Unitary representations of $U(p,q)$ and Generalized Robinson-Schensted Algorithms

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ABSTRACT

In Part I of this thesis, we locate a (conjecturally complete) set of unitary representations in the admissible dual of $U(p,q)$. In a little more detail, Barbasch and Vogan have used the theory of Kazhdan-Lusztig cells to parametrize the irreducible Harish-Chandra modules with integral infinitesimal character in terms of their annihilators and associated varieties. Vogan has conjectured that the weakly fair cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ exhaust the unitary dual of $U(p,q)$ for the kinds of infinitesimal character that they can have. Here we compute the annihilators and associated varieties of these modules, thus locating them in the admissible dual. In particular, this determines all coincidences among these modules and gives their Langlands parameters. We conclude Part I with some evidence for the conjecture.

In Part II, we interpret some of the combinatorics which arise in the Barbasch-Vogan parametrization in terms of the geometry of the generalized Steinberg variety. This leads to a study of geometric cells, which are exactly analogous to Kazhdan-Lusztig cells, except that one begins with a topological action of the complex Weyl group instead of the coherent continuation action. We compute the structure of geometric cells for type A real groups; more precisely, we compute Springer's generalized Robinson-Schensted algorithm for these groups, and compare the computation to the Barbasch-Vogan parametrization.

Thesis Committee: Bertram Kostant
George Lusztig
David A. Vogan, Jr. (Thesis Advisor)

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And, nothing himself, he beholds
Nothing that is not there and the nothing that is.

Wallace Stevens, *The Snow Man*

INTRODUCTION

The primary existing approach to enumerating the unitary representations of a real Lie group is an external one: one works algebraically and first classifies a larger, more tractable class of objects—the irreducible Harish-Chandra modules—and then tries to determine which of these are unitary. Such arguments typically take the following form. One constructs a large, conjecturally complete catalog of known unitary representations, and proceeds with a program consisting of three parts: finding a suitable parametrization of irreducible Harish-Chandra modules; identifying the parameters of the catalog of known unitary representations; and then proving that all other parameters give rise to non-unitary representations. The best kind of result allows one to "see" why parameters not on the list of unitary ones produce a failure of unitarity.

The problem, historically, with this strategy is that all known classifications of Harish-Chandra modules had nothing to do with unitarity. This is the case, for example, with the Langlands classification: families of unitary representations appear whose Langlands parameters don't look like families. But recently, Salamanca and Vogan have discovered a weakened version of the Vogan-Zuckerman classification which *does* behave well with respect to unitarity ([SaV]).

In practice, their results (which are still partly conjectural) reduce the unitary classification problem to the study of the compact set of Harish-Chandra modules whose infinitesimal character is in the convex hull of ρ (see the concluding remarks in Section 8, for example). Such a reduction is a powerful means to organize bottom-layer arguments. For instance, Salamanca has given a very elegant proof of a conjecture of Vogan and Zuckerman describing unitary representations with regular integral infinitesimal character ([Sa2]). In terms of the program described above, the catalog of known unitary representations consists of the modules $A_{\mathfrak{q}}(\lambda)$; these are the ($K$-finite vectors) of the representations attached to elliptic coadjoint orbits (more details will be recalled below). The parametrization is by Vogan's theory of lowest K-type, and the exhaustion result essentially follows from a reduction (in the spirit of [SaV]) to small infinitesimal character ultimately relying on the Dirac operator inequality.

It is instructive to see where the assumption of regular infinitesimal character comes into the argument of [Sa2]. In the case of singular integral infinitesimal character, we still have a large list of unitary representations for $U(p,q)$: Vogan's unitarizability theorem applies to certain $A_{\mathfrak{q}}(\lambda)$—the weakly fair ones of Definition 2.4, for instance—and, for $U(p,q)$, these are all still irreducible. Vogan has conjectured that the list is complete.

Conjecture 0.1 (Vogan). *The cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$ in the weakly fair range exhaust the unitary Harish-Chandra modules for $U(p,q)$ whose infinitesimal character is a weight-translate of ρ.*

(The kind of integral infinitesimal character is explained in Remark 2.2.) When applying the approach of [Sa2] to the conjecture, the arguments break down for difficult reasons: the shape of the set of lowest K-types of the $A_{\mathfrak{q}}(\lambda)$ with singular infinitesimal character is complicated and seemingly intractable, and the second part of the program (locating the known unitary representations in the K-type parametrization) cannot be repaired. For representations with small infinitesimal character, the Dirac inequality frequently gives no information when applied to a lowest K-type. Examples show that it can even be inconclusive on *all* K-types of a non-unitary representation with infinitesimal character in the convex hull of ρ (see Example

6.13), and so the third piece of the program (the exhaustion step) also breaks down. This suggests we need to overhaul the entire approach, beginning with the parametrization, if we are to make progress on the singular case.

In their fundamental study of complex groups ([BV2], [BV3]), Barbasch and Vogan establish deep relationships between the theory of nilpotent orbits, primitive ideals, and Kazhdan-Lusztig cells (among other things). Their methods apply to real groups, though this setting introduces substantial complications. A reasonable place to start is by looking at a relatively uncomplicated real group like $U(p,q)$. Applying their techniques in [BV4], they obtained a beautiful parametrization of Harish-Chandra modules for $U(p,q)$ in terms their annihilators and asymptotic supports (which are classified in terms of tableaux—see Sections 4 and 5). Part of what makes the Barbasch-Vogan parametrization appealing is that it is well suited for translation arguments passing from regular to singular infinitesimal character.

In Part I of this thesis, we identify the mediocre $A_{\mathfrak{q}}(\lambda)$ for $U(p,q)$ in terms of the Barbasch-Vogan parametrization. (The set of mediocre $A_{\mathfrak{q}}(\lambda)$, as defined in Definition 2.4, properly includes the set of weakly fair $A_{\mathfrak{q}}(\lambda)$.) The answer given in Corollary 6.11 is quite complicated, and is perhaps discouraging if one hopes to carry out the unitarity program detailed above. On the other hand, Theorem 6.9 and Corollary 6.11 determine all coincidences among the mediocre $A_{\mathfrak{q}}(\lambda)$ for $U(p,q)$, and combined with Garfinkle's algorithm ([G]), it gives their Langlands parameters (and hence their lowest K-types). This kind of information is interesting in its own right.

The techniques used in proving Corollary 6.11 are, in some sense, independent of explicit computations with tableaux. The proof is roughly a reduction to the case of maximal $\mathfrak{q}$, which suggests generalizations to other classical groups. McGovern has recently extended part of the analysis of [BV4] to all classical groups in [Mc1], though there are difficult combinatorial issues still to be resolved. For example, McGovern's results suggest that the next interesting case to consider is $Sp(p,q)$, a group (like $U(p,q)$) whose Cartan subgroups are all connected. For $Sp(p,q)$, however, no explicit analog of the Barbasch-Vogan parametrization is known, though we will return to this below.

The point of departure for Part II is the following observation. Using the Beilinson-Bernstein parametrization, we can think of irreducible Harish-Chandra modules for $U(p,q)$ (with infinitesimal character ρ) as parametrized by the orbits of $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ on the complex flag variety $GL(p+q,\mathbb{C})/B$. The Barbasch-Vogan parametrization can then be interpreted as a map from these orbits to same-shape pairs of certain kinds of tableaux (as in Section 5). As we mentioned above, Garfinkle [G] computed this map explicitly. The resulting algorithm is very reminiscent of the Robinson-Schensted algorithm which takes elements of the symmetric group S_n to same-shape pairs of standard Young tableaux of size n. If we think in terms of Bruhat cells, S_n parametrizes the diagonal $GL(n,\mathbb{C})$ orbits on pairs of flags for $GL(n,\mathbb{C})$ (i.e. the $K_{\mathbb{C}}$ orbits on $G_{\mathbb{C}}/B$ for $G = GL(n,\mathbb{C})$). In turn, we can think of these parametrizing irreducible Harish-Chandra modules for $GL(n,\mathbb{C})$. Now it is well-known that the Robinson-Schensted algorithm essentially computes the left and right annihilators of these Harish-Chandra modules. On the other hand, Steinberg discovered a completely different occurrence of the Robinson-Schensted algorithm using the elementary geometry of the Steinberg variety of triples for $GL(n,\mathbb{C})$. Hence, in the context of $GL(n,\mathbb{C})$, the Robinson-Schensted algorithm has a representation theoretic and geometric interpretation. One is led to ask if the (representation theoretic) Barbasch-Vogan map has a geometric analog.

We give an affirmative answer as follows. Springer (unpublished) has shown how to parametrize $K_{\mathbb{C}}\backslash G_C/B$ (for arbitrary real G) in terms of nilpotent $K_{\mathbb{C}}$ orbits on the Cartan $\mathfrak{p}_{\mathbb{C}}$ and irreducible components of the Springer fiber. For $U(p,q)$, the parametrization reduces to a 'generalized Robinson-Schensted algorithm' (see Remark 12.4) which maps $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ orbits on the flag variety for $GL(p+q,\mathbb{C})$ to certain same-shape pairs of tableaux. In Theorem 12.6 we compute the map explicitly, and show it coincides with the Barbasch-Vogan parametrization. Using Vogan's duality theorem, we obtain similar results for $SU^*(2n)$ and $GL(n,\mathbb{R})$. The persistence of the coincidence of the representation theoretic and geometric maps may perhaps lead one to speculate that something deeper is going on here. We will avoid such difficult questions, and only remark that some evidence indicates that the geometric map for $Sp(p,q)$ might again coincide with a representation theoretic parametrization of the admissible dual. (Recall that this is what is needed to duplicate the results of Part I for $Sp(p,q)$.)

The thesis is organized as follows. Part I begins by fixing notation in Section 1, and treating (mostly) well-known results about the $A_{\mathfrak{q}}(\lambda)$ modules in Section 2. Of particular interest are Theorem 2.1(b)(iv) which gives a larger range of irreducibility for the $A_{\mathfrak{q}}(\lambda)$ modules than is typically considered, and Proposition 2.8 which provides a (conjectural) description of the unitarily small representations of $U(p,q)$ whose infinitesimal character is a weight translate of ρ. In Section 3, we recall the classification of primitive ideals in $\mathfrak{gl}(n,\mathbb{C})$ and prove a weak statement describing their behavior under cohomological induction. In Section 4, we first recall a few deep facts about asymptotic supports and associated varieties. We describe these invariants abstractly for the modules $A_{\mathfrak{q}}(\lambda)$, and then make that description explicit for $U(p,q)$. An appendix detailing a direct calculation of Richardson orbits for $\mathfrak{u}(p,q)$ concludes Section 4. In Section 5, we precisely state the Barbasch-Vogan parametrization and identify the parameters of the $A_{\mathfrak{q}}(\lambda)$ in the good range. We accomplish this by a simple application of our results in Sections 3 and 4, side-stepping any intermediate consideration of Langlands parameters. In Section 6, we state our main results (Theorem 6.9 and Corollary 6.11): an explicit computation and identification of the annihilators and associated varieties of the mediocre $A_{\mathfrak{q}}(\lambda)$. We prove Theorem 6.9 in Section 7 by carefully understanding the effect of certain wall crossing translation functors from regular to singular infinitesimal character. The main tool is an early theorem of Vogan's ([V3]) which keeps track of a distinguished constituent of certain wall crosses. In Section 8, we prove that any mediocre $A_{\mathfrak{q}}(\lambda)$ is isomorphic to a weakly fair one. (As explained in Remark 2.7, this is a small piece of Conjecture 0.1.) Section 8 finishes with an illustrative example involving certain ladder representations of $U(p,q)$, as well as a sketch of Conjecture 0.1 for $U(p,1)$.

Part II begins by fixing a slightly more convenient set of notations in the setting of an arbitrary real reductive group G. In Section 10, we describe Springer's parametrization of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$, the so-called generalized Robinson-Schensted correspondence (as explained in Remark 12.4). The parametrization is a refinement of a partition of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$ into disjoint subsets called geometric cells. We explain the "cell" terminology in Section 11 by recalling results of Tanisaki [Ta], and discuss how the elements of a geometric cell index a basis of a representation of W. In Section 12, we restrict our attention to $G = U(p,q)$, and state our main result (Theorem 12.6) relating Springer's parametrization to the Barbasch-Vogan parametrization; analogous results for $SU^*(2n)$ and $GL(n,\mathbb{R})$ are also given. In particular, this computes (real) Richardson orbits for these groups (Remark 12.8). We conclude Section 12 by giving a representation theoretic interpretation of a shape-preserving involution on the set of standard Young tableaux first studied by Spaltenstein (Corollary 12.10). In

Section 13, we work out an explicit description of $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$ for the relevant groups, and write down Vogan's duality on the level of orbits. We give explicit calculations (mostly due to Garfinkle [G]) of annihilators of Harish-Chandra modules in Section 14. In Section 15, we assemble the results of Sections 13–14 to prove Theorem 12.6. We conclude by applying our results to give an elementary computation of some associated varieties for the type A groups under consideration.

PART I: ANNIHILATORS AND ASSOCIATED VARIETIES OF $A_{\mathfrak{q}}(\lambda)$ MODULES FOR $U(p,q)$

1. NOTATION AND STRUCTURE THEORY

Let $G = U(p,q)$ be the group of complex linear transformations of $\mathbb{C}^{p+q}$ preserving a Hermitian form defined by a matrix with p pluses and q minuses on the diagonal. (We will be a little more precise about the arrangement of the signs below.) Let $K \cong U(p) \times U(q)$ be the fixed points of the Cartan involution of inverse conjugate transpose. Let θ be the differentiated involution and let $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$ be the corresponding decomposition. Here and elsewhere, we denote real Lie algebras with naught subscripts and their corresponding complexifications by deleting the subscript; for example $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$, with $n = p+q$.

Fix the diagonal torus $T \subset K$ with Lie algebra $\mathfrak{t}_o$, and set $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_o$. Write $\Delta(\mathfrak{g},\mathfrak{t})$ for the roots of $\mathfrak{t}$ in $\mathfrak{g}$ and make the standard choice of positive roots,

$$\Delta^+ = \{e_i - e_j \mid i < j\}.$$

Let $\alpha_i = e_i - e_{i+1} \in \Delta(\mathfrak{g},\mathfrak{t})$ denote the ith simple root, and Σ denote the collection of all simple roots. With these choices, a weight $\nu = (\nu_1, \dots, \nu_n) \in \mathfrak{t}^*_{\mathbb{R}}$ is dominant if $\nu_1 \geq \cdots \geq \nu_n$.

Let $\mathfrak{b}$ be the Borel subalgebra corresponding to Δ^+. Write $W \simeq S_n$ for the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$, and let w_o denote the long element in W. For a dominant $\nu \in \mathfrak{t}^*$ and $w \in W$ define Verma modules by

$$M_{\mathfrak{b}}(w\nu) = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{ww_o\nu-\rho}),$$

and denote their unique irreducible quotients by $L_{\mathfrak{b}}(w\nu)$. The definition is arranged so that $L_{\mathfrak{b}}(\nu) = M_{\mathfrak{b}}(\nu)$ and $L_{\mathfrak{b}}(w_o\nu)$ is finite-dimensional (if ν is integral and regular).

We will need a very explicit description of (representatives of K-conjugacy classes of) θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$; this is standard and well-known (see, for example, [V8, Example 4.5] for omitted details). Let $\{(p_1,q_1), \dots, (p_r,q_r)\}$ be an ordered sequence of pair of positive integers (not both zero). Set $p = \sum_i p_i$, $q = \sum_i q_i$, and $n_i = p_i + q_i$. Define $U(p,q)$ with respect to the Hermitian form defined by a diagonal matrix consisting of p_1 pluses, then q_1 minuses, then p_2 pluses, and so on. Let $\mathfrak{l}$ denote the block diagonal subalgebra

$$\mathfrak{gl}(n_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_r,\mathbb{C}),$$

let $\mathfrak{u}$ denote the strict block upper-triangular subalgebra, and write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Then $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. As the ordered sequences of pairs range over all

$$\{(p_1,q_1), \dots (p_r,q_r)\}, \qquad \sum p_i = p, \; \sum q_i = q,$$

the $\mathfrak{q}$ constructed in this way exhaust the K conjugacy classes of θ-stable parabolic subalgebras for $\mathfrak{g}$.

1.1. Notation for θ-stable parabolics. Whenever we speak of (the K-conjugacy class of) a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ attached to a sequence $\{(p_1,q_1), \dots, (p_r,q_r)\}$, we shall always mean the one described above. In the coordinates given above, any unitary one-dimensional $(\mathfrak{l}, L \cap K)$-module, restricted to T, has differential

$$\lambda = (\overbrace{\lambda_1, \dots, \lambda_1}^{n_1=p_1+q_1}, \dots\dots, \overbrace{\lambda_r, \dots, \lambda_r}^{n_r=p_r+q_r}) \in \mathfrak{t}^*_{\mathbb{R}},$$

with each $\lambda_i \in \mathbb{Z}$.

1.2. **Translation functors, τ-invariants, and primitive ideals.** Let F be a finite-dimensional irreducible representation of $\mathfrak{g}$ with extremal weight μ. For any weight $\gamma \in \mathfrak{t}^*$, let P_γ denote the projection, defined on the category of $Z(\mathfrak{g})$-finite $U(\mathfrak{g})$ modules, onto generalized infinitesimal character γ. Define the translation functor

$$\psi_\nu^{\nu+\mu}(X) = P_{\nu+\mu} \circ (F \otimes \cdot\,) \circ P_\nu(X).$$

Certain translation functors will arise frequently for us, and we give them special names. Suppose $\nu \in \mathfrak{t}_{\mathbb{R}}^*$ is dominant, integral, and regular, and for a simple root α, let μ_α be an extremal weight of a finite-dimensional representation so that $\nu_\alpha = \nu + \mu_\alpha$ is still dominant but lies exactly on the α wall: $\langle \nu_\alpha, \beta \rangle = 0$ if and only if $\beta = \pm\alpha$. We denote the corresponding translation functor $\psi_\nu^{\nu_\alpha}$ by ψ_α. Given an irreducible $U(\mathfrak{g})$ module X with infinitesimal character ν we define its τ-invariant

$$\tau(X) = \{\alpha \in \Sigma \mid \psi_\alpha(X) = 0\}.$$

(Neither μ_α nor ψ_α is well-defined, but the τ-invariant definition is.) Next let

$$\mu_i = e_{i+1} + \cdots + e_n,$$

and consider the finite-dimensional representation of $\mathfrak{g}$ with highest weight μ_i. We will write (somewhat sloppily) T_i for $\psi_\nu^{\nu+\mu_i}$, suppressing the dependence on ν (which we no longer assume is dominant). We let T_i^k denote the k-fold composition of T_i.

A two-sided ideal in the enveloping algebra $U(\mathfrak{g})$ is called a primitive ideal if it is the annihilator of a simple $U(\mathfrak{g})$ module. A primitive ideal is said to have infinitesimal character ν if it contains the maximal ideal in $Z(\mathfrak{g})$ corresponding to ν. Denote the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character ν by $\mathrm{Prim}(U(\mathfrak{g}))_\nu$. If ν is dominant, regular, and integral, we define the τ-invariant of $I = \mathrm{Ann}(X) \in \mathrm{Prim}(U(\mathfrak{g}))_\nu$ to be the subset of simple roots consisting of those α for which $\psi_\alpha(X)$ is zero. (The fact that this is well-defined on $\mathrm{Prim}(U(\mathfrak{g}))_\nu$ is not immediate, but it is easy.)

1.3. **Tableau notation.** Given a partition $n = n_1 + \cdots + n_k$ with the n_i decreasing, we may attach a left justified arrangement of n boxes with n_i boxes in the ith row. Call such an arrangement a Young diagram of size n. If $\nu = (\nu_1, \ldots, \nu_n)$ is an n-tuple of real numbers, a ν-quasitableau is defined to be *any* arrangement of $\nu_1, \ldots, \nu_n$ in a Young diagram of size n. The underlying diagram of a quasitableau is called its shape. If a ν-quasitableau satisfies the condition that the entries weakly increase across rows and strictly increase down columns, it is said to be a ν-tableau. A ν-tableau whose entries strictly increase across rows is called standard. If $\nu = (1, 2, \ldots, n)$, then a standard ν-tableau is called a Young tableau. Replacing 'increasing' by 'decreasing' in the definition of a ν-tableau defines a ν-antitableau.

Although admittedly it seems a little silly, we will need to switch between two sets of data: the data of a Young tableau together with a decreasing n-tuple $\nu = (\nu_1 \geq \cdots \geq \nu_n)$; and the data of a ν-antitableau. Given a decreasing n-tuple ν and a Young tableau S_{Y}, we get a ν-antitableau by changing the ith entry of S_{Y} to ν_i. For the converse construction, we need to adopt the convention that given two occurrences of an identical entry in a ν-antitableau, one is said to be larger than the other if it occurs strictly to the left of the other. Then given a ν-antitableau S_{A} we first order $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, and construct a Young tableau S_{Y} from S_{A} as follows. Locate the the largest occurrence (in the sense of the convention just mentioned) of ν_1 in S_{A} and relabel it '1'. If $\nu_2 = \nu_1$, then locate the next largest occurrence of ν_1 in S_{A} and relabel it '2'; if $\nu_2 < \nu_1$, locate the largest occurrence of ν_2 and relabel it '2'.

Continuing in this way, we obtain a Young tableau S_{Y}. For example,

$$S_{\mathrm{A}} = \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}\ , \qquad S_{\mathrm{Y}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\ .$$

We call S_{Y} the underlying tableau of S_{A}.

A skew diagram is any diagram obtained by removing a smaller Young diagram from a larger one that contains it. A skew column is a skew tableau whose shape consists of at most one box per row and whose entries strictly increase when moving down in the diagram. A skew column is called difference-one if its consecutive entries (when moving down the column) *decrease* by exactly one when moving down the column.

A signed Young tableau of signature (p, q) is an equivalence class of Young diagrams whose boxes are filled with p pluses and q minuses so that the signs alternate across rows; two signed Young diagrams are equivalent if they can be made to coincide by interchanging rows of equal length. (Note that the equivalence relation preserves shapes.) A skew column of a signed tableau is any arrangement of pluses and minuses in a skew diagram consisting of at most one box per row.

2. The modules $A_{\mathfrak{q}}(\lambda)$

In this section, we recall the definition and properties of the modules $A_{\mathfrak{q}}(\lambda)$. Most of the material in this section is standard, but part of Theorem 2.1b(iv) is new (see Remark 2.3).

We adopt the notation of [KV] for our cohomological induction functors, and return for the moment to the setting of an arbitrary reductive $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the complexification of a maximally compact θ-stable Cartan subalgebra $\mathfrak{h}_o$. Choose a θ-stable system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ containing $\mathfrak{h}$ with $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$. A one-dimensional unitary $(\mathfrak{l}, L \cap K)$-module $\mathbb{C}_\lambda$ is determined by $\lambda \in \mathfrak{h}^*$, its differential restricted to $\mathfrak{h}$. Define $\mathbb{C}_\lambda^\# = \mathbb{C}_\lambda \otimes_{\mathbb{C}} \bigwedge^{top} \mathfrak{u}$ viewed as a $(\bar{\mathfrak{q}}, L \cap K)$ module and form

$$\mathcal{L}_j(\mathbb{C}_\lambda) = (\Pi_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})_j (\mathrm{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(\mathbb{C}_\lambda^\#));$$

here Π_j is the derived Bernstein functor. For $S = \dim(\mathfrak{u} \cap \mathfrak{k})$, write $A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$.

Here are the main properties of these modules:

Theorem 2.1. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic and let $\mathbb{C}_\lambda$ be a one-dimensional unitary $(\mathfrak{l}, L \cap K)$ module; set $S = dim(\mathfrak{u} \cap \mathfrak{k})$.*

(a) *$\mathcal{L}_j(\mathbb{C}_\lambda)$ has infinitesimal character $\lambda + \rho$.*

(b) *Suppose $\mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\mathbb{C}_{\lambda + t\rho(\mathfrak{u})}^\#)$ is irreducible for all $t \geq 0$; then:*

- (i) *$\mathcal{L}_j(\mathbb{C}_\lambda) = 0$ for $j \neq S$.*
- (ii) *$A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$ is a unitarizable $(\mathfrak{g}, K)$ module.*
- (iii) *If the infinitesimal character $\lambda + \rho$ is regular, then $A_{\mathfrak{q}}(\lambda)$ is nonzero and irreducible.*
- (iv) *Suppose further that $G = U(p, q)$; then $A_{\mathfrak{q}}(\lambda)$ is either irreducible unitary or zero.*

Remark 2.2. Part (a) says that (for G linear and $\mathrm{rank}(G) = \mathrm{rank}(K)$) the infinitesimal character of an $A_{\mathfrak{q}}(\lambda)$ module is always a translate of ρ by a weight of a finite dimensional representation of $\mathfrak{g}$. This explains the infinitesimal character condition in Conjecture 0.1. Note that the zero infinitesimal character $\gamma = (0, \ldots, 0)$ is of this form if and only if n is odd. We will return to this in Example 5.9.

