in order to express the graphs as join-meet polynomials, nevertheless the proof of inequality works just the same no matter where these terminal vertices are placed.

DEFINITION 4.3. Given an Arguesian identity $I : P \overset{E}{\equiv} Q$, the **dual identity** $\check{I} : \check{P} \overset{E}{\equiv} \check{Q}$ is the identity obtained from I by dualizing polynomials P and Q. It is clearly an Arguesian identity.

THEOREM 4.7. *Arguesian identities are lattical.*

PROOF. Let $I : P \overset{E}{\equiv} Q$ be an Arguesian identity, where P is type I and Q is type II, which implies the geometric theorem $R \Longleftrightarrow S$. By lemma 4.6, the left semi-analog of I holds, which gives a lattical proof of $R \Longrightarrow S$.

It remains to prove the opposite implication. Notice that the dual $\check{I}$: $\check{P} \equiv \check{Q}$ is an Arguesian identity for which $\check{P}$ is type II and $\check{Q}$ is type I, hence, by lemma 4.6, the following inequality holds

$$\check{A} \wedge \check{Q} \leq \check{A} \wedge \check{P}, \tag{4.15}$$

for every covector A. Inequality (4.15) implies theorem

$$\check{S} \Longrightarrow \check{R}. \tag{4.16}$$

By taking the dual theorem of (4.16), we finally obtain $S \Longrightarrow R$, as wanted. □

2. Examples and Open Problems

For completeness, we are going to work out a complex example starting from the geometric theorem.

THEOREM 4.8. *Given points $a,\dots,f$ and hyperplanes $A,\dots,F$ in a 5-dimensional projective space, the point f lies on the join of line*

$$(e\vee((bc\wedge D)\vee(ad\wedge E)))\wedge F$$

with plane

$$((((ab\wedge A)\vee cf)\wedge B)\vee de)\wedge C$$

if and only if the intersection of plane

$$(((((CF\vee e)\wedge E)\vee d)\wedge AB)\vee a)\wedge D$$

with the 3-space

$$((((BC\vee f)\wedge DF)\vee c)\wedge A)\vee b$$

lies on E.

PROOF. The proof is just a corollary of the following Arguesian identity:

$$(((((CF\vee e)\wedge E)\vee d)\wedge AB)\vee a)\wedge DE\wedge(((((BC\vee f)\wedge DF)\vee c)\wedge A)\vee b)\overset{E}{\equiv}$$
$$\overset{E}{\equiv}(((((ab\wedge A)\vee cf)\wedge B)\vee de)\wedge C)\vee f\vee((e\vee((bc\wedge D)\vee(ad\wedge E)))\wedge F), \tag{4.17}$$

which is built from the following incidence matrix:

	A	B	C	D	E	F
a	*	*	*		*	*
b	*	*	*	*		*
c		*	*	*		*
d			*		*	*
e			*			*
f		*	*			

(4.18)

□

According to proposition 4.2, the identity (4.6) has the lattice semi-analog

$$f\wedge(BC\vee(DF\wedge(c\vee(A\wedge(b\vee((((((CF\vee e)\wedge E)\vee d)\wedge AB)\vee a)\wedge DE))))))$$
$$\leq f\wedge((((((ab\wedge A)\vee cf)\wedge B)\vee de)\wedge C)\vee((e\vee((bc\wedge D)\vee(ad\wedge E)))\wedge F)). \tag{4.19}$$

which, if it holds, implies the same geometric theorem as the "only if" part of theorem 4.8. Furthermore, lattice identity 4.19 is stronger then the identity 4.6 in GC, since the former holds for every $a,\dots,f,A,\dots,F$, no matter if they are distinct, and no matter if they are vector, covectors or subspaces of any dimension whatsoever. Although inequality (4.19) holds thanks to lemma 4.6, we will show once again, for more clarity, how the proof is actually derived. The left hand side of (4.19) has the series-parallel equivalent shown in figure 4, where S and T are the terminal vertices We

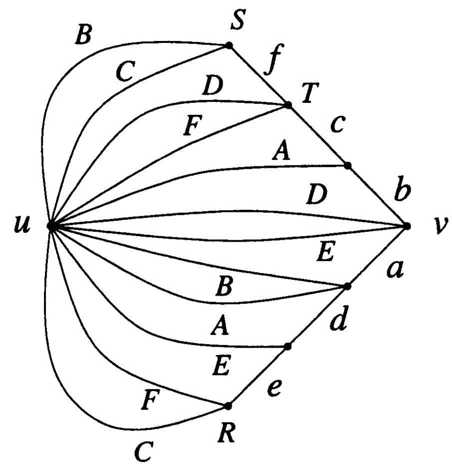


FIGURE 4. Series-parallel translation of LHS of (4.19)

can now double each vector edge and detach all the points except S and R, to obtain the graph of figure 5. By enlarging the circle given by the vector

