Stability of Algebraic Manifolds

by

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ABSTRACT

In this paper, we study the stability of non-singular projective varieties. We will prove a geometric criterion for a non-singular projective variety to be GIT stable in the Hilbert scheme, and then relate the Gieseker-Mumford stability of polarized manifolds to the behavior of heat kernels. We will also discuss the stability notion used by Viehweg, and state some new semi-positivity results of Hodge bundles. In the end, we find some of the methods we used to understand the semi-positivity can be applied to study various vanishing theorems.

Thesis Supervisor: I. M. Singer
Title: Professor of Mathematics, Institute Professor

Acknowledgment

When I came Boston five years ago, I knew little about the current development of modern mathematics. It is very lucky that one year later, I began to attend professor Shing-Tung Yau's student seminar. I found it is such an exciting mathematics world for me. I really want to thank professor Shing-Tung Yau for his numerous advice in my graduate studies. It is after I met him that I learned most of my knowledge about modern geometry.

I would like to thank professor I. M. Singer for his financial support and helpful conversations during my research. There are many other people gave me also a lot of help and encouragement. I would like to take this opportunity to thank them: professor Hung Cheng, Victor Guillemin, Joe Harris, Steven Kleiman, Jun Li, Gang Tian, Eckart Viehweg, ... , etc.

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Contents

1 Introduction and Discussion

The main theme of this paper is to try to understand the geometric meaning of stability of algebraic manifolds for the study of moduli problem. The results are divided into several parts, however, they are closely related to each other.

Heat Kernel and Gieseker-Mumford Stability. In Geometric Invariant Theory, the notion of stability for any polarized projective variety is introduced. However to check the stability is usually a difficult problem, see [Mu], [Gi] and [V1]. It is therefore very interesting to describe the meaning of stability by geometric data of the polarized projective varieties. In this paper we will in particular show that the Gieseker-Mumford stability of a polarized smooth projective variety (as used by them in [Gi], [Mu]) is related to the existence of a special metric on the polarized line bundle.

In early 80's, Yau conjectured the relation between notions of stability of manifolds and existence of special metrics such as Kahler-Einstein metrics. During my graduate studies, Yau suggested me to work towards this direction. From chapter 2 to chapter 4, I will deal with the case of Gieseker-Mumford stability. Similar problems have been studied before. In Tian's recent work ([T1], [T2]), he studied the relation between Kahler-Einstein metric and stability extensively. The notion of stability used by Tian is different from those used by Gieseker and Mumford. However we will see with modifications his methods can still be used in the study of the stability of polarized manifold in the sense of Gieseker and Mumford.

Another motivation comes from the work on Mumford stability of vector bundles by Donaldson ([Do1],[Do2]), and by Uhlenbeck and Yau ([UY]). They proved that Mumford stability is equivalent to the existence of Hermitian-Einstein metric. So the meaning of stability of a vector bundle is clearly described by its geometry. We would like to call this correspondence the HKDUY correspondence (Hitchin-Kobayashi-Donaldson-Ulenbeck-Yau).

Our first main result is an interesting geometric criterion for a smooth projective subvariety of $\mathbb{C}P^N$ to be GIT stable in the Hilbert scheme.

Theorem 1.1 *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety. Assume there exists $\sigma \in SL(N+1,\mathbb{C})$ such that*

$$\frac{1}{vol(M)} \int_{\sigma(M)} \left(\frac{z_i \cdot \overline{z}_j}{|z_0|^2 + \ldots + |z_N|^2} \right) \omega_{FS}^n = \frac{1}{N+1} \delta_{ij}$$

where ω_{FS} is the Fubini-Study metric and $[z_0, \cdots, z_N]$ is the homogeneous coordinates of $\mathbb{C}P^N$. Then the Hilbert point of M is (GIT) stable if its stabilizer with respect to the action of $SL(N+1,\mathbb{C})$ is finite.

Apply this theorem, we can study the stability of polarized manifolds. For any polarized manifold (M, L) with fixed Hilbert polynomial, choose a large number k (depends on Hilbert polynomial), we can embed M into some $\mathbb{C}P^N$ by L^k. Then we can talk about the stability of (M, L) by consider the GIT stability of the corresponding Hilbert point. This is the stability notion for polarized manifolds used by Gieseker and Mumford. More precise definition will be given in chapter 2.

Theorem 1.2 *Let (M, L) be a polarized manifold. For any large number k, if there exists a metric g (depending on k) on L such that $B_k(z) = B_k(z, g, Ric(g))$ is pointwise constant function on M, then the k-th Hilbert point of (M, L) is (GIT) stable if it has finite stabilizer and consequently (M, L) is Gieseker-Mumford stable.*

Here $B_k(z)$ is the limiting function (as time goes to infinity) of heat kernel for Hermitian line bundle L^k. The definition of $B_k(z)$ is given in chapter 4. We don't know whether the converse of this theorem is true yet. However at least for a large class of polarized manifolds, the converse is true.

The proof of these two theorems will occupy the next three chapters. In chapter 2, we introduce the Gieseker-Mumford stability. We will also try to reduce the problem of checking stability. The definition of Gieseker-Mumford stability depends on the Hilbert scheme and the universal family which is usually "very singular". In chapter 3, we will try to deal with this difficulty. And we will use (singular) Riemann-Roch to introduce a functional D_M which is closely related to Gieseker-Mumford stability. Actually the definition of D_M is motivated by K-energy in the study of Kahler-Einstein metric, and also by Donaldson functional in the study of stability of vector bundles. In chapter 4, we will prove these two theorems and relate the Gieseker-Mumford stability to the existence of good metrics on polarized line bundles.

Weak Positivity and Viehweg Stability. In chapter 5, we'll introduce briefly Viehweg's approach to moduli space of polarized projective varieties with semi-ample canonical line bundle. Compared with Gieseker and Mumford's approach, he used only an open part of the usual Hilbert scheme and he choosed a different ample line bundle over this (quasi-projective) Hilbert scheme. In his approach, the stability will follow directly from the weak positivity of some vector bundles such as the direct images of tensor powers of relative dualizing sheaves. So he shift the difficulty of checking stability to weak positivity. Therefore if one wants to understand Viehweg's approach, he has to understand those weak positivity. Roughly speaking, a vector bundle over a quasi-projective variety is weak positive if we can find a suitable compactification of the quasi-projective variety such that the vector bundle can be extended to be a semi-positive vector bundle.

For the direct images of relative dualizing sheaves, the positivity has been studied by Fujita, Kawamata, Kollar, Viehweg, etc. It is natural to study it from the view-point of Variation of Hodge Structures since the direct images of relative dualizing sheaves is one of the Hodge bundles. We can give a geometric proof of the following known theorem.

Theorem 1.3 *Let $\bar{Y}$ be a smooth complex projective variety and D be a divisor of normal crossing on $\bar{Y}$. Consider a polarized VHS of weight b on $Y = \bar{Y} - D$ with unipotent local monodromies. Let H be the underlying local system and let $\mathcal{H}$ be the canonical extension of H on $\bar{Y}$. Then the lowest filtration $F^n(\mathcal{H}) = \mathcal{H}^{n,b-n}$ is a semi-positive vector bundle over $\bar{Y}$*

This theorem was proved before by Kawamata in [Ka2]. His proof depends on the theory of limiting mixed Hodge structures. Our approach is to use Hormander's L^2 estimate for Hodge bundles equiped with Hodge metric. Schmid's Nilpotent Orbit theorem is used to understand the asymptotical behavior of Hodge metric. If we consider a family of polarized Calabi-Yau manifolds then we can get new semi-positivity results for Hodge bundles. One of them is the following theorem.

Theorem 1.4 *Let $f : X \to Y$ be a family of n-dimensional polarized Calabi-Yau manifolds. Assume Y is Zariski open subset of a smooth complex projective variety $\bar{Y}$ such that $D = \bar{Y} - Y$ is a normal crossing divisor. Let $H = R^n f_*(\mathbb{C})_{prim}$ and assume the local monodromies of H around D are unipotent. Let $\mathcal{H}$ be the canonical extension of H on $\bar{Y}$. Then $\mathcal{H}^{n-1,1} \otimes \mathcal{H}^{n,0}$ is semi-positive vector bundle on $\bar{Y}$, and $det^2(\mathcal{H}^{n-1,1}) \otimes \mathcal{H}^{n,0}$ is nef line bundle on $\bar{Y}$.*

We hope this theorem will also have some implications to the moduli space of Calabi-Yau manifolds.

In this paper, we will not include the detail of the proofs of both of these two theorems since they are essentially very similar as the proofs of vanishing theorems which we will discuss in chapter 6.

Discussion of Vanishing Theorems. Various vanishing theorems play important roles in algebraic geometry. In the development of geometry of higher dimensional varieties (especially in Mori's minimal model program), Kawamata-Viehweg vanishing theorem (see [Ka1], [V3]) is one of the cornerstones. Our motivation to study vanishing theorems, however, are obtained from an attempt to understand Viehweg's work on moduli space, especially on weak positivity. We found our geometric methods used to understand weak positivity can be applied to study vanishing theorems too. By this way, we can prove the following result.

Theorem 1.5 *Let X be a smooth complex projective variety, F be a holomorphic line bundle over X, and A be a subvariety of X. Assume some positive multiple mF can be written as $mF = L + D$ where $D = \sum_{i=1}^{r} \nu_i D_i$ is an effective normal crossing divisor ($\nu_i \geq 0$), and L satisfies for some number $k \geq 1$, $B_{|kL|} \subset A$ and $\Psi_{|kL|}$ restricted on $X \backslash A$ has at most $b-$dimensional fibers. Then for any $p + q \geq n + max(b, dim(A)) + 1$*

$$H^q(X, \ \Omega^p(logD) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1 + [\frac{\nu_i}{m}])D_i)) = 0$$

This theorem can be viewed as Kawamata-Viehweg type generalization of Shiffman-Sommese's theorem ([SS], theorem 3.37). In particular when L is ample, it generalizes Akizuki-Kodaira-Nakano vanishing theorem.

Esnault and Viehweg studied vanishing theorems extensively in [EV1], [EV2], [EV3]. They built up general algebraic methods to various vanishing theorems. Though theorem 1.5 are not written explicitly in their papers, however, it can also been deduced from their results. In this paper we try to use analytic approach here. Our proof is quite elementary and depends on a careful study of L^2 cohomology on the open manifold $X \backslash D$. One advantage of this method is it can be applied to study vanishing theorems on Kahler manifold instead of projective variety. Another advantage is it can be applied to study higher rank vector bundles. By the same method, we can deduce a generalization of Nakano's vanishing theorem which is not known before.

Theorem 1.6 *Let X be a compact complex manifold of dimension n and F be a Nakano positive vector bundle on X. If D is a simple normal crossing divisor, then*

$$H^q(X, \ K_X \otimes F \otimes \mathcal{O}(D)) = 0 \qquad for \quad q \geq 1$$

We can also generalize Kawamata-Viehweg vanishing theorem slightly to the case of higher rank vector bundles (see theorem 6.2). The detail will be found in chapter 6.

We will try to apply our methods to study vanishing theorems on singular varieties in the future.

2 Gieseker-Mumford Stability

In algebraic geometry people frequently need to consider the moduli problem of polarized varieties, *i.e.*, we consider moduli functor

$$\Im : \quad \text{Schemes}/\mathbb{C} \longrightarrow \text{Sets}$$

for the following objects:

$$\Im(\mathbb{C}) = \{(\Gamma, \mathcal{H}) \mid \Gamma \text{ is a projective variety}, \mathcal{H} \text{ ample line bundle on } \Gamma\}$$

$\mathcal{H}$ is called a polarization of Γ, and $(\Gamma, \mathcal{H})$ is a polarized variety. (M, L) is called a polarized manifold when Γ is smooth. If the canonical sheaf ω_Γ is an ample line bundle then usually people choose $\mathcal{H}$ to be ω_Γ, and Γ is called canonically polarized. Also we identify $(\Gamma, \mathcal{H})$ with $(\Gamma', \mathcal{H}')$ if there isomorphism $\tau : \Gamma \to \Gamma'$ such that $\tau^*(\mathcal{H}') \cong \mathcal{H}$.

It turns out we should fix some numerical invariants (Hilbert polynomial) first, in order to "split" $\Im$ into smaller pieces. Recall for any line bundle $\mathcal{H}$ over Γ, the Euler-Poincare characteristic $\chi(\Gamma, \mathcal{H}^m)$ is a polynomial of m. Fix a polynomial $h'(T) \in \mathbb{Q}[T]$ of degree n, then we can consider the moduli problem for

$$\Im_{h'}(\mathbb{C}) = \{(\Gamma, \mathcal{H}) \mid (\Gamma, \mathcal{H}) \in \Im, \chi(\Gamma, \mathcal{H}^m) = h'(m) \quad \forall m \geq 1\}$$

People are interested in proving the existence of moduli space. If using Geometric Invariant Theory, then the essential point is the study of stability.

2.1 Moduli Space and Gieseker-Mumford Stability

Let's introduce the approach Gieseker and Mumford used to study moduli space. Later in chapter 5, we will discuss the approach used by Viehweg.