Remark 2.3. Assertion (iv) is a special feature of the $U(p,q)$ setting; in general such an $A_{\mathfrak{q}}(\lambda)$ need not be irreducible or zero. In the case of general G, Chapter 8 of [KV] provides sufficient conditions from which to conclude (iv). More precisely, under

Hypothesis 1:

$$A_{\mathfrak{q}}(\lambda') \text{ is irreducible; and}$$

$$\psi_{\lambda'+\rho}^{\lambda+\rho}(\mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\mathbb{C}_{\lambda'}^{\#})) = \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\mathbb{C}_{\lambda}^{\#}) \text{; and}$$

Hypothesis 2:

$$\text{The Kostant problem for } \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\mathbb{C}_{\lambda}^{\#}) \text{ has a positive solution;}$$

one can conclude $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero. The second hypothesis is subtle in general, but it certainly holds if the closure of the (complex) Richardson orbit $\mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{zero})$ is normal and equivariantly simply connected. Of course this is always the case for $\mathfrak{gl}(n,\mathbb{C})$, and hence we obtain (iv) when $G = U(p,q)$ and λ in the weakly fair range. For general G, the relevant orbit closures can fail to be normal and simply connected and we see, at least morally, why the irreducibility result can fail in general.

Under the assumptions of Theorem 2.1b(iv), Hypothesis 1 can be verified by taking $\lambda'+\rho$ is dominant and regular (so that Theorem 2.1(b)(iii) implies $A_{\mathfrak{q}}(\lambda')$ is irreducible) and applying Lemma 2.13. When λ is in the weakly fair range of Definition 2.4, Lemma 2.13 holds for general G, as a consequence of [KV, Lemma 8.39]. Outside the weakly fair range, the lemma (and hence Theorem 2.1b(iv)) are apparently new, although the proof is surprisingly simple.

Perhaps more importantly for us is that the work of Sections 5-7 allows us to deduce Theorem 2.1b(iv) from Theorem 2.1b(iii), without referencing Chapter 8 in [KV]. (To be fair, our proof is extremely complicated and far from conceptual.) However, it is reasonable to expect that the arguments of Sections 5-7 can be used to detect reducibility of singular $A_{\mathfrak{q}}(\lambda)$ modules for classical groups other than $U(p,q)$.

We set aside the definition of certain ranges of positivity for λ and $\mathfrak{q}$.

Definition 2.4. A one-dimensional unitary $(\mathfrak{l}, L\cap K)$-module λ said to be in the mediocre range for $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ if $\mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\mathbb{C}_{\lambda+t\rho(\mathfrak{u})}^{\#})$ is irreducible for all $t \geq 0$. We say that λ is in the (weakly) good range if $\lambda+\rho$ is (weakly) dominant; and say that λ is in the (weakly) fair range if $\lambda+\rho(\mathfrak{u})$ is (weakly) dominant.

A module $A_{\mathfrak{q}}(\lambda)$ is said to be good, or in the good range, if λ is in the good range for $\mathfrak{q}$. Similar terminology applies for weakly good, fair, weakly fair, and mediocre.

The fair range is easily seen to contain the good range, and it's not too hard to see that the mediocre range contains the fair range (see [KV, Theorem 5.105]). The next lemma makes these ranges explicit in the $U(p,q)$ setting (from which the containments become obvious).

Lemma 2.5. *Recall Notation 1.1, and let* $\mathfrak{q}$ *correspond to* $\{(p_1,q_1),\ldots,(p_r,q_r)\}$, *and set* $n_i = p_i+q_i$. *Fix a one-dimensional unitary* $(\mathfrak{l}, L\cap K)$*-module*

$$\lambda = (\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1=p_1+q_1},\ldots\ldots,\overbrace{\lambda_r,\ldots,\lambda_r}^{n_r=p_r+q_r}).$$

(a) λ *is in the good range for* $\mathfrak{q}$ *if and only if*

$$\lambda_i - \lambda_{i+1} > -1, \qquad \textit{for all } i.$$

(b) λ *is in the fair range range for* $\mathfrak{q}$ *if and only if*

$$\lambda_i - \lambda_{i+1} > -\frac{n_i + n_{i+1}}{2}, \qquad \textit{for all } i.$$

(c) λ *is in the mediocre range for* $\mathfrak{q}$ *if and only if*

$$\lambda_i - \lambda_j \geq -\max(n_i, n_j) - \sum_{i<k<j} n_k, \qquad \textit{for all } i < j.$$

Remark 2.6. The weakly good and fair ranges are characterized by relaxing the strict inequalities in (a) and (b) to weak ones. Parts (a) and (b) follow directly from the definitions. Part (c) is deeper; it is proved in Satz 4 and Corollar 4 of [Ja]. Finally note that the condition in (c) isn't transitive, so we need to consider all pairs $i < j$.

Remark 2.7. Since the fair range is properly contained in the mediocre range, Theorem 2.1 suggests that we are perhaps excluding some unitary representations by restricting our attention to the weakly fair range. Conjecture 0.1 says that this should not be the case and, in fact, in Theorem 8.1, we prove that any mediocre $A_{\mathfrak{q}}(\lambda)$ is isomorphic to a weakly fair one; so we obtain no new unitary representations inside the mediocre range (but outside the fair range). On the other hand, even if λ is not mediocre for $\mathfrak{q}$ (i.e. even if the generalized Verma module is reducible) $A_{\mathfrak{q}}(\lambda)$ may still be a nonzero irreducible unitary representation. Vogan's conjecture still says that such $A_{\mathfrak{q}}(\lambda)$ should coincide with modules in the weakly fair range, but this appears to be a more subtle issue.

Before turning to more detailed matters about the $A_{\mathfrak{q}}(\lambda)$ modules below, we discuss which ones have their infinitesimal character in a central translate of $\overline{W\rho}$, the convex hull of the Weyl group orbit of ρ. (The irreducible unitary ones with this property are conjecturally unitarily small in the sense of [SaV].) For general λ, the condition is complicated, but in the weakly fair range, the complications magically disappear.

Proposition 2.8. *Retain the notations of Lemma 2.5.*

(a) *Choose* $\sigma \in S_r$ *so that* $\lambda_{\sigma(1)} \geq \cdots \geq \lambda_{\sigma(r)}$. *Then* $\lambda \in \overline{W\rho}$ *(modulo the center of* $\mathfrak{g}$*) if and only if*

$$\lambda_{\sigma(i)} - \lambda_{\sigma(i+1)} \leq \frac{(n_{\sigma(i)} + n_{\sigma(i+1)})}{2}, \qquad \textit{for all } i.$$

(b) *If* λ *is in the weakly fair range for* $\mathfrak{q}$, $\lambda+\rho \in \overline{W\rho}$ *(modulo the center of* $\mathfrak{g}$*) if and only if*

$$\langle \lambda, \alpha \rangle \leq 0, \ \textit{for all } \alpha \in \Delta^+;$$

or, explicitly, if and only if $\lambda_i - \lambda_{i+1} \leq 0$ *for all* i.

Said differently, given the Salamanca-Vogan conjecture and Conjecture 0.1, the $A_{\mathfrak{q}}(\lambda)$ modules with

$$-\langle \rho(\mathfrak{u}), \alpha \rangle \leq \langle \lambda, \alpha \rangle \leq 0$$

conjecturally exhaust the unitarily small representations of $U(p,q)$ whose infinitesimal character is a weight translate of ρ.

Sketch. For part (a), we can clearly assume that λ is dominant and σ is the identity. We can also assume (by modifying λ by a central element) that the sum of the entries of λ is zero; i.e. that λ lives in the dual of the semisimple piece of the diagonal Cartan subalgebra. (We no longer are assuming that the entries of λ are integers, of course.) In order for this kind of λ

to live in $\overline{W\rho}$, it must be inside each codimension-one face of $\overline{W\rho}$ which contains the point ρ. We can characterize such faces as follows. Given a simple reflection s_i, let $S(i) \simeq S_{i-1} \times S_{n-i}$ be the subgroup of S_n generated by the simple reflections other than s_i. (That is, $S(i)$ is the Weyl group of the levi factor of a maximal parabolic subgroup.) Then the codimension-one faces containing ρ are precisely the convex hulls $\overline{S(i)\rho}$. The condition that λ lie inside the ith such face is exactly the ith condition given in part (a), thus completing the sketch of the first part. (The reader is encouraged to draw the rank two picture.)

For part (b), suppose we admit the the following fact: if λ is a one-dimensional $(\mathfrak{l}, L \cap K)$ module, then $\lambda \in \overline{W\rho}$ (modulo center) if and only if $\lambda + \rho(\mathfrak{l}) \in \overline{W\rho}$ (modulo center). Given this, we can conclude that $\lambda + \rho$ is in $\overline{W\rho}$ (modulo center) if and only if $\lambda + \rho(\mathfrak{u})$ is. Then part (b) follows by noting that if λ is in the weakly fair range for $\mathfrak{q}$, $\lambda + \rho(\mathfrak{u})$ is dominant (so that the condition in part (a) can be applied to $\lambda + \rho(\mathfrak{u})$ taking σ to be the identity).

Thus it remains to prove the assertion at the beginning of the previous paragraph. For simplicity, we throw out the center and work only with the semisimple part $\mathfrak{g}_{ss} \simeq \mathfrak{sl}(p+q, \mathbb{C})$. Set $P = (\mathfrak{l} \cap \mathfrak{h}_{ss})^* \cap \overline{W\rho}$; we are to prove

$$(*) \qquad \lambda \in P \text{ if and only if } \lambda + \rho(\mathfrak{l}) \in \overline{W\rho}.$$

(The condition that λ be an $(\mathfrak{l}, L \cap K)$ module in the original claim was in fact superfluous.) The first observation is that P is the convex hull of the 'extremel' $\lambda \in P$,

$$P = \text{convex hull of } \{\mu_i := \frac{1}{|W_{\mathfrak{l}}|} \sum_{w \in W_{\mathfrak{l}}} w w_i \rho \mid w_i \in W\}.$$

So it suffices to show that each $\mu + \rho(\mathfrak{l}) \in \overline{W\rho}$. We argue below that we can take $\mu + \rho(\mathfrak{l})$ to be dominant; assume this for a moment. If this is the case, we can apply [SaV, Proposition 2.12(b),(c)] to see that $\mu + \rho(\mathfrak{l}) \in \overline{W\rho}$ if and only if

$$\langle \mu + \rho(\mathfrak{l}), \alpha \rangle \le \langle \rho, \alpha \rangle,$$

for all positive roots α. Now if α is a simple root of $\mathfrak{l} \cap \mathfrak{t}$ in $\mathfrak{l}$, then one can see easily that equality holds. On the other hand, if $\alpha \in \Delta(\mathfrak{u})$, the condition follows from the weakly fair hypothesis, and the lemma follows.

It remains to reduce to the case of $\mu + \rho(\mathfrak{l})$ dominant. Clearly it is enough to verify the following assertion: if $\mu + \rho(\mathfrak{l})$ is not dominant, then there exists $w \in W$ so that $w(\mu + \rho(\mathfrak{l}))$ is dominant and

$$w(\mu + \rho(\mathfrak{l})) = \mu' + \rho(\mathfrak{l}')$$

for some (possibly different) levi factor $\mathfrak{l}'$ and some μ' of the form

$$\mu' := \frac{1}{|W_{\mathfrak{l}'}|} \sum_{w \in W_{\mathfrak{l}}} w w' \rho, \qquad \text{for some } w' \in W.$$

This is not obvious (but not difficult). We leave the details to the reader. □

Returning to more immediate questions, the good $A_{\mathfrak{q}}(\lambda)$ with infinitesimal character ρ are (almost) parametrized by the set of θ-stable parabolics, but there is some repetition. In our $U(p,q)$ setting, for instance, a simple induction in stages argument shows that coincidences arise from adjacent compact factors (of the same signature) in $\mathfrak{l}_o$.

Lemma 2.9. *Suppose that* $\mathfrak{q}'$ *corresponds to* $\{(p'_1, q'_1), \ldots, (p'_{r+1}, q'_{r+1})\}$ *(as in Notation 1.1) and that for some* $i \leq r$, $q'_i = q'_{i+1} = 0$. *Let* $\mathfrak{q}$ *correspond to the sequence* $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ *obtained by combining the ith and* $(i+1)$*st entries:*

$$(p_j, q_j) = \begin{cases} (p'_j, q'_j) & \text{if } j < i, \\ (p'_i + p'_{i+1}, 0) & \text{if } j = i, \\ (p'_{j-1}, q'_{j-1}) & \text{if } j > i. \end{cases}$$

Then $A_{\mathfrak{q}}(\mathbb{C}_{triv}) \simeq A_{\mathfrak{q}'}(\mathbb{C}_{triv})$. *The analogous statement holds if* $p'_i = p'_{i+1} = 0$.

These are the only coincidences that can arise, however.

Proposition 2.10. *The good* $A_{\mathfrak{q}}(\lambda)$ *for* $U(p,q)$ *with infinitesimal character* ρ *are parametrized by ordered sequences of pairs of integers*

$$\{(p_1, q_1), \ldots, (p_r, q_r)\} \text{ with } \sum p_i = p, \sum q_i = q,$$

so that no adjacent pairs are of the form $(p_i, 0), (p_{i+1}, 0)$ *or* $(0, q_i), (0, q_{i+1})$. *The correspondence takes a sequence to* $A_{\mathfrak{q}}(\mathbb{C}_{triv})$ *where* $\mathfrak{q}$ *is defined as in Notation 1.1.*

Example 2.11. In the case of $U(p,1)$, the parameters appearing in the proposition are all of the form

$$\{(i,0), (p-i-j, 1), (j, 0)\}, \quad 0 \leq i, j \leq p\ , i+j \leq p;$$

here if the pair $(0,0)$ appears we ignore it. For future reference, we denote the above set of pairs by $[i,j]$ and the corresponding $A_{\mathfrak{q}}(\mathbb{C}_{triv})$ as $X[i,j]$.

We now record a few results, specific to the $U(p,q)$ setting, describing the effect of translation functors on the $A_{\mathfrak{q}}(\lambda)$.

Lemma 2.12. *Let* $\mathfrak{q}$ *correspond to an ordered sequence* $\{(p_1, q_1), \ldots, (p_r, q_r)\}$, *let* λ *be in the good range for* $\mathfrak{q}$, *and set* $n_i = p_i + q_i$. *Consider a simple root* $\alpha = e_{k+1} - e_k$ *and write (uniquely)*

$$k = \sum_{i \leq j} n_j + l, \text{ with } 0 \leq l < n_{j+1}.$$

Then $\alpha \in \tau(A_{\mathfrak{q}}(\lambda))$ *(Notation 1.2) if and only if one of the following conditions holds*

(a) $l \geq 1$; *or*

(b) *If* $l = 0$, *the consecutive entries* $(p_j, q_j), (p_{j+1}, q_{j+1})$ *are of the form* $(p_j, 0), (p_{j+1}, 0)$ *or of the form* $(0, q_j), (0, q_{j+1})$.

In particular, if the sequence $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ *is of the kind described in Proposition 2.10, case (b) can never occur, and* $\tau(A_{\mathfrak{q}}(\lambda))$ *consists of the simple roots of* $\mathfrak{t}$ *in* $\mathfrak{l}$.

Pf. One can certainly prove the lemma by understanding the Langlands parameters of the good $A_{\mathfrak{q}}(\lambda)$ (as is done in [VZ]) and then applying Vogan's τ invariant calculation of [V3]. This requires some fairly serious bookkeeping, so we sketch an alternative proof.

The ideas given below together with induction in stages reduce the Lemma to the case when $\mathfrak{q}$ is maximal and of the form of Proposition 2.10. So let $\mathfrak{q}$ be associated to the sequence $\{(p_1, q_1), (p_2, q_2)\}$ with neither both p's nor both q's zero. The roots of the form specified by condition (a) in the lemma are exactly the $\alpha_k = e_k - e_{k+1}$ with $k \neq p_1 + q_1$. Now the translation functor ψ_α commutes with the derived Bernstein functor, so the composition factors of $\psi_\alpha(A_{\mathfrak{q}}(\lambda))$ are of the form $\Pi_S(Z)$ where Z is a composition factor of $\psi_\alpha(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathbb{C}_\lambda))$.

By a Mackey isomorphism, this latter module admits a generalized Verma flag which can be explicitly examined to conclude that if condition (a) is satisfied, $\psi_\alpha(A_{\mathfrak{q}}(\lambda))$ is zero.

To finish the sketch we need to show that for $k = p_1+q_1$, $\alpha_k \notin \tau(A_{\mathfrak{q}}(\lambda))$. If this were the case, the τ invariant would consist of all simple roots. Lemma 3.2 then implies that the shape of the annihilator of $A_{\mathfrak{q}}(\lambda)$ is a single column, which (by the 'same-shape' result of [BV1]) in turn implies that the asymptotic support of $A_{\mathfrak{q}}(\lambda)$ is zero. Since we have assumed that $\mathfrak{q}$ is associated to $\{(p_1, q_1), (p_2, q_2)\}$, Proposition 4.4 and Lemma 4.6 imply that either $p_1 = p_2 = 0$ or $q_1 = q_2 = 0$, which contradicts our original assumption on the p's and q's. $\square$

The next lemma will be the basis of moving from good to worse ranges. The first assertion is Lemma 7.1 below; the second follows from the first using the ideas of the preceding proof.

Lemma 2.13. *Let λ' be in the good range for $\mathfrak{q}$, let $A \subset \Sigma - \tau(A_{\mathfrak{q}}(\lambda'))$, and recall Notation 1.2. Suppose*

$$\lambda = \lambda' + \sum_{i \in A} k_i \mu_i \qquad \textit{(with each } k_i \geq 0)$$

is in the mediocre range for $\mathfrak{q}$, and set $\nu = \lambda' + \rho$. Let T be any translation functor of the form

$$T = \prod_{i \in A} T_i^{k_i},$$

with the factors of the product being taken in any order. Then

$$T(\mathrm{ind}_{\bar{\mathfrak{q}}}(\mathbb{C}^{\#}_{\lambda'})) = \mathrm{ind}_{\bar{\mathfrak{q}}}(\mathbb{C}^{\#}_{\lambda}),$$

and $T(A_{\mathfrak{q}}(\lambda')) = A_{\mathfrak{q}}(\lambda)$.

3. Primitive ideals in $\mathfrak{gl}(n, \mathbb{C})$

We begin with a convenient choice of Joseph's parametrization of primitive ideals in $\mathfrak{gl}(n, \mathbb{C})$.

Theorem 3.1. *For a dominant integral $\nu = (\nu_1 \geq \cdots \geq \nu_n) \in \mathfrak{t}^*$, the set $Prim(U(\mathfrak{g}))_\nu$ is in bijection with the set of ν-antitableau (Notations 1.2, 1.3).*

We now describe how we want the parametrization of the theorem to work. Duflo's theorem asserts that the map from (the involutions of) W to $\mathrm{Prim}(U(\mathfrak{g}))_\nu$ sending w to $\mathrm{Ann}_{U(\mathfrak{g})}(L(w\nu))$ is surjective. Assume now that ν is non-singular, dominant, and integral; we treat the singular case in a moment. Joseph proved that

$$\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{b}}(w\nu)) = \mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{b}}(w'\nu))$$

if and only if $RS(w) = RS(w')$; here $RS(w)$ denotes the the 'counting' tableau of the Robinson-Schensted algorithm (see [Sag]). We obtain a ν-antitableau by changing the ith entry to ν_i, thus describing the parametrization of the theorem in the regular case. (At first glance the 'antitableau' parametrization appears like a ridiculous complication. It does, however, have the significant advantage of making the statements of our main theorems much cleaner.)

The singular case follows from the translation principle as discussed after Theorem 3.4 below. In order to state that theorem, we need to first consider τ-invariants on the level of tableaux. Janzten first showed that, for ν regular and integral, $\tau(L_{\mathfrak{b}}(w\nu))$ coincides with the combinatorial definition of $\tau(w)$ coming from the Bruhat order. Combined with an easy observation about the Robinson-Schensted algorithm, this implies that one can read off the τ invariant of a primitive ideal (with regular integral infinitesimal character) from its tableau:

Lemma 3.2. *Let ν be dominant, integral, and regular, and fix $I \in Prim(U(\mathfrak{g}))_\nu$. Then $\alpha = e_i - e_{i+1}$ is in the τ-invariant of I if and only if ν_{i+1} is strictly below ν_i in the tableau corresponding to I (by the procedure of the previous paragraph).*

The tableau condition comes up sufficiently often that we set it aside in a definition.

Definition 3.3. Let ν be dominant, regular, and integral. The simple root $\alpha = e_i - e_{i+1}$ is said to be in the τ-invariant of a ν-standard tableau S if and only if ν_{i+1} is strictly below ν_i in S; or, equivalently, if and only if $i+1$ is strictly below i in the underlying tableau of S.

Now we isolate the relevant version of the translation principle. (See [KV, Chapter 7] and the references given there.)

Theorem 3.4. *Let ν be regular, integral, and dominant, and recall Notation 1.2. Suppose there is a finite-dimensional representation with extremal weight μ so that $\nu' = \nu + \mu$ is again dominant (but potentially singular). Let $A = \{\alpha \in \Sigma \mid \langle \nu', \alpha \rangle = 0\}$.*

(a) *The translation functor $\psi_\nu^{\nu'}$ establishes a bijection between irreducible $U(\mathfrak{g})$ modules with infinitesimal character ν whose τ invariants are contained in the complement of A and irreducible $U(\mathfrak{g})$ modules with infinitesimal character ν'.*

(b) *$\psi_\nu^{\nu'}$ is well-defined on the level of primitive ideals and defines a bijection between primitive ideals with infinitesimal character ν whose τ-invariants are contained in the complement of A and primitive ideals with infinitesimal character ν'.*

To complete the description of the parameterization of Theorem 3.1, consider a primitive ideal I' of dominant (but potentially singular) infinitesimal character ν'. Let ν be as in Theorem 3.4; then there is a unique primitive ideal I of infinitesimal character ν with $\psi_\nu^{\nu'}(I) = I'$. We have already described a ν-standard tableau parametrizing I. Under Theorem 3.1, I' is parameterized by the unique ν'-standard tableau whose underlying tableau coincides with that of I. Notice that in terms of the parameterization of Theorem 3.1, the tableau of $I' = \psi_\nu^{\nu'}(I)$ is obtained by changing the coordinates of the tableau corresponding to I from ν to ν'. We shall see in Lemma 7.13 that much more complicated translation functors can be described in this way.

To be absolutely explicit, we summarize how to go from a ν-antitableau S_{A} to the highest weight module whose annihilator it parametrizes. Take $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, and construct the underlying Young tableau S_{Y} using the procedure described in Notation 1.3. Consider the set of elements of S_n whose Robinson-Schensted counting tableau is S_{Y}. For a given one of these elements, say w, S_{A} parametrizes the annihilator of the highest weight module $L_{\mathfrak{b}}(w\nu)$.

In Section 5, we will need some weak information about how annihilators behave under cohomological induction. The next lemma is that kind of statement.

Lemma 3.5. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a maximal θ-stable parabolic for $U(p,q)$ with $\mathfrak{l}_o = \mathfrak{u}(p_1, q_1) \oplus \mathfrak{u}(p_2, q_2)$; set $n_i = p_i + q_i$. Let $X' \otimes X''$ be an irreducible $(\mathfrak{l}, L \cap K)$-module with infinitesimal character ν satisfying*

$$\langle \nu + \rho(\mathfrak{u}), \alpha \rangle > 0 \qquad \textit{for all } \alpha \in \Delta(\mathfrak{u}).$$

Set

$$X = \mathcal{L}_S(X' \otimes X'').$$

Then X is irreducible and the first n_1 boxes of the underlying tableau of $\mathrm{Ann}(X)$ *coincide with the underlying tableau of* $\mathrm{Ann}(X')$ *(Notation 1.3, Theorem 3.1).*

Sketch. By [KV, Theorem 8.2], X is irreducible. The results of [V1] imply that the first n_1 boxes of the underlying tableau of $\mathrm{Ann}(X)$ are characterized by applying sequences of wall-crossing translation functors to X. The walls in question correspond to the first $n_1 - 1$ simple roots of $\mathfrak{g}$. Using the ideas of the proof of Lemma 2.12, it follows that the wall crossing information is identical for X and X'. The lemma follows.