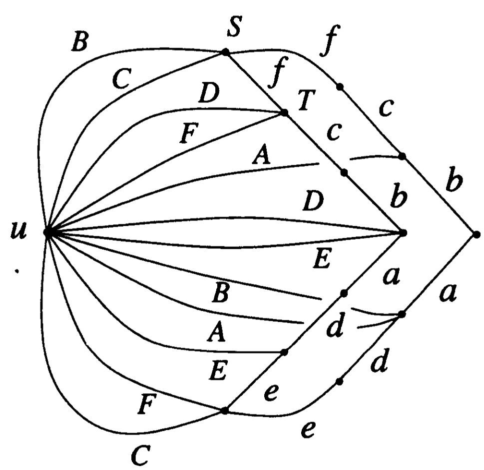


FIGURE 5

edges, and using transitivity on covector edges, we can derive the graph of figure 6, that is the series-parallel graph correspondent to the right hand side of (4.19), as desired.

If the reader tried to build an Arguesian identities from some incidence matrix, she must have noticed that not any incidence matrix will produce an Arguesian identity. The matrix must satisfy some suitable properties,

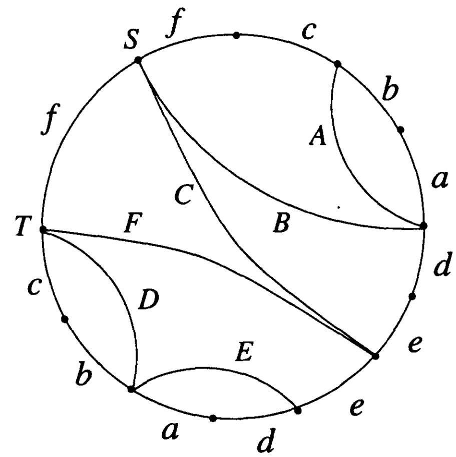


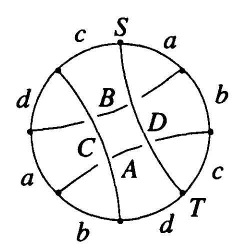
FIGURE 6. Series-parallel translation of RHS of (4.19)

that are subtly expressed in theorem 4.3. We will call **admissible** a matrix for which an Arguesian identity can be built.

It is natural to wonder what happens if we start from a non-admissible matrix. Take the following matrix for example:

$$\begin{array}{c|cccc|} & A & B & C & D \\ a & & * & * & * \\ b & * & & * & * \\ c & * & * & & * \\ d & * & * & * & \end{array} \qquad (4.20)$$

If we try to draw a graph for a potentially type I Arguesian polynomial with the same method (procedure 4.1) as before, we obtain the following,



which is not series-parallel anymore, for any choice of terminal vertices at the ends of a maximal arc. This is what motivated us for introducing invariant operations both in a GC algebra and in a linear lattice which are not series-parallel, yet are a generalization of them. The issue of understanding the new class of identities we could get with this generalization is still open.

In [**9**], Haiman showed that there exist Arguesian lattices (lattices in which Desargues inequality holds) which are not linear. It is an open question weather Arguesian identities hold in any Arguesian lattice.

DEFINITION 4.4. An admissible matrix T is **self-dual** if there exists a permutation of the columns and the rows which transforms T into a symmetric matrix.

Geometric theorem implied by self-dual matrices are clearly self-dual. Examples of self-dual theorems are Desargues', third identity's, and equation (4.14), as can be easily verified. Matrix (4.18), on the other hand, is not self-dual.

For step 3, the only Arguesian identities, up to a permutation of the vector and covector sets, are Desargues' and third identity, as previously remarked, and the geometric theorems they imply can be shown to be equivalent.

Step 4 still bears a highly geometric relevance, for its interpretation in 3-dimensional projective space, and a complexity still under control, so that a classification can be found without too much effort, as next theorem shows. For step greater than 4 a classification seems quite challenging.

THEOREM 4.9. *There exist exactly 7 irreducible Arguesian identities in step 4. Identities 1-5 are self-dual and the last two are dual one of the other. The corresponding matrices are given by:*

$$1: \begin{vmatrix} * & * & & \\ * & & * & \\ & * & & * \\ & & * & * \end{vmatrix} \qquad 2: \begin{vmatrix} & * & * & \\ * & * & * & \\ * & * & & * \\ & & * & * \end{vmatrix} \qquad 3: \begin{vmatrix} * & * & * & * \\ * & * & * & \\ * & * & & \\ * & & & * \end{vmatrix}$$

$$4: \begin{vmatrix} * & * & & * \\ * & * & & \\ & & * & * \\ * & & * & * \end{vmatrix} \qquad 5: \begin{vmatrix} * & * & * & * \\ * & & * & * \\ * & * & * & \\ * & * & & \end{vmatrix} \qquad 6: \begin{vmatrix} * & * & * & * \\ * & * & & \\ * & & * & \\ & & * & * \end{vmatrix}$$

and corresponding identities are graphically shown in figure 7. Identity 1 is what Haiman calls first higher Arguesian identity *in* [**8**].