By Matsusaka's Big Theorem, $\Im_{h'}$ is bounded, so we can choose a large number $\mu_0 \geq 1$ depending only on h', such that for all $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$ we have

$$\begin{aligned} &\mathcal{H}^\mu \text{ is very ample, for all } \mu \geq \mu_0 \\ &H^i(\Gamma, \mathcal{H}^\mu) = 0, \quad \text{for all } i \geq 1, \mu \geq \mu_0 \end{aligned} \tag{1}$$

Therefore for all $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$ we can use $\mathcal{H}^\mu$ $(\mu \geq \mu_0)$ to embedd Γ as a closed subvariety of a fixed projective space $\mathbb{C}P^N$, for $N = h'(\mu) - 1$. This embedding is not canonical, it depends on the choice of a basis of $H^0(\Gamma, \mathcal{H}^\mu)$. Let $h(T) = h'(\mu T)$ be a polynomial in $\mathbb{Q}[T]$. Grothendieck proved there is a scheme $Hilb_h$ (the so called Hilbert scheme) parametrize all the subschemes of $\mathbb{C}P^N$ with fixed Hilbert polynomial h, and over $Hilb_h$ there is a universal family $Univ_h$ given as

$$\begin{array}{ccc} Univ_h & \xrightarrow{\subset} & Hilb_h \times \mathbb{C}P^N \\ {\scriptstyle g}\downarrow & & \\ Hilb_h & & \end{array} \tag{2}$$

Definition 2.1 *For any projective subvariety $X \subset \mathbb{C}P^N$ with Hilbert polynomial $h \in \mathbb{Q}[T]$, the Hilbert point of X is the corresponding point $[X] \in Hilb_h$. For any polarized variety $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$, let $\mu \geq \mu_0$ and consider an embedding $e_\mu : \Gamma \to \mathbb{C}P^N$ by $\mathcal{H}^\mu$, then the Hilbert point of $e_\mu(\Gamma) \subset \mathbb{C}P^N$ is called (one of) the μ-th Hilbert point of $(\Gamma, \mathcal{H})$.*

Group $G = SL(N+1, \mathbb{C})$ acts on $\mathbb{C}P^N$ naturally, and consequently G will acts on $Univ_h$ and $Hilb_h$ equivariantly. Let ν_0 be a large number depending on Hilbert polynomial h, Grothendieck proved on $Hilb_h$ there is an ample line bundle given by

$$\mathfrak{L} = det(g_*(\pi_2^* \mathcal{O}(\nu))) \qquad \nu \geq \nu_0 \tag{3}$$

where $\pi_2 : Univ_h \to \mathbb{C}P^N$ is the projection. $\mathfrak{L}$ is G-linearized, by this we mean the action of G on $Hilb_h$ can be lifted to the geometric bundle $\mathfrak{L}$. Therefore we may apply GIT to Hilbert scheme with respect to the action of group G and the line bundle $\mathfrak{L}$. Let's recall the definition of stable points from [Mu], [V2].

Definition 2.2 *A point $x \in H = Hilb_h$ is called (GIT) stable with respect to $G, \mathfrak{L}$ and the given linearisation, or $x \in H(\mathfrak{L})^s$, if for some $m \geq 1$, there exists a section $t \in \Gamma(Hilb_h, \mathfrak{L}^m)^G$ such that*

1) *$H_t = H - V(t)$ is affine, where V(t) denotes the zero locus of t.*

2) $x \in H_t$, *or in other terms,* $t(x) \neq 0$.

3) *The induced action of* G *on* H_t *is closed.*

And $(\Gamma, \mathcal{H})$ *is called Gieseker-Mumford stable if when* μ *is very large, there exists* $\nu_0 \geq 1$ *such that for any* $\nu \geq \nu_0$*, the* μ*-th Hilbert points of* $(\Gamma, \mathcal{H})$ *in* $Hilb_h$ *is (GIT) stable with respect to* G *and* $\mathfrak{L} = det(g_* \pi_2^* \mathcal{O}(\nu))$.

As we see "stability" depends on the choice of a G-linearized line bundle. Actually do not like Gieseker and Mumford, Viehweg choosed a different ample line bundle on some quasi-projective subscheme of Hilbert scheme. Our formulation of stability is the same as used by Gieseker and Mumford, therefore we call it Gieseker-Mumford stability.

2.2 Simple Propositions for Stability

Now pick up a polarized manifold (M, L) in $\Im_{h'}(\mathbb{C})$, we will try to understand the stability of M from the differential geometric view point. Notice Hilbert scheme $Hilb_h$ and the universal family $Univ_h$ are usually singular, and this will be one of the difficulties for us to apply differential geometric method later. Let's do some reduction first, in order to simplify the problem a little.

Since the Hilbert scheme $H = Hilb_h$ is complete and $\mathfrak{L}$ is ample line bundle over H, we can give a more geometric description for stable points on H. Assume $\mathfrak{L}^m$ is very ample for some $m \geq 1$, then we embedd H into a projective space $\mathbb{C}P^M$, such that $\mathfrak{L}^m = \mathcal{O}_{\mathbb{C}P^M}(1)|_H$. Since $\mathfrak{L}$ is G-linearized, G acts on $\mathbb{C}P^M$ by a rational representation $G \to SL(\mathbb{C}^{M+1})$, and the embedding is G equivariant. Let $\theta : \mathbb{C}^{M+1} - \{0\} \to \mathbb{C}P^M$ be the projection, and $\hat{H}$ be the affine cone over H, *i.e.*, the closure of $\theta^{-1}(H)$ in $\mathbb{C}^{M+1}$.

Proposition 2.1 $x \in H(\mathfrak{L})^s$ *if and only if for all points* $\hat{x} \in \theta^{-1}(x)$*, the orbit of* $\hat{x}$ *in* $\hat{H}$ *is closed and the stablizer of* x *is finite.*

This proposition is well known, so we omit its proof. This proposition can be transformed into better versions for doing analysis later. Give a Hermitian metric $\|\cdot\|$ on

$\mathcal{O}_{\mathbb{C}P^M}(1)$ over $\mathbb{C}P^M$. Fix a point $x \in H$, define a function $F_x : G \to \mathbb{R}$ by

$$F_x(\sigma) = -log(\|\sigma(\hat{x})\|^{\frac{2}{m}}), \qquad \text{for} \quad \sigma \in G \tag{4}$$

where $\hat{x}$ is a fixed lifting of x to the fiber of $\mathcal{O}_{\mathbb{C}P^M}(1)$ at x. Then proposition 2.1 is the same as the following.

Proposition 2.2 *$x \in H(\mathfrak{L})^s$ if and only if F_x is a proper function on G, i.e., for any $c_1, c_2 \in \mathbb{R}$ the set*

$$\{\sigma \in G \mid c_1 \leq F(\sigma) \leq c_2\}$$

is compact subset of G with respect to Hausdorff topology.

For some technique reason, let's reduce this proposition a little further. For any $x \in H$ we have morphism

$$\tau_x : G \to Hilb_h \tag{5}$$

given by $\tau_x(\sigma) = \sigma(x)$. Notice the Hilbert scheme $Hilb_h$ is complete, so we can choose $\bar{G}$, a smooth compactification of G, such that τ_x extends to a morphism

$$\tau : \bar{G} \to Hilb_h \tag{6}$$

Use τ to pull back the universal family $Univ_h$ over $Hilb_h$, then we get a flat family of varieties $\bar{\Sigma}$ over $\bar{G}$

$$\begin{array}{ccc} \bar{\Sigma} & \xrightarrow{\subset} & \bar{G} \times \mathbb{C}P^N \\ {\scriptstyle f}\big\downarrow & & \\ \bar{G} & & \end{array} \tag{7}$$

Let $i : \bar{\Sigma} \to \bar{G} \times \mathbb{C}P^N$ be the inclusion, and use $\bar{\pi}_1, \bar{\pi}_2$ to denote the projection of $\bar{G} \times \mathbb{C}P^N$ to $\bar{G}$ and $\mathbb{C}P^N$ respectively. In general τ is not a flat morphism, however since the family $Univ_h$ is bounded (see [V2], for example), we know if ν_0 is very large, then for all fibers Γ of $g : Univ_h \to Hilb_h$ we have

$$H^i(\Gamma, \ \mathcal{O}_\Gamma(\nu)) = 0 \qquad \text{for} \quad i \geq 1, \ \nu \geq \nu_0 \tag{8}$$

Therefore we can apply Cohomology and Base Change Theorem, then for all $\nu \geq \nu_0$,

$$\tau^*(\mathfrak{L}) = \tau^*(\det(g_*(\pi_2^*\mathcal{O}(\nu)))) = \det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(\nu))) \tag{9}$$

Consequently proposition 2.2 now becomes the following proposition which will be used to check the Gieseker-Mumford stability of polarized manifolds.

Proposition 2.3 *Let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold. Let μ_0 be given as in (1). Then for any $\mu \geq \mu_0$, the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(\pi_2^*\mathcal{O}(v)))$ $(\nu \geq \nu_0)$ if and only if F_M is a proper function on G. Where $F_M : G \to \mathbb{R}$ is defined by*

$$F_M(\sigma) = -log(\|\sigma(\hat{x})\|^2)$$

and $\|\cdot\|$ is any Hermitian metric on $\mathfrak{L}_o = \det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(v)))$ over $\bar{G}$.

The difference between this proposition and proposition 2.2 is we know the definition of F_M depends now only on the family $f : \bar{\Sigma} \to \bar{G}$ as given in (7), however in proposition 2.2, F_x depends on the line bundle $\mathfrak{L}$ which is defined from the universal family over the Hilbert scheme. So in some sense, we are able to "forget" about Hilbert scheme and the universal family which are usually very singular, and pay attention only to the subfamily $f : \bar{\Sigma} \to \bar{G}$. There are still singularities on $\bar{\Sigma}$, but notice all the singular points are contained in $f^{-1}(\bar{G} - G)$.

3 Singular Riemann-Roch

In order to study the behavior of F_M, we are going to use Riemann-Roch to relate the information on $\bar{G}$ to each fibers of the family $\bar{\Sigma}$ (see [Do1], [T1], [T2]). By this way we will introduce a functional D_M which is similar to Donaldson functional in the study of stability of vector bundles ([Do1], [Do2]), and also similar to K-energy in the study of Kahler-Einstein metrics ([T1], [T2]). It is defined on the set of Kahler metrics on M, and unlike F_M, the definition of this functional D_M depends only on the geometry of M. We will prove that F_M can be bounded from below by D_M (see lemma 3.6), and thus properness of D_M will imply Gieseker-Mumford stability. We will prove this estimate by differential geometric method.

3.1 Deal With Singular Fibers: Some Intersection Theory

In our situation, the family $f : \bar{\Sigma} \to \bar{G}$ has still singular fibers. This force us to use Singular Riemann-Roch of Baum-Fulton-MacPherson. We expect the singualr fibers will play a minor role. First let's recall singular Riemann-Roch theorem from [Ful]. It tells us for any variety X, we can associate a homomorphisorm from the Grothendick group of coherent sheaves to the Chow ring on X:

$$\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}$$

This homomorphism will in particular satisfy the following properties:

1) (Covariance). If $f : X \to Y$ is proper, $\alpha \in K_0(X)$, then $f_*\tau_X(\alpha) = \tau_Y f_!(\alpha)$.

2) (Module). If $\alpha \in K_0(X), \beta \in K^0(X)$ (Grothendick group of locally free sheaves), then $\tau_X(\beta \otimes \alpha) = ch(\beta) \cap \tau_X(\alpha)$

3) (Top Term) If V is a closed subvariety of X, with $dim(V) = n$, then

$$\tau_X(\mathcal{O}_V) = [V] + (\text{terms of dimension} < n)$$

Using the homomorphism τ, we know then the Todd class for a general variety X can be defined by

$$Td(X) = \tau_X(\mathcal{O}_X) \in A_*(X)_{\mathbb{Q}} \tag{10}$$

And for any $\beta \in K^0(X)$, $\tau_X(\beta)$ can be writen as

$$\tau_X(\beta) = ch(\beta) \cap Td(X)$$

Let's return to our case, consider the family of varieties $f : \bar{\Sigma} \to \bar{G}$ given in (7). Recall $\mathcal{L}_0$ is the determinant line bundle $\det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(v)))$. For simplicity, we denote the line bundle $\bar{\pi}_2^*\mathcal{O}(1)$ over $\bar{G} \times \mathbb{C}P^N$ by L. Apply the covariance of Riemann-Roch to $f : \bar{\Sigma} \to \bar{G}$, then we get

$$f_*\tau_{\bar{\Sigma}}(i^*(L^\nu)) = \tau_{\bar{G}}(f_! i^*(L^\nu)) \tag{11}$$

Apply the vanishing results (8), when $\nu \geq \nu_0$ the right hand side of the above equation can be simplified to the following

$$f_!(i^*(L^\nu)) = f_*(i^*(L^\nu)) \tag{12}$$

Also by properties of Riemann-Roch, the left hand side of (11) can be writen as

$$\begin{aligned} f_*\tau_{\bar{\Sigma}}(i^*(L^\nu)) &= f_*(ch(i^*(L^\nu)) \cap \tau_{\bar{\Sigma}}(\mathcal{O}_{\bar{\Sigma}})) \qquad \text{(Module)} \\ &= f_*(ch(i^*(L^\nu)) \cap ([\bar{\Sigma}] + \text{terms of lower dimension})) \qquad \text{(Top Term)} \end{aligned} \tag{13}$$

Now let $\tilde{\Sigma}$ be a desingularization of $\bar{\Sigma}$. We want to write down Riemann-Roch by using smooth varieties $\tilde{\Sigma}$ and $\bar{G} \times \mathbb{C}P^N$ in stead of $\bar{\Sigma}$ since we need to do some analysis later.

$$\begin{tikzcd} \tilde{\Sigma} \arrow[d, "\pi"] \arrow[dr, "s"] \arrow[dd, bend right, "g"'] & \\ \bar{\Sigma} \arrow[r, "i"] \arrow[d, "f"] & \bar{G} \times \mathbb{C}P^N \\ \bar{G} & \end{tikzcd} \tag{14}$$

Notice we have the following simple relation after desingularization

$$[\bar{\Sigma}] = \pi_*[\tilde{\Sigma}]$$

Since $i^*(L^\nu)$ is a line bundle over $\bar{\Sigma}$, by the Projection Formular for Chow groups we get

$$\pi_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) = ch(i^*(L^\nu)) \cap \pi_*[\tilde{\Sigma}] \tag{15}$$

Recall we use $\bar{\pi}_1, \bar{\pi}_2$ to denote the projections of $\bar{G} \times \mathbb{C}P^N$ to $\bar{G}$ and $\mathbb{C}P^N$ respectively. Combine the results of (13), (15), and notice $g_* = \bar{\pi}_{1*}s_*$, then the left hand side of (11) becomes

$$\begin{aligned} f_*(\tau_{\bar{\Sigma}}(i^*(L^\nu))) &= g_*(ch(s^*(L^\nu)) \cap ([\tilde{\Sigma}] + \text{terms of lower dimension})) \\ &= g_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) + \bar{\pi}_{1*}(ch(L^\nu) \cap [Z]) \end{aligned} \tag{16}$$

where $[Z]$ is a cycle of $\bar{G} \times \mathbb{C}P^N$ surpported in $\bar{\Sigma}$, and

$$\dim(Z) \leq n + r - 1, \qquad r = \dim(\bar{G}) \tag{17}$$

Here $n = \dim(\bar{\Sigma}) - \dim(\bar{G})$ is the dimension of fibers.