We isolate the precise statement that we will need in a corollary, which follows by induction using an easy induction in stages argument taking $X' = A_{\mathfrak{q}'}(\lambda')$ and X'' an appropriate one-dimensional representation.

Corollary 3.6. *Let $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ be the θ-stable parabolic corresponding to the ordered set $\{(p_1, q_1), \ldots, (p_r, q_r)\}$. Let $r \geq s$ and let $\mathfrak{q}' \subset \mathfrak{gl}(n', \mathbb{C})$ be the θ-stable parabolic corresponding to $\{(p_1, q_1), \ldots, (p_s, q_s)\}$. Suppose $\mathbb{C}_\lambda$ is a one-dimensional $(\mathfrak{l}, L \cap K)$-module in the good range for $\mathfrak{q}$, and let $\mathbb{C}_{\lambda'}$ denote the $(\mathfrak{l}', L' \cap K')$ module obtained by restriction. Then the first n' boxes of the underlying tableau of* $\mathrm{Ann}(A_{\mathfrak{q}}(\lambda))$ *coincide with the underlying tableau of* $\mathrm{Ann}(A_{\mathfrak{q}'}(\lambda'))$.

4. Asymptotic supports and associated varieties for the $A_{\mathfrak{q}}(\lambda)$

For the moment we return to the general setting of an arbitrary reductive group G. Given an irreducible Harish-Chandra module X, one is led to the study of singularities of its distribution characters at the identity. The relevant notion is due to Barbasch and Vogan ([BV1]); roughly speaking, the distribution character of X has an asymptotic expansion whose leading term is a real linear combination of (Fourier transforms of canonical measures on) real nilpotent orbits, all of the same dimension. This linear combination, denoted $\mathcal{AS}(X)$, is called the asymptotic cycle of X; the closure of the union of the orbits appearing in $\mathcal{AS}(X)$ is called the asymptotic support of X and is denoted $\mathrm{AS}(X)$.

Nilpotent orbits also arise naturally through Vogan's construction of the associated variety of X ([V2], [V7]). Using a good filtration on X, one forms the associated graded object which turns out to be a finitely generated module over $S(\mathfrak{g}/\mathfrak{k})$ and therefore corresponds to an algebraic cycle in $(\mathfrak{g}/\mathfrak{k})^*$. This cycle is called the associated variety of X and is denoted $\mathcal{AV}(X)$; it is an *integral* linear combination of nilpotent $K_{\mathbb{C}}$ orbits on $\mathfrak{p}$. The union of terms appearing in $\mathcal{AV}(X)$ is denoted $\mathrm{AV}(X)$.

Although the asymptotic support is a purely analytic invariant, and the associated variety is a purely algebraic one, the next result indicates the formal possibility that the two may be related.

Proposition 4.1 (The Kostant-Sekiguchi bijection; see [CMc]). *The set of nilpotent coadjoint orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}^*$ is in bijection with the set of real nilpotent coadjoint orbits of G on $\mathfrak{g}_o^*$.*

Let Φ denote the bijection of the theorem (taking G orbits to $K_{\mathbb{C}}$ ones), and extend Φ in the obvious way to linear combinations and unions of orbits. Barbasch and Vogan conjectured the following result; a proof has been announced by Schmid and Vilonen [ScVi].

Theorem 4.2. $\Phi(\mathcal{AS}(X)) = \mathcal{AV}(X)$.

In particular, note that the coefficients appearing in $\mathcal{AS}(X)$ are all integers, and that $\Phi(\mathrm{AS}(X)) = \mathrm{AV}(X)$.

A natural question to ask is how asymptotic supports behave under cohomological induction. In particular, we can ask for the asymptotic support of an $A_{\mathfrak{q}}(\lambda)$. Since the asymptotic support of a finite-dimensional representation is zero, one expects $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$ to be (somehow) induced from the zero orbit. A precise statement appears in Proposition 4.4, but first we need to define a notion of induction for real orbits.

Definition 4.3. Let $\mathcal{O}_{\mathfrak{l}}$ be a real nilpotent coadjoint orbit for L in $\mathfrak{l}_o^*$. Suppose L is a Levi subgroup of G and $\mathfrak{q}$ is a θ-stable parabolic of $\mathfrak{g}$ with $\mathfrak{l}_o = \mathfrak{q} \cap \bar{\mathfrak{q}}$. Let $\mathcal{O}_{\mathfrak{l},\theta} = \Phi(\mathcal{O}_{\mathfrak{l}})$ be the corresponding $(K \cap L)_{\mathbb{C}}$ orbit in $\mathfrak{p}^* \cap \mathfrak{l}^*$ (Proposition 4.1). Then $K_{\mathbb{C}} \cdot (\mathcal{O}_{\mathfrak{l},\theta} + (\mathfrak{u}^* \cap \mathfrak{p}^*))$ has a unique open $K_{\mathbb{C}}$ orbit $\mathcal{O}_{\mathfrak{g},\theta}$. We define

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \Phi^{-1}(\mathcal{O}_{\mathfrak{g},\theta}).$$

(Note that the induced orbit depends on $\mathfrak{q}$ and not just $\mathfrak{l}$.)

The following proposition was conjectured in [BV4]; for $U(p,q)$, Barbasch and Vogan knew a proof based on explicit computation. The general statement given below is well-known to experts.

Proposition 4.4. *Let λ be in the good range for $\mathfrak{q}$. Then $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$ is the closure of the Richardson orbit $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$.*

Pf. Let $X = A_{\mathfrak{q}}(\lambda)$, let $Q \subset G_{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{q}$, and consider the closed $K_{\mathbb{C}}$ orbit $\mathcal{O}$ of the identity coset of the partial flag variety $G_{\mathbb{C}}/Q$. Let $\Delta(X)$ denote the (partial flag) $\mathcal{D}$-module localization of X at dominant regular infinitesimal character $\lambda + \rho$. Then [BoBr, Corollary 1.9] implies that $\mathrm{AV}(X)$ is the image under the moment map of the support of the characteristic cycle of $\Delta(X)$. Since $\mathcal{O}$ is closed, it is not difficult to see that the support of the characteristic cycle is the conormal bundle of $\mathcal{O}$ (see [Ch], for instance). The image of the conormal bundle is the $K_{\mathbb{C}}$ saturation of $\mathfrak{u}^* \cap \mathfrak{p}^*$ which is just the closure of the orbit $\Phi^{-1}(\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero}))$ appearing in Definition 4.3. Since $\mathrm{AV}(X) = \Phi(\mathrm{AS}(X))$ (Theorem 4.2), the proposition is proved.

Now we return to the $\mathfrak{u}(p,q)$ setting to record some explicit results.

Lemma 4.5 (see [CMc], 9.3.3). *Nilpotent orbits in $\mathfrak{u}(p,q)$ are parametrized by signed Young diagrams of signature (p,q) (Notation 1.3).*

We conclude this section with a lemma that gives the results of certain orbit inductions on the level of tableaux. In its statement, an empty row is to be interpreted as ending with both plus and minus signs.

Lemma 4.6. *Let $\mathcal{O}_1$ be a nilpotent orbit in $\mathfrak{u}(p_1,q_1)$ corresponding to the signed tableau T_1. Let $\mathcal{O}_{zero}$ be the zero orbit in $\mathfrak{u}(p_2,q_2)$. Let $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{gl}(p+q,\mathbb{C})$ be associated to the sequence $\{(p_1,q_1),(p_2,q_2)\}$. Then the signed tableaux of signature (p,q) corresponding to the induced orbit*

$$\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_1 \times \mathcal{O}_{zero})$$

is obtained by adding r pluses and s minuses, from top to bottom, to the row-ends of T_1 so that

(a) *at most one sign is added to each row-end; and*

(b) *the signs of the resulting diagram must alternate across rows.*

(The resulting diagram may not necessarily have rows of decreasing length, but one can choose a tableau equivalent to T_1 so that the result does have rows of decreasing length.)

Sketch. One may prove Lemma 4.6 as follows. First note that $\mathcal{O}_1$ is itself Richardson so we can write $\mathcal{O}_1 = \mathrm{ind}_{\mathfrak{q}_1'}^{\mathfrak{g}_1}(\mathcal{O}_{zero})$; then an appropriate induction in stages argument shows that the lemma computes $\mathrm{ind}_{\mathfrak{q}'}^{\mathfrak{g}}(\mathcal{O}_{zero})$ for some $\mathfrak{q}'$ (possibly) different than $\mathfrak{q}$. Hence the lemma reduces to the computation of Richardson orbits. As remarked at the end of the proof of Proposition 4.4, this computation amounts to composing the Kostant-Sekiguchi bijection with the computation of the moment map image of a certain conormal bundle. A. Yamamoto [Ya] has given an algorithm to perform this latter computation in terms of signed tableau. Tracking through these steps gives the algorithm of the lemma. $\square$

Remark 4.7. At best, this sketch again requires substantial bookkeeping. In particular, one needs to understand the $K_{\mathbb{C}}$ orbits on $G_{\mathbb{C}}/B$ which parametrize the $A_{\mathfrak{q}}(\lambda)$ modules. This by itself is rather involved — it follows from Vogan and Zuckerman's description of the $A_{\mathfrak{q}}(\lambda)$ Langlands parameters [VZ] and an application of the Matsuki correspondence. Instead, we give a straightforward proof of Lemma 4.6 in the Appendix below.

Remark 4.8. Garfinkle ([G]) has given a map taking Langlands parameters of Harish-Chandra modules for $U(p,q)$ with trivial infinitesimal character to signed tableaux. A number of people have conjectured that her algorithm in fact computes associated varieties, and McGovern ([Mc2]) has independently checked that this is indeed the case. In any event, we have given enough details above to give an explicit proof: the algorithm of Lemma 4.6 coincides with Garfinkle's algorithm for $A_{\mathfrak{q}}(\lambda)$ modules, and the general case is reduced to this by the Harish-Chandra cell structure described in the beginning of Section 5.

Example 4.9. We continue the example of $U(p,1)$ initiated in 2.11. If $p \geq 2$, there are four nilpotent orbits in $\mathfrak{u}(p,1)$. They are parametrized by the signed tableaux

$$\begin{array}{|c|}\hline - \\ \hline + \\ \hline \vdots \\ \hline + \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline + & - \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline + & - & + \\ \hline + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \, .$$

and we will abbreviate these tableaux by their top row. We can apply the algorithm of Lemma 4.6 to the compute the Richardson orbits corresponding to the asymptotic support of the good $A_{\mathfrak{q}}(\lambda)$ with infinitesimal character ρ parametrized in Example 2.11. Using the notation established there we get

$$\begin{aligned} \mathrm{AS}(X[0,0]) &= \boxed{-} \; ; \\ \mathrm{AS}(X[i,0]) &= \boxed{+}\boxed{-} \; , \quad i \neq 0; \\ \mathrm{AS}(X[0,j]) &= \boxed{-}\boxed{+} \; , \quad j \neq 0; \\ \mathrm{AS}(X[i,j]) &= \boxed{+}\boxed{-}\boxed{+} \; , \quad i,j \neq 0. \end{aligned}$$

This concludes the example.

We will need the following technical lemma for applications below.

Lemma 4.10. *Let $\mathfrak{q}$ be attached to the sequence $\{(p_1,q_1),\ldots,(p_r,q_r)\}$. For $j \leq r$, let $\mathfrak{q}(j)$ be the θ-stable parabolic of $\mathfrak{g}(j) = \mathfrak{gl}(\sum_{i\leq j}(p_i+q_i),\mathbb{C})$ attached to the subsequence $\{(p_1,q_1),\ldots,(p_j,q_j)\}$. Let S be the (equivalence class of) signature (p,q) tableau corresponding to $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$. For any representative $\hat{S}$ of S, write $\hat{S} = \coprod \hat{S}_i$ for the partition of $\hat{S}$ into disjoint skew columns (Notation 1.3) obtained by requiring*

$$\mathrm{shape}(\coprod_{i\leq j} \hat{S}_i) = \mathrm{shape}(\mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero})).$$

Then for any $\hat{S}$, we have the following conclusion:

$$\coprod_{i\leq j} \hat{S}_i = \mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero});$$

or, equivalently, if a row is skipped in the arrangement of $\hat{S}_j$ in $\hat{S}^j$, then all row-ends in $\hat{S}^j$ on or below the first skipped row and above the last row of $\hat{S}_j$ must have the same parity. In particular, the number of plus (or minus) signs in each $\hat{S}_j$ is independent of the choice of $\hat{S}$.

Sketch. The first assertion is not as obvious as it may seem. The main point is that in the lemma $\hat{S}$ is fixed, yet at each stage the algorithm of Lemma 4.6 potentially requires rows to be interchanged (cf. the parenthetic comment concluding the statement of Lemma 4.6). The reason this introduces no complications is as follows. Write $\hat{T}^j$ for any representative of $\mathrm{ind}_{\mathfrak{q}(j)}^{\mathfrak{g}(j)}(\mathcal{O}_{zero})$. Suppose $\hat{T}^j$ has several rows of length m, with at least one length m row ending $+$ and at least one ending in $-$. Then the corresponding rows of $\hat{T}^{j-1}$ either *all* have length $m-1$ or *all* have length m. Given this observation, the first assertion follows. The equivalence of the two conditions in the statement of the lemma follows by examining the algorithm of Lemma 4.6. The final assertion is clear. $\square$

4.1. **Appendix: Richardson orbit calculations for $U(p,q)$.** In this section, we give a self-contained proof of Lemma 4.6. The steps below provide a model for analogous computations with other classical groups.

First we need to describe the Proposition 4.1 explicitly for $U(p,q)$. Given a nilpotent $K_{\mathbb{C}}$ orbit $\mathcal{O}_\theta$ in $\mathfrak{p}^*$, we describe the signed tableau T that corresponds to the image $\mathcal{O}$ of $\mathcal{O}_\theta$ under the Kostant-Sekiguchi bijection. Identify $\mathfrak{p}^*$ with $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^q,\mathbb{C}^p) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^p,\mathbb{C}^q)$, and consider a nilpotent element $X = (A,B)$ of $\mathfrak{p}^*$. The action of $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ is given by

$$(k_1,k_2)\cdot(A,B) = (k_1Ak_2^{-1}, k_2Bk_1^{-1}).$$

Now the Kostant-Sekiguchi bijection preserves $G_{\mathbb{C}}$ orbits, so the Jordan form of X gives the shape of the tableau T we are seeking. In fact, we have

$$\begin{aligned}
&(1) & \dim(\ker(A)) &= \text{number of} - \text{'s in the first column of } T;\\
&(2) & \dim(\ker(BA)) &= \text{number of} - \text{'s in the first 2 columns of } T;\\
&(3) & \dim(\ker(ABA)) &= \text{number of} - \text{'s in the first 3 columns of } T;\\
&(4) & &\vdots\\
&(5) & \dim(\ker(B)) &= \text{number of} +\text{'s in the first column of } T;\\
&(6) & \dim(\ker(AB)) &= \text{number of} +\text{'s in the first 2 columns of } T;\\
&(7) & &\vdots
\end{aligned}$$

This data characterizes T.

We begin the proof of Lemma 4.6 by reducing to $\mathfrak{u}(p_1, q_1)$ being quasisplit (i.e. $|p_1 - q_1| \leq 1$) and $\mathcal{O}_1$ being parametrized by a tableau consisting of a single row. (Such orbits are called principal, since they are real forms of the principal complex orbit.) Suppose $X_1 = (A_1, B_1) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{q_1}, \mathbb{C}^{p_1}) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{p_1}, \mathbb{C}^{q_1})$ is a nilpotent parametrized by a tableau T_1 whose rows we denote $T_1^{(1)}, \ldots, T_1^{(l)}$. Then X_1 is $GL(p_1, \mathbb{C}) \times GL(q_1, \mathbb{C})$ conjugate to a pair of matrices of the form

$$(A_1^{(1)} \oplus \cdots \oplus A_1^{(l)}, B_1^{(1)} \oplus \cdots \oplus B_1^{(l)}),$$

where each pair $(A_1^{(i)}, B_1^{(i)})$ (viewed as a representative of a nilpotent orbit for some quasisplit $\mathfrak{u}(p_1^{(i)}, q_1^{(i)})$) is parametrized by the single row $T_1^{(i)}$. Using this fact, the following arguments for principal orbits can easily be adapted to the general case.

So assume that $q_1 = p_1 - 1$ and that

$$X_1 = (A_1, B_1) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{p_1 - 1}, \mathbb{C}^{p_1}) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{p_1} \mathbb{C}^{p_1 - 1})$$

represents the unique $(K_1)_{\mathbb{C}}$ orbit whose signed tableau is a single row. (The $q_1 = p_1 + 1$ case is identical, and the $p_1 = q_1$ case follows in the same way as the other two.) Now the signed tableau of X_1 has a plus sign in its first column and no minus signs. By the above classification, we conclude that

(8) A_1 is injective with a one dimensional cokernel, and

(9) B_1 has a one dimensional kernel and is surjective.

Let $p = p_1 + q_1$ and $q = q_1 + q_2$, and let $\mathfrak{q}$ be the upper triangular parabolic attached to $\{(p_1, q_1), (p_2, q_2)\}$. Contrary to our usual convention, let $G = U(p, q)$ be defined with respect to the standard signature (p, q) form and embed $G_1 = U(p_1, q_1)$ using the first p_1 coordinates along with the coordinates numbered $p+1, \ldots, p+q_1$. We are trying to compute $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_1 \times \mathcal{O}_{zero})$ which, by Definition 4.3, amounts to locating the largest $K_{\mathbb{C}}$ orbit of the form

$$\begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \end{array}$$

where $(A,B) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^q,\mathbb{C}^p) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^p,\mathbb{C}^q)$ is of the form

$$A = \begin{array}{c c} & \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \end{array} & \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \end{array}; \quad B = \begin{array}{c c} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \end{array} & \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \end{array}.$$

Assume that both p_2 and q_2 are nonzero. (The zero cases follow in the same way.) Keeping in mind that $p_1 = q_1 + 1$ and the conclusions of equations (8) and (9), it's easy to see that

$$\dim(\ker(A)) \geq q_2 - 1,$$

and that this lower bound can be obtained for an appropriate choice of A_2. Similarly,

$$\dim(\ker(B)) \geq p_2 + 1,$$

and for some B_2, this lower bound can be obtained. Now for such choices of A and B, one can check directly that the Jordan form of X is the partition $(p_1+q_1+1, 1^{p_2+q_2})$. Combining this with the parametrization of equations (1) through (6), we arrive at the algorithm of Lemma 4.6. This completes the proof. □

5. The Barbasch-Vogan parametrization and the good range.

In [V6] and [BV4], the definition of Kazhdan-Lusztig cells is adapted to the real case giving an equivalence relation on the set of Harish-Chandra modules with infinitesimal character ρ. Equivalence classes contain modules with the same asymptotic support and, for $U(p,q)$ each class contains a canonically defined $A_{\mathfrak{q}}(\lambda)$. Thus, by Proposition 4.4, the asymptotic support of *any* Harish-Chandra module for $U(p,q)$ with trivial infinitesimal character is irreducible. In fact, Barbasch and Vogan proved that cells are completely characterized by the signed tableau corresponding to the asymptotic support of any element in the cell. Moreover, all such tableaux arise in this way.

As in the complex case, the elements of a Harish-Chandra cell parametrize an integral basis for a subquotient of the coherent continuation representation. (The subquotient is minimal with respect to the property of being spanned by irreducible characters.) By a counting argument, Barbasch and Vogan showed that all subquotients in question are irreducible. (McGovern [Mc1] has subsequently shown that this phenomenon is a consequence of the fact that all irreducible representations of S_n are special.) In any case, Barbasch and Vogan also proved that the subquotient corresponding to the cell parametrized by a signed tableau $T_\pm$ is simply the irreducible representation of S_n given (in Young's notation) as shape$(T_\pm)$. The dimension of this representation is, by Theorem 3.1, the number of primitive ideals with trivial infinitesimal character. Since annihilators of elements in a given cell must be distinct (at least in S_n setting), we arrive at the following theorem for infinitesimal character $\nu = \rho$. The general case follows from a translation principle (cf. Theorem 3.4).

Theorem 5.1 ([BV4]). *Suppose that $\nu \in \rho + \mathbb{Z}^n$ is a weight lattice translate of the infinitesimal character of the trivial representation. The map assigning an irreducible Harish-Chandra module for $U(p,q)$ with infinitesimal character ν to the pair consisting of its annihilator and its asymptotic support is an injection. On the level of tableaux (Section 3 and 4), the map assigns a ν-antitableau and a signature (p,q) signed tableau (of the same shape) to each irreducible module of infinitesimal character ν, and any such pair arises in this way.*

All this can be see very explicitly in the rank one case. See Example 5.6 below.

Remark 5.2. When $\nu = \rho$, the map described in the theorem is formally analogous to the Robinson-Schensted algorithm arising in the computation of cells for $\mathfrak{sl}(n, \mathbb{C})$. In Part II, Section 12, we make this analogy more precise in terms of the geometry of the generalized Steinberg variety.

The main goal of this paper is to identify the parameters of the weakly fair $A_{\mathfrak{q}}(\lambda)$, and we need to start by identifying the good $A_{\mathfrak{q}}(\lambda)$. For a fixed regular integral infinitesimal character ν, Garfinkle described an algorithm taking Langlands parameters to pairs of tableaux and proved that the algorithm computes annihilators by a very detailed and complicated combinatorial calculation with the generalized τ-invariant [G]. Moreover, [VZ] explicitly gives the Langlands parameters of the good $A_{\mathfrak{q}}(\lambda)$, so combining these results one obtains a tableau characterization of the good $A_{\mathfrak{q}}(\lambda)$. This is entirely tractable, but we choose to avoid these relatively ponderous references and side-step the issue of Langlands parameters by a simple application of the results of Sections 3 and 4. We take that up now.

The idea is to build up the tableaux parameters of an $A_{\mathfrak{q}}(\lambda)$ step-by-step from the simple factors of $\mathfrak{l}_o$. The inductive proof is quite simple but the notation for the general case is a little overwhelming. We indicate the inductive procedure in the following example, and leave it to the reader to formulate the general proof of the theorem which follows.

Example 5.3. Let $\mathfrak{q} \subset \mathfrak{gl}(8, \mathbb{C})$ correspond to $\{(2,2),(1,3)\}$, and let

$$\lambda = (\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2)$$

be in the good range for $\mathfrak{q}$. Let $\lambda^{(1)} = (\lambda_1, \ldots, \lambda_1)$ be the indicated character of $\mathfrak{gl}(4, \mathbb{C})$, and similarly for $\lambda^{(2)}$. Set $\nu = \lambda + \rho$ and $\mu = \lambda^{(1)} + \rho(\mathfrak{gl}(4, \mathbb{C}))$. We compute the tableau parameters $(S, S_{\pm})$ giving the annihilator and associated variety of $A_{\mathfrak{q}}(\lambda)$. To illustrate the induction, we write

$$A_{\mathfrak{q}}(\lambda) = \mathcal{L}(\mathbb{C}_{\lambda^{(1)}} \otimes \mathbb{C}_{\lambda^{(2)}}) = \mathcal{L}(A_{\mathfrak{q}^{(1)}}(\lambda^{(1)}) \otimes \mathbb{C}_{\lambda^{(2)}});$$

here $\mathfrak{q}^{(1)} = \mathfrak{gl}(4, \mathbb{C})$. Now $A_{\mathfrak{q}^{(1)}}(\lambda^{(1)}) = \mathbb{C}_{\lambda^{(1)}}$, so its tableau parameters are

$$\begin{array}{|c|}\hline \mu_1 \\ \hline \mu_2 \\ \hline \mu_3 \\ \hline \mu_4 \\ \hline \end{array} \quad , \quad \begin{array}{|c|}\hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline \end{array} \; .$$

Proposition 4.4 abstractly computes $\mathrm{AS}(A_{\mathfrak{q}}(\lambda))$, and Lemma 4.6 does so explicitly. Using the algorithm of the lemma, we obtain

$$S_{\pm} = \mathrm{AS}(A_{\mathfrak{q}}(\lambda)) \quad = \quad \begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline - & + \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, .$$

Now S must have the same shape as $S_\pm$, and Corollary 3.6 tells us the first four coordinates of S; so far, then, we know that S looks like

$$S = \begin{array}{|c|c|} \hline \nu_1 & \\ \hline \nu_2 & \\ \hline \nu_3 & \\ \hline \nu_4 & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

Finally τ-invariant considerations (Lemma 2.12 and Lemma 3.2) imply that the remaining coordinates $\nu_5, \ldots, \nu_8$ must be sequentially entered moving strictly down S; there is only one way to do this:

$$S = \begin{array}{|c|c|} \hline \nu_1 & \nu_5 \\ \hline \nu_2 & \nu_6 \\ \hline \nu_3 & \nu_7 \\ \hline \nu_4 & \multicolumn{1}{c}{} \\ \cline{1-1} \nu_8 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

This completes the inductive computation of $(S, S_\pm)$ for $A_\mathfrak{q}(\lambda)$.