□

From the incidence matrices, or from the graphs of figure 7, we can derive the equivalent geometric lattical theorems. Using a dual basis $A = b'c'd'$, $B = a'c'd'$, $C = a'b'd'$, and $D = a'b'c'$, moreover, we can express the theorems in terms of configurations of points. The following three examples can be derived from matrices 1, 4, and 6, respectively.

THEOREM 4.10. *Given points $a, b, c, d, a', b', c', d'$ in a 3-dimensional real projective space, then the points $ab \cap b'c'd'$, $bd \cap a'b'd'$, $cd \cap a'c'd'$, and $ac \cap a'b'c'$ lie on a common plane if and only if the planes $b'c'a$, $b'd'b$, $a'b'd$, and $a'c'c$ meet at a common point.*

□

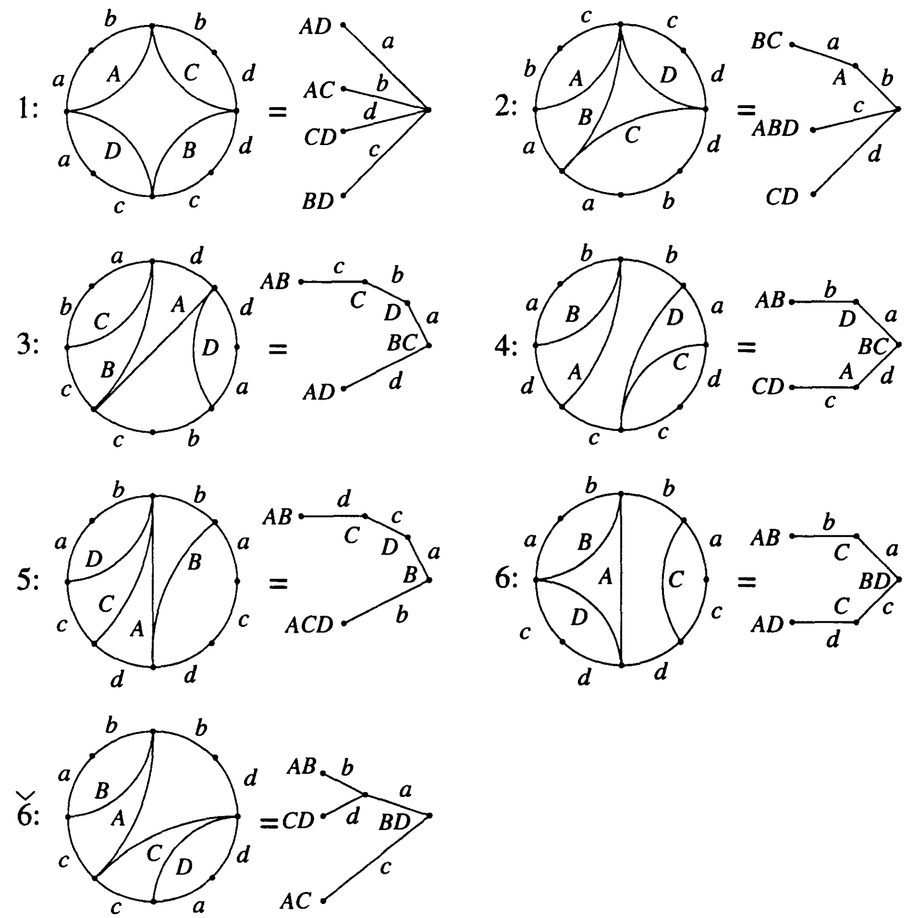


FIGURE 7. Classification of Arguesian identities in $\mathbb{P}^3$

THEOREM 4.11. *Let points $a, b, c, d, a', b', c', d'$ be given in a 3-dimensional real projective space. The points $((ab\wedge a'c'd')\vee d)\wedge b'c'd'$, $((cd\wedge a'b'd')\vee a)\wedge a'b'c'$, b and c lie on a common plane if and only the line given by the intersections of the planes $(c'd'b\wedge a'b'c')\vee a$ with $(a'b'c\wedge b'c'd')\vee d$ intersects the line $a'd'$.*

□

THEOREM 4.12. *Let points $a, b, c, d, a', b', c', d'$ be given in a 3-dimensional real projective space. Then the line determined by the points $ab\wedge a'c'd'$ and $cd\wedge a'b'c'$ meets the line determined by the intersection of the planes $(ac\wedge a'b'd')\vee bd$ and $b'c'd'$ if and only if the line determined by the intersection of the planes $(c'd'b\wedge a'b'd')\vee a$ and $(b'c'd\wedge a'b'd')\vee c$ meets the line $a'c'$.*

□

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