Now $\mathbb{C}P^N$ has a filtration $\mathbb{C}P^N \supset \mathbb{C}P^{N-1} \supset \cdots \supset \mathbb{C}P^1$ by linear subspaces, and each $\mathbb{C}P^k - \mathbb{C}P^{k-1} = \mathbb{C}^k$ is affine. This means $\mathbb{C}P^N$ has a cellular decomposition. It follows (see [Ful], for example) that for any m, we have a surjective morphism of Chow groups

$$\bigoplus_{k+l=m} A_k(\bar{G}) \otimes A_l(\mathbb{C}P^N) \to A_m(\bar{G} \times \mathbb{C}P^N)$$

In particular, this implies

$$[Z] = [C_1] \times [D_1] + \cdots + [C_r] \times [D_r] \tag{18}$$

Where $[C_i]$'s are cycles on $\bar{G}$ and $[D_i]$'s are cycles on $\mathbb{C}P^N$. Assume among $[C_1], \cdots, [C_r]$, only $[C_1], \cdots, [C_s]$ are in $Z_{r-1}(\bar{G}), r = \dim(\bar{G})$. From (11), (12), (16) and (18), by comparing the corresponding parts in $A_{r-1}(\bar{G})$, we get

$$\begin{aligned} &\frac{1}{(n+1)!} g_*(c_1(s^*L^\nu)^{n+1}) + \bar{\pi}_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s}([C_k] \times [D_k]))_{r-1} \\ &= c_1(det(f_* i^*(L^\nu))) + \frac{1}{2} c_1(\bar{G}) \\ &= c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}) \end{aligned} \tag{19}$$

where $(\cdot)_{r-1}$ means the $(r-1)$-dimensional part of this cycle.

Now notice $\tilde{\Sigma}, \bar{G}$ are smooth varieties, and $g, \bar{\pi}_1$ are holomorphic maps, so we can compute the terms in this equation by using differential geometric methods. Of course then we will have to deal with those $[C_k]$ and $[D_k]$ terms. In the following lemma, we have a simple but useful observation about those $[C_k]$ terms.

Lemma 3.1 *There are cycles $[D_k](1 \leq k \leq s)$ on $\mathbb{C}P^N$, and $(r-1)$-dimensional cycles $[C_k](1 \leq k \leq s)$ on $\bar{G}$, such that*

$$\frac{1}{(n+1)!} g_*(c_1(s^*L^\nu)^{n+1}) + \bar{\pi}_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s}([C_k] \times [D_k]))_{r-1} = c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}) \quad (20)$$

And we may choose $C_k(1 \leq k \leq s)$ to be divisors of $\bar{G}$ surpported in $\bar{G} - G$.

Proof. Assume $[D_k]$ is b_k-dimensional cycle of $\mathbb{C}P^N$. Notice that for all $0 \leq i \leq N$, $A_i(\mathbb{C}P^N)$ is a free abelian group generated by $i-$dimensional linear subspace $\mathbb{C}P^i$ of $\mathbb{C}P^N$. Therefore in (19) we may assume b_k is different from each other, and $b_1 < b_2 < \cdots < b_s$. Notice

$$\begin{aligned}
&\bar{\pi}_{1*}\left(ch(L^\nu) \cap \sum_{k=1}^{s}[C_k] \times [D_k]\right)_{r-1} \\
&= \sum_{k=1}^{s} \bar{\pi}_{1*}\left(ch(L^\nu) \cap [D_k]\right)_0 [C_k] \\
&= \sum_{k=1}^{s} \frac{\nu^{b_k}}{b_k!}\left(c_1(L)^{b_k} \cap [D_k]\right)_0 [C_k] \\
&= \sum_{k=1}^{s} \lambda_k \nu^{b_k}[C_k]
\end{aligned} \quad (21)$$

where λ_k are some constants. Now by (19), we get

$$\frac{\nu^{n+1}}{(n+1)!} g_*(c_1(s^*L)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k}[C_k] = c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}) \quad (22)$$

Notice when restricted on G, $\mathfrak{L}_0$ and $K_{\bar{G}}$ are trivial line bundles because of the G action. By the following exact sequence

$$A_*(\bar{G} - G) \to A_*(\bar{G}) \to A_*(G) \to 0$$

we conclude there exist divisors Y and Y_0, such that they are supported in $\bar{G} - G$, and

$$\frac{\nu^{n+1}}{(n+1)!} g_*(c_1(s^*L)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k} [C_k] = [Y] + \frac{1}{2}[Y_0] \tag{23}$$

Also recall, by (17), we know

$$(r-1) + b_k = \dim(C_k \times D_k) \leq (n + r - 1)$$

Therefore $b_1 < b_2 < \cdots < b_s < (n+1)$. Choose $\nu = 1, 2, \cdots, s+1$ in (23). By solving a non-degenerate $(s+1) \times (s+1)$ system of linear equations we find for every k, $\lambda_k[C_k]$ may be represented by divisors with surpport in $\bar{G} - G$, *i.e.*, we may assume C_k is a divisor with surpport in $\bar{G} - G$. This proves lemma 3.1.

3.2 Logarithmic Green Current

Now let's begin to use differential geometric method. Give the Hermitian metric on $s^*(L)$ over $\tilde{\Sigma}$ and the Hermitian metric on L over $\bar{G} \times \mathbb{C}P^N$ by using the standard Euclidean metric on the hyperplane bundle over $\mathbb{C}P^N$. Let ω_{FS} be Fubini-Study metric on $\mathbb{C}P^N$, then the curvature of $s^*(L)$ is $s^*\bar{\pi}_2^*(\omega_{FS})$, and the curvature of L is $\bar{\pi}_2^*(\omega_{FS})$. We will also fix a Hermitian metric $\|\cdot\|$ on $\mathcal{L}_0$. Denote the curvature of this Hermitian line bundle by $R(\|\cdot\|)$. Fix a Hermitian metric $\|\cdot\|_{\bar{G}}$ on $K_{\bar{G}}$ and its curvature is denoted by $R(\|\cdot\|_{\bar{G}})$. Assume $[C_k]$ is Poincare dual to a smooth differential form α_k on $\bar{G}$, and $[D_k]$ is Poincare dual to a smooth differential form β_k on $\mathbb{C}P^N$.

We want to write the equation in lemma 3.1 as equality of currents. For this purpose, let's recall the Green current which was used by Gillet-Soulé ([GS]) in their study of Arakelov geometry. If X is any n-dimensional smooth projective (complex) variety, and

$Y \subset X$ a closed irreducible subvariety of codimension p, then there exists a $(p-1,p-1)$-current ψ (the so called Green current), and a smooth closed (p,p)-form ω on X, such that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\psi) + \delta_Y = \omega$$

Here δ_Y is the current representing integration on Y. What's important is that we can choose ψ to be given by smooth differential form on $X - Y$ which is of logarithmic type along Y. By Hironaka's theorem on the resolution of sigularities, there exists a proper morphism

$$\pi: \quad \tilde{X} \longrightarrow X$$

such that $\tilde{X}$ is smooth, $E = \pi^{-1}(Y)$ is a divisor with normal crossings, and when restricted on $\tilde{X} - E$, π is an isomorphism. Then ψ is of logarithmic type along Y means near each $x \in \tilde{X}$, if $z_1 \cdots z_k = 0$ $(0 \leq k \leq n)$ is the local equation of E then there exists ∂ and $\bar{\partial}$ closed smooth forms α_i and a smooth form β such that

$$\pi^*(\psi) = \sum_{i=1}^{k}(\log|z_i|^2)\alpha_i + \beta$$

Then ψ is called the logarithmic Green current of the subvariety $Y \subset X$. Using this kind of logarithmic Green current, we can write the results in lemma 3.1 into equalities of currents.

Lemma 3.2 *There is a measurable function θ_ν (depending on ν), such that as currents*

$$\begin{aligned} &\frac{\nu^{n+1}}{(n+1)!}g_*(s^*\bar{\pi}_2^*(\omega_{FS}^{n+1})) + \bar{\pi}_{1*}(exp(\bar{\pi}_2^*\omega_{FS}) \wedge \sum_{k=1}^{s}\alpha_k \wedge \beta_k)_{r-1} \\ &= \frac{\sqrt{-1}}{2\pi}R(\|\cdot\|) + \frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}}) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta_\nu \end{aligned} \tag{24}$$

θ_ν is smooth function when restrict on G, and is bounded from above by a constant on G. Here $(\cdot)_{r-1}$ means the $(r-1,r-1)$ part of a differential form.

Proof Let $[Z] \in A_{n+r-1}(\tilde{\Sigma})$ and $[Y] \in A_{r-1}(\bar{G})$ be cycles such that $[Z] = c_1(s^*L^\nu)^{n+1}$, and $[Y] = (c_1(\mathcal{L}_0) + \frac{1}{2}c_1(\bar{G}))$. By lemma 3.1 we get equality between cycles.

$$\frac{\nu^{n+1}}{(n+1)!}g_*[Z] + \sum_{k=1}^{s} \lambda_k \nu^{b_k}[C_k] = [Y]$$

This equation, in terms of currents, are

$$\frac{\nu^{n+1}}{(n+1)!}g_*(\delta_Z) + \sum_{k=1}^{s} \lambda_k \nu^{b_k}\delta_{C_k} = \delta_Y$$

Let ψ_Z, ψ_Y and ψ_{C_k} be the logarithmic Green current of $Z \subset \tilde{\Sigma}$, $Y \subset \bar{G}$ and $C_k \subset \bar{G}$ respectively. Then we find (24) is true for some measurable function θ_ν given by

$$\theta_\nu = \frac{\nu^{n+1}}{(n+1)!}g_*(\psi_Z) + \sum_{k=1}^{s} \lambda_k \nu^{b_k}\psi_{C_k} - \psi_Y$$

Here θ_ν is smooth on $G - g_*(Z) - Y - (C_1 + \cdots + C_s)$ and has at most logarithmic growth along $Y + g_*(Z) + (C_1 + \cdots + C_s) + (\bar{G} - G)$. However, every term other than $\partial\bar{\partial}\theta_\nu$ in (24) is smooth differential form on G, therefore by the regularity of $\bar{\partial}$ operator, θ_ν can be extended to be a smooth function on G. In (24), $g_*(s^*\bar{\pi}_2^*(\omega_{FS}^{n+1}))$ is a positive $(1,1)$ current on $\bar{G}$, other terms except $\partial\bar{\partial}\theta_\nu$ are smooth differential forms on $\bar{G}$. Therefore for example by Green's Formular

$$\theta_\nu(x) = \frac{1}{V}\int_{\bar{G}} \theta_\nu(y)\omega_y^r - \frac{1}{V}\int_{\bar{G}} G(x,y)\Delta\theta_\nu(y)\omega_y^r \tag{25}$$

we can show θ_ν is bounded from above on G. Notice here we used the fact that L^1 norm of θ_ν on $\bar{G}$ is finite, since θ_ν has at most logarithmic growth along $\bar{G} - G$.

3.3 Secondary Characteristic Classes Type Computations

The right hand side of the equation in lemma 3.2 will contain $\partial\bar{\partial}F_M$ term when restricted on G, thus we may expect to recover some information about F_M from it.

Fix a reference point $0 \in G \subset \bar{G}$, let M_0 be the fiber of $g : \tilde{\Sigma} \to \bar{G}$ over 0, then M_0 is isomorphic to M. Let's identify M_0 with M, and let $\omega = s^*\bar{\pi}_2^*(\omega_{FS})|_{M_0}$. Denoted by

$P(M,\omega)$ the set of all Kahler metrics on M in the same cohomology class as ω. We'll define a functional on $P(M,\omega)$.

Definition 3.1 *D_M is defined to be a functional from $P(M,\omega)$ to $\mathbb{R}$. For any $\omega' \in P(M,\omega)$, let $\omega' = \omega + \partial\bar{\partial}\varphi$ for some smooth function φ. Then $D_M(\omega')$ is defined by*

$$D_M(\omega') = \int_0^1 \int_M \dot{\varphi}_t \omega_t^n \wedge dt \tag{26}$$

Here $\omega_t = \omega + \partial\bar{\partial}\varphi_t$ $(0 \leq t \leq 1)$ is a smooth path from ω to ω' in $P(M,\omega)$.

It is straight forward to check that $D_M(\omega')$ is well defined, *i.e.* it is independent of the choice of a path ω_t in $P(M,\omega)$.

Now since we have indentified M with M_0, then M becomes a subvariety in $\mathbb{C}P^N$. We know for any $\sigma \in G$, $g^{-1}(\sigma(0))$ can be indentified with $\sigma(M) \subset \mathbb{C}P^N$. Then we let

$$\omega_\sigma = \sigma^*(\omega_{FS}|_{\sigma(M)}) \in P(M,\omega) \tag{27}$$

For convenience, let's make another simple definition though it is not essential.

Definition 3.2 *Bergman metrics of $M \subset \mathbb{C}P^N$ is defined by*

$$Berg(M) = \{\omega_\sigma | \sigma \in G\} \subset P(M,\omega)$$

Now use Bergman metrics, D_M can be considered as a functional defined on Berg(M). D_M can also be considered as a functional on G by

$$D_M(\sigma) = D_M(\omega_\sigma), \qquad \text{for any } \sigma \in G \tag{28}$$

Eventually we will show that in order to prove $F_M(\sigma)$ is proper it is enough to show $D_M(\sigma)$ is proper.