Theorem 5.4. *Let* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ *corresponds to* $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ *(Notation 1.1), let* $\mathbb{C}_\lambda$ *be a one-dimensional* $(l, L \cap K)$*-module in the good range for* $\mathfrak{q}$ *(Definition 2.4), and let*

$$\nu = (\nu_1^{(1)}, \ldots, \nu_{p_1+q_1}^{(1)}, \ldots\ldots, \nu_1^{(r)}, \ldots, \nu_{p_r+q_r}^{(r)}) = \lambda + \rho.$$

The tableau parameters (Theorem 5.1) of $A_\mathfrak{q}(\lambda)$ *are obtained inductively as follows. Start with the empty pair of tableaux and assume that the* $(s-1)$ *step has been completed giving a pair* $(S^{(s-1)}, S_\pm^{(s-1)})$*.* $S_\pm^{(s)}$ *is obtained by adding* p_s *pluses and* q_s *minuses to* $S_\pm^{(s-1)}$ *according to the algorithm of Lemma 4.6;* $S^{(s)}$ *is the tableau of the same shape of* $S_\pm^{(s)}$ *obtained by adding the coordinates* $\nu_1^{(s)}, \ldots, \nu_{p_s+q_s}^{(s)}$ *sequentially from top to bottom in the remaining unspecified boxes.*

Remark 5.5. Because of the good range condition on λ, the algorithm of Theorem 5.4 automatically produces a ν-antitableau S. But even if λ isn't good, the algorithm still produces a ν-quasitableau. Theorem 6.9 describes how to straighten this quasitableau into a ν-antitableau which corresponds to the annihilator of $A_\mathfrak{q}(\lambda)$.

Example 5.6. Consider again $U(p,1)$ and recall Examples 2.11 and 4.9. Recall the $A_\mathfrak{q}(\lambda)$ modules $X[i,j]$ of infinitesimal character $\rho = (\rho_1, \ldots, \rho_n)$. Using Theorem 5.4, we can compute annihilators of these modules.

$$\mathrm{Ann}(X[0,0]) = \begin{array}{|c|} \hline \rho_1 \\ \hline \vdots \\ \hline \rho_n \\ \hline \end{array}$$

$$\mathrm{Ann}(X[i,0]) = \begin{array}{|c|c|} \hline \rho_1 & \rho_{k_i} \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \qquad k_i = i+1; i \neq 0$$

$$\mathrm{Ann}(X[0,j]) = \begin{array}{|c|c|} \hline \rho_1 & \rho_{l_j} \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \qquad l_j = n+1-j; j \geq 1$$

$$\operatorname{Ann}(X[i,j]) = \begin{array}{|c|c|c|} \hline \rho_1 & \rho_{k_i} & \rho_{l_j} \\ \hline \vdots & & \\ \cline{1-1} \vdots & & \\ \cline{1-1} \end{array}, \qquad k_i = i+1; l_j = n+1-j; i,j \neq 0.$$

An easy count of Langlands parameters (or, equivalently of the $K_{\mathbb{C}}$ orbits described below) shows that the modules $X[i,j]$ exhaust the irreducible Harish-Chandra modules with infinitesimal character ρ. By Theorem 2.1, the $X[i,j]$ are all unitary, and so we have verified Conjecture 0.1 explicitly for infinitesimal character ρ. (This case was handled originally by Baldoni-Barbasch [BaBa].)

Since all Cartan subgroups of $U(p,1)$ are connected, the irreducible Harish-Chandra modules with infinitesimal character ρ are parametrized by the $K_{\mathbb{C}}$ orbits on the complex flag variety $G_{\mathbb{C}}/B$. For $U(p,1)$ it is very easy to understand the $K_{\mathbb{C}}$ orbits and their closure relationships explicitly; this is done in [Ya] for instance. Below we give the Hasse diagram for these $K_{\mathbb{C}}$ orbits, replacing the orbit vertices by the Harish-Chandra modules $X[i,j]$ that they parametrize.

$$\begin{tikzcd}
 & & X[0,0] \arrow[dl, dash] \arrow[dr, dash] & & \\
 & X[1,0] \arrow[dl, dash] \arrow[dr, dash] & & X[0,1] \arrow[dl, dash] \arrow[dr, dash] & \\
X[2,0] & & X[1,1] & & X[0,2] \\
 & & \vdots & & \\
X[p,0] & X[p-1,1] & \cdots & X[1,p-1] & X[0,p]
\end{tikzcd}$$

Of course, $X[0,0]$ is the trivial representation, and it is the unique module with zero asymptotic support. On the other hand, using the results and notation of Example 4.9, we see that the nontrivial modules $X[j,0]$, $1 \le j \le p$, are precisely the modules with asymptotic support $\begin{array}{|c|c|}\hline + & - \\ \hline\end{array}$. Similarly, the nontrivial modules $X[0,j]$, $1 \le j \le p$, are exactly those with asymptotic support $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$. Finally, the remaining representations $X[i,j]$ with $i,j \neq 0$ all have asymptotic support $\begin{array}{|c|c|c|}\hline + & - & + \\ \hline\end{array}$.

Note that in all four cases, the cardinality of the corresponding subset of Harish-Chandra modules coincides with the dimension of an irreducible representation of S_n. We can sharpen this observation as follows. The coherent continuation representation of W on the Grothendieck group of Harish-Chandra modules for $U(n,1)$ with infinitesimal character ρ can be worked out (essentially from scratch) in terms of the $K_{\mathbb{C}}$ orbit parametrization given above; this is done in [C], for instance. If $\alpha \in \tau(X)$, then $s_\alpha(\Theta(X)) = -\Theta(X)$, of course; since we have computed the annihilators of the $X[i,j]$, $\tau(X[i,j])$ is easy to read off from its tableau by Lemma 3.2. On the other hand, if $\alpha \notin \tau(X)$, then there is an upward line emanating from X in Figure 1, which ends in a module X' with $\alpha \in \tau(X')$. In this case $s_\alpha(\Theta(X))$ is the linear combination $\Theta(X) + \Theta(X') + \Theta(X'')$; here, X'' is the the module colinear with these two and immediately below X. (If X is in the bottom row of the Hasse diagram, (that is, if it is a discrete series) then there is nothing below X and we take $\Theta(X'') = 0$.) We can then explicitly verify the following general fact due to Barbasch and Vogan: the modules with associated variety $\mathcal{O}$ form an integral basis for the subquotient of the coherent continuation representation defined by

$$\mathrm{Span}\{\Theta(X) \mid \mathrm{AS}(X) \subset \bar{\mathcal{O}}\}/\mathrm{Span}\{\Theta(X) \mid \mathrm{AS}(X) \subsetneq \bar{\mathcal{O}}\}.$$

Moreover one can check directly (by computing dimensions and a few character values, for instance) that this subquotient is isomorphic to the representation of S_n parametrized by the Young diagram whose shape is the shape of $\mathcal{O}$. This concludes the example.

In terms of the program described in the introduction, the more important kind of result is determining when a pair of tableaux actually parametrizes an $A_{\mathfrak{q}}(\lambda)$. Such a statement follows by formally examining the algorithm of Theorem 5.4. (Corollary 5.7 is restated a little more cleanly in Corollary 5.12).

Corollary 5.7. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*$ be dominant, integral, and regular. Consider a pair $(S_\pm, S)$ consisting of a signature (p,q) signed tableau and a ν-antitableau. Partition S into disjoint union of difference-one skew columns (Notation 1.3) $S_1, \ldots S_m$ ordered by their maximal entry, and let $\hat{S}_{\pm,j}$ denote the corresponding skew columns of a representative $\hat{S}_\pm$ of $S_\pm$. Set $S^k = \coprod_{i \le k} S_i$ and similarly define $\hat{S}^k_\pm$. Assume that each S^k is itself a tableau. Then $(S_\pm, S)$ parametrizes a good $A_{\mathfrak{q}}(\lambda)$ for $U(p,q)$ if and only if there is a representative $\hat{S}_\pm$ of $S_\pm$ such that the following condition holds for all j: if a row is skipped in the arrangement of S_j in S, then all row-ends in $\hat{S}^j_\pm$ on or below the first skipped row and above the last row of S_j must have the same parity.*

Moreover, $\mathfrak{q}$ and λ can be read off from the S_j as follows: $\mathfrak{q}$ corresponds to the ordered sequence of pairs of integers obtained from the number of plus and minus signs in $\hat{S}_{\pm,j}$; and $\lambda = \nu - \rho$. (The data of $\mathfrak{q}$ and λ is independent of the choice of representative $S_\pm$.)

Sketch. Suppose $(S, S_\pm) = (\mathrm{Ann}(A_{\mathfrak{q}}(\lambda)), \mathrm{AV}(A_{\mathfrak{q}}(\lambda)))$, for some λ in the good range for $\mathfrak{q}$. We are to find a partition of S and a representative $\hat{S}_\pm$ of $S_\pm$ satisfying the requirements of the corollary. Theorem 5.4 gives a partition of $S = \coprod S_i$ into disjoint difference-one skew columns, and Lemma 4.10 implies that the corresponding partition of $\hat{S}_\pm$ (for any choice of representative $\hat{S}_\pm$) satisfies the required conditions.

Conversely, if such a partition and representative of $(S, S_\pm)$ are given, Theorem 5.4 clearly implies $(S, S_\pm) = (\mathrm{Ann}(A_{\mathfrak{q}}(\lambda)), \mathrm{AV}(A_{\mathfrak{q}}(\lambda)))$, where $\mathfrak{q}$ and λ are defined as in the second paragraph of the corollary. The final parenthetical assertion follows from the concluding assertion in the statement of Lemma 4.10. $\square$

Remark 5.8. The partition in the corollary may not be unique; the failures of uniqueness correspond exactly to the adjacent pairs condition in Lemma 2.9.

Example 5.9. Theorem 5.1 implies that there is an irreducible Harish-Chandra module (of infinitesimal character ρ, say) with empty τ-invariant if and only if $|p - q| \le 1$, i.e. if and only if $G = U(p,q)$ is quasisplit. We thus recover a special case of an early theorem of Vogan's ([V2]) asserting that G has a large representation if and only if G is quasisplit. If $p = q$ there are two such modules of infinitesimal character ρ:

$$(\mathrm{Ann}(X_1), \mathrm{AS}(X_1)) = \left(\boxed{\rho_1}\boxed{\rho_2}\boxed{\cdot}\boxed{\rho_n}\,,\ \boxed{+}\boxed{-}\boxed{\cdot}\boxed{-}\right) \ ; \text{ and}$$

$$(\mathrm{Ann}(X_2), \mathrm{AS}(X_2)) = \left(\boxed{\rho_1}\boxed{\rho_2}\boxed{\cdot}\boxed{\rho_n}\,,\ \boxed{-}\boxed{+}\boxed{\cdot}\boxed{+}\right) .$$

If $|p-q| = 1$, we can assume $p = q+1$, and then there is just one such module

$$(\mathrm{Ann}(X), \mathrm{AS}(X)) = \left(\boxed{\rho_1}\boxed{\rho_2}\boxed{\cdot}\boxed{\cdot}\boxed{\rho_n}\,,\ \boxed{+}\boxed{-}\boxed{\cdot}\boxed{-}\boxed{+}\right) .$$

Using Corollary 5.7, it is easy to see that X_1, X_2, and X are all $A_{\mathfrak{q}}(\lambda)$ with, in fact, $\mathfrak{l}_o$ compact. Hence these are all large discrete series representations.

Now if $|p-q| = 1$, then ρ is the highest weight of an irreducible representation of $GL(n,\mathbb{C})$; so Theorem 3.4 implies that the limit of discrete series

$$\left(\boxed{0}\boxed{0}\boxed{\cdot}\boxed{\cdot}\boxed{0}\,,\ \boxed{+}\boxed{-}\boxed{\cdot}\boxed{-}\boxed{+}\right)$$

is the unique irreducible Harish-Chandra module with zero infinitesimal character. It is of course unitary, as predicted by Conjecture 0.1.

On the other hand, if $p = q$ there is a unique irreducible Harish-Chandra module X with zero infinitesimal character. Remark 2.2 says that X cannot be an $A_{\mathfrak{q}}(\lambda)$, but this doesn't necessarily mean that it is nonunitary. In fact, Vogan's results in [V2] characterize X as the unique irreducible unitary principal series with zero infinitesimal character. (It's worth noting that the asymptotic support of X is reducible,

$$\mathrm{AS}(X) = \boxed{+}\boxed{-}\boxed{\cdot}\boxed{+}\boxed{-} \cup \boxed{-}\boxed{+}\boxed{\cdot}\boxed{-}\boxed{+} .$$

The reader is invited to think of the case of $U(1,1)$ where the indicated asymptotic support is both halves of the nilpotent cone.)

Now we introduce a little more notation designed to rewrite the statement of Corollary 5.7 in a more compact form which generalizes.

Definition 5.10. Let $S_1, \ldots, S_r$ be a set of disjoint difference-one skew columns of a ν-quasitableau S, and suppose $S = \coprod S_i$. Then we say that the S_i form a partition of S into difference-one skew columns if $S^j = \coprod_{a \leq j} S_i$ is a quasitableau for each $j = 1, \ldots, r$.

If $\hat{S}_\pm$ is a representative of a signed tableau $S_\pm$ of the same shape of S, any partition of S into skew columns induces a partition $\hat{S}_\pm = \coprod \hat{S}_{\pm,i}$ of $\hat{S}_\pm$ into skew columns. Let (p_i, q_i) denote the number of plus and minus signs in $\hat{S}_{\pm,i}$, and let $\mathfrak{q}^j$ be the θ-stable parabolic corresponding to the ordered sequence $\{(p_1,q_1), \ldots, (p_j,q_j)\}$ (as in Notation 1.3) of the appropriate $\mathfrak{g}^j = \mathfrak{gl}(n^j,\mathbb{C})$. We say that the partition $S = \coprod S_i$ is consistent with (the representative) $\hat{S}_\pm$ if

$$\hat{S}^j_\pm = \coprod_{i \leq j} \hat{S}_{\pm,i} = \mathrm{ind}_{\mathfrak{q}^j}^{\mathfrak{g}^j}(\mathcal{O}_{zero}), \qquad \text{for all } j.$$

Using Lemma 4.6, we obtain an explicit condition for $S = \coprod S_i$ to be consistent with $\hat{S}_\pm$: if a row is skipped in the arrangement of S_j in S, then all row-ends in $\hat{S}^j_\pm$ on or below the first skipped row and above (and including) the last row of S_j must have the same parity.

Suppose we are given a partition of a ν-antitableau S into difference-one skew columns, $S = \coprod S_j$, consistent with $\hat{S}_\pm = \coprod \hat{S}_{\pm,i}$. To this data, we may attach a θ-stable parabolic

$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ to the sequence $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ as above; and we obtain a unitary one-dimensional representation, $\mathbb{C}_\lambda$, of $\mathfrak{l}$ as follows. Set $[\nu]$ equal to the n-tuple of numbers obtained by concatenating the entries appearing in the skew columns $S_1, \ldots, S_r$, and view $[\nu]$ as a functional on $\mathfrak{t}$; then set $\lambda = [\nu] - \rho$. Note that $\mathfrak{q}$ and λ constructed in this way are independent of the choice of representative $\hat{S}_\pm$. (Clearly λ is independent of the choice; the last sentence of Lemma 4.10 implies that $\mathfrak{q}$ is too.) We say that $\mathfrak{q}$ and λ are associated to the partition of $S = \coprod S_i$ consistent with $S_\pm$.

Finally, we translate the ranges of positivity of Definition 2.4 to the level of tableau. (The point is that if λ is in the, say, mediocre range for $\mathfrak{q}$, and λ and $\mathfrak{q}$ are associated to some partition $S = \coprod S_i$, we want to define the columns S_i to be mediocre.) If $i < j$, two difference-one skew columns, S_i and S_j in a partition of S are said to be in mediocre position if either of the following conditions is satisfied: the smallest entry in S_i is greater than or equal to the smallest entry in S_j; or the largest entry in S_i is greater than or equal to the largest entry of S_j. The skew columns are said to be in (weakly) fair position if the average of the entries in S_i is (weakly) greater than the average of the entries in S_j. Similarly, S_i and S_j are said to be in (weakly) good position if the smallest entry in the S_i is (weakly) larger than the largest entry in S_j. Finally, we say that S_i and S_j are in nice position if *both* the smallest entry in S_i is greater than or equal to the smallest entry in S_j, and the largest entry in S_i is greater than or equal to the largest entry of S_j. (We have not encountered the nice condition before, but it will be important in the combinatorics of Section 7.) The entire partition is called mediocre, fair, good, or nice if all pairs of its skew columns are in the specified position.

Remark 5.11. We have the following implications on the ranges defined above:

$$\begin{tikzcd}
\text{good} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{fair} \arrow[r, Rightarrow] & \text{mediocre} \\
 & \text{nice} \arrow[u, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

With the above definitions, Corollary 5.7 becomes:

Corollary 5.12. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*$ be dominant, integral, and regular. Let X be an irreducible Harish Chandra module for $U(p,q)$ of infinitesimal character ν and consider $(S, S_\pm) = (\mathrm{Ann}(X), \mathrm{AS}(X))$ (Theorem 5.1). Then $X \cong A_\mathfrak{q}(\lambda)$ if and only if there is a partition of S into difference-one skew columns consistent with a representative of $S_\pm$ so that $\mathfrak{q}$ and λ are associated to this partition (Definition 5.10).*

We will generalize this in Corollary 6.11 below.

6. Statement of main theorems

As mentioned in Remark 5.5, the algorithm of Theorem 5.4 has an obvious analog outside the good range. But when $A_\mathfrak{q}(\lambda)$ is no longer good, there is nothing to guarantee that the quasitableau produced is in fact an antitableau. Sometimes it is, and in these cases, the algorithm of Theorem 5.4 does produce the annihilator of $A_\mathfrak{q}(\lambda)$ (this has to be proved, of course). But sometimes the quasitableau is not an antitableau, and we need a way to convert it into the one parameterizing the corresponding annihilator. In order to do this, we must move outside the class of $A_\mathfrak{q}(\lambda)$ modules to a larger class of representations that still retains most of the nice translation properties of the $A_{\mathfrak{q}}(\lambda)$'s. Combinatorially, this procedure introduces an equivalence relation on the the set of partitions of ν-quasitableaux into difference-one skew columns. Then given X, we can conclude that X is isomorphic to a weakly fair $A_{\mathfrak{q}}(\lambda)$ if and only if there exists a suitably consistent representative in the equivalence class of some nice partition of $\mathrm{Ann}(X)$. Moreover, the equivalence relation will keep track of all coincidences and vanishing among the weakly fair (and, in fact, mediocre) $A_{\mathfrak{q}}(\lambda)$.

Now we make these matters more precise, and begin to describe the equivalence relation. As a first step we need to define a rough measure of the size and singularity of two adjacent columns in a partition of S into difference-one skew columns. (The manner in which 'size' is to be interpreted is discussed in the remark following the definition.)

Definition 6.1. Given two adjacent columns $C = S_j$ and $D = S_{j+1}$ of a partition of S into difference-one skew columns (Definition 5.10), we first define an integer depending only on the shape of C and D in the following way. Label the entries of C and D (moving sequentially down each skew column) as $c_1, \ldots, c_k$, and $d_1, \ldots, d_l$. For $1 \leq m \leq \min(k, l)$ define a condition

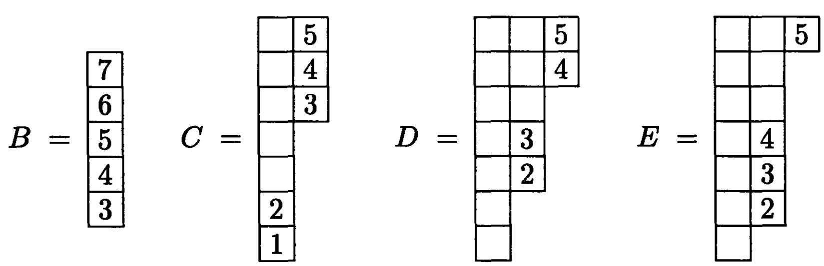
$$\begin{aligned}\text{condition } m: \quad & c_{k-m+i} \text{ is strictly left of } d_i \text{ in } S \\ & \text{for } 1 \leq i \leq m.\end{aligned}$$

Define the overlap of C and D, denoted $\mathrm{overlap}(C, D)$ to be the largest $m \leq \min(k, l)$ so that condition m holds. (If condition m never holds, define $\mathrm{overlap}(C, D) = 0$.)

The singularity of C and D, denoted $\mathrm{sing}(C, D)$, is defined to be the number of pairs of identical entries among the c_i and d_j. This is an integer which depends on the entries of C and D (and is independent of shape of C and D).

Remark 6.2. When $S = C \coprod D$ consists of only two skew columns, then it is easy to see that the overlap is simply the number of rows of length two in S, and hence is precisely related to the size of the variety associated to the primitive ideal corresponding to S. In the general case, an analogous interpretation of $\mathrm{overlap}(C, D)$ exists, but since we do not need the precise statement, we leave it to the reader to formulate.

Example 6.3. Consider



Then $\mathrm{overlap}(B, C) = \mathrm{sing}(B, C) = 3$; $\mathrm{overlap}(C, D) = \mathrm{sing}(C, D) = 4$; and $\mathrm{overlap}(C, E) = 3 < \mathrm{sing}(C, E) = 4$.

With these definitions in hand, we can now define the equivalence relation. The reader is encouraged to read Example 6.6 concurrently with the definition.

Definition 6.4. We define an equivalence relation on the set consisting of mediocre partitions of ν-quasitableau into difference-one skew columns (Notation 1.3, Definition 5.10) together with the formal quasitableau 0. Suppose $S = \coprod S_i$ is a mediocre partition of a ν-quasitableau into difference-one skew columns. The equivalence relation will be generated by replacing adjacent skew columns S_i, S_{i+1} in weakly mediocre position with S_i', S_{i+1}' in nice position (Definition 5.10); here $R = S_i \coprod S_{i+1}$ and $R' = S_i' \coprod S_{i+1}'$ have identical shape and the entries of R' are a permutation of those of R. The resulting equivalence is given by

$$S = \coprod_j S_j \sim S' = \coprod_{j<i} S_j \coprod S_i' \coprod S_{i+1}' \coprod_{j>i+1} S_j.$$

We now describe $R' = S_i' \coprod S_{i+1}'$. Since we will constantly refer to this procedure, we set it aside.

Procedure 6.5. Suppose that S_i and S_{i+1} have lengths r and s respectively, and recall Definition 6.1.

(a) If

$$\operatorname{overlap}(S_i, S_{i+1}) > \operatorname{sing}(S_i, S_{i+1}),$$

or

$$\operatorname{overlap}(S_i, S_{i+1}) = \operatorname{sing}(S_i, S_{i+1}) < \min(r, s),$$

Then $R' = R$.

(b) If

$$\operatorname{overlap}(S_i, S_{i+1}) < \operatorname{sing}(S_i, S_{i+1}),$$

then S is defined to be equivalent to 0, the (formal) zero tableau.

(c) Assume

$$\operatorname{overlap}(S_i, S_{i+1}) = \operatorname{sing}(S_i, S_{i+1}) = \min(r, s).$$

We begin by describing a rearrangement, R' of the coordinates of $R = S_i \coprod S_{i+1}$. Let $a_1, \ldots, a_r$ denote the sequential entries of S_i and $b_i, \ldots, b_s$ likewise for S_{i+1}. Assume that the b's are a subset of the a's (the opposite case is described below), and write them as

$$\begin{array}{ll} a_1, \ldots, a_{l+1}, \ldots, a_{l+s}, \ldots, a_r, & \quad a_{l+i} = b_i \\ \qquad\quad b_1, \ldots, b_s. & \end{array}$$

Place $a_1, \ldots, a_l$ in the first l boxes of S_i. Place a_{l+1} in the next box of S_i, say χ, for which there is a box of R weakly above and to strictly to the right of χ. Place b_1 in the top-most such box of R which is above and to the right of χ. Continue entering the pairs $(a_{l+2}, b_2), \ldots, (a_{l+s}, b_s)$ in this way. Finally enter the coordinates $a_{l+s+1}, \ldots, a_s$ sequentially in the remaining boxes of R. Note that the overlap assumption guarantees that this procedure is well defined. Let R' denote the skew tableau so obtained.

(d) Keep the assumption on overlap and singularity as in the previous case, but suppose that the a's are a subset of the b's. Write them as

$$\begin{array}{ll} \qquad\quad a_1, \ldots, a_r & \\ b_1, \ldots, b_{l+1}, \ldots, b_{l+r}, \ldots, b_s & \quad a_i = b_{l+i}. \end{array}$$

Informally, we compose the algorithm of the previous case with an automorphism of $\mathfrak{g}$ coming from the Dynkin diagram. More precisely, place $b_s, \ldots, b_{l+r}$ sequentially from bottom to top in the boxes of S_{i+1}. Place b_{l+r} in the the lowest box, say χ, of S_{i+1} for which there is a box of R weakly below and strictly left of χ. Place a_s in the bottom-most such box. Continue adding the pairs of coordinates in this way. Conclude by entering the remaining coordinates $b_1, \ldots, b_l$ sequentially in the remaining boxes of R. Let R' denote the resulting tableau.

To finish the definition, we must give a partition $R' = S'_i \coprod S'_{i+1}$ into difference-one skew columns in nice position. We construct S'_{i+1} as follows. Its last entry consists of the the smallest entry, say c, in R'. (Recall if two entries appear in a skew tableau, the smaller one is the one that occurs strictly right of the other). Its next to last entry consist of the smallest occurrence of $c-1$ which is strictly above and weakly to the right of the last entry of S'_{i+1}. We continue in this way until we run out of room in R'. This defines S'_{i+1} and S'_i is defined to be what remains. It's not hard to see that the resulting $R' = S'_i \coprod S'_{i+1}$ is actually a nice partition of R' into difference-one skew columns.

Example 6.6. Recall the difference-one skew columns C, D, E of Example 6.3. Consider the partitions

$$S = \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 3 & \\ 3 & 2 & \\ 2 & & \\ 1 & & \end{array} = B \coprod C \coprod D \ ; \text{ and}$$

$$T = \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & \\ 5 & 3 & \\ 4 & 4 & \\ 3 & 3 & \\ 2 & 2 & \\ 1 & & \end{array} = B \coprod C \coprod E \ .$$

(a) Referring to Example 6.3, we know $\text{overlap}(B, C) = \text{sing}(B, C) < \min(5, 3)$; Hence Procedure 6.5(a) applies to $R = B \coprod C$ to give $R' = R$. Moreover, the partition described in the last paragraph of Definition 6.4 is $R' = B \coprod C$.
(b) Again referring to Example 6.3, we see that Procedure 6.5(b) applies to $R = C \coprod E$ to give zero. Hence T is equivalent to the zero tableau.
(c) Finally consider $R = C \coprod D$. Applying Procedure 6.5(c) and the definition of the partition of R', we obtain

$$R' = C' \coprod D' = \begin{array}{cc} \ & 5 \\ \ & 4 \\ \ & \\ \ & \\ \ & \\ 3 & \\ 2 & \end{array} \coprod \begin{array}{ccc} \ & \ & 5 \\ \ & \ & 4 \\ \ & 3 & \\ \ & 2 & \\ \ & 1 & \\ \ & & \\ \ & & \end{array} .$$

We thus obtain

$$B\coprod C\coprod D = \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 2 & \\ 3 & 2 & \\ 2 & & \\ 1 & & \end{array} \quad \sim \quad \begin{array}{ccc} 7 & 5 & 5 \\ 6 & 4 & 4 \\ 5 & 3 & \\ 4 & 2 & \\ 3 & 1 & \\ 3 & & \\ 2 & & \end{array} = B\coprod C'\coprod D'.$$

Now applying Procedure 6.5(a) to $B\coprod C'$ shows that $S \sim 0$, since $\mathrm{overlap}(B, C') = 2 < \mathrm{sing}(B, C') = 3$.

The next lemma will be used frequently. Its proof amounts to the fact that the mediocre condition was defined not only for adjacent pairs of columns, but all pairs (see Definition 5.10 and Remark 2.6).

Lemma 6.7. *Let $S = \coprod S_j$ be a mediocre partition of a ν-quasitableau into difference-one skew columns. If $S \sim S' = \coprod S'_j$, then the partition of S' is again mediocre.*

The following very technical lemma will be useful below. (The reader is encouraged to skip it, and refer back when necessary.)

Lemma 6.8. *Let $T = T_1 \coprod T_2 \coprod T_3$ be a partition of a ν-antitableau into difference-one skew columns. Assume T_1 and T_2 are in nice position, T_2 and T_3 are in mediocre position, and that T_1 contains the largest entry of T while T_3 contains the smallest. Suppose that Procedure 6.5 applied to $T_2 \coprod T_3$ gives $T'_2 \coprod T'_3$. Then $T_1 \coprod T'_2 \coprod T'_3$ is nice.*

Sketch. By hypothesis T'_2 and T'_3 are in nice position, and since T_1 contains the largest entry of T by hypothesis, we need only verify that the smallest entry of T_1 is greater than or equal to the smallest entry of T'_2. Write t_i for the smallest entry of T_i, and similarly for t'_i. The nice hypothesis on T_1 and T_2 implies $t_1 \geq t_2$, so it is enough to verify the following claim: t_2 occurs in T'_2. Since t_3 is the smallest entry in T (by hypothesis), we know $t_2 \geq t_3$. So either t_2 is larger than every entry in T_3, or t_2 occurs in T_3. In the former case, Procedure 6.5(a) applies to give $T'_2 = T_2$, and we conclude t_2 occurs in T'_2 as claimed. In the latter case, t_2 occurs twice in $T'_2 \coprod T'_3$. We conclude t_2 must occur in T'_2, thus proving the claim and hence the lemma. $\square$

We can now state our main results.

Theorem 6.9. *Let $\mathfrak{q}$ be a θ-stable parabolic and $\mathbb{C}_\lambda$ be a one-dimensional $(\mathfrak{l}, L\cap K)$-module in the mediocre range for $\mathfrak{q}$. Let $\nu = \lambda + \rho = (\nu_1, \ldots, \nu_n)$, and construct a ν-quasitableau S, together with a partition into difference-one skew columns $S = \coprod S_i$, as in Theorem 5.4. Then there is a algorithm (described below) to locate a distinguished $S' = \coprod S'_i$ equivalent (in the sense of Definition 6.4) to $S = \coprod S_i$ such that either: $S' = 0$; or S' is actually a ν-antitableau and $\coprod S'_i$ is a nice partition into difference-one skew columns (Definition 5.10) with*

$$\mathrm{overlap}(S'_i, S'_{i+1}) \geq \mathrm{sing}(S'_i, S'_{i+1}) \quad \textit{for all } i \textit{ (Definition 6.1).}$$

The module $A_\mathfrak{q}(\lambda)$ is nonzero if and only if the latter case holds and in this case,

$$\mathrm{Ann}(A_\mathfrak{q}(\lambda)) = S'.$$

We describe the algorithm of the theorem. Let $\mathfrak{q}$ be associated to the sequence of pairs of positive integers $\{(p_1, q_1), \ldots, (p_r, q_r)\}$, let λ be in the mediocre range for $\mathfrak{q}$, and let $\nu = \lambda + \rho$. The algorithm is defined inductively in terms of r; we consider the cases $r \leq 3$, leaving the general statement to the reader. When $r = 1$, $A_{\mathfrak{q}}(\lambda) = \mathbb{C}_\lambda$, and the theorem is trivial. If $r = 2$, Theorem 5.4 gives a mediocre partition $S = S_1 \coprod S_2$ of a ν-quasitableau into difference-one skew columns. Using Procedure 6.5, we obtain $S \sim S' = S_1' \coprod S_2'$ or $S \sim 0$. In the former case S' is actually a ν-antitableau and $S' = S_1' \coprod S_2'$ is a nice partition of S' into difference-one skew columns whose singularity does not exceed their overlap, as required in the theorem.

Next suppose $r = 3$. Again Theorem 5.4 gives a mediocre partition $S = S_1 \coprod S_2 \coprod S_3$ of a ν-quasitableau into difference-one skew columns. By the $r = 2$ case and Lemma 6.7, we may assume S_1 and S_2 are in nice position. Applying Procedure 6.5 to $R = S_2 \coprod S_3$, we obtain either $S \sim 0$ or $S \sim S' = S_1 \coprod S_2' \coprod S_3'$, the partition being mediocre by Lemma 6.7. In the latter case, S_2' and S_3' are in nice position and their singularity does not exceed their overlap; but S_1 and S_2' are only mediocre. So apply Procedure 6.5 to $R = S_1 \coprod S_2'$ which gives $S \sim S'' = S_1'' \coprod S_2'' \coprod S_3'$. Again if S'' is nonzero, then S_1'' and S_2'' are in nice position (with the correct overlap condition), but S_2'' and S_3' are only mediocre, so we can again apply Procedure 6.5 as before, and continue in this way.

We claim that the see-saw algorithm must eventually produce either zero or a nice partition (with the correct overlap conditions) of a ν-antitableau equivalent to S. To see this notice that when two mediocre columns $T_i \coprod T_{i+1}$ are exchanged for good ones $T_i' \coprod T_{i+1}'$ by Procedure 6.5, the largest entry of $T_i \coprod T_{i+1}$ always resides in T_i' and the smallest entry in T_{i+1}'. (This follows immediately from the definition.) So after two see-saws, either $S \sim 0$ or $S \sim T_1 \coprod T_2 \coprod T_3$ with the largest entry of S contained in T_1 and the smallest entry in T_3. We can assume T_1 and T_2 are in nice position, and then Lemma 6.8 implies that S is equivalent to a nice parition (or zero) as claimed. The algorithm of the theorem in the $r = 3$ case is complete.

It is clear how the algorithm works for $r > 3$. (Convergence follows by induction and the $r = 3$ case.)

Remark 6.10. When λ is in the weakly good range for $\mathfrak{q}$, Theorem 3.4 and Lemma 2.13 imply that the annihilator of $A_{\mathfrak{q}}(\lambda)$ can be computed from Theorem 5.4 and τ-invariant considerations. Theorem 6.9 reduces to such considerations since the overlap requirement on the nice partition is nothing but a τ-invariant condition in this case.

Formally inverting the statement of Theorem 6.9, we obtain:

Corollary 6.11. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{t}^*_{\mathbb{R}}$ be dominant and integral. Suppose $(S, S_\pm)$ is a pair consisting of a ν-antitableau and a signature (p, q) tableau of the same shape (Notation 1.3). Then $(S, S_\pm) = (\mathrm{Ann}(X), \mathrm{AV}(X))$ for a mediocre $X \cong A_{\mathfrak{q}}(\lambda)$ if and only if the following condition holds: there exists a nice partition $S = \coprod S_i$ into difference-one skew columns with*

$$\mathrm{overlap}(S_i, S_{i+1}) \geq \mathrm{sing}(S_i, S_{i+1}) \quad \textit{for all } i$$

such that the partition $S = \coprod S_i$ is equivalent (in the sense of Definition 6.4) to a mediocre partition $S' = \coprod S_i'$ so that the partition of S' is consistent with $S_\pm$ and so that $\mathfrak{q}$ and λ are associated to this partition of S' (Definition 5.10).

A practical illustration of how Theorem 6.9 and Corollary 6.11 detect coincidences among the mediocre $A_{\mathfrak{q}}(\lambda)$ modules is given in Example 8.2 below.

Remark 6.12. When ν is dominant and regular Theorem 6.9 and Corollary 6.11 reduce to Theorem 5.4 and Corollary 5.12.

Example 6.13. Consider the following pair of tableau for $SU(3,2)$,

$$S = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & -1 \\ \hline -1 \\ \cline{1-1} \end{array} \quad , \quad S_{\pm} = \begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline + \\ \cline{1-1} \end{array} .$$

There are six nice partitions of S into difference-one skew columns. The most interesting one is

$$S = \begin{array}{|c|} \hline 1 \\ \hline \end{array} \ \amalg \ \begin{array}{|c|c|} \hline & 1 \\ \hline 0 \\ \cline{1-1} -1 \\ \cline{1-1} \end{array} \ \amalg \ \begin{array}{|c|c|} \hline & \\ \hline & -1 \\ \hline \\ \cline{1-1} \end{array} ;$$

three others are obtained by breaking the second skew column into two or three columns, and the two more have $S_1 = \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array}$. The equivalence classes of these last five all are singletons, while the first partition is equivalent to the mediocre partition

$$S = \begin{array}{|c|c|} \hline 1 & \\ \hline 0 & \\ \hline -1 \\ \cline{1-1} \end{array} \ \amalg \ \begin{array}{|c|c|} \hline & 1 \\ \hline & \\ \hline \\ \cline{1-1} \end{array} \ \amalg \ \begin{array}{|c|c|} \hline & \\ \hline & -1 \\ \hline \\ \cline{1-1} \end{array} .$$

Since none of the these partitions is consistent with $S_{\pm}$, Theorem 6.9 allows us to conclude that the Harish-Chandra module X corresponding to $(S, S_{\pm})$ is not a weakly mediocre (or weakly fair for that matter) $A_{\mathfrak{q}}(\lambda)$. Vogan's conjecture then predicts that X should not be unitary. Using Garfinkle's algorithm, one can obtain the Langlands parameters of X, and using the Vogan-Knapp minimal K type formula ([K]), one can check that X is spherical. One may verify that the Dirac operator inequality (as stated, for example, in [SaV]) is inconclusive on the trivial K type. One might try to find other K types of X close to the trivial one, and see if the Dirac inequality gives any information there. We have done this for all other possible K types, and verified again that the Dirac inequality is inconclusive. So we must use a different tack to prove non-unitarity. Using the ideas of the proof of Theorem 6.9, it's not too hard to show that $X \cong \mathcal{L}_S(X_L)$ where $\mathfrak{q}$ is attached to the sequence, $\{(2,1),(1,1)\}$ and X_L, as a representation of $S(U(2,1) \times U(1,1))$, is $\det^{-1} \boxtimes (\mathbb{C}^2_{std} \otimes \det)$. Any non-degenerate invariant hermitian form on the two-dimensional representation X_L can be arranged to be positive on one $L \cap K$ type and negative on the other. Now $\mathcal{L}_S$ preserves signatures in this case, so its enough to find two K types of X arising from the distinct $L \cap K$ types in the Blattner multiplicity formula. The $S(U(3) \times U(2))$ types $(3,0,0|1,-4)$ and $(0,0,0|0,0)$ are two such, and we can conclude non-unitarity.

This example generalizes beyond $SU(3,2)$ to $SU(p,p-1)$. The tableaux in question are

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & \cdot & 1 \\ \hline 0 & -1 & -1 & \cdot & -1 \\ \hline -1 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|} \hline + & - & + & \cdot & \cdot \\ \hline - & + & - & \cdot & \cdot \\ \hline + \\ \cline{1-1} \end{array} .$$

These tableaux do not parametrize a mediocre $A_{\mathfrak{q}}(\lambda)$. The infinitesimal character of the corresponding Harish-Chandra module is strictly smaller in length than ρ_c (if $p > 3$), so the Dirac inequality is necessarily inconclusive on all K types. Using Garfinkle's algorithm and the Vogan-Knapp formula, one can again verify that this is a spherical representation. By modifying the argument given for $SU(3,2)$, one can prove that this representation is nonunitary.

Actually, the above two examples are illustrations of a general (unpublished) result of Barbasch's stating that any unitary spherical representation is a weakly fair $A_{\mathfrak{q}}(\lambda)$. The above argument for $SU(3,2)$, however, can be then turned around to prove that the two dimensional standard representation of $U(1,1)$ is nonunitary. Of course this is obvious, but it provides a hint of a general technique for proving nonunitarity.

7. Proof of Theorem 6.9

In this section, we prove Theorem 6.9. To compute the annihilators of the mediocre $A_{\mathfrak{q}}(\lambda)$, we will use the following strategy. Given such an $A_{\mathfrak{q}}(\lambda)$, we can pull apart the overlaps of λ to obtain a good λ', and then use Lemma 2.13 to move from the good $A_{\mathfrak{q}}(\lambda')$ (where we have complete information about its annihilator and asymptotic support) to our module of interest, $A_{\mathfrak{q}}(\lambda)$. On the surface, the program seems hopeless. The translation functor, T, defined in the lemma is complicated; it is a sequence of multiple wall crossing functors, so it appears as though we need very detailed information about the coherent continuation representation in order to understand T. But Lemma 2.13 says that $T(A_{\mathfrak{q}}(\lambda')) = A_{\mathfrak{q}}(\lambda)$ which implies that whatever intermediate complications involved in computing $T(A_q(\lambda'))$ must all disappear in the final answer.

We are going to compute T by inductively applying the T_i's (of Notation 1.2) to $A_{\mathfrak{q}}(\lambda')$, and we first need to describe the effect of the translations T_i on generalized Verma modules inside the mediocre range.

Lemma 7.1. *Let $\mathfrak{q} = l \oplus \mathfrak{u}$ be the block upper triangular parabolic subgroup of $\mathfrak{gl}(n,\mathbb{C})$ with $\mathfrak{l} = \oplus_{j=1}^{k} \mathfrak{gl}(n_j,\mathbb{C})$. Let i be of the form $i = \sum_{j\leq l} n_j$, and let*

$$\mu_i = e_{i+1} + \cdots + e_n.$$

Suppose that $\mathbb{C}_\eta$ is a character of $\mathfrak{l}$ with η and $\eta + \mu_i$ in the mediocre range for $\mathfrak{q}$ (Definition 2.4). Set $\nu = \eta + \rho$ and $\nu' = \eta + \mu_i + \rho$, and consider the translation functor $T_i = \psi_\nu^{\nu'}$ as in Notation 1.2. Let $M(\eta)$ denote the (normalized) generalized Verma module

$$M(\eta) = \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(C_\eta \otimes \bigwedge\nolimits^{top}(\mathfrak{u})).$$

Then

$$T_i(M(\eta)) = M(\eta + \mu_i).$$

Pf. The proof is an induction on k, the number of factors of $\mathfrak{l}$. We describe the base case when $\mathfrak{l} = \mathfrak{gl}(n_1,\mathbb{C}) \oplus \mathfrak{gl}(n_2,\mathbb{C})$, giving enough details so that the reader can complete the induction. In this case, $i = n_1$, and without loss of generality (by, say, Theorem 3.4) we may take

$$\eta = (\overbrace{0,\ldots,0}^{n_1},\overbrace{t,\ldots,t}^{n_2}),$$

the mediocre hypothesis on $\eta + \mu_i$ implying $t < \max(n_1, n_2)$. As outlined in the proof of Lemma 3.2, $T_{n_1}(M(\eta))$ admits a filtration with generalized Verma quotients $M(\kappa)$ characterized by

(a) κ is of the form $\eta+\lambda$ where λ is a highest weight (for $\mathfrak{t}\cap\mathfrak{l}$ in $\mathfrak{l}$) of an irreducible constituent of the finite-dimensional $\mathfrak{g}$ module F^{μ_i} of extremal weight μ_i restricted to $\mathfrak{l}$; and
(b) $\kappa+\rho \in W(\eta+\mu_i+\rho)$.

Concretely, F^{μ_i} is nothing but $\bigwedge^{n_2}(\mathbb{C}^n)$, so the restriction in (a) is easy to compute: the highest weights λ are exactly

$$\lambda_{l,m} = (\overbrace{1,\dots,1,\underbrace{0,\dots,0}_{l}}^{n_1},\overbrace{1,\dots,1,\underbrace{0,\dots,0}_{m}}^{n_2}), \qquad \text{with } l+m = n_1.$$

Now, using the mediocre assumption on t, one can check directly that $\lambda = \lambda_{0,n_2} = \mu_i$ is the unique highest weight satisfying the requirements (a) and (b) above (hence giving the conclusion that $T(M(\eta)) = M(\eta+\mu_i)$, and completing the $k=2$ case). Since this is the crux of the matter, we give the argument in detail.

Write

$$\eta+\rho = (\rho_1,\dots,\rho_{n_1}, t+\rho_{n_1+1},\dots,t+\rho_n), \qquad \rho_j = \frac{n-2j+1}{2}.$$

First assume that $n_1 \geq n_2$; together with the mediocre assumption on $t+\mu_i$, this implies that

$$\rho_1 > t+\rho_{n_1+1}.$$

Hence the value ρ_1+1 does not appear as a coordinate of $\eta+\mu_i+\rho$. Thus if $\eta+\lambda_{l,m}+\rho \in W(\eta+\mu_i+\rho)$, we must have that the first coordinate of $\eta+\lambda_{l,m}+\rho$ is ρ_1+1. Hence the first coordinate of $\lambda_{l,m}$ is 0. This implies $\lambda_{l,m} = \lambda_{0,n_2}$, as claimed.

On the other hand, if $n_1 < n_2$, then the mediocre hypothesis on $\eta+\mu_i$ implies that

$$\rho_{n_1} > t+\rho_n.$$

Thus the value $t+\rho_n$ does not appear as an entry of $\eta+\mu_i+\rho$. So if $\eta+\lambda_{l,m}+\rho \in W(\eta+\mu_i+\rho)$, the last coordinate of $\lambda_{l,m}$ must be 1. Hence $\lambda_{l,m} = \lambda_{0,n_2}$, as claimed.

The general case follows by using the $k=2$ arguments on adjacent Levi factors of $\mathfrak{l}$ and proceeding inductively. $\square$

Corollary 7.2. *Retain the notations of the previous lemma, and for $1 \leq l \leq m \leq n$, let*

$$r = \sum_{j\leq l} n_j, \qquad s = \sum_{j\leq m} n_j$$

Suppose η, $\eta+\mu_r$, $\eta+\mu_s$, and $\eta+\mu_r+\mu_s$ are in the mediocre range for $\mathfrak{q}$. Then

$$T_rT_s(M(\eta)) = M(\eta+\mu_r+\mu_s) = T_sT_r(M(\eta))$$

The previous lemma and corollary complete the proof of Lemma 2.13. We will, however, need a strengthened version of Corollary 7.2.

Lemma 7.3. *Retain the notations and assumptions of the previous lemma and corollary, and let F^r and F^s denote the irreducible representations of $\mathfrak{g}$ with extremal weights μ_r and μ_s. Then the translation functors $T_r(T_s(M(\eta))) = T_s(T_r(M(\eta)))$ can be computed as*

$$P(M(\eta)\otimes F^r\otimes F^s),$$

where P denotes the projection on infinitesimal character $\eta+\rho+\mu_r+\mu_s$ (as in Notation 1.2).

Pf. All of the ideas of the general setting are captured in the case when $\mathfrak{l}$ is the sum of three blocks. So assume $\mathfrak{l} = \oplus_{i=1}^{3}\mathfrak{gl}(n_1, \mathbb{C})$, with $r = n_1$ and $s = n_1 + n_2$. Write $T_{rs}(M(\eta))$ instead of $P(M(\eta) \otimes F^r \otimes F^s)$. As in the proof of Lemma 7.1, $T_{rs}(M(\eta))$ admits a filtration with generalized Verma quotients $M(\kappa)$ characterized by

(a) κ is of the form $\eta + \lambda$ where λ is a highest weight (for $\mathfrak{t} \cap \mathfrak{l}$ in $\mathfrak{l}$) of an irreducible constituent of $F^r \otimes F^s$ restricted to $\mathfrak{l}$; and
(b) $\kappa + \rho \in W(\eta + \mu_r + \mu_s + \rho)$.