Now we try to derive information of D_M from (24). Let's do some computation first. The following lemma is in fact Bott-Chern secondary characteristic classes type arguments.

Lemma 3.3 *For any smooth $2(r-1)$ form ϕ with compact surpport in G*

$$\int_{\tilde{\Sigma}} s^*\bar{\pi}_2^*(\omega_{FS}^{n+1}) \wedge g^*(\phi) = \int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma) \wedge \partial\bar{\partial}\phi \tag{29}$$

Proof Let $G(M) = g^{-1}(G) \subset G \times \mathbb{C}P^N$. Now define $\psi : G \times M \to G(M) \subset \tilde{\Sigma}$ by sending (σ, x) to $(\sigma, \sigma(x))$. Let H be the Hermitian metric on $\psi^* s^*(L)$ by pulling back the Hermitian metric on $s^*(L)$ and the curvature is denoted by $R(H)$. Let $H_0 = pr_2^*(H|_M)$ be another Hermitian metric on $\psi^* s^*(L) \cong pr_2^*(L|_M)$, where $pr_2 : G \times M \to M$ is the projection. Define a path of Hermitian metrics $H_t (0 \le t \le 1)$ on $\psi^* s^*(L)$ over $G \times M$ from H_0 to H, such that

$$H_t = e^{\varphi_t} H_0 \qquad \text{and} \quad \varphi_t = t \cdot \log(\frac{H(\sigma, x)}{H_0(\sigma, x)})$$

Then by straightforward computation we have

$$\begin{aligned}
\text{LHS of (29)} &= \int_{G\times M} (\frac{\sqrt{-1}}{2\pi} R(H_t))^{n+1} \wedge pr_1^*(\phi) \\
&= \int_{G\times M} \int_0^1 \frac{\sqrt{-1}}{2\pi}(n+1)\partial\bar{\partial}\dot{\varphi}_t \wedge (\frac{\sqrt{-1}}{2\pi} R(H_t))^n \wedge pr_1^*(\phi) \\
&= \int_{G\times M} \int_0^1 \frac{\sqrt{-1}}{2\pi}(n+1)\dot{\varphi}_t \wedge (\frac{\sqrt{-1}}{2\pi} R(H_t))^n \wedge pr_1^*(\partial\bar{\partial}\phi) \\
&= \int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma) \wedge \partial\bar{\partial}\phi
\end{aligned} \tag{30}$$

Therefore the lemma is proved.

Note $K_{\bar{G}}$ is trivial on G, so we may pick up a meromorphic section s_0 of $K_{\bar{G}}$, and when restricted on G, s_0 is a nonzero holomorphic section of K_G. By Poincare-Lelong lemma

$$\frac{\sqrt{-1}}{2\pi} R(\|\cdot\|_{\bar{G}}) = \delta_{Y_0} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|s_0\|_{\bar{G}}^2) \tag{31}$$

where Y_0 is a divisor of $\bar{G}$ surpported in $\bar{G} - G$. Similarly we can choose a divisor Y (depends on ν) surpported in $\bar{G} - G$ too, such that

$$\frac{\sqrt{-1}}{2\pi} R(\|\cdot\|) = \delta_Y + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}F_M(\sigma) \tag{32}$$

Here F_M is defined in proposition 2.3.

Lemma 3.4 *For any smooth* $2(r-1)$ *form* ϕ *with compact surpport in* G

$$\int_{\bar{G}}(\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)+\frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}}))\wedge\phi=\int_{G}(\frac{\sqrt{-1}}{2\pi}F_M(\sigma)-\frac{\sqrt{-1}}{4\pi}log(\|s_0\|^2_{\bar{G}}))\wedge\partial\bar{\partial}\phi \tag{33}$$

Proof (31) is true in the sense of current, so we get

$$\begin{aligned}\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|_{\bar{G}})\wedge\phi&=\int_{\bar{G}}\left(\delta_{Y_0}-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|s_0\|^2_{\bar{G}})\right)\wedge\phi\\&=-\int_{G}\frac{\sqrt{-1}}{2\pi}\log(\|s_0\|^2_{\bar{G}})\wedge\partial\bar{\partial}\phi\end{aligned} \tag{34}$$

By (32), similar arguments shows

$$\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)\wedge\phi=\int_{G}\frac{\sqrt{-1}}{2\pi}F_M(\sigma)\wedge\partial\bar{\partial}\phi \tag{35}$$

Add these two results together, then the lemmar will follows.

Using lemma 3.1, we can prove something similar to Lemma 3.3 and Lemma 3.4. Fix a Hermitian metric $\|\cdot\|$ on $\mathcal{O}_{\bar{G}}(C_k)$, then by Poincare-Lelong Lemma again as what we did before we can deduce the following Lemma.

Lemma 3.5 *For any smooth* $2(r-1)$ *form* ϕ *with compact surpport in* G

$$\int_{\bar{G}\times\mathbb{C}P^N}exp\,(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k\wedge\bar{\pi}_1^*(\phi)=-\int_{G}\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}log(\|s_k\|^2)\wedge\partial\bar{\partial}\phi \tag{36}$$

where s_k *is the section of* $\mathcal{O}_{\bar{G}}(C_k)$ *defining* C_k*, and* λ_k *is the constant given by 21.*

Since the proof is similar as before, we omit it. Remember we proved in lemma 3.1 that C_k is supported in $\bar{G}-G$.

3.4 Analytic Criterion to Check Stability

Now from computations of last section and lemma 3.2, we conclude there is a holomorphic function R on G, such that

$$F_M(\sigma)-\frac{\nu^{n+1}}{n!}D_M(\sigma)+\sum_{k=1}^{s}\lambda_k\nu^{b_k}\log(\|s_k\|^2)-\frac{1}{2}\log(\|s_0\|^2)+\theta_\nu=\log|R|^2 \tag{37}$$

We can show R is actually a constant function. This follows from an observation of Tian in [T1]. For reader's convenience, let's write out the detail. Let us denoted by $\{[z_{ij}, \omega] | 0 \leq i, j \leq n\}$ be the homogenious coordinates of $\mathbb{C}P^{(N+1)^2}$. Then we can use W, a projective subvariety of $\mathbb{C}P^{(N+1)^2}$, to compactify $G = SL(N+1, \mathbb{C})$ natually. Where W is given by

$$W = \{[z_{ij}, \omega]_{0 \leq i,j \leq N} | det(z_{ij}) = \omega^{N+1}\}$$

Then by using the definition of D_M and lemma 3.2, some easy computation shows that R has at most polynomial growth near $W \setminus G$, *i.e.*, there exist constants $l > 0, C > 0$, such that

$$|R(\sigma)| \leq C \cdot d(\sigma, W \setminus G)^{-l}$$

where $d(\sigma, W \setminus G)$ denotes the distance from σ to $W \setminus G$ with respect to the Study-Fubini metric of $\mathbb{C}P^{(N+1)^2}$. Therefore R extends to be a meromorphic function on W. Notice that W is normal and $W \setminus G$ is irreducible. Since R is nonzero everywhere in G, it follows that R has to be a constant, otherwise the divisor $W \setminus G$ will be linearly equivalent to zero. Also recall we already showed that θ_ν is bounded from above, and consequently from (37) we get the following lemma.

Lemma 3.6 *There are constants $C' > 0$ such that for ν large enough*

$$F_M(\sigma) \geq \frac{\nu^{n+1}}{n!} D_M(\sigma) - \sum_{k=1}^{s} \lambda_k \nu^{b_k} log(\|s_k\|^2) + \frac{1}{2} log(\|s_0\|^2) - C' \tag{38}$$

Here λ_k and $0 \leq b_k \leq n$ are constants.

Therefore eventually we can established an analytic criterion for the stability of a smooth subvariety.

Proposition 3.1 *Let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold. Let μ_0 be given as in (1). For any $\mu \geq \mu_0$, if D_M is a proper function on $Berg(M)$ then μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(i^* \bar{\pi}_2^* \mathcal{O}(v)))$ for very large ν.*

Proof By (29) D_M is a pluri-subharmonic function on G. And from its definition D_M has logarithmic growth along $\bar{G} - G$. If we know D_M is proper, then there will exists constants $\delta > 0$ and $C > 0$, such that

$$D_M(\sigma) \geq \delta \cdot \log(d(\sigma, W \setminus G)^{-1}) - C$$

Then by (38) we know for ν large enough, $F_M(\sigma)$ will be proper. Consequently by proposition 2.3 then μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable.

Notice this functional D_M is closely related to the K-energy functional defined by Mabuchi for the study of Kahler-Einstein metric. Recall the K-energy ν_ω is a functional from $P(M, \omega)$ to $\mathbb{R}$, and in the case when $c_1(M) > 0$, for any $\omega' \in P(M, \omega)$ we define

$$\nu_\omega(\omega') = \int_0^1 \int_M \dot{\varphi}_t(s(\omega_t) - n)\omega_t^n \wedge dt$$

where $\omega_t = \omega + \partial\bar{\partial}\varphi_t$ $(0 \leq 1)$ is a path from ω to ω' in $P(M, \omega)$. Actually from [T1], [T2] the properness of this K-energy will implies the existence of Kahler-Einstein metric. In our case the Gieseker-Mumford stability of variety does not relate to Kahler-Einstein metric directly.

This functional D_M is also similar to Donaldson functional for vector bundles, see [Do2] where the relation between Donaldson functional and family index theorem is explained.

4 Heat Kernel and Gieseker-Mumford Stability

Recall HKDUY correspondence ([Do1], [Do2], [UY]) says Mumford stability of complex vector bundle is equivalent to the existence of Hermitian-Einstein metric on this vector bundle. Suggested by this correspondence, we will also try to relate the Gieseker-Mumford stability of polarized manifold (M, L) with existence of some good metric. Due to some technique difficulty, up to now we can only succeed to show one side of this story is true, *i.e.*, existence of a good metric implies the Gieseker-Mumford stability. Along the way, we also get an interesting criterion for the Hilbert point of a smooth projective subvariety of $\mathbb{C}P^N$ to be (GIT) stable.

4.1 Geometric Criterion for Gieseker-Mumford Stability

First let's try to find equation satisfied by the critical points of D_M. Let $\sigma \in G$ be a critical point of D_M. Let $s(t)(-\epsilon < t < \epsilon)$ be a path in $G = SL(N+1, \mathbb{C})$ and $s(0) = \sigma$. We will denote $D_M(s(t))$ by $D_M(t)$, and denote $\omega_{s(t)}$ by ω_t when there is no confusion. Recall

$$\omega_t = \omega + \partial\bar{\partial}\varphi_t \tag{39}$$

where φ_t is function on M, and for any $z = [z_0, \cdots, z_N] \in M$

$$\varphi_t(z) = \log\left(\frac{\|s(t)\cdot z\|^2}{\|z\|^2}\right)$$

From the definition of D_M in (26), by straightforward computation, we get

$$D_M(\tau) = D_M(0) + \int_0^\tau \int_M \frac{\partial}{\partial t}(\varphi_t)\omega_t^n \wedge dt \tag{40}$$

Consequently by simple computation, we find σ is critical point of D_M on G if and only if it satisfies the following equation

$$\frac{1}{Vol(M)}\int_{\sigma(M)}\left(\frac{z_i \cdot \bar{z}_j}{|z_0|^2 + \cdots + |z_N|^2}\right)\omega_{FS}^n = \frac{1}{N+1}\delta_{ij} \tag{41}$$

Lemma 4.1 *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety, and its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $G = SL(N+1, \mathbb{C})$. If D_M has a critical point, then D_M is a proper function on G, and there exist constant $\delta > 0$ and $C > 0$ such that*

$$D_M(s) \geq \delta \cdot log(d(s, \bar{G} \setminus G)^{-1}) - C \tag{42}$$

Here $d(s, \bar{G} \setminus G)$ is the distance of s to $\bar{G} \setminus G$ with respect to a smooth metric on $\bar{G}$.

Proof For any $s \in G = SL(N+1, \mathbb{C})$, let $s^* s = U^* \Lambda^2 U$, here U is a unitary matrix and Λ is a real diagonal matrix. Then by the definition, $D_M(s) = D_M(\Lambda \cdot U)$. Let $\phi : \mathbb{C}^N \times U(N+1, \mathbb{C}) \to G$ be a surjective map such that for any $(z_1, \cdots, z_N, U) \in \mathbb{C}^N \times U(N+1, \mathbb{C})$,

$$\phi(z_1, \cdots, z_N, U) = \Lambda \cdot U, \quad \text{for } \Lambda = \mathrm{diag}(z_0, \cdots, z_N)$$

here $z_0 = (z_1 \cdots z_N)^{-1}$. Then we need only to prove the pull back function $\phi^*(D_M)$ on $\mathbb{C}^N \times U(N+1, \mathbb{C})$ is a proper function. Fix any $U \in U(N+1, \mathbb{C})$, let $\varphi = \phi^*(D_M)|_{\mathbb{C}^N \times \{U\}}$, then by (29) φ is a pluri-subharmonic function on $\mathbb{C}^N$. What's more, notice the complex Hessian of φ is nonzero everywhere and φ is invariant under the obvious action of torus $S^1 \times \cdots \times S^1$ on $\mathbb{C}^N$. Simple computation shows φ is a strict convex function of $(\log|z_1|, \cdots, \log|z_N|)$, *i.e.*, for all $(z_1, \cdots, z_N) \in \mathbb{C}^N$,

$$\left(\frac{\partial^2 \varphi}{\partial \log|z_i| \partial \log|z_j|} \right) > 0$$

Consequently since φ has a critical point, straightforward computation shows there exist constant $\delta > 0$ and $C > 0$ such that

$$\varphi(z_1, \cdots, z_N) \geq \delta \cdot \log(|z_1|^2 + \cdots + |z_N|^2) - C$$

Thus the lemma is proved.