Using the fact that $F^r = \bigwedge^{n_2+n_3}(\mathbb{C}^n)$ and $F^s = \bigwedge^{n_2}(\mathbb{C}^n)$, it is not difficult to see that

$$F^r \otimes F^s = \bigoplus F[2^l 1^m],$$

where the sum is over all pairs l and m with $2l + m = 2n_2 + n_3$ and $0 \le l \le n_2$; here $F[2^k 1^l]$ is the finite dimensional representation of $\mathfrak{gl}(n, \mathbb{C})$ with highest weight

$$2(e_1 + \cdots e_k) + (e_{k+1} + \cdots e_{k+l}).$$

We thus see that the $\mathfrak{l}$ highest weights of $F^r \otimes F^s$ restricted to $\mathfrak{l}$ are all of the form

$$\lambda = (\overbrace{\underbrace{2,\dots,2}_{l_1},\underbrace{1,\dots,1}_{m_1},0,\dots,0}^{n_1} \mid \overbrace{\underbrace{2,\dots,2}_{l_2},\underbrace{1,\dots,1}_{m_2},0,\dots,0}^{n_2} \mid \overbrace{\underbrace{2,\dots,2}_{l_3},\underbrace{1,\dots,1}_{m_3},0,\dots,0}^{n_3}),$$

with

$$2(l_1 + l_2 + l_3) + (m_1 + m_2 + m_3) = 2n_2 + n_3.$$

Arguing as in Lemma 7.1 (and using the mediocre hypothesis crucially), we conclude that the only λ of the above form which satisfies the requirements of (a) and (b) above is

$$\lambda = (\overbrace{0,\dots,0}^{n_1},\overbrace{1,\dots,1}^{n_2},\overbrace{2,\dots,2}^{n_3}).$$

Hence the lemma amounts to proving that the $\mathfrak{l}$ representation F with highest weight λ occurs exactly once in the restriction of $F^r \otimes F^s$ to $\mathfrak{l}$. To see this, note that λ is an extremel weight for $F^r \otimes F^s$, and hence F occurs at most once. Clearly λ is extremel for $\mathfrak{l}$, and so we conclude F occurs exactly once. $\square$

We then obtain the following corollary which is absolutely essential in what follows.

Corollary 7.4. *Retain the notations and assumptions of Corollary 7.2, and suppose that Y is an irreducible Harish-Chandra module with*

$$\mathrm{Ann}(M(\eta)) \subset \mathrm{Ann}(Y).$$

Then

$$T_r T_s(Y) = T_s T_r(Y).$$

Now we describe how to compute the $T_i(Y)$ in terms of the coherent continuation representation. The computation in part (b) can be envisioned as first passing to regular infinitesimal character, then crossing a sequence of walls, and finally pushing to a different sequence of walls.

Lemma 7.5. *Let Y be an irreducible Harish-Chandra module with dominant infinitesimal character ν. Let ν' denote a dominant representative of $\nu + \mu_i$, write W^ν for the stabilizer in W of ν ,and similarly for $W^{\nu'}$. Let W_o denote any choice of representatives for the cosets $W^\nu/(W^\nu \cap W^{\nu'})$. Finally let Θ be the unique coherent family based at Y. Then*

$$T_i(Y) = \sum_{w \in W_o} \Theta(w\nu').$$

In particular, if Y_{reg} is an irreducible Harish-Chandra module of (dominant) regular infinitesimal character ν_{reg} with $\psi^{\nu}_{\nu_{reg}}(Y_{reg}) = Y$ (as in Theorem 3.4(a)), then

$$T_i(Y) = \sum_{w \in W_o} \psi^{\nu'}_{\nu_{reg}}(w^{-1} \cdot \Theta(Y_{reg})).$$

Pf. Let F^i denote the finite dimensional representation with extremel weight $\nu' - \nu$, i.e. $F^i = \bigwedge^{n-i} \mathbb{C}^n$. From the definition of a coherent family, we have

$$T_i(Y) = \sum \Theta(\nu + \gamma),$$

where γ is a weight of F^i and $\nu + \gamma = w\nu'$ for some $w \in W$. Hence we are to determine when $w\nu' - \nu$ is a weight of F^i. Obviously this is the case if $w \in W^{\nu'}$, and it is easy to see that the same is true if $w \in W^\nu$. On the other hand, one can check directly that if $w \notin W^\nu W^{\nu'}$, then $w\nu' - \nu$ cannot be a weight of F^i. Finally notice that $\{w\nu' \mid w \in W_o\}$ coincides with $\{w\nu' \mid w \in W^\nu W^{\nu'}\}$. The first assertion of the lemma follows. The second assertion is clear. $\square$

Since ν' is very singular, many terms of the form $\psi^{\nu'}_{\nu_{reg}}(w^{-1} \cdot \Theta(Y_{reg}))$ will vanish in the expression for $T_i(Y)$. In fact, in practice we will only need to compute the action of a single w^{-1} on $\Theta(Y_{reg})$, so the computation becomes tractable. In any event, Lemma 7.5 suggests that we need to know something about the coherent continuation representation, and the next lemma provides that kind of information.

Lemma 7.6. *Let α and β be consecutive adjacent simple roots spanning a subroot system $A_2 \subset A_{n-1}$. Let X be an irreducible Harish-Chandra module with nonsingular integral infinitesimal character, and suppose $\beta \in \tau(X)$ while $\alpha \notin \tau(X)$. Then $s_\alpha \Theta(X)$ contains a unique irreducible constituent X' such that $\beta \notin \tau(X)$ and $\alpha \in \tau(X)$. Moreover, in this setting, we have the following conclusions:*

(a) *The tableau S' parameterizing* $\mathrm{Ann}(X')$ *is explicitly computable as a 'hook exchange' of the tableau S parameterizing* $\mathrm{Ann}(X)$. *More precisely, write $\beta = e_{k-1} - e_k$ and $\alpha = e_k - e_{k+1}$. Assume X has (dominant) infinitesimal character $\nu = (\nu_1, \ldots, \nu_n)$. The τ invariant assumptions on X imply (cf. the comments preceding Definition 3.3) that the coordinates $\nu_{k-1}, \nu_k, \nu_{k+1}$ are arranged in one of two relative configurations:*

$$\begin{array}{ccc} & & \boxed{\nu_{k-1}} \\ & \boxed{\nu_{k+1}} & \\ \boxed{\nu_k} & & \end{array} \qquad \textit{or} \qquad \begin{array}{ccc} & & \boxed{\nu_{k+1}} \\ & \boxed{\nu_{k-1}} & \\ \boxed{\nu_k} & & \end{array} .$$

Then S' coincides with S except that in the first case, the coordinates ν_{k-1} and ν_k are interchanged; and in the second case, the coordinates ν_k and ν_{k+1} are interchanged.

(b) *In particular, if γ is a simple root orthogonal to α and β, then $\gamma \in \tau(X)$ if and only if $\gamma \in \tau(X')$.*

Pf. The first statement is Theorem 3.10(b) in [V3]. Part (a) is explained very carefully in the statement of [V1, Theorem 3.2]. Part (b) is elementary (though it obviously follows from part (a) and Lemma 3.2). $\square$

As the equivalence relation of Section 6 suggests, we are going to essentially reduce to the case of two columns; this is the setting of the next lemma.

Lemma 7.7. *Let X be an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ. Suppose $S = \mathrm{Ann}(X)$ has a partition $S = S_1 \coprod S_2$ into difference-one skew columns of size n_1 and n_2 in good position (Definition 5.10). Let $i = n_1$ and $T = T_i$ (Notation 1.2 or as in Lemma 7.1). Set*

$$\nu(k) = \nu + k\mu_i$$

and consider the $\nu(k)$-quasitableau $S(k) = S_1 \coprod S_2(k)$ where $S_2(k)$ denotes the skew column obtained by adding k to each entry of S_2. Suppose that the partition of $S(k)$ is mediocre. Then $T^k(X)$ is nonzero if and only if $\mathrm{overlap}(S_1, S_2) \geq \mathrm{sing}(S_1, S_2(k))$. In this case, $T^k(X)$ is irreducible and the annihilator $\mathrm{Ann}(T^k(X))$ is obtained from $S(k)$ by Procedure 6.5.

Pf. By the translation principle (Theorem 3.4), it is enough to treat the case when $\nu = \rho$. (In this case the condition that $S(k) = S_1 \coprod S_2(k)$ be mediocre is equivalent to requiring $k \leq \max(n_1, n_2)$.) The proof of the lemma follows from a complicated induction on k. The case $k = 1$ is essentially treated by Theorem 3.1 (see especially the comments following Theorem 3.4). In a little more detail, if $\mathrm{overlap}(S_1, S_2) = 0$, then from Definition 6.1 and Lemma 3.2, we conclude that the simple reflection s through $e_{n_1} - e_{n_1+1}$ is in the τ-invariant of X. Hence $T(X)$ is zero, and this is exactly what Procedure 6.5(b) gives. On the other hand, if $\mathrm{overlap}(S_1, S_2) > 0$, then $s \notin \tau(X)$, and the paragraph following Theorem 3.4 implies that $\mathrm{Ann}(T(X)) = S_1 \coprod S_2(1)$; this agrees with Procedure 6.5(a).

We will describe the $k = 2$ case and sketch how to reduce the $k = 3$ case to the $k = 2$ one. The formidable details of the general induction are left to the reader. For future reference, we let $A(k) = \{\alpha \text{ simple } \mid \langle \alpha, \rho(k) \rangle = 0\}$, the walls on which $\rho(k)$ is singular. Then $\mathrm{sing}(S_1, S_2(k)) = \#A(k)$. We also write $\widetilde{\rho}(k)$ for a dominant representative of the Weyl group orbit of $\rho(k)$.

We describe the $k = 2$ case now. We apply Lemma 7.5 to compute $T^2(X)$; to apply the lemma, we take $Y = T(X)$, $Y_{reg} = X$, and $W_o = \{e, s_\gamma\}$ where s_γ is the reflection in the simple root $\gamma = e_{n_1} - e_{n_1+1}$. Hence

$$T(Y) = \psi_\rho^{\widetilde{\rho}(2)}(s_\gamma \cdot \Theta(X) + \Theta(X)) = \psi_\rho^{\widetilde{\rho}(2)}(s_\gamma \cdot \Theta(X)), \tag{10}$$

with the last equality following because $A(2) \cap \tau(X)$ is nonempty by hypothesis. Hence we are interested in locating constituents X' of $s_\gamma \cdot \Theta(X)$ so that $\tau(X') \cap A(2)$ is empty.

Label the simple roots near γ as follows:

$$\overset{\alpha}{\bullet} \text{———} \overset{\beta}{\bullet} \text{———} \overset{\gamma}{\bullet} \text{———} \overset{\delta}{\bullet} \text{———} \overset{\epsilon}{\bullet}$$

(Of course, some of these vertices need not exist on the Dynkin diagram, so ignore them if they don't.) There are several possibilities for $A(2)$; either $A(2) = \{\beta\}$, $\{\delta\}$, or $\{\beta, \delta\}$.

Assume $A(2) = \{\beta\}$ (the $A(2) = \{\delta\}$ case being identical by symmetry, i.e. by composing with an outer A_n automorphism coming from the Dynkin diagram). Then Lemma 7.6(a) implies that there is a unique constituent X' of $s_\gamma \cdot \Theta(X)$ with $\beta \notin \tau(X')$, i.e. with $\tau(X') \cap A(2)$ empty; moreover the underlying tableau of $\mathrm{Ann}(X')$ is obtained by a hook exchange also described in Lemma 7.6(d). The remarks following Theorem 3.4 and Equation (10) imply that the underlying tableau of $\mathrm{Ann}(T^2(X))$ coincides with that of $\mathrm{Ann}(X')$, and hence we have computed the annihilator of X'. On the other hand, in this case necessarily $n_1 = 1$, so Procedure 6.5(c) applies and gives non-zero $\rho(2)$-tableau. A direct check shows that this tableau is indeed $\mathrm{Ann}(X')$.

Now assume $A(2) = \{\beta, \delta\}$. By the above, we know that there is a unique constituent X' of $s_\gamma \cdot \Theta(X)$ with $\beta \notin \tau(X')$, and we know its underlying tableau. There are two possibilities here: either $\delta \in \tau(X')$, in which case $T^2(X)$ is zero (by Equation (10) and τ-invariant considerations); or $\delta \notin \tau(X')$, in which case the underlying tableau of $\mathrm{Ann}(X')$ coincides with that of $\mathrm{Ann}(T^2(X))$. We can distinguish between these two case by explicitly examining the hook exchange giving $\mathrm{Ann}(X')$.

There are two possibilities for the *relative* positions of the coordinates $n_1 - 1, n_1, n_1 + 1$, and $n_1 + 2$ in the underlying tableau of X; either

$$\begin{array}{|c|c|}\hline n_1 - 1 & n_1 + 1 \\ \hline n_1 & n_1 + 2 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline n_1 - 1 & n_1 + 1 \\ \hline n_1 \\ \cline{1-1} n_1 + 2 \\ \cline{1-1} \end{array} \; .$$

In the first case $\mathrm{overlap}(S_1, S_2) \geq 2$, and in the second $\mathrm{overlap}(S_1, S_2) = 1$. The hook exchange of Lemma 7.6(d) interchanges the coordinates n_1 and n_1+1. Thus, by Lemma 3.2, $\delta \notin \tau(X)$ if and only if we are in the first case. Hence $T^2(X) \neq 0$ if and only if $\mathrm{overlap}(S_1, S_2) \geq 2 = k$, and in this case one may verify that the hook-exchanged underlying tableau of $\mathrm{Ann}(X')$ coincides with the one given in Procedure 6.5. The $k = 2$ case is thus completed.

Now consider the $k = 3$ case. There are five possibilities for $A(3)$:

$$A(3) = \{\alpha\}, \{\epsilon\}, \{\alpha, \gamma\}, \{\gamma, \epsilon\}, \text{ or } \{\alpha, \gamma, \epsilon\}.$$

The first case is handled exactly as the the first case treated in the $k = 2$ case. If $A(3) = \{\alpha\}$, then since $k = 3 \leq \max(n_1, n_2)$, necessarily $A(2) = \{\beta\}$. Using Lemma 7.5 to compute $T(Y)$ with $Y = T^2(X)$, we can take $Y_{reg} = X'$ as defined in the $k = 2$ case above, and $W_o = \{e, s_\beta\}$. By τ-invariant considerations,

$$T(Y) = \psi_\rho^{\tilde{\rho}(3)}(s_\beta \cdot \Theta(X)),$$

and so we are to locate constituents of $s_\beta \cdot \Theta(X)$ that do not contain α in their τ-invariants. Again using Lemma 7.6, there is a unique such constituent of $s_\beta \cdot \Theta(X')$ with $\alpha \notin \tau(X'')$; the underlying tableau of X'' is computed by a hook exchange from that of X', and one can check explicitly that the underlying tableau of X'' is the one described by Procedure 6.5. The case $A(3) = \{\epsilon\}$ is handled in exactly the same way (or by symmetry).

Next note that by symmetry, the case of $A(3) = \{\alpha, \gamma\}$ is identical to the case of $A(3) = \{\gamma, \epsilon\}$, so assume now that $A(3) = \{\alpha, \gamma\}$ or $\{\alpha, \gamma, \epsilon\}$. In both of these cases, necessarily we have $A(2) = \{\beta, \delta\}$. Hence, using Lemma 7.5 to compute $T(Y), Y = T^2(X)$, we can take

$Y_{reg} = X'$ (as defined in the $k = 2$ case above) and $W_o = \{e, s_\beta, s_\delta, s_\delta s_\beta\}$. We are thus interested in constituents X'' of $w \cdot \Theta(X')$, $w \in W_o$, with $\tau(X'') \cap A(3)$ empty.

Now in either case at hand, $\alpha \in A(3)$. Hence if a constituent Z of $w \cdot \Theta(X')$ is to survive the translation $\psi_\rho^{\widetilde{\rho}(3)}$, we must have $\alpha \notin \tau(Z)$. Since $\alpha \in \tau(X')$ and α is orthogonal to s_δ, Lemma 7.6(b) implies that either $w = s_\beta$ or $s_\delta s_\beta$. In any event, Lemma 7.6 implies that there is a unique constituent Z of $s_\beta \Theta(X)$ with $\alpha \notin \tau(Z)$; a hook exchange on $\mathrm{Ann}(X')$ computes the underlying tableau of Z. Either by explicitly examining the underlying tableau of Z or by an elementary calculation with the coherent continuation representation (along the lines of the proofs of Lemma 7.6(b),(c) in [V3]) one sees that $\gamma \notin \tau(Z)$. Since $\gamma \in A(3)$, we conclude that $\psi_\rho^{\widetilde{\rho}(3)}(s_\beta \cdot \Theta(X)) = 0$.

Thus it remains to find constituents X'' of $s_\delta s_\beta \cdot \Theta(X')$, with $\tau(X'') \cap A(3)$ empty. By the above, all such constituents arise in $s_\delta \cdot \Theta(Z)$. In fact, we are exactly in the setting of the $k = 2$ case with Z taking the place of $T(X)$ and the root δ taking the place of γ. This is essentially the inductive step.

In the case that $A(3) = \{\alpha, \gamma\}$, Lemma 7.6(a) says that there is always a constituent X'' of $s_\delta(Z)$ with $\tau(X'') \cap A(3)$ empty; its underlying tableau is computable in terms of a hook exchange on the underlying tableau of Z (i.e. two hook exchanges on the underlying tableau of X'). The underlying tableau of X'' coincides with the underlying tableau of $T^3(X)$ and one may verify directly that this is the underlying tableau of the $\rho(3)$-tableau produced by Procedure 6.5.

In the case that $A(3) = \{\alpha, \gamma, \epsilon\}$, then $T^3(X) = 0$ if and only if the X'' described in the previous paragraph has $\epsilon \in \tau(X'')$. A direct inspection of tableaux reveals that this is equivalent to requiring $\mathrm{overlap}(S_1, S_2) = 2$. In the case that $\mathrm{overlap}(S_1, S_2) \geq 3$, $\epsilon \notin \tau(X'')$ and $T^3(X) \neq 0$. Its underlying tableau is that of $\mathrm{Ann}(X'')$ and hence may be computed as in the previous paragraph and can be seen to coincide with Procedure 6.5. □

Remark 7.8. Consider a particular example of the lemma. Let $\mathfrak{q}$ be a maximal parabolic, and let $X = A_{\mathfrak{q}}(\mathbb{C}_{triv})$. The root γ is the unique simple root not contained in $\mathfrak{l}$, and the assumption on k implies that $\lambda = k\mu_i$ is in the mediocre range for $\mathfrak{q}$. The lemma gives a sharp condition on k guaranteeing that $T^k(X)$ is nonzero irreducible, and it computes its annihilator. By Lemma 2.13, $A_{\mathfrak{q}}(\lambda) = T^k(X)$, and we thus obtain a special case of Theorem 6.9. Closer inspection reveals that we have proved more: we have, in fact, deduced a special case of Theorem 2.1b(iv) using only irreducibility in the good range (Theorem 2.1b(iii)).

We need to extend this two column case to the case of adjacent skew columns in a partition of S. The arguments given above carry over to this case, so long as the adjacent columns do not interact with the rest of the tableau.

Definition 7.9. Suppose S is a ν-antitableau of size n and $S = \coprod S_i$ is a partition into skew columns. The adjacent columns S_j and S_{j+1} are said to be isolated if:

(a) the entries of S_i, $i < j$, are strictly greater than the entries of $S_j \coprod S_{j+1}$; and
(b) the entries of S_i, $i > j+1$, are strictly smaller than the entries of $S_j \coprod S_{j+1}$.

Here is the more general two column result.

Proposition 7.10. *Let X be an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ, and let S be the ν-tableau corresponding to* $\mathrm{Ann}(X)$ *(Theorem 3.1). Suppose S has a partition into skew columns $S = \coprod S_i$ and that S_j and S_{j+1}*

are difference-one and in good position (Definition 5.10). Let the column S_i have length n_i and set

$$\gamma = e_t - e_{t+1}, \qquad t = \sum_{i \le j} n_i.$$

For any integer k, let $S_{j+1}(k)$ be the skew column obtained by adding k to every entry of S_{j+1}. Assume that S_j and $S_{j+1}(k)$ are isolated (in the sense of Definition 7.9) in $\coprod_{i \le j} S_i \coprod_{i \ge j+1} S_i(k)$. Let

$$\nu(k) = \nu + k\mu_l,$$

set $R(k) = S_j \coprod S_{j+1}(k)$ be the indicated skew $\nu(k)$-quasitableau, and assume S_j and $S_{j+1}(k)$ are in mediocre position. Then $T^k_\gamma(X)$ is nonzero if and only if

$$\operatorname{overlap}(S_j, S_{j+1}) \ge \operatorname{sing}(S_j, S_{j+1}(k)).$$

In this case, $T^k_\gamma(X)$ is irreducible and the tableau $S(k)$ which parametrizes $\operatorname{Ann}(T^k_\gamma(X))$ is given by

$$S(k) = \coprod_{i \le j-1} S_i \coprod R'(k) \coprod_{i \ge j+2} S_i(k),$$

where $R'(k)$ is skew tableau obtained from $R(k)$ using Procedure 6.5.

Sketch. Again the translation principle reduces the lemma to the case of $\nu = \rho$. The arguments of Lemma 7.7 extend to this setting, but not immediately so, since the geometry of the adjacent columns S_j and S_{j+1} is more complicated. To prove the proposition one needs to understand the computation of $\operatorname{Ann}(T^k_\gamma(X))$ of the previous lemma in terms of explicit hook exchanges. These are precisely the hook exchanges that appear in the computation of $\operatorname{Ann}(T^k_\gamma(X))$ in the more general setting of the proposition. Since hook exchanges only depend on the relative position of the entries in the tableau, and since the relative positions are essentially the same in both the two column and adjacent column setting, the proof goes through. We leave the details to the reader. $\square$

Remark 7.11. Again taking X to be an appropriate $A_{\mathfrak{q}}(\lambda)$, we recover a special case of Theorem 6.9.

We have thus completed a description of the two column case. To pass to the general case, we need to prove a statement describing how to combine more than two columns and, in order to do so, we need to gain control over the formula in Lemma 7.5. In the case of Lemma 7.7, we were able to do this using Lemma 7.6. Here is the generalization that we need.

Lemma 7.12. *Let $\alpha_1 = e_1 - e_2, \dots, \alpha_l = e_l - e_{l+1}$ span a root subsystem $A_l \subset A_{n-1}$, and write $W(l)$ for the corresponding Weyl subgroup. Let X be an irreducible Harish-Chandra module with nonsingular integral infinitesimal character, and suppose*

$$\alpha_1, \dots, \alpha_{l-1} \notin \tau(X), \text{ but } \alpha_l \in \tau(X).$$

Then there is a unique constituent X' of $\sum_{w \in W(l)} w \cdot \Theta(X)$ such that

$$\alpha_2, \dots, \alpha_l \notin \tau(X), \text{ but } \alpha_1 \in \tau(X).$$

Moreover, X' is actually a constituent of $s_{\alpha_1} \cdots s_{\alpha_{l-1}} \cdot \Theta(X)$, and:

(a) *The underlying tableau of X' can be explicitly computed by iterating hook exchanges through the coordinates*

$$e_{l-2-i}, e_{l-1-i}, e_{l-i}, \qquad i = 0, \ldots, l-3,$$

on the underlying tableau of X.

(b) *If γ is orthogonal to $\alpha_1, \ldots, \alpha_l$, then $\gamma \in \tau(X)$ if and only if $\gamma \in \tau(X')$.*

Pf. The statement follows by induction on l, the base case $l = 2$ being treated by Lemma 7.6. The induction is complicated to write down, but all the ideas are contained in the proof of the $l = 2$ case. We refer the reader to the details of Theorem 3.10(b) in [V3]. □

Now we can prove a statement about 'nice' multi-column translations. On level of tableaux, these translations are easy to compute: one simply changes the coordinates of the infinitesimal character. (This generalizes the comments following Theorem 3.4.)

Lemma 7.13. *Suppose X is an irreducible Harish-Chandra module whose infinitesimal character ν is a weight translate of ρ, and let S be the ν-antitableau corresponding to* Ann(X). *Suppose $S = \coprod_{i=1}^{m+1} S_i$ is a good partition of S into $m+1$ difference-one skew columns S_i of length n_i. For $i = 1, \ldots, m$ set*

$$\gamma_i = e_{t_i} - e_{t_i+1}, \qquad t_i = \sum_{j \leq i} n_i.$$

Consider integers $k_1, \ldots, k_m$, and define

$$\nu(k_1, \ldots, k_m) = \nu + \sum_{i \leq m} k_i(e_{t_i+1} + \cdots + e_n).$$

As usual, for any integer k, let $S_i(k)$ denote the skew column obtained from S_i by adding k to every entry, and consider the $\nu(k_1, \ldots, k_m)$-tableau

$$S(k_1, \ldots, k_m) = \coprod S_i(l_i), \qquad l_i = \sum_{j \leq i} k_j.$$

Suppose that this partition is nice (Definition 5.10). Then $\mathrm{Ann}(T_{\gamma_m}^{k_m} \circ \cdots \circ T_{\gamma_1}^{k_1}(X))$ *is nonzero if and only if*

$$\mathrm{overlap}(S_i, S_{i+1}) \geq \mathrm{sing}(S_i(l_i), S_{i+1}(l_{i+1})) \qquad \textit{for all } i;$$

in this case,

$$\mathrm{Ann}(T_{\gamma_m}^{k_m} \circ \cdots \circ T_{\gamma_1}^{k_1}(X)) = S(k_1, \ldots, k_m).$$

Pf. Again the lemma reduced to the case $\nu = \rho$. (In this case the nice hypothesis is equivalent to $k_i \leq \min(n_{i-1}, n)$ for all i, and the condition for nonvanishing of the translation functor is that $\mathrm{overlap}(S_i, S_{i+1}) \leq k_i$, for all i.) The proof is an extremely complicated double induction on m and k_m using the ideas in the proof of Lemma 7.7. The idea is to use Lemma 7.5 to compute successive application of T_{γ_m}. By using using Lemma 7.12, we can reduce matters to locating the constituents of a single $w \cdot \Theta(Y)$ that have the correct τ-invariants. The 'nice' assumption of the lemma guarantees that we may proceed exactly as in the proof of Lemma 7.7 to locate a unique such constituent. The annihilator of this constituent can be explicitly computed by hook exchanges, the result of which is given in the statement of the lemma. We omit the horrendous details. □

Remark 7.14. By Theorem 5.4, the previous lemma applies with $X = A_{\mathfrak{q}}(\mathbb{C}_{triv})$, and (using Lemma 2.13) we recover another special case of Theorem 6.9. (When there are only two columns, this is subsumed by Lemma 7.7.) Moreover, we have deduced from Theorem 2.1(b)(iii) that any nice $A_{\mathfrak{q}}(\lambda)$ is nonzero and irreducible.

Now we have amassed all the tools to prove Theorem 6.9 for the weakly fair range. The proof is an induction on r; the $r = 1$ case is trivial, and the $r = 2$ case is Lemma 7.7 (see Remark 7.8). So consider the $r = 3$ case; we are trying to compute $\mathrm{Ann}(A_{\mathfrak{q}}(\lambda))$ for λ in the mediocre range. Taking $X = A_{\mathfrak{q}}(\lambda')$ for an appropriate λ' (in the good range) of the form

$$\lambda = \lambda' + k_1\mu_{\gamma_1} + k_2\mu_{\gamma_2},$$

we are to compute $T_1^{k_1}T_2^{k_2}(X)$, where $T_i = T_{\gamma_i}$. (By changing λ', we can assume both k_1 and k_2 are positive.) Use Theorem 5.4 to compute $\mathrm{Ann}(X) = S_1 \coprod S_2 \coprod S_3$. Lemma 7.7 (that is, the $r = 2$ case) computes

$$\mathrm{Ann}(T_2^{k_2}(X)) = S_1 \coprod S_2' \coprod S_3',$$

where S_2' and S_3' are in nice position and obtained by applying Procedure 6.5 and the rest of the definition of the equivalence relation to $S_2 \coprod S_2(k_2)$. By Lemma 7.13, we can write $T_2^{k_2}(X)$ as $S_2^{m_2}(X')$ for an appropriate $S_2 = T_{i_2}$, where

$$\mathrm{Ann}(X') = S_1 \coprod S_2' \coprod S_3'(-m_2).$$

(It is important to note that X' may not be an $A_{\mathfrak{q}}(\lambda)$ module — this is the sense in which we must move outside the class of $A_{\mathfrak{q}}(\lambda)$ modules.) Taking m_2 large enough, we can assume S_1 and S_2' are isolated in the sense of Definition 7.9.

Now we are interested in computing the annihilator of $T_1^{k_1}T_2^{k_2}(X)$. By the above this is $T_1^{k_1}S_2^{m_2}(X')$, and by Corollary 7.4, this is $S_2^{m_2}T_1^{k_1}(X')$. Since the columns S_1 and S_2' of $\mathrm{Ann}(X')$ are isolated, we can use Proposition 7.10 to compute $T_1^{k_1}(X')$; the result is

$$\mathrm{Ann}(T_1^{k_1}T_2^{k_2}(X)) = \mathrm{Ann}(S_2^{m_2}(X'')),$$

where

$$\mathrm{Ann}(X'') = S_1' \coprod S_2'' \coprod S_3'(-m_2+k_1);$$

here $S_1' \coprod S_2''$ is obtained by applying Procedure 6.5 to $S_1 \coprod S_2'(k_1)$. Now the second two columns are in nice position, so we can isolate them using Lemma 7.13, interchange the order of translation, and use Lemma 7.7 on the first two columns, and so on.

It is clear that we are obtaining the see-saw algorithm described after Theorem 6.9. As remarked there, the algorithm must eventually either produce zero or a nice partition (which we know how to put together using Lemma 7.13). This finishes the $r = 3$ case. It is clear that the arguments just described suffice to handle the general case, and thus the proof of Theorem 6.9 is complete. We have also deduced Theorem 2.1b(iv) from Theorem 2.1b(iii).

8. Evidence for conjecture 0.1

In this section, we prove a small small piece of Conjecture 0.1 (see Remark 2.7).

Theorem 8.1. *Let λ be in the mediocre range for $\mathfrak{q}$ (Definition 2.4). Then there exists λ' in the weakly fair range for some $\mathfrak{q}'$, so that*

$$A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{q}'}(\lambda').$$

As one might expect, we are going to reduce the theorem to the case of maximal $\mathfrak{q}$. This case turns out to follow from a simple application of Lemma 7.7, as the next example illustrates.

Example 8.2. Let $(p, q) = (5, 2)$, let $\mathfrak{q}$ be attached to $\{(3,2), (2,0)\}$ and let $\lambda = (1, \ldots, 1, 5, 5)$, which is outside the weakly fair range, but inside the mediocre range for $\mathfrak{q}$. Theorem 5.4 attaches the following partition to $A_{\mathfrak{q}}(\lambda)$,

$$\begin{ytableau} 4 & 3 \\ 3 & 2 \\ 2 \\ 1 \\ 0 \end{ytableau} = \begin{ytableau} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{ytableau} \ \coprod\ \begin{ytableau} \ & 3 \\ \ & 2 \\ \ \\ \ \\ \ \end{ytableau},$$

and Lemma 7.7 guarantees that this is the annihilator of $A_{\mathfrak{q}}(\lambda)$. On the other hand, we can compute the associated variety directly from Proposition 4.4 and Lemma 4.6, giving

$$\mathrm{AV}(A_{\mathfrak{q}}(\lambda)) = \begin{ytableau} - & + \\ - & + \\ + \\ + \\ + \end{ytableau}.$$

Thus we are looking for a weakly fair $A_{\mathfrak{q}'}(\lambda')$ with the above tableau parameters.

The idea is to move the first column of the above partition past the second one, in order to move from the mediocre range to the fair range. To make this precise, we notice that the previous partition is equivalent (in the sense of Definition 6.4) to the following one

$$(*) \qquad \begin{ytableau} 3 & 4 \\ 2 & 3 \\ 2 \\ 1 \\ 0 \end{ytableau} = \begin{ytableau} 3 \\ 2 \end{ytableau} \ \coprod\ \begin{ytableau} \ & 4 \\ \ & 3 \\ 2 \\ 1 \\ 0 \end{ytableau}.$$

In fact, we skipped a step in this last assertion: Procedure 6.5(b)-(c) give that both of the preceding partitions are equivalent to the nice partition

$$\begin{ytableau} 4 & 3 \\ 3 & 2 \\ 2 \\ 1 \\ 0 \end{ytableau} = \begin{ytableau} 4 \\ 3 \\ 2 \end{ytableau} \ \coprod\ \begin{ytableau} \ & 3 \\ \ & 2 \\ \ \\ 1 \\ 0 \end{ytableau}.$$

Incidentally, we see directly that this partition is the one that Theorem 5.4 attaches to $A_{\mathfrak{q}''}(\lambda'')$ where $\mathfrak{q}''$ corresponds to $\{(1,2), (0,4)\}$ and $\lambda'' = (1,1,1,3,\ldots,,3)$; Lemma 7.7 implies that the tableau is in fact $\mathrm{Ann}(A_{\mathfrak{q}''}(\lambda''))$. Moreover we can check directly from

Lemma 4.6 that $\mathrm{AV}(A_{\mathfrak{q}''}(\lambda'')) = \mathrm{AV}(A_{\mathfrak{q}}(\lambda))$. Hence we conclude that $A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}''}(\lambda'')$ which is in the fair (in fact, nice) range. So we have shown that the mediocre $A_{\mathfrak{q}}(\lambda)$ is isomorphic to a fair one. To make the conclusion generalize, however, we need to return to the partition given in $(*)$.

Now the partition $(*)$ is weakly fair, and if it is to correspond to some $A_{\mathfrak{q}'}(\lambda')$ with $\mathrm{AV}(A_{\mathfrak{q}}(\lambda)) = \mathrm{AV}(A_{\mathfrak{q}'}(\lambda'))$, Lemma 4.6 implies that $\mathfrak{q}'$ must be attached to $\{(0,2),(5,0)\}$. The data of $\mathfrak{q}'$ and the infinitesimal character imply that $\lambda' = (0,0,3,\ldots,3)$, which is in the weakly fair range for $\mathfrak{q}$. In fact, one can check directly that this partition is the one that Theorem 5.4 attaches to the weakly fair $A_{\mathfrak{q}'}(\lambda')$. Hence we conclude that $A_{\mathfrak{q}}(\lambda) \cong A_{\mathfrak{q}'}(\lambda')$, as desired.

The second argument given in the example easily leads to a general two column result.

Lemma 8.3. *Let $\mathfrak{q}$ be the maximal parabolic attached to $\{(p_1,q_1),(p_2,q_2)\}$, set $n_i = p_i + q_i$. Suppose*

$$\lambda = (\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1},\overbrace{\lambda_2,\ldots,\lambda_2}^{n_2})$$

is inside the mediocre range (but outside the weakly fair range) for $\mathfrak{q}$ and that $A_{\mathfrak{q}}(\lambda) \neq 0$; explicitly (using Definition 2.4 and Lemma 7.7) these conditions become

$$-\frac{n}{2} < \lambda_1 - \lambda_2 < -max(n_1,n_2);$$

$$\text{if } p_1+q_1 \geq p_2+q_2 \text{ then } p_1 \geq q_2 \text{ and } p_2 \geq q_1; \text{ and}$$

$$\text{if } p_1+q_1 \leq p_2+q_2 \text{ then } p_1 \leq q_2 \text{ and } p_2 \leq q_1.$$

Set

$$\lambda' = (\overbrace{\lambda_2,\ldots,\lambda_2}^{n_2},\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1}) + (\overbrace{-n_1,\ldots,-n_1}^{n_2},\overbrace{n_2,\ldots,n_2}^{n_1}),$$

and

(a) *if $p_1+q_1 \geq p_2+q_2$, let $\mathfrak{q}'$ be attached to*

$$\{(q_2,p_2),(p_1+p_2-q_2,q_1+q_2-p_2)\};$$

(b) *if $p_1+q_1 \geq p_2+q_2$, let $\mathfrak{q}'$ be attached to*

$$\{(p_1+p_2-q_1,q_1+q_2-p_1),(q_1,p_1)\};$$

Then λ' is in the weakly fair range for $\mathfrak{q}'$ and $A_{\mathfrak{q}}(\lambda) \cong A_{\mathfrak{q}'}(\lambda')$.

(Note that the hypothesis on the range of λ and the non-vanishing of $A_{\mathfrak{q}}(\lambda)$ imply that the sequence to which $\mathfrak{q}'$ is attached consists of pairs of nonnegative integers, as it must.)

An inductive argument using induction in stages now completes the proof of Theorem 8.1. The induction is not as trivial as it may first seem; in the multicolumn case, the application of Lemma 8.3 to two columns changes the relative position of other columns with respect to the original two. We leave the details to the reader.

Example 8.4. Let $\mathfrak{q}$ be attached to the sequence $\{(1,0),(p-1,q)\}$, assume $q > 1$, and let

$$\lambda = (\lambda_1,\overbrace{\lambda_2,\ldots,\lambda_2}^{p+q-1})$$

with

$$p+q-1 \leq \lambda_1 - \lambda_2 < \frac{n}{2}.$$

The lemma shows that $A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}'}(\lambda')$ where $\mathfrak{q}'$ is attached to the sequence $\{(p, q-1), (0,1)\}$, and

$$\lambda' = (\lambda_2, \overbrace{\lambda_1, \ldots, \lambda_1}^{p+q-1}) + (\overbrace{1, \ldots, 1}^{p+q-1}, -p-q+1).$$

Now $\mathfrak{u} \cap \mathfrak{p}$ is an irreducible as a representation of $L \cap K$, and similarly for $\mathfrak{q}'$. Hence the Blattner formula implies that both modules $A_{\mathfrak{q}}(\lambda)$ and $A_{\mathfrak{q}'}(\lambda')$ are ladder representations whose (multiplicity-free) K type spectrums can be explicitly computed. The result of the computation shows the K types of both modules coincide, and hence one verifies directly that $A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}'}(\lambda')$.

We conclude this section by sketching a proof of Conjecture 0.1 for $U(p,1)$. By directly applying Theorem 6.9, one deduces the following result.

Corollary 8.5. *Suppose X is an irreducible Harish-Chandra module for $U(p,1)$ whose infinitesimal character ν is a weight translate of ρ, and suppose ν is contained in a central translate of the convex hull of the Weyl group orbit of ρ. Then X is isomorphic to a weakly fair (in fact, weakly good) $A_{\mathfrak{q}}(\lambda)$ module.*

Now suppose X is any irreducible Harish-Chandra module for $U(p,1)$ whose infinitesimal character is a weight translate of ρ. If X is unitary, then the main conjecture of [SaV] implies that X is cohomologically induced (in the good range) from an irreducible unitary representation X_L (on a Levi factor L) whose infinitesimal character is a central translate of the convex hull of the $W_{\mathfrak{l}}$ orbit of $\rho_{\mathfrak{l}}$. By the corollary, X_L is a weakly good $A_{\mathfrak{q}}(\lambda)$. Since good range cohomological induction takes weakly good $A_{\mathfrak{q}}(\lambda)$ to weakly good $A_{\mathfrak{q}}(\lambda)$, we conclude that any unitary representation of $U(p,1)$ is a weakly fair (in fact, weakly good) $A_{\mathfrak{q}}(\lambda)$ module, verifying Conjecture 0.1. (This case was treated by [BaBa].)

The arguments of the preceding paragraph show how to reduce Conjecture 0.1 to considerations inside the convex hull of $W\rho$.

PART II: ROBINSON-SCHENSTED ALGORITHMS FOR REAL GROUPS

9. NOTATION FOR PART II

9.1. **General Notation.** In Part II, we will adhere to the notations outlined in Section 1, except in the following cases. We will take G to be an arbitrary linear real reductive group with maximal compact subgroup K. We write $K_{\mathbb{C}}$ and $G_{\mathbb{C}}$ for the corresponding complexifications. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the complexified Cartan decomposition, and let B be a Borel subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$. (We now write $\mathfrak{h}$ for our Cartan since it may no longer have a compact real form.) Write Δ^+ for the roots of $\mathfrak{h}$ in $\mathfrak{u}$ and let $\rho = \rho(\Delta^+)$. Let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ and write w_o for the long element. Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}$ and write $\mathcal{N}_\theta$ for $\mathcal{N} \cap \mathfrak{p}$. Given $N \in \mathcal{N}$, we let X^N denote the fixed points of $1+N$ on the complex flag variety $X = G_{\mathbb{C}}/B$.

Given $v \in K_{\mathbb{C}} \backslash X$, and a $K_{\mathbb{C}}$ equivariant local system ϕ on v, the Beilinson-Bernstein theory produces an irreducible Harish-Chandra module for G (with infinitesimal character ρ) which we denote $L_G(v, \phi)$. When ϕ is trivial, we write $L_G(v)$ instead. Finally we let $\widehat{G}_\rho$ denote the set of irreducible Harish-Chandra modules for G with infinitesimal character ρ.

Next we carefully define some real forms of interest. Let $V \simeq \mathbb{C}^n$ be spanned by vectors $e_1, \ldots, e_n$. For $p+q = n$, define a form $\langle\ ,\ \rangle$ on V via

$$\langle \sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{n} b_j e_j \rangle = \sum_{i \leq p} \overline{a_i} b_i - \sum_{i \geq p+1} \overline{a_i} b_i.$$

The group $U(p,q)$ is defined to be the subset of $GL(n,\mathbb{C})$ preserving this form. (Unlike Part I, we will think of a choice of form being fixed once and for all.) With this definition $K = U(p) \times U(q)$ (embedded block diagonally) and $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$. Next, $SU^*(2n)$ is defined to be $SL(n,\mathbb{H})$ viewed as $2n$ dimensional complex matrices via the identification $\mathbb{H} = \mathbb{C} \oplus k\mathbb{C}$. In this setting, $K = U(n,\mathbb{H}) \simeq Sp(n)$, and $K_{\mathbb{C}} = Sp(2n,\mathbb{C})$ defined with respect to the symplectic form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Finally, if $G = GL(n,\mathbb{R})$, then $K = O(n)$ and $K_{\mathbb{C}} = O(n,\mathbb{C})$.

9.2. **Notation for S_n.** Write $\Sigma(n)$ for the set of involutions in S_n and define

$$\Sigma_0(n) = \{\sigma \in \Sigma \mid \sigma(i) \neq i \text{ for all } i\}$$

to be the set of involutions without fixed points. Let

$$\begin{aligned}\Sigma_\pm(n) = \{(\sigma, \epsilon) \in \Sigma \times \{+,-\}^n \mid\ & \epsilon_i = + \text{ if } \sigma(i) > i \text{ and} \\ & \epsilon_i = - \text{ if } \sigma(i) < i\}.\end{aligned}$$

We view $\Sigma_\pm(n)$ as the set of involution in S_n with signed fixed points. (The definition arranges a convenient normalization of the signs for the non-fixed points.) Finally we write $\Sigma_\pm(p,q)$ for the subset of $\Sigma_\pm(p+q)$ whose elements have exactly p of the ϵ_i's labeled $+$ (and q labeled $-$). The reader may find the pictures in Example 13.8 useful.

9.3. (More) tableau notation. We need to name a few things appearing in Notation 1.3. Let $\mathcal{D}(n)$ denote the set of Young diagrams of size n. We write $\mathcal{D}_e(n)$ for those diagrams whose rows are all even. Given $D \in \mathcal{D}_e(n)$, the transpose D^{tr} is a diagram in which each row length occurs an even number of times. We denote this set $\mathcal{D}_e^{tr}(n)$. We write $\mathcal{T}(n)$ for the set of Young tableaux of size n, and correspondingly $\mathcal{T}_e(n)$ and $\mathcal{T}_e^{tr}(n)$ for those tableaux of the indicated shape. Finally write $\mathcal{T}_\pm(p,q)$ for the set of signature (p,q) signed Young tableaux.

10. Springer's parametrization of $K_\mathbb{C}\backslash G_\mathbb{C}/B$

Begin by considering the generalized Steinberg variety

$$M = \{(N, gB) \in \mathcal{N}_\theta \times G_\mathbb{C}/B \mid N \in Ad(g)\mathfrak{u}\}.$$

(When G is itself a complex Lie group, $G_\mathbb{C}$ is diffeomorphic to $G \times G$, $K_\mathbb{C}$ is the diagonal copy of G, and M is the familiar Steinberg variety of triples.) Write $\mathrm{Irr}(M)$ for the irreducible components of M. Once we use an invariant bilinear form to identify $\mathfrak{p}$ and $\mathfrak{p}^*$, it is easy to see that M is the union of the conormal bundles of $K_\mathbb{C}$ orbits on $X = G_\mathbb{C}/B$. Given $v \in K_\mathbb{C}\backslash X$, the conormal bundle $T_v^*(X)$ need not be irreducible (since $K_\mathbb{C}$ need not be connected). It is clear, however, that $T_v^*(X)$ is pure of dimension equal to the dimension of X. It is also clear that $K_\mathbb{C}$ acts on $\mathrm{Irr}(M)$, and the each $T_v^*(X)$ is a single orbit in $K_\mathbb{C}\backslash\mathrm{Irr}(M)$. Hence we conclude that M has pure dimension $\dim(X)$ and that $K_\mathbb{C}\backslash\mathrm{Irr}(M)$ is parametrized by $K_\mathbb{C}\backslash X$.

On the other hand, we can consider the subset $M_{N,C}$ of M consisting of the closure of the $K_\mathbb{C}$ saturation of $N \times C \subset M$; here C is an irreducible component of the Springer fiber X^N. Now all such C have dimension equal to $\frac{1}{2}[\dim(Z_{G_\mathbb{C}}(N)) - \mathrm{rank}(G_\mathbb{C})]$, and hence we conclude that closure of the $K_\mathbb{C}$ orbit $M_{N,C}$ is a single $K_\mathbb{C}$ orbit on $\mathrm{Irr}(M)$ of pure dimension

$$\dim(K_\mathbb{C}) - \dim(Z_{K_\mathbb{C}}(N)) + \frac{1}{2}[\dim(Z_{G_\mathbb{C}}(N)) - \mathrm{rank}(G_\mathbb{C})].$$

A result of Kostant-Rallis ([KoR, Proposition 5]) insures that $\frac{1}{2}\dim(Z_{G_\mathbb{C}}(N)) - \dim(Z_{K_\mathbb{C}}(N))$ is equal, independent of N, to $\frac{1}{2}\dim(G_\mathbb{C}) - \dim(K_\mathbb{C})$. Applying this to the formula for $\dim(M_{N,C})$ we see that the $M_{N,C}$ each have dimension $\dim(X)$, and hence exhaust $K_\mathbb{C}\backslash\mathrm{Irr}(M)$.

A final point to consider is that the closures $M_{N,C}$ need not be distinct. To take this into account we need to consider the component group $A_{K_\mathbb{C}}(N)$ of the centralizer $Z_{K_\mathbb{C}}(N)$. We then obtain the following result.

Proposition 10.1 (Springer). *The set of $K_\mathbb{C}$ orbits on $\mathrm{Irr}(M)$ is parametrized by $K_\mathbb{C}\backslash X$ and by pairs consisting of a $K_\mathbb{C}$-orbit $K_\mathbb{C} \cdot N$ in $\mathcal{N}_\theta$ and an orbit of $A_{K_\mathbb{C}}(N)$ on the set of irreducible components of X^N.*

Hence we conclude that there is a bijection $K_\mathbb{C}\backslash X$ and pairs consisting of a $K_\mathbb{C}$-orbit $K_\mathbb{C} \cdot N$ in $\mathcal{N}_\theta$ and an orbit of $A_{K_\mathbb{C}}(N)$ on the set of irreducible components of X^N. Write Φ_1 for the map which takes $K_\mathbb{C}\backslash X$ to $K_\mathbb{C}\backslash\mathcal{N}_\theta$.

From the discussion preceding the proposition, we can describe Φ_1 as follows. Recall the moment map $\mu : T^*(X) \longrightarrow \mathfrak{g}^*$. Given an orbit $v \in K_\mathbb{C}\backslash X$, we can consider its conormal bundle $T_v^*(X)$ inside $T^*(X)$, and from the definition of μ it is not difficult to see that the moment map image $\mu(T_v^*(X))$ actually lives in $\mathcal{N}_\theta$. Since μ is proper and $T_v^*(X)$ is irreducible, $\mu(T_v^*(X))$ is an irreducible $K_\mathbb{C}$ equivariant subvariety of $\mathcal{N}_\theta$. Since the number of $K_{\mathbb{C}}$ orbits on $\mathcal{N}_\theta$ is finite, $\mu(T_v^*(X))$ is the closure of a single $K_{\mathbb{C}}$ orbit. In this way we obtain the element $\Phi_1(v) \in K_{\mathbb{C}}\backslash\mathcal{N}_\theta$.

The fibers of Φ_1 thus give an interesting partition of $K_{\mathbb{C}}\backslash X$ into disjoint subsets. For a fixed $\mathcal{O} \in K_{\mathbb{C}}\backslash\mathcal{N}_\theta$, we call $\Phi_1^{-1}(\mathcal{O})$ a geometric cell of $K_{\mathbb{C}}$ orbits for G. The terminology is suggestive, and will be explained in the next section.