Now we get our first main theorems stated in the introduction.

Theorem 4.1 *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety, and its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $SL(N+1,\mathbb{C})$. Then $[M] \in Hilb_h$ is (GIT) stable if there exists $\sigma \in SL(N+1,\mathbb{C})$, such that (41) holds.*

This theorem says for the Hilbert point of $M \subset \mathbb{C}P^N$ to be (GIT) stable, M must have a lot of symmetry.

4.2 Relate Gieseker-Mumford Stability to Heat Kernel

Now let's try to translate the results to be the existence of a good metric. In order to characterize the metric we need a definition.

Definition 4.1 *Let (M,ω) be a compact Kahler manifold, and let L be a holomorphic line bundle with a Hermitian metric g. Then we define $B_k(z) = B_k(z,g,\omega)$ to be a function on M, and for any $z \in M$*

$$B_k(z,g,\omega) = \sum_{i=0}^{N} \|s_i(z)\|_g^2 \tag{43}$$

Here $s_0,\cdots,s_N$ is any orthonormal frame of $H^0(M,L^k)$.

It is easy to check that $B_k(z)$ is independent of choice of the orthonormal frame $s_0,\cdots,s_N$. Also we should point out that $B_k(z)$ is closely related to the so called distortion function discussed before by Kempf and Ji.

Now let $(M,L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold. Consider an embedding $e_k : M \to \mathbb{C}P^N$ such that $e_k^*\mathcal{O}(1) = L^k$ for some $k \geq \mu_0$ (μ_0 is given in (1)). Assume (41) is true for some $\sigma \in SL(N+1,\mathbb{C})$ then the k-th Hilbert point of (M,L) is (GIT) stable. Notice if we pull back the standard Euclidean metric on $\mathcal{O}(1)$ over $\mathbb{C}P^N$ by the mapping

$$\sigma \cdot e_k : M \longrightarrow \mathbb{C}P^N$$

then we get a Hermitian metric $\|\cdot\|$ on L^k, and the curvature of this metric is given by

$$\frac{\sqrt{-1}}{2\pi} R(\|\cdot\|) = e_k^*\sigma^*(\omega_{FS})$$

Now let's choose $g = \|\cdot\|^{\frac{1}{k}}$ to be the Hermitian metric on L, and $\omega = Ric(g)$ to be the Kahler metric on M, then $\{e_k^*\sigma^*(z_i) \mid 0 \leq i \leq N\}$ will be holomorphic sections of L^k on M. What's more, we can check that

$$\|z_i\|_g = \frac{|z_i|^2}{|z_0|^2 + \cdots + |z_N|^2} \tag{44}$$

Therefore by (41), $\{e_k^*\sigma^*(z_i) \mid 0 \leq i \leq N\}$ is orthonormal frame of $H^0(M, L^k)$ with respect to g and $\omega = Ric(g)$. Consequently we conclude from the explicit expression of g

$$B_k(z, g, Ric(g)) = \sum_{i=0}^{N} \frac{|z_i|^2}{|z_0|^2 + \cdots + |z_N|^2} = 1$$

In particular $B_k(z, g, Ric(g)$ is a pointwise constant function on M.

Conversely assume there exists Hermitian metric g for L such that for othonormal basis $s_0, \cdots, s_N$ of $H^0(M, L^k)$, $B_k(z) = B_k(z, g, Ric(g))$ is a pointwise constant function. Then we have a cannonical embedding of M into $\mathbb{C}P^N$ by

$$z \to [s_0(z), \cdots, s_N(z)]$$

We can check that (41) is satisfied when $\sigma = id$, therefore the k-th Hilbert point of (M, L) is stable in the Hilbert scheme $Hilb_h$. So we have established one of our main theorems of this paper.

Theorem 4.2 *Let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold, let μ_0 be a large number given by (1). For any $k \geq \mu_0$, if there exists a Hermitian metric g (depends on k) on L over M such that $B_k(z) = B_k(z, g, Ric(g))$ is pointwise constant function on M, then the k-th Hilbert point of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(\pi_2^*\mathcal{O}(v)))$ for all large enough ν as long as the stabilizer of the Hilbert point is finite. And consequently, (M, L) is Gieseker-Mumford stable.*

This kind of metric deserves a further study. In fact, we should point out the function $B_k(z) = B_k(z, g, \omega)$ is related to the heat kernel. If we denote $H_t(z, w)$ to be the heat kernel with respect to the $\bar{\partial}$-Laplacian operator on $C^\infty(M, L^k)$, then $B_k(z)$ is precisely the limit function of $H_t(z, z)$ when the time t goes to infinity.

We don't know if the converse of this theorem is true or not yet. The main reason is in the estimate (38), we only used the highest order term (with respect to ν). However, certainly for a large class of polarized manifolds, the converse of this theorem is true.

5 Weak Positivity and Viehweg Stability

In this chapter, let's consider Viehweg's approach to moduli problem. We will introduce Viehweg's work in the first part, then in the second part we will include some new results about semi-positivity of Hodge bundles. These results are obtained during an attempt to understand Viehweg's work on weak positivity of the direct images of tensor powers of relative dualizing sheaves. In [V1], [V2], these kind of weak positivity was used to deduce the stability of smooth projective variety with semi-ample cannonical line bundle and consequently to prove the existence of coarse quasi-projective moduli space.

5.1 Viehweg Stability and Relations to Weak Positivity

Fix a polynomial $h' \in \mathbb{Q}[T]$ of degree n. As in chapter 2, let's use $\Im_{h'}$ to denote the moduli functor for polarized varieties with fixed Hilbert polynomial h'. Viehweg considered the moduli problem for polarized varieties with semi-ample cannonical line bundle. In particular this include smooth projective varieties with ample cannonical line bundle and polarized Calabi-Yau manifolds. For simplicity let's concentrate on these two cases, and let $\Im'_{h'}$, $\Im''_{h'}$ be the moduli functors respectively. Therefore the objects of $\Im'_{h'}$ will be

$$\begin{aligned}\Im'_{h'}(\mathbb{C}) =&\{\Gamma \mid \Gamma \text{ is a smooth projective variety,} \\ &\omega_\Gamma \text{ is ample, and } \chi(\Gamma, \omega_\Gamma^m) = h'(m), \forall m \geq 1\}\end{aligned}$$

In this case, Γ is cannonically polarized. The objects of $\Im''_{h'}$ will be

$$\begin{aligned}\Im''_{h'}(\mathbb{C}) =&\{(\Gamma, \mathcal{H}) \mid \Gamma \text{ is a Calabi-Yau manifold,} \\ &\mathcal{H} \text{ ample line bundle, and } \chi(\Gamma, \mathcal{H}^m) = h'(m), \forall m \geq 1\}\end{aligned}$$

It's well known $\Im'_{h'}$ and $\Im''_{h'}$ are locally closed, separated and bounded. Use boundedness, there exists $\mu_0 > 0$ such that for all $\Gamma \in \Im'_{h'}(\mathbb{C})$ (or all $(\Gamma, \mathcal{H}) \in \Im''_{h'}(\mathbb{C})$ in the second case), ω_Γ^μ (or $\mathcal{H}^\mu$ in the second case) will be very ample and without higher cohomology for all $\mu \geq \mu_0$. Then in both cases, we can use ω_Γ^μ (or $\mathcal{H}^\mu$) to embedd all these Γ as subvariety of a fixed $\mathbb{C}P^N$, for $N = h'(\mu) - 1$. By locally closedness, in both cases, all these subvarieties will be parametrized by a quasi-projective scheme H, the Hilbert scheme of cannonically polarized manifolds (or the Hilbert scheme of polarized Calabi-Yau manifold in the second case). Let $h \in \mathbb{Q}[T]$ be a polynomial given by $h(T) = h'(\mu T)$. Let $Hilb_h$ be the Hilbert scheme parametrize all subvarieties of $\mathbb{C}P^N$ with fixed Hilbert polynomial h as we used in chapter 2, then actually H is a locally closed subscheme of $Hilb_h$. And thus there is also a universal family $\mathfrak{X}$ over H given by.

$$
\begin{array}{ccccc}
\mathfrak{X} & \xrightarrow{\subset} & Univ_h & \xrightarrow{\subset} & Hilb_h \times \mathbb{C}P^N \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\
H & \xrightarrow{\subset} & Hilb_h & &
\end{array}
\tag{45}
$$

H is $G = SL(N+1, \mathbb{C})$ invariant. As in [Mu], *Prop*.5.4, or [V2], *Prop*.7.7, in order to prove the existence of coarse moduli scheme for $\Im'_{h'}$ and $\Im''_{h'}$, we need only to show there exists a geometric quotient of H by G. By using GIT, after choosing a G-linearized ample line bundle $\mathfrak{L}_0$ on H, the problem is reduced to show all points in H are stable with respect to G and $\mathfrak{L}_0$. In chapter 2, we know that $\mathfrak{L} = \det(g_*(\pi_2^*\mathcal{O}(\nu)))$ is an ample line bundle on $Hilb_h$. However, Viehweg choosed a different ample line bundle over H. For moduli problem of cannonically polarized manifolds, he choosed $\mathfrak{L}_0$ to be

$$\mathfrak{L}_0 = \det(f_*\omega^{\nu}_{\mathfrak{X}|H})$$

Here ν is very large number. And for the moduli problem of Calabi-Yau manifolds, he choosed $\mathfrak{L}_0$ to be

$$\mathfrak{L}_0 = f_*\omega_{\mathfrak{X}|H}$$

Now we can study the property of H with respect to G and $\mathfrak{L}_0$.

Definition 5.1 *For any $\Gamma \in \Im'_{h'}(\mathbb{C})$, we say Γ is Viehweg stable if its Hilbert point in H is a (GIT) stable point with respect to G and $\mathfrak{L}_0 = det(f_*\omega^{\nu}_{\mathfrak{X}|H})$. For any $(\Gamma, \mathcal{H}) \in \Im''_{h'}(\mathbb{C})$, we say Γ is Viehweg stable if its Hilbert point in H is a (GIT) stable point with respect to G and $\mathfrak{L}_0 = f_*\omega_{\mathfrak{X}|H}$.*

Since Viehweg used a different ample line bundle $\mathfrak{L}_0$ instead of $\mathfrak{L}$, and apply GIT to H instead of $Hilb_h$, therefore Viehweg stability is different from Gieseker-Mumford stability. In [V1], [V2] Viehweg proved in fact $H = H(\mathfrak{L})^s$, consequently we know there exists quasi-projective muduli space in both cases. However, we should point out it is hard to prove the ampleness of $\mathfrak{L}_0$.

It is realized in [V1], [V2] that stability and the ampleness of $\mathfrak{L}_0$ is related to certain positivity of direct images of powers of relative dualizing sheaves which he called weak positivity. Let's recall the definition of weak positivity from [V2].

Definition 5.2 *Let Y be a quasi-projective variety and $\mathcal{G}$ be a vector bundle on Y. We say $\mathcal{G}$ is weak positive on Y if for any ample line bundle $\mathcal{H}$ on H, any $\alpha > 0$, there exists $\beta_0 > 0$ such that $S^{\alpha\beta}(\mathcal{G}) \otimes \mathcal{H}^{\beta}$ is generated by global sections on Y for any $\beta \geq \beta_0$.*

When Y is projective, weak positive is the same as semi-positive. And when $\mathcal{G}$ is a line bundle and Y is projective, then weak positivity means nef (numerically effective). In general, a vector bundle $\mathcal{G}$ is weak positive on quasi-projective variety Y means we can find a projective compactification $\bar{Y}$ of Y, such that $\mathcal{G}$ can be extended to be a semi-positive vector bundle over $\bar{Y}$.

For the moduli problem of cannonically polarized manifolds, if we can show $f_*(\omega^{\nu}_{\mathfrak{X}|H})$ is weak positive on H for all large $\nu > 0$, then we can show directly that $\mathfrak{L}_0 = \det(f_*\omega^{\nu}_{\mathfrak{X}|H})$ is ample on H for very large ν since it is not hard to show

$$\mathfrak{L}|_H = \det(f_*\omega^{\nu\mu}_{\mathfrak{X}|H}) \otimes \det{}^{-1}(f_*\omega^{\mu}_{\mathfrak{X}|H})$$

And recall this line bundle $\mathfrak{L}$ is the ample line bundle on $Hilb_h$ used by Gieseker and Mumford. This kind of weak positivity is actually ture for any family of cannonically polarized manifolds instead of $\mathfrak{X}$, and we can check stability directly using this kind of weak positivity (see [V1], [V2]). For the moduli problem of Calabi-Yau manifolds, similar thing is true, mainly in order to prove the ampleness and check stability, one need only to prove some kind of weak positivity.

So if we try to understand Viehweg's approach, we have to understand the weak positivity. We can give geometric proof of a special case. Let's consider the weak positivity of direct images of relative dualizing sheaves. In this case, under some condition, we may even compactify the family $f : \mathfrak{X} \to H$ and prove the semi-positivity of the direct images of relative dualizing sheaves. This is stronger than weak positivity. This kind of semi-positivity has been studied before by many people such as Fujita, Kawamata, Kollár, Viehweg, etc. It turns out we can reprove this result by a geometric method. And in the case of Calabi-Yau manifolds we can deduce new semi-positivity results for Hodge bundles. These results will be introduced in the following subsection.

5.2 Some Results about Semi-Positivity of Hodge Bundles

The semi-positivity of the directly images of relative dualizing sheaves can be studied naturally in the frame of Variation of Hodge Structure. Since the directly images of relative dualizing sheaves is exactly a Hodge bundle. One typical theorem is the following theorem proved by Kawamata.

Theorem 5.1 *Let $\bar{Y}$ be a smooth complex projective variety and D be a divisor of normal crossing on $\bar{Y}$. Consider a polarized VHS of weight b on $Y = \bar{Y} - D$ with unipotent local monodromies. Let H be the underlying local system and let $\mathcal{H}$ be the canonical extension of H on $\bar{Y}$. Then the lowest filtration $F^n(\mathcal{H}) = \mathcal{H}^{n,b-n}$ is a semi-positive vector bundle over $\bar{Y}$*

When the VHS comes from a geometric situation, this theorem has a clear geometric meaning. Let $f : X \to Y$ be a family of polarized n dimensional smooth projective varieties. Assume there are smooth compactifications $\bar{X}$ and $\bar{Y}$ for X and Y respectively such that f extends to a morphism $\bar{f} : \bar{X} \to \bar{Y}$, and $D = \bar{Y} - Y$ is a normal crossing divisor. Let $H = R^d f_*(\mathbb{C})_{prim}$ $(d \leq 2n)$, then H forms a polarized VHS according to Griffiths. Assume the local monodromies of H around D are unipotent. Let $\mathcal{H}$ be the canonical extension of H on $\bar{Y}$. If we choose $d = n$ then by the discription of [Ka1],

$$F^n(\mathcal{H}) = \mathcal{H}^{n,0} = \bar{f}_*(\omega_{\bar{X}|\bar{Y}})$$

Consequently by theorem 0.1, we know $\bar{f}_*(\omega_{\bar{X}|\bar{Y}})$ is semi-positive. If we choose $d = n + k$, then we get

$$F^n(\mathcal{H}) = \mathcal{H}^{n,k} = R^k \bar{f}_*(\omega_{\bar{X}|\bar{Y}})$$

Therefore $R^k \bar{f}_*(\omega_{\bar{X}|\bar{Y}})$ is semi-positive too.

As we point out in section 5.1, semi-positivity of the direct images of relative dualizing sheaves is related to the stability of polarized manifold and the existence of moduli space.

Kawamata's proof of theorem 0.1 depends on the theory of limiting mixed Hodge structures. Our proof is to use Hormander's L^2 estimate of $\bar{\partial}$ operator study L^2 cohomology of Hodge bundles equiped with Hodge metrics on the open manifold Y. We will use Griffiths' curvature computation of Hodge bundles, and will use Schmid's Nilpotent Orbit theorem to analyze the asymptotical behavior of Hodge metric. By the same method, if we consider a family of polarized Calabi-Yau manifolds then we can get new semi-positivity results for Hodge bundles.

Theorem 5.2 *Let $f : X \to Y$ be a family of n dimensional polarized Calabi-Yau manifolds. Assume Y is Zariski open subset of a smooth complex projective variety $\bar{Y}$ such that $D = \bar{Y} - Y$ is a normal crossing divisor. Let $H = R^n f_*(\mathbb{C})_{prim}$ and assume the local monodromies of H around D are unipotent. Let $\mathcal{H}$ be the canonical extension of H on $\bar{Y}$. Then for any $k (0 \le k \le n)$, $(\mathcal{H}^{n,0})^s \otimes \mathcal{H}^{k,n-k}$ is semi-positive on $\bar{Y}$ for $s = dim(H^{k,n-k})$.*

When $k = n - 1$ theorem 0.2 has an interesting stronger form. Since in this case we can use Tian's description of Weil-Petersson metric in to compute the curvature of Hodge bundle $H^{n-1,1} \otimes H^{n,0}$. It turns out we can get more sharp results.

Theorem 5.3 *Under the same condition as theorem 0.2, $\mathcal{H}^{n-1,1} \otimes \mathcal{H}^{n,0}$ is semi-positive vector bundle on $\bar{Y}$, and $det^2(\mathcal{H}^{n-1,1}) \otimes \mathcal{H}^{n,0}$ is nef line bundle on $\bar{Y}$.*

The proof of these theorems are similar. The samilar techniques can be used to study vanishing theorems in next chapter.

6 Vanishing Theorems

Though the main theme of this paper is to try to understand the stability notion comes from algebraic geometry, however, I still would like to include one more chapter about vanishing theorems. The reason is because various vanishing theorems are closely related to the study of weak positivity in Viehweg's work. Also the techniques we used in the proof of vanishing theorems can be applied to study the semi-positivity of Hodge bundles. We don't write out the detail of proofs for semi-positivity of Hodge bundles, but instead we choose to write out the detail of proofs of vanishing theorems.

Vanishing theorems have been studied extensively before. One classic theorem is the Akizuki-Kodaira-Nakano vanishing theorem [AN]. It says if F is an ample line bundle over a compact complex n dimensional manifold X, then

$$H^{p,q}(X, F) = H^q(X, \ \Omega^p \otimes F) = 0 \qquad \text{for} \quad p + q \geq n + 1$$

When $p = n$, this is the Kodaira vanishing theorem. If X is assumed to be smooth complex projective variety then Kodaira vanishing theorem has an important generalization given by Kawamata and Viehweg ([Ka1],[V3]). Their result says if some positive multiple mF can be written as $mF = L + D$ where L is a nef and big line bundle and D an effective normal crossing divisor, then

$$H^q(X, \ K_X \otimes F \otimes \mathcal{O}(-[\frac{1}{m}D])) = 0 \qquad \text{for} \quad q \geq 1$$

Esnault-Viehweg built up general algebraic methods to various vanishing theorems ([EV1], [EV2], [EV3]). They gave various generalization Akizuki-Kodaira-Nakano vanishing theorem to the case of logarithmic differential forms. Here depends on analytic method, we will try to generalize it too.

Recall the definition of $b-$ampleness of Sommese (see [SS]). Recall for $k \geq 1$, let $B_{|kL|}$ be the base locus of $|kL|$, we can define the canonical holomorphic map

$$\Psi_{|kL|} : X \backslash B_{|kL|} \to \mathbb{C}P^N \qquad N = \dim \Gamma(X, kL) - 1$$

Then L is $b-$ample just means for some number k, $B_{|kL|}$ is empty (*i.e.* kL is generated by global sections on X) and the fibers of $\Psi_{|kL|}$ is at most $b-$dimensional. Using these terminologies, we can prove the following result.

Theorem 6.1 *Let X be a smooth complex projective variety, F be a holomorphic line bundle over X, and A be a subvariety of X. If some positive multiple mF can be written as $mF = L + D$ where $D = \sum_{i=1}^{r} \nu_i D_i$ is an effective normal crossing divisor ($\nu_i \geq 0$), and L satisfies for some number $k \geq 1$, $B_{|kL|} \subset A$ and $\Psi_{|kL|}$ restricted on $X \backslash A$ has at most $b-$dimensional fibers, then for any $p + q \geq n + max(b, dim(A)) + 1$,*

$$H^q(X,\ \Omega^p(logD) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1 + [\frac{\nu_i}{m}])D_i)) = 0$$

As we mentioned in the introduction, this theorem can be deduced from Esnault-Viehweg's results. Our method is different, and depends on Hormander's L^2 estimate. In fact, Kawamata-Viehweg vanishing theorem can also be proved by using L^2 estimate (see [Del], [Nad]). They choosed singualr metric on L and then apply L^2 estimate on X. Our simple idea here is instead of working on the compact manifold X, we will do all the estimate on the open manifold $Y = X \setminus D$. By the same method, we can derive a generalization of Kawamata-Viehweg vanishing theorem. Before that, let's give a definition.

Definition 6.1 *Let X be a compact Kahler manifold with Kahler form ω. We say a vector bundle V over X is almost Nakano semi-positive, if for any $\epsilon > 0$, there exists a Hermitian metric on V, such that its curvature form is bounded from below by $-\epsilon\omega \otimes Id_V$ in the sense of Nakano.*

In particular, any Nakano semi-positive vector bundle is almost Nakano semi-positive. Using this notation, then we have

Theorem 6.2 *Let X be a smooth complex projective variety. Let F be a line bundle over X, such that some positive multiple mF can be written as $mF = L + D$ where L is a* *nef line bundle and D an effective normal crossing divisor. Then for any almost Nakano semi-positive holomorphic vector bundle V over X,*

$$H^q(X,\ K_X \otimes F \otimes V \otimes \mathcal{O}(-[\frac{1}{m}D])) = 0 \qquad \text{for } q \geq n - \nu(L) + 1$$

Here $\nu(L)$ is the numerical dimension of L.

Similar result as this theorem is also obtained in [Ca] recently by different methods. For vector bundles of higher rank, we have the Nakano vanishing theorem ([Nak]). Based on the same idea and technique, we will study also Nakano's vanishing theorem. One of the results we can get is the following.

Theorem 6.3 *Let X be a compact complex manifold of dimension n and F be a Nakano positive vector bundle on X. If D is a simple normal crossing divisor, then*

$$H^q(X,\ K_X \otimes F \otimes \mathcal{O}(D)) = 0 \qquad \text{for} \quad q \geq 1$$

The organization of this chapter is as follows: section 6.1 to section 6.3 are the proof of theorem 6.1 and 6.3. The main technique point is the proof of a L^2 Dolbeault lemma stated in section 6.1. We will give only a brief sketch of the proof for theorem 6.3 in section 6.3 since it is quite similar as the proof of theorem 6.1. In section 6.4 we will prove theorem 6.2.

6.1 L^2 Cohomology and L^2 Dolbeault Lemma.

From now on to section 6.3, we will try to prove theorem 6.1. However, we will assume L is ample and $A = \emptyset$. After we prove this special case, we will deal with the general case in section 6.3.

Let X be a compact Kahler manifold of dimension n as given in theorem 1, and let Y be the complement of the effective normal crossing divisor D. we will consider the L^2 cohomology on Y. In order to define L^2 cohomology we first assign Y a Kahler metric and F a Hermitian metric.

Since L is ample, there exists a Hermitian metric $\|\cdot\|_L$ on L such that its curvature $R(\|\cdot\|_L)$ is a positive $(1,1)$ form on X. Choose any Hermitian metric $\|\cdot\|_{D_i}$ for $\mathcal{O}_X(D_i)$ and let s_i be the defining section. Let $\|\cdot\|_D$ be the induced Hermitian metric on $\mathcal{O}_X(D)$. Then we will give F the Hermitian metric $h_{\alpha,F}$ over Y defined by

$$h_{\alpha,F} = \prod_{i=1}^{r} \|s_i\|_{D_i}^{-(2\delta+\frac{2\nu_i}{m})}(\log(\epsilon\|s_i\|_{D_i}^2))^{2\alpha}\|\cdot\|_D^{\frac{2}{m}} \otimes \|\cdot\|_L^{\frac{2}{m}} \tag{46}$$

Where $\epsilon > 0, \delta > 0$ and $\alpha > 0$ are constants to be determined later. We will choose α to be very large, while will choose ϵ and δ to be very small (if we know $\nu_i > 0$ for all $1 \leq i \leq r$, then actually we can simply choose $\delta = 0$). Let

$$c_1(D_i) = -\partial\bar{\partial}\log\|s_i\|_{D_i}^2$$

Denote the curvature of $h_{\alpha,F}$ be $R(h_{\alpha,F})$. Then

$$\begin{aligned} R(h_{\alpha,F}) = \frac{1}{m}R(\|\cdot\|_L) - \sum_{i=1}^{r}\delta\cdot c_1(D_i) + \sum_{i=1}^{r}\frac{\alpha\cdot c_1(D_i)}{\log(\epsilon\|s_i\|_{D_i}^2)} \\ + \sum_{i=1}^{r}\frac{\alpha\cdot\partial(\log(\|s_i\|_{D_i}^2))\wedge\bar{\partial}(\log(\|s_i\|_{D_i}^2))}{(\log(\epsilon\|s_i\|_{D_i}^2))^2} \end{aligned} \tag{47}$$

Since $R(\|\cdot\|_L)$ is positive $(1,1)$ form on the compact manifold X, straightforward computation shows for any fixed $\alpha > 0$, when ϵ and δ is small enough, $R(h_{\alpha,F})$ is positive definite everywhere on the open manifold Y. Therefore we can give Y the Kahler metric $\omega_{\alpha,Y}$ defined by

$$\omega_{\alpha,Y} = R(h_{\alpha,F}) \qquad \text{for } \alpha > 0 \tag{48}$$

Let's describe the asymptotical behavior of $\omega_{\alpha,Y}$ near D. Take any local coordinate chart $(U; z_1, \cdots, z_n)$ for X such that the locus of D is given by $z_1\cdots z_k = 0$, and $Y \cap U = \triangle_{\frac{1}{2}}^{*k} \times \triangle_{\frac{1}{2}}^{l}$, where $\triangle_{\frac{1}{2}}$ is the open disk of radius $\frac{1}{2}$ and $\triangle_{\frac{1}{2}}^{*}$ is the punctured open disk of radius $\frac{1}{2}$ in complex plane. Then we find $\omega_{\alpha,Y}$ is quasi-isometric to the following Poincare type metric $\omega_{\alpha,P}$ on $U^* = \triangle_{\frac{1}{2}}^{*k} \times \triangle_{\frac{1}{2}}^{l}$.

$$\omega_{\alpha,P} = \sqrt{-1}\sum_{i=1}^{k}\frac{\alpha\cdot dz_i\wedge d\bar{z}_i}{|z_i|^2(\log|z_i|^2)^2} + \sqrt{-1}\sum_{i=k+1}^{n} dz_i\wedge d\bar{z}_i$$

And actually the bound between $\omega_{\alpha,Y}$ and $\omega_{\alpha,P}$ is independent of α. Notice when $\alpha = 1$ we get the usual Poincare metric on $\Delta^{*k}_{\frac{1}{2}} \times \Delta^{l}_{\frac{1}{2}}$ which we denote by ω_P.

Using the Kahler metric $\omega_{\alpha,Y}$ on Y and the Hermitian metric $h_{\alpha,F}$ on F, we can define the L^2 norms of any F valued (p,q) froms. Consider the sheaf $\Omega^{p,q}_{(2)}(X,F)$ on X such that on any open subset U of X, the sections of $\Omega^{p,q}_{(2)}(X,F)$ consist of F valued (p,q) froms φ with measurable coefficients such that the L^2 norm of both φ and $\bar{\partial}\varphi$ are intergrable on any compact subset of U. The global sections of $\Omega^{p,*}_{(2)}(X,F)$ with $\bar{\partial}$ operator form a complex,

$$\Gamma(\Omega^{p,0}_{(2)}(X,F)) \xrightarrow{\bar{\partial}} \Gamma(\Omega^{p,1}_{(2)}(X,F)) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Gamma(\Omega^{p,n}_{(2)}(X,F)) \to 0$$

and the associate cohomology groups $H^{p,*}_{(2)}(Y,F)$ of this complex $\{\Gamma(\Omega^{p,*}_{(2)}(X,F)), \bar{\partial}\}$ are the Dolbeault L^2 cohomology groups with coefficients in F.

Let's point out some facts which will be used later. Notice that $\omega_{\alpha,Y}$ is a complete Kahler metric on Y with finite volume. Simple calculation shows if φ is a L^2 integrable F valued (p,q) form and f is a smooth function on X then both $f\varphi$ and $\bar{\partial}(f\varphi)$ will still be L^2 integrable. Consequently we conclude $\Omega^{p,q}_{(2)}(X,F)$ is a fine sheaf on X and therefore we get the vanishing of its sheaf cohomology groups,

$$H^i(X,\ \Omega^{p,q}_{(2)}(F)) = 0 \qquad \text{for } p,q \geq 0,\ i \geq 1 \tag{49}$$

Now let's introduce the L^2 estimate which is essentially due to Hormander [Ho], and Andreotti-Vesentini [AV]. Here we will use the version suitable for our purpose as stated in [De2].

Theorem 6.4 *([Ho], [AV]) Let (M,ω) be a complete Kahler manifold of dimension n. Let F be a holomorphic Hermitian vector bundle of rank r over M, and assume the curvature operator $A = [iR(F), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}(T^*_M) \otimes F$, $q \geq 1$. Then for any form $g \in L^2(M, \Lambda^{p,q}(T^*_M) \otimes F)$ satisfying $\bar{\partial} g = 0$ and $\int_M \langle A^{-1}g, g\rangle \omega^n < +\infty$, there exists $f \in L^2(M, \Lambda^{p,q-1}(T^*_M) \otimes F)$ such that $\bar{\partial} f = g$ and*

$$\int_M |f|^2 \omega^n \leq \int_M \langle A^{-1}g, g\rangle \omega^n$$

For our complete Kahler manifold $(Y, \omega_{\alpha,Y})$ the Kahler form $\omega_{\alpha,Y}$ is precisely the curvature of the Hermitian line bundle $(F, h_{\alpha,F})$. Consequently we know the curvature operator of F restricted on $\Lambda^{p,q} \otimes F$ is given by

$$[iR(F), \Lambda] = (p+q-n) \cdot Id$$

Therefore applying theorem 6.4, we conclude

$$H^{p,q}_{(2)}(Y, F) = 0 \qquad \text{for } p+q \geq n+1 \tag{50}$$

Eventually we will see the vanishing of this L^2 cohomology will imply our theorem 6.1.

By (50), in order to prove theorem 6.1, we need only to prove the L^2 Dolbeault cohomology $H^{p,q}_{(2)}(Y, F)$ equals the sheaf cohomology of the holomorphic vector bundle $\Lambda^{p,0}(\log D) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1+[\frac{\nu_i}{m}])D_i)$ on X given in the theorem. This will follows from the following L^2 Dolbeault lemma.

Lemma 6.1 *When $\alpha > 0$ is very large, there is a resolution of $\Lambda^{p,0}(logD) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1+[\frac{\nu_i}{m}])D_i)$ by fine sheaves on X given by*

$$0 \to \Lambda^{p,0}(logD) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1+[\frac{\nu_i}{m}])D_i) \to \Omega^{p,*}_{(2)}(X, F) \tag{51}$$

Some case of this kind of L^2 Dolbeault lemma (when $p = 0$) was known to S. Zucker [Zu] and A. Fujiki [Fuj] by direct computations. Our proof will depends on the L^2 estimate of $\bar{\partial}$ operator and some curvature computations. Also for the proof of theorem 6.2 let's point out, when $p = n$ this lemma is true for any $\alpha > 0$, and similar results are true even δ is a small negative constant. These facts can be proved quickly by the same arguments we are going to give.

From section 6.1, we already know $\Omega^{p,q}_{(2)}(X, F)$ are fine sheaves on X. We need only to check the exactness of (51). Let $(U; z_1, \cdots, z_n)$ be the local coordinate chart of X as before, and let e be a trivializing section of F on U. Denote ξ_i be

$$\xi_i = \begin{cases} \frac{1}{z_i} dz_i & 1 \leq i \leq k \\ dz_i & k+1 \leq i \leq n \end{cases} \tag{52}$$

Let s be a section of $\Omega^{p,0}_{(2)}(X, F)$ on U. If furthermore $\bar{\partial} s = 0$, then we write

$$s(z) = \sum_{|I|=p} \lambda_I(z)(\xi_{i_1} \wedge \cdots \wedge \xi_{i_p}) \otimes e$$

where $I = (i_1, \cdots, i_p)$ is a multiple index and $\lambda_I(z)$ is a holomorphic function on $U^* = U \cap Y = \triangle^{*k}_{\frac{1}{2}} \times \triangle^l_{\frac{1}{2}}$. By definition s is L^2 integrable on $U^*_r = \triangle^{*k}_r \times \triangle^l_r \subset U \cap Y$ for any $0 < r < \frac{1}{2}$.

To simplify the computation notice that our Kahler metric $\omega_{\alpha,Y}|_{U^*}$ is quasi isometric to the usual Poincare metric ω_P and the Hermitian metric $h_{\alpha,F}|_{U^*}$ is quasi-isometric to the following Hermitian metric

$$h_\alpha = \prod_{i=1}^{k} |z_i|^{-(2\delta + \frac{2\nu_i}{m})}(\log|z_i|^2)^{2\alpha} \| \cdot \|^2_F \tag{53}$$

Here $\| \cdot \|_F$ is a smooth Hermitian metric of F and we may assume $\|e(z)\|_F \equiv 1$. Now s is L^2 integrable with respect to $\omega_{\alpha,Y}$ and $h_{\alpha,F}$ if and only if it is L^2 integrable with respect to ω_P and h_α. So we will replace $\omega_{\alpha,Y}$ and $h_{\alpha,F}$ by ω_P and h_α when we talk about L^2 integrability. Let's compute the L^2 norm of $s(z)$ on $U^*_r = \triangle^{*k}_r \times \triangle^l_r$ with respect to ω_P and h_α. If we denote $\{i_1, \cdots, i_p\} \cap \{1, \cdots, k\} = \{i_1, \cdots, i_{p'}\}$, then we get

$$\|s\|^2_{L^2(U^*_r)} = \sum_{|I|=p} \int_{U^*_r} |\lambda_I(z)|^2 (\log|z_{i_1}|^2)^2 \cdots (\log|z_{i_{p'}}|^2)^2 \prod_{j=1}^{k} |z_j|^{-(2\delta + \frac{2\nu_j}{m})} (\log|z_j|^2)^{2\alpha} \omega^n_P$$

Assume the Laurent series representation of $\lambda_I(z)$ on $\triangle^{*k}_{\frac{1}{2}} \times \triangle^l_{\frac{1}{2}}$ is given by

$$\lambda_I(z) = \sum_{\beta=-\infty}^{+\infty} c_{I\beta}(z_{k+1}, \cdots, z_n) z_1^{\beta_1} \cdots z_k^{\beta_k}$$

here $c_{I\beta}(z_{k+1}, \cdots, z_n)$ is holomorphic function on $\triangle^l_{\frac{1}{2}}$. Then by Parseval summation formular we find s is L^2 on $U^*_r = \triangle^{*k}_r \times \triangle^l_r$ if and only if s has removable singularities and vanishes with order at least $(1 + [\frac{\nu_i}{m}])$ along D_i. This means s is a section of $\Lambda^{p,0}(\log D) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^{r}(1 + [\frac{\nu_i}{m}])D_i)$ on U.

Conversely if we choose s to be any holomorphic section of $\Lambda^{p,0}(\log D)\otimes F\otimes\mathcal{O}(-\sum_{i=1}^{r}(1+[\frac{\nu_i}{m}])D_i)$ on U, then it is straightforward to check s is L^2 integrable on U_r^* for any $0<r<\frac{1}{2}$. Therefore we proved (51) is exact at $\Omega_{(2)}^{p,0}(X,F)$ for any real number α.

For the exactness of (51) at $\Omega_{(2)}^{p,q}(X,F)$ for $q\geq 1$, we need to show for any $g\in L^2(U^*,\Lambda^{p,q}\otimes F)$ with respect to ω_P and h_α, if $\bar\partial g=0$ then on some $U_\epsilon^*=\triangle_\epsilon^{*k}\times\triangle_\epsilon^l$ $(0<\epsilon<\frac{1}{2})$ we can find f such that

$$\bar\partial f=g,\quad \text{and } f\in L^2(U_\epsilon^*,\Lambda^{p,q-1}\otimes F) \tag{54}$$

For doing this, I will deform the Kahler metric ω_P to be a complete Kahler metric on $U_r^*=\triangle_r^{*k}\times\triangle_r^l$, and then apply L^2 estimate to solve (54).

6.2 Deformation of Metrics and Curvature Computations.

Fix $0<r<\frac{1}{2}$, and deform the Kahler metric ω_P to be a new Kahler metric $\tilde\omega_P$ on $U_r^*=\triangle_r^{*k}\times\triangle_r^l$ given by

$$\tilde\omega_P=\omega_P+\partial\bar\partial(\psi_1+\cdots+\psi_n) \tag{55}$$

where ψ_i is a function given by

$$\psi_i(z)=\frac{1}{r^2-|z_i|^2}$$

Then $\tilde\omega_P$ is a complete Kahler metric on $U_r^*=\triangle_r^{*k}\times\triangle_r^l$ which can be seen easily from its expression

$$\begin{aligned}\tilde\omega_P=&\sqrt{-1}\sum_{i=1}^{k}(\frac{1}{|z_i|^2(\log|z_i|^2)^2}+\frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3})\,dz_i\wedge d\bar z_i\\&+\sqrt{-1}\sum_{i=k+1}^{n}(1+\frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3})\,dz_i\wedge d\bar z_i\end{aligned} \tag{56}$$

We choose also a new Hermitian metric $\tilde h_\alpha$ for F defined by

$$\tilde h_\alpha=\exp(-\sum_{i=1}^{n}(\alpha|z_i|^2+\alpha\psi_i))h_\alpha \tag{57}$$

So $\tilde{h}_\alpha$ is given explicitly by

$$\tilde{h}_\alpha = \prod_{i=1}^{k} |z_i|^{-(2\delta+\frac{2\nu_i}{m})}(\log|z_i|^2)^{2\alpha} \prod_{i=1}^{n} \exp(-\alpha|z_i|^2 - \alpha\psi_i)\|\cdot\|_F^2 \tag{58}$$

Here $\|\cdot\|_F$ is a smooth Hermitian metric on F, and recall $\|e(z)\|_F \equiv 1$ for a generating holomorphic section e of F on U. Denote the curvature of $\tilde{h}_\alpha$ to be $R(\tilde{h}_\alpha)$ then by (53), (55) and (57)

$$R(\tilde{h}_\alpha) \geq \alpha \cdot \tilde{\omega}_P \tag{59}$$

Notice $\tilde{\omega}_P$ is the direct product of metrics on $\triangle_r^*$ or $\triangle_r$. Denote the i-th component of $\tilde{\omega}_P$ to be $\tilde{\omega}_i$, then from (56) straightforward computation shows its curvature $R_i = -\mathrm{Ric}(\tilde{\omega}_i)$ on U_r^* satisfies

$$|R_i| = |\partial\bar{\partial}\log(\tilde{\omega}_i)| \leq C\tilde{\omega}_i \qquad \text{for some } C > 0 \tag{60}$$

Consequently by (59) and (60), the curvature of $F \otimes K^{-1}$ equipped with the Hermitian metric induced from $\tilde{\omega}_P$ and $\tilde{h}_\alpha$ will satisfy for some $C > 0$,

$$R(F \otimes K^{-1}) = R(\tilde{h}_\alpha) + \mathrm{Ric}(\tilde{\omega}_P) \geq (\alpha - C)\tilde{\omega}_P \tag{61}$$

We will show that the curvature of $E = \Lambda^{p,0} \otimes F \otimes K^{-1}$ with respect to $\tilde{h}_\alpha$ and $\tilde{\omega}_P$ is positive in the sense of Nakano. For this reason let's denote L to be the line bundle $F^{\frac{1}{p}} \otimes K^{-\frac{1}{p}}$ on U_r^*, then

$$E = (T^* \otimes L) \wedge \cdots \wedge (T^* \otimes L)$$

where T^* is the holomorphic cotangent bundle of U_r^*. In terms of orthonormal holomorphic frames $\{f_i\}$ $(1 \leq i \leq n)$ of T^* given by

$$f_i = \begin{cases} \left(\frac{1}{|z_i|^2(\log|z_i|^2)^2} + \frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3}\right)^{-\frac{1}{2}} dz_i & 0 \leq i \leq k \\ \left(1 + \frac{r^2+|z_i|^2}{(r^2-|z_i|^2)^3}\right)^{-\frac{1}{2}} dz_i & k+1 \leq i \leq n \end{cases} \tag{62}$$

we find the curvature of T^* is

$$R(T^*) = \sum_{i=1}^{n} c_{i\bar{i}}\, dz_i \wedge d\bar{z}_i \otimes f_i \otimes f_i^* \tag{63}$$

where $c_{i\bar{i}} = R_i$ is bounded by constant on U_r^* from (60). Now if we denote the corresponding orthonormal holomorphic frame of $T^* \otimes L$ to be e_λ and denote the curvature of L to be

$$R(L) = \frac{1}{p} R(F \otimes K^{-1}) = \sum_{i=1}^{n} b_{i\bar{i}}\, dz_i \wedge d\bar{z}_i$$

then the curvature of $T^* \otimes L$ is

$$R(T^* \otimes L) = \sum_{i,\lambda=1}^{n} (\delta_{i\lambda} c_{i\bar{i}} + b_{i\bar{i}})\, dz_i \wedge d\bar{z}_i \otimes e_\lambda \otimes e_\lambda^* \tag{64}$$

From (60), (61) and (64), it is then easy to check that $R(T^* \otimes L)$ is Nakano positive if $\alpha > C(1+p)$. And consequently $R(E)$ is also Nakano positive with respect to $\tilde{\omega}_P$.

6.3 On Akizuki-Kodaira-Nakano's Theorem and Nakano's Theorem

We will prove theorem 6.1 and 6.3 in this section which are generalizations of Akizuki-Kodaira-Nakano vanishing theorem and Nakano vanishing theorem respectively. Let's prove theorem 6.1 under the assumption L is ample and $A = \emptyset$. First let's establish the L^2 Dolbeault lemma. Let Λ be the contraction operator with respect to $\tilde{\omega}_P$. $R(E)$ is Nakano positve implies there exist $c > 0$ such that

$$A = [iR(E), \Lambda] \geq c \cdot Id \qquad \text{on } \Lambda^{n,q} \otimes E \tag{65}$$

Now let's solve (54). Notice $\tilde{h}_\alpha$ decays to zero exponentially when $|z_i|$ goes to r. Since $g \in L^2(U^*, \Lambda^{p,q} \otimes F)$ with respect to ω_P and h_α, from (55 and (57) we find $g \in L^2(U^*, \Lambda^{p,q} \otimes F)$ with respect to $\tilde{\omega}_P$ and $\tilde{h}_\alpha$. Recall that $E = \Lambda^{p,0} \otimes F \otimes K^{-1}$, so g can be considered as L^2 integrable E valued (n,q) form on U_r^* with respect to $\tilde{\omega}_P$ and $\tilde{h}_\alpha$.

Notice $(U_r^*, \tilde{\omega}_P)$ is a complete Kahler manifold, so by (65) and theorem 6.4 we can find $f \in L^2(U_r^*, \Lambda^{n,q-1} \otimes E) = L^2(U_r^*, \Lambda^{p,q-1} \otimes F)$ with respect to $\tilde{\omega}_P$ and $\tilde{h}_\alpha$ such that $\bar{\partial} f = g$. Now from the explicit expressions of $\tilde{\omega}_P$ and $\tilde{h}_\alpha$, we conclude that $f \in L^2(U_{\frac{r}{2}}^*, \Lambda^{p,q-1} \otimes F)$ with respect to ω_P and h_α. Thus (54) is ture for $\epsilon = \frac{1}{2}r$, therefore we established the L^2 Dolbeault lemma.

From L^2 Dolbeault lemma, we get resolution of $\Lambda^{p,0}(\log D) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^r (1+[\frac{\nu_i}{m}])D_i)$ by fine sheaves $\Omega_{(2)}^{p,q}$ $(0 \leq q \leq n)$ on X, therefore

$$H^q(X, \ \Lambda^{p,0}(\log D) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^r (1+[\frac{\nu_i}{m}])D_i)) = H_{(2)}^{p,q}(Y, F)$$

However from (50) the L^2 cohomology group $H_{(2)}^{p,q}(Y, F)$ vanishes, so we conclude

$$H^q(X, \ \Lambda^{p,0}(\log D) \otimes F \otimes \mathcal{O}(-\sum_{i=1}^r (1+[\frac{\nu_i}{m}])D_i)) = 0 \quad \text{for } p+q \geq n+1$$

And this finishes the proof of theorem 6.1 in the special case.

For the proof of theorem 6.1 in general, we need a theorem of Hironaka in [Hir]. For reader's convinnience, we state the theorem.

Lemma 6.2 *([Hir]) Let X be a smooth complex projective variety. Let $\mathfrak{L}$ be an ample line bundle over X, and A a subvariety. Then for any holomorphic map $f : X \backslash A \to \mathbb{C}P^M$, there exists a number $c \geq 1$ such that the generic element of $\Gamma(X, c\mathfrak{L})$ doesn't vanish on any positive dimensional irreducible component of $f^{-1}(y)$, for all $y \in f(X \backslash A)$.*

Now let's prove theorem 6.1 in the general case. It's a standard hypersection argument. We will do induction on $r = max(b, dim(A))$. When $r = 0$, L is actually ample (see [SS]), so the conclusion for this case then follows from the previou special case. Now assume theorem 6.1 is true for some $r \geq 1$. Fix an ample line bundle $\mathfrak{L}$ on X. Using Hironaka's theorem and Bertini's theorem, we can choose a smooth divisor Y in $|c\mathfrak{L}|$ for some $c \geq 1$, such that $\dim(A \cap Y) \leq \dim(A) - 1$ and $\Psi_{|kL|}$ when restricted on $Y \backslash A$ has at most $(b-1)-$dimensional fibers. Therefore by induction, we have for

$p+q \leq n - max(b, dim(A)) - 1$

$$H^q(Y,\ \Omega^p_Y(\text{log}D) \otimes F^{-1} \otimes \mathcal{O}([\frac{1}{m}D])) = 0$$

Now notice we have an exact sequence of sheaves on X

$$0 \to \Omega^p_X(log(D+Y)) \otimes \mathcal{O}(-Y) \to \Omega^p_X(logD) \to \Omega^p_Y(logD) \to 0$$

Using the associated long exat sequence of sheave chomology groups, we know theorem 6.1 in this case is reduced to show that for any $p+q \leq n - max(b, dim(A)) - 1$

$$H^q(X,\ \Omega^p_X(\text{log}D+Y) \otimes F^{-1} \otimes \mathcal{O}(-Y + [\frac{1}{m}D])) = 0$$

However this result can be proved by applying the dual version of the special case of theorem 6.1 (which we have established) to $m(F+2Y) = (L+mY) + (D+mY)$.

The proof of theorem 6.3 will be similar as theorem 6.1 and actually even simpler. We will give only the outlines.

Since F is Nakano positive, there exists a Hermitian metric $\|\cdot\|_F$ on F over X such that the curvature $R(F)$ is Nakano positive. As in section 6.1, we give F a new Hermitian metric h_F over Y defined by

$$h_F = \prod_{i=1}^{r} \|s_i\|^{2\delta}_{D_i} (\text{log}\epsilon \|s_i\|^2_{D_i})^2 \|\cdot\|^2_F$$

Here ϵ and δ are small positive constants. We fix a Kahler metric ω_Y on Y which is of Poincare type along D. Then we find if ϵ and δ are small enough then F (with the Hermitian metric h_F) is Nakano positive on Y, which is a complete Kahler manifold with the Kahler form ω_Y. Consequently, we know the curvature operator of F satisfies for some constant $c > 0$

$$A = [iR(F), \Lambda] \geq c \cdot Id \qquad \text{on } \Lambda^{n,q} \otimes F$$

Apply theorem 6.4, we get

$$H^q_{(2)}(Y,\ K_Y \otimes F) = 0 \qquad \text{for } q \geq 1$$

Restricted on each coordinate chart U of X such that $U \cap Y = \triangle_{\frac{1}{2}}^{*k} \times \triangle_{\frac{1}{2}}^{l}$, the Kahler metric ω_Y is quasi isometric to the usual Poincare metric ω_P, and h_F is also quasi isometric to

$$\tilde{h} = \prod_{i=1}^{k} |z_i|^{2\delta} (\log|z_i|^2)^2 \| \cdot \|_F^2$$

We may assume F restricted on $\triangle_{\frac{1}{2}}^{*k} \times \triangle_{\frac{1}{2}}^{l}$ is just a direct sum of r copies of a holomorphic line bundle L, and thus $\tilde{h}$ is just a product of Hermitian metric on L. Now notice our L^2 Dolbeault lemma is true for any $\alpha > 0$ when $p = n$. Therefore we can establish the corresponding L^2 Dolbeault lemma for this line bundle L, then consequently we will have a resolution of $K_X \otimes F \otimes \mathcal{O}(D)$ by fine sheaves on X

$$0 \to K_X \otimes F \otimes \mathcal{O}(D) \to \Omega_{(2)}^{n,*}(X, F)$$

Therefore we conclude

$$H^q(X,\ \ K_X \otimes F \otimes \mathcal{O}(D)) = H_{(2)}^{n,q}(Y, F) = 0$$

Thus we proved theorem 6.3.

6.4 A Generalization of Kawamata-Viehweg Vanishing Theorem.

In this section we will give a proof of theorem 6.2. In this section, X is n dimensional complex projective variety.

Let's prove theorem 6.2. First, let's prove a special case when L is ample on X. Give F over $Y = X - D$ the Hermitian metric $h_{\alpha,F}$ as we did in section 6.1. We still use the Kahler metric $\omega_{\alpha,Y} = R(h_{\alpha,F})$ on Y which is of Poincare type along D. Since V is almost Nakano semi-positive, fix any Kahler form ω on X, for any $\epsilon > 0$, there exists a Hermitian metric h_V on V over X such that its curvature form $R(V)$ is bounded from below by $-\epsilon\omega \otimes Id_V$ in the sense of Nakano. Notice $\omega \leq C\omega_{\alpha,Y}$ on Y for some constant

$C > 0$. Then the curvature operator of $F \otimes V$ restricted on $\Lambda^{n,q}(F \otimes V)$ satisfies

$$[iR(F \otimes V), \Lambda] \geq (qr - C\epsilon) \cdot Id$$

Here $r = rank(V)$ and Λ is the contraction operator with respect to $\omega_{\alpha,Y}$. Therefore if we choose ϵ very small, we know the curvature operator $[iR(F \otimes V), \Lambda]$ is positive definite on $\Lambda^{n,q}(F \otimes V)$ for any $q \geq 1$. Study the L^2 cohomology on Y, then by theorem 6.4

$$H^q_{(2)}(Y,\ K_X \otimes F \otimes V) = 0 \qquad \text{for } q \geq 1$$

Since V locally is a trivial Hermitian vector bundle on X, the L^2 Dolbeault lemma in section 6.1 implies we still have a corresponding L^2 Dolbeault lemma for $F \otimes V$. Then consequently, we conclude

$$H^q(X,\ K_X \otimes F \otimes V \otimes \mathcal{O}(-[\frac{1}{m}D])) = H^q_{(2)}(Y,\ K_X \otimes F \otimes V) = 0 \qquad \text{for } q \geq 1$$

Let's prove theorem 6.2 in general. Actually, we may assume L is a nef and big line bundle (*i.e.*, $\nu(L) = n$). Since the general case when L is assumed to be nef only can be reduced to this case by a standard hypersection argument. Then let's prove theorem 6.2 under the assumption when L is only nef and big. We state a lemma first. This lemma is a well-kown fact, however for the convenience of the reader we write out the proof.

Lemma 6.3 *Given any line bundle L over X, if L is big then there exists a positive number c such that*

$$cL = L' + D' \tag{66}$$

where L' is an ample line bundle over X and D' is an effective divisor.

Proof Fix a non-singular ample divisor A and denote $L' = \mathcal{O}(A)$. Consider the exact sequence

$$0 \to H^0(X,\ cL - A) \to H^0(X,\ cL) \to H^0(A,\ cL)$$

By Siegel's theorem, $\dim H^0(A,\ cL) = O(c^{n-1})$. L is big implies when c is very large $\dim H^0(X,\ cL) \geq \epsilon c^n$ for some constant $\epsilon > 0$. Thus for a large c, $H^0(X,\ cL - A) \neq 0$ and consequently $cL - A = \mathcal{O}(D')$ for some effective divisor D'.

Now let's go back to the proof of theorem 6.2. Choose blowing-up $\pi : Y \to X$ such that $\pi^*(D') + \pi^*(D) + \sum_{i=1}^k E_i$ is a normal crossing divisor. Here $E_i (1 \leq i \leq k)$ are the exceptional divisors of blowing-up. Since L' is ample over X we know that

$$\tilde{L} = \pi^*(L') - \sum_{i=1}^{k} \lambda_i E_i \tag{67}$$

is an ample line bundle over Y for some $\lambda_i > 0$. Therefore for any number $a \geq 1$ by (66) and (67) we have decomposition

$$acm\pi^*(F) = \underbrace{((a-1)c\pi^*(L) + \tilde{L})}_{ample} + (\pi^*(D') + ac\pi^*(D) + \sum_{i=1}^{k} \lambda_i E_i) \tag{68}$$

Notice $(\pi^*(D') + ac\pi^*(D) + \sum_{i=1}^k \lambda_i E_i)$ is effective divisor of normal crossing over Y. Therefore we can apply our vanishing theorem for the special case to (68). If we choose a to be a very large number then we know

$$[\frac{1}{acm}(\pi^*(D') + ac\pi^*(D) + \sum_{i=1}^{k} \lambda_i E_i)] = [\frac{1}{m}\pi^*(D)]$$

Consequently by our result, we get

$$H^q(Y,\ K_Y \otimes \pi^*(F \otimes V) \otimes \mathcal{O}(-[\frac{1}{m}\pi^* D])) = 0 \qquad \text{for} \quad q \geq 1 \tag{69}$$

However since $\pi : Y \to X$ is blowing-up so we have

$$R^i \pi_* K_Y = \begin{cases} K_X & i = 0 \\ 0 & i \geq 1 \end{cases} \tag{70}$$

Therefore by the projection formular we know

$$R^i \pi_*(K_Y \otimes \pi^*(F \otimes V) \otimes \mathcal{O}(-[\frac{1}{m}\pi^* D])) = \begin{cases} K_X \otimes F \otimes V \otimes \mathcal{O}(-[\frac{1}{m}D]) & i = 0 \\ 0 & i \geq 1 \end{cases} \tag{71}$$

Then theorem 6.2 follows directly from (69) and (71) by standard use of Leray spectral sequence.

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