Remark 10.2. In the complex case, X is a product of two flag varieties for G, and there is a natural refinement of geometric cells into left and right cells. McGovern [Mc3] (following van Leeuwen [vL]) has given a complete description of left and right geometric cells in the complex classical case.

11. Geometric cells and Weyl group representations

In this section, we take the opportunity to recall a few of the remarkable properties of geometric cells and, in particular, describe their relation with Weyl group representations. (This material is not needed elsewhere in Part II, and may be completely omitted.) The only potential novelty of this section is the conjecture discussed after Theorem 11.2. Though not absolutely essential, in this section we assume that G is connected.

In [Ro], Rossmann gives an action of the Weyl group on the top Borel-Moore homology group $H^{\text{top}}(M, \mathbb{C})$ of the Steinberg variety M. (When G is complex, Rossmann shows that the construction coincides with an earlier one given by Kazhdan-Lusztig in [KazL].) Now the fundamental classes of the conormal bundles $T_v^*(X)$, $v \in K_{\mathbb{C}}\backslash X$, give a basis for $H^{\text{top}}(M, \mathbb{C})$. In analogy with the case of the coherent continuation representation (cf. the beginning of Section 5), one considers subquotients of the Rossmann action which are minimal with respect to being spanned by fundamental classes of conormal bundles. In this way one obtains a partition of $K_{\mathbb{C}}\backslash X$ into disjoint subsets, which one would like to call some sort of cells. Tanisaki [Ta, Lemma 2.11] showed that the cells obtained in this way are precisely the geometric cells defined at the end of the previous section. (Actually, he dealt only with the complex case, but the result is true in general.)

Thus the elements of a fixed geometric cell index a basis of a representation of W. Since neither Rossmann's (nor Kazhdan-Lusztig's) description of the action is particularly easy to describe, we give an alternate description based on ideas of Hotta [Ho], Joseph [Jo], and D. King. To be more precise, the construction of Hotta [Ho] applies to only a special case which we describe in Theorem 11.2. (A plausible generalization is discussed after the theorem.)

To begin, fix $\mathcal{O}^{K_{\mathbb{C}}} = K \cdot N \in K_{\mathbb{C}}\backslash\mathcal{N}_\theta$, and write $\mathcal{O}^{G_{\mathbb{C}}}$ for the $G_{\mathbb{C}}$ orbit through N. Now the natural inclusion of centralizers

$$Z_{K_{\mathbb{C}}}(N) \longrightarrow Z_{G_{\mathbb{C}}}(N)$$

induces a map on the level of component groups

$$A_{K_{\mathbb{C}}}(N) \longrightarrow A_{G_{\mathbb{C}}}(N).$$

Write $A(N)$ for the image of this map; it corresponds to a subgroup $H(N) \subset Z_{G_{\mathbb{C}}}(N)$ which contains the connected component of the identity $Z^{\circ}_{G_{\mathbb{C}}}(N)$. Hence we may consider the orbit cover

$$\widetilde{\mathcal{O}}^{G_{\mathbb{C}}} = G_{\mathbb{C}}/H(N) \xrightarrow{\pi} \mathcal{O}^{G_{\mathbb{C}}}.$$

The next lemma identifies the elements of the geometric cell $\Phi_1^{-1}(\mathcal{O}^{K_{\mathbb{C}}})$ with the intrinsic geometry of $\widetilde{\mathcal{O}}^{G_{\mathbb{C}}}$. To state the lemma, we need some notation. Recall the fixed Borel B

with nilradical $\mathfrak{n}$, write $\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n}$ for $\pi^{-1}(\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{n})$, and let $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n})$ denote the set of its irreducible components. (More intrinsically, such irreducible components exhaust the B-stable Lagrangian subvarieties of $\widetilde{\mathcal{O}}^{G_\mathbb{C}}$ — these have been studied recently in the context of the orbit method (see [Mi], for instance) — but we will not need any of this here.) The following lemma goes back Spaltenstein [Spa1].

Lemma 11.1. *Fix notation as in the previous two paragraphs. Then there is a natural bijection from the set* $\mathrm{Irr}(\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n})$ *to the set of* $A(N)$ *orbits on* $\mathrm{Irr}(X^N)$. *(By Proposition 10.1, this latter set is in natural correspondence with the elements of the geometric cell* $\Phi_1^{-1}(\mathcal{O}^{K_\mathbb{C}})$.*)*

Sketch. As indicated, the parenthetical assertion follows from Proposition 10.1, once we observe that the $A(N)$ and $A_{K_\mathbb{C}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide. For the first assertion, let η_1 denote the projection $G_\mathbb{C} \longrightarrow X$, and write $G_\mathbb{C}^N$ for $\eta_1^{-1}(X^N)$. Then $\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n} \simeq H(N)\backslash G_\mathbb{C}^N$; write η_2 for the projection of $G_\mathbb{C}^N$ onto $\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n}$. If C is an irreducible component of X^N, then $\eta_2(\eta_1^{-1}(C))$ is an irreducible component of $\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n}$. It is straightforward to check that this correspondence implements the bijection of the lemma. □

Now we impose an additional assumption. Suppose that the $A_{K_\mathbb{C}}(N)$ surjects onto $A_{G_\mathbb{C}}(N)$; i.e. assume that $H(N) = Z_{G_\mathbb{C}}(N)$, so that $\widetilde{\mathcal{O}}^{G_\mathbb{C}} = \mathcal{O}^{G_\mathbb{C}}$. (For instance, this is always satisfied if $A_{G_\mathbb{C}}(N)$ is trivial.) Under this assumption, Lemma 11.1 implies that the elements of the geometric cell $\Phi_1^{-1}(\mathcal{O}^{K_\mathbb{C}})$ are in natural correspondence with $\mathrm{Irr}(\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{n})$.

We are going to attach a polynomial on $\mathfrak{h}$ to each element of $\mathrm{Irr}(\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{n})$. Through the W action on $\mathfrak{h}$, W will act on the span of these polynomials; this will be the representation of W indexed by the cell $\Phi^{-1}(\mathcal{O}^{K_\mathbb{C}})$. The idea, due to Joseph [Jo] (who attributes the idea to D. King), is to measure the growth of the $\mathfrak{h}$-weight spaces of the ring of functions $R(\overline{U})$ on the closure an element of $U \in \mathrm{Irr}(\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{n})$. More precisely, $\mathfrak{h}$ acts locally finitely on $R(\overline{U})$ with weights of the form $\sum_{\alpha\in\Delta^+} \mathbb{N}\alpha$. For a fixed $H \in \mathfrak{h}$ and $j \in \mathbb{N}$, write

$$R(\overline{U})_j^H = \{f \in R(\overline{U}) \mid H \cdot f = jf\}.$$

If H satisfies $\alpha(H) \geq 0$ for all $\alpha \in \Delta^+$, we can write

$$\sum_{j=0}^{k} \dim(R(\overline{U})_j^H) = \pm \left(\frac{p_U(H)}{\prod_{\alpha\in\Delta^+} \alpha(H)} \right) k^{\dim(U)} + \text{ terms of lower order in } k;$$

here $p_U \in S(\mathfrak{h}^*)$ is a polynomial on $\mathfrak{h}$. We then have the following theorem which can be extracted from [Ho] and [Ro].

Theorem 11.2. *Fix* $\mathcal{O}^{K_\mathbb{C}} = K_\mathbb{C}{\cdot}N \in K_\mathbb{C}\backslash\mathcal{N}_\theta$, *write* $\mathcal{O}^{G_\mathbb{C}} = G_\mathbb{C}{\cdot}N$, *and assume* $\widetilde{\mathcal{O}}^{G_\mathbb{C}} = \mathcal{O}^{G_\mathbb{C}}$. *Then the span of the polynomials* p_U *as* U *ranges over* $\mathrm{Irr}(\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{n})$ *is the* W *representation attached to the geometric cell* $\Phi_1^{-1}(\mathcal{O}^{K_\mathbb{C}})$ *by Rossmann's action; moreover, the basis element* p_U *corresponds to fundamental class of the conormal bundle to the orbit* $v \in \Phi_1^{-1}(\mathcal{O}^{K_\mathbb{C}})$ *associated to* U *by Lemma 11.1. Finally, the representation coincides with the* $A_{G_\mathbb{C}}(N)$ *invariants in Springer's* W *action (twisted, as usual, by the sign character) on* $H^{\mathrm{top}}(X^N, \mathbb{C})$; *the basis element* p_U *corresponds to the sum of the fundamental classes of elements in the* $A_{G_\mathbb{C}}(N)$ *orbit on* $\mathrm{Irr}(X^N)$ *attached to* U *by Lemma 11.1.*

Probably the assumption that $\widetilde{\mathcal{O}}^{G_\mathbb{C}} = \mathcal{O}^{G_\mathbb{C}}$ is not necessary, as long as one considers all irreducible components $U \in \mathrm{Irr}(\widetilde{\mathcal{O}}^{G_\mathbb{C}} \cap \mathfrak{n})$ and replaces $A_{G_\mathbb{C}}(N)$ by $A(N)$. I have not checked the necessary details to make this conclusion.

Finally, it is worth noting that in the complex case, Kashiwara and Tanisaki [KaT] have shown that the characteristic cycle functor relates the Kazhdan-Lusztig basis of the coherent continuation representation with the geometric cell basis of the topological W action. In this way, Tanisaki was able to give examples in B_3 and C_3 of highest weight modules with reducible characteristic cycles. Kashiwara and Saito have reportedly given an example in A_7. At present, it seems that no analogous results are known to detect reducible characteristic cycles of representations of real groups.

12. Robinson-Schensted algorithms for type A real groups.

We are going to specialize the statement of Proposition 10.1 in the case of $U(p,q)$, $SU^*(2n)$, and $GL(n,\mathbb{R})$ to obtain explicit maps from $K_\mathbb{C}\backslash X$ to certain kinds of tableaux. (The resulting maps are called generalized Robinson-Schensted algorithms, as explained in Remark 12.4.) In order to make everything explicit, we first have to parametrize $K_\mathbb{C}\backslash\mathcal{N}_\theta$ and $A_{K_\mathbb{C}}(N)\backslash\mathrm{Irr}(X^N)$ by tableaux.

The tableau parametrizations of $K_\mathbb{C}\backslash\mathcal{N}_\theta$ are well-known. We have already discussed the case of $U(p,q)$ in Section 4; in that case $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{T}_\pm(p,q)$. If $G = GL(n,\mathbb{R})$, then the Jordan form of $N \in \mathcal{N}_\theta$ is a complete invariant for the action of $K_\mathbb{C}$. Hence $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{D}(n)$ (Notation 9.3). When $G = SU^*(2n)$ the Jordan form (over $\mathbb{H}$) identifies $K_\mathbb{C}\backslash\mathcal{N}_\theta$ with $\mathcal{D}(n)$. We prefer, however, to consider the Jordan form over $\mathbb{C}$; this amounts to duplicating each row to a get a tableaux whose row lengths all occur an even number of times, thus parametrizing $K_\mathbb{C}\backslash\mathcal{N}_\theta$ by $\mathcal{D}_e^{tr}(2n)$. Since we will need to refer to these results below, we set them off in a lemma.

Lemma 12.1. *We have the following parametrizations of $K_\mathbb{C}\backslash\mathcal{N}_\theta$:*

(a) *For $G = U(p,q)$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{T}_\pm(p,q)$.*
(b) *For $G = SU^*(2n)$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{D}_e^{tr}(2n)$.*
(c) *For $G = GL(n,\mathbb{R})$, $K_\mathbb{C}\backslash\mathcal{N}_\theta$ is parametrized by $\mathcal{D}(n)$.*

We turn to a tableau parametrization of $A_{K_\mathbb{C}}(N)\backslash\mathrm{Irr}(X^N)$. In each of the above cases, we claim that $A_{K_\mathbb{C}}(N)$ acts trivially on $\mathrm{Irr}(X^N)$. This is obvious if $G_\mathbb{R} = U(p,q)$ since $A_{K_\mathbb{C}}(N)$ is trivial. For $SU^*(2n)$ or $GL(n,\mathbb{R})$, the component groups need not be trivial, so we argue as follows. Clearly the $A_{K_\mathbb{C}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide with those of the image of $A_{K_\mathbb{C}}(N)$ in $A_{G_\mathbb{C}}(N)$. In fact, $A_{G_\mathbb{C}}(N)$ acts trivially on $\mathrm{Irr}(X^N)$. To see this, note that the $A_{G_\mathbb{C}}(N)$ orbits on $\mathrm{Irr}(X^N)$ coincide with the orbits of the component group $A_{G'_\mathbb{C}}(N)$ for any connected $G'_\mathbb{C}$ with Lie algebra $\mathfrak{g}$. If $G'_\mathbb{C}$ is adjoint, the component group is trivial, so the claim follows.

Now if X is the flag variety for $GL(n,\mathbb{C})$, then $\mathrm{Irr}(X^N)$ is parametrized by standard Young tableau of size n; we see this as follows. Given $N \in \mathcal{N}$, choose $F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^n) \in X^N$ generically (that is, in the open part of an irreducible component). We obtain a tableau T of size n (whose shape is the Jordan form of N) by requiring that the first j boxes of T coincide with the Jordan form of N restricted to F_j. We write $T = \gamma(N,F)$ and also denote the corresponding map $\mathrm{Irr}(X^N) \longrightarrow \mathcal{T}(n)$ by γ. (Actually, there is a twist of γ which gives an equally natural parametrization of $\mathrm{Irr}(X^N)$; we return to this following Theorem 12.6.)

Proposition 10.1 then reduces to the following explicit statements.

Corollary 12.2. *Recall Notations 9.1 and 9.3.*

(a) *If $G = U(p,q)$, Proposition 10.1 gives a bijection (Φ_1^a, Φ_2^a) between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{T}_{\pm}(p,q) \times \mathcal{T}(p+q)$.*

(b) *If $G = SU^*(2n)$, Proposition 10.1 gives a bijection (Φ_1^b, Φ_2^b) between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{D}_e^{tr}(2n) \times \mathcal{T}_e^{tr}(2n)$.*

(c) *If $G = GL(n,\mathbb{R})$, Proposition 10.1 gives a bijection (Φ_1^c, Φ_2^c) between $K_{\mathbb{C}}\backslash X$ and the same-shape subset of $\mathcal{D}(n) \times \mathcal{T}(n)$.*

(Clearly Φ_1^b and Φ_1^c are redundant, but we choose to keep them to preserve the analogy.)

Remark 12.3. To be absolutely explicit, we summarize how to compute the bijections appearing in the corollary. Begin with a fixed $v \in K_{\mathbb{C}}\backslash X$, and choose a flag F in v. Consider the moment map image of the corresponding fiber of the conormal bundle to v in X,

$$\mu(T_v^*(X)|_F).$$

Let N be a generic nilpotent in the image. The $K_{\mathbb{C}}$ orbit through N is parametrized by the combinatorial set of data that appears as the image of the various Φ_1's. Applying the map γ to N and F (as described just before the corollary) we then get a tableau in $\mathcal{T}(n)$. The resulting tableau defines Φ_2.

Remark 12.4. If we take $G = GL(n,\mathbb{C})$, then Proposition 10.1 reduces to a bijection from S_n to same-shape pairs of standard Young tableau of size n. Steinberg ([St]) discovered that the bijection is the Robinson-Schensted algorithm and, motivated by this fact, Springer calls the bijections appearing in parts (a)–(c) (or, more generally, in Proposition 10.1) generalized Robinson-Schensted algorithms. Since we will need it in Section 15, we recall a few details of Steinberg's calculation. If $G = GL(n,\mathbb{C})$, then $G_{\mathbb{C}}/B$ consists of two copies of the flag manifold for G, $K_{\mathbb{C}}$ acts as the diagonal $GL(n,\mathbb{C})$, and $K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$ is parametrized by W. Let F be the standard flag in $\mathbb{C}^n$ and let F^w denote the flag obtained from the permutation w of the standard basis vectors; then (F, F^w) in the $K_{\mathbb{C}}$ orbit corresponding to w. Write $A(w)$ for the permutation matrix in $GL(n,\mathbb{C})$ given by w, and let $\mathfrak{n}$ be the upper triangular nilradical in a single copy of $\mathfrak{gl}(n,\mathbb{C})$. Following to Remark 12.3, let N be a generic nilpotent in $Ad(A(w))\mathfrak{n} \cap \mathfrak{n}$. (We leave it to the reader that verify that $(N, -N)$ is generic in the appropriate moment map image.) Then Steinberg's result states that $(\gamma(N,F), \gamma(N,F^w))$ is the pair of tableau attached to w by the Robinson-Schensted algorithm.

Remark 12.5. To understand the *existence* of the bijections appearing in the corollary, one need not make reference to Proposition 10.1. Consider part (b) for example. In Proposition 13.1 below, we will see that $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma_0(2n)$. Now S_{2n} acts on $\Sigma_0(2n)$ by conjugation, and the isotropy group at a fixed $\sigma \in \Sigma_0(2n)$ is isomorphic to $W(C_n) \simeq (\mathbb{Z}/2)^n \rtimes S_n$, the Weyl group of type C_n. (To get the standard realization of $W(C_n)$, let σ interchange 1 with $2n$, 2 with $2n-1$, and so on.) The induced representation $\mathrm{ind}_{W(C_r)}^{S_{2n}}(\mathbb{C}_{\mathrm{triv}})$ decomposes as

$$\mathbb{C}[S_{2n}/W(C_n)] = \bigoplus_{\pi \in \mathcal{D}_e^{tr}(2n)} E_\pi.$$

(We are using Young's parametrization of $\widehat{S}_{2n}$ in terms of $\mathcal{D}(2n)$.) Since $\Sigma_0(2n)$ parametrizes $K_{\mathbb{C}}\backslash X$, we can conclude that $K_{\mathbb{C}}\backslash X$ indexes a basis for $\mathbb{C}[S_{2n}/W(C_n)]$. Young's dimension formula for E_π then gives the existence of the bijection appearing in part (b) of the corollary. We leave it to the reader to use Proposition 13.1 to give similar abstract interpretations of the bijections appearing in the corollary. (See [BV4], for instance.)

The main point is that the bijections appearing in Corollary 12.2 are very reminiscent of the Barbasch-Vogan parametrization of Section 5. We can make this precise as follows.

Theorem 12.6. *For $G = U(p,q)$, $SU^*(2n)$, or $GL(n,\mathbb{R})$, the generalized Robinson-Schensted algorithms (Φ_1, Φ_2) (of Corollary 12.2 and Remark 12.3) compute annihilators and associated varieties of Harish-Chandra modules for G. More precisely, we have:*

(a) *Let $G = U(p,q)$, fix $v \in K_{\mathbb{C}}\backslash X$, and let $L_G(v)$ be the irreducible Harish-Chandra module associated to the trivial local system on v. Recall the tableau parametrizations of Theorem 3.1 and Lemma 12.1. We have*

$$(\Phi_1^a(v), \Phi_2^a(v)) = (\mathrm{AV}(L_G(v)), \mathrm{Ann}(L_G(v)).$$

(b) *Let $G = SU^*(2n)$, and consider $v \in K_{\mathbb{C}}\backslash X$. Then*

$$(\Phi_1^b(v), \Phi_2^b(v)) = (\mathrm{AV}(L_G(v)), \mathrm{Ann}(L_G(v)).$$

(c) *For $G = GL(n,\mathbb{R})$, there are either one or two $K_{\mathbb{C}}$ equivariant local systems $\phi_{\pm}$ on any given orbit $v \in K_{\mathbb{C}}\backslash X$ such that*

$$(\Phi_1^c(v), \Phi_2^c(v)) = (\mathrm{AV}(L_G(v,\phi)), \mathrm{Ann}(L_G(v,\phi)).$$

Remark 12.7. Explicit computations of (Φ_1, Φ_2) (or, more precisely, the right-hand sides of the above equalities) are given in Theorems 14.1, 14.2, and 15.6. In particular, it is possible to describe the local system(s) ϕ appearing in part (c) very explicitly. (We give enough details to do this in the comments following 15.6.)

Remark 12.8. If $L_G(v) = A_{\mathfrak{q}}(\lambda)$, then Proposition 4.4 and its proof imply that $\Phi_1(v)$ is the ($K_{\mathbb{C}}$ orbit corresponding to the) Richardson orbit $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{zero})$. To locate which v in fact parametrize $A_{\mathfrak{q}}(\lambda)$ modules, one must refer to the the Langlands parameter computations in [VZ, Section 6]. This requires some reasonably involved bookkeeping, but is quite tractable in practice.

The proof of Theorem 12.6, which we defer until Section 15, is entirely empirical: we simply work out both sides of the equality in Theorem 12.6, and show they coincide. One consolation prize in the course of the proof is an explicit identification of $K_{\mathbb{C}}\backslash X$, and a statement of Vogan's duality on the level of orbits. (None of this is new, but none of it is written down anywhere.) We also make some results of Garfinkle [G] a little more explicit.

To conclude this section we dispense with a slight ambiguity concerning the parametrization $\gamma : \mathrm{Irr}(X^N) \longrightarrow \mathcal{T}(n)$. There is another equally natural choice of this parametrization obtained as follows. Given $N \in \mathcal{N}$, and a generic $F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^n) \in U \in \mathrm{Irr}(X^N)$, we can build a tableau T' (whose shape is the Jordan form of N) by requiring the first j boxes coincide with the Jordan form of N viewed as a nilpotent endomorphism of F_n/F_{n-j}. Write γ' for the resulting map $\mathrm{Irr}(X) \longrightarrow \mathcal{T}(n)$.

Actually, it is better to think of γ' as follows. Given a flag $F = (F_i)$, define $F_i^\vee = (F_n/F_{n-i})^*$; here $*$ denotes vector space dual. Clearly $F_i^\vee \hookrightarrow F_{i+1}^\vee$, so we have constructed a dual flag $F^\vee = (F_i^\vee)$. (Despite the notation, this has nothing to do with Vogan's duality described below.) Now if $1 + N$ fixes F, and $N^\vee$ denotes the transpose endomorphism of F_n^*, then it is clear that $1 + N^\vee$ fixes $F^\vee$. From the definitions, it's easy to verify that $\gamma'(N,F) = \gamma(N^\vee, F^\vee)$.

In any case, it was Spaltenstein who apparently first noticed that the map $\gamma'\gamma^{-1}$ gives an interesting (shape-preserving) involution on $\mathcal{T}(n)$; Douglass has recently computed it [Do]. With some effort, his computation simplifies to the following result, which will be crucial in the course of our proof of Theorem 12.6.

Proposition 12.9. *For $T \in \mathcal{T}(n)$ write $T = RS(\sigma)$ for a unique $\sigma \in \Sigma(n)$. Then*

$$\gamma'\gamma^{-1}(T) = RS(w_o\sigma w_o).$$

In particular, both $RS(\sigma)$ and $RS(w_o\sigma w_o)$ have the same shape.

We can interpret Spaltenstein's involution in a more representation theoretic setting as follows. Write τ for the type A diagram automorphism. It induces an involution, say $L \mapsto {}^{\tau}L$, on the set of irreducible $\mathfrak{g}$ modules by composing the action of $\mathfrak{g}$ with any automorphism of $\mathfrak{g}$ coming from τ.

Corollary 12.10. *Let L be an irreducible $\mathfrak{g}$ module with trivial infinitesimal character. Then, in terms of Theorem 3.1,*

$$\mathrm{Ann}({}^{\tau}L) = \gamma'\gamma^{-1}(\mathrm{Ann}(L)).$$

Pf. If $\mathrm{Ann}(L) = \mathrm{Ann}(L_{\mathfrak{b}}(w\rho))$, then $\mathrm{Ann}({}^{\tau}L) = \mathrm{Ann}({}^{\tau}L_{\mathfrak{b}}(w))$; so Duflo's Theorem (see the comment before Theorem 3.1) reduces the proof to the $L = L_{\mathfrak{b}}(w\rho)$ case. Since $\mathrm{Ann}({}^{\tau}L_{\mathfrak{b}}(w\rho)) = \mathrm{Ann}(L_{\mathfrak{b}}(w_o w w_o \rho))$, the corollary now follows immediately from Proposition 12.9. ☐

Hence, the ambiguity concerning γ and γ' amounts to a choice of orientation of the type A Dynkin diagram. Explicit examples of the corollary are given at the end of Section 14.

13. $K_{\mathbb{C}}$-ORBITS ON X, VOGAN DUALITY

In order to prove Theorem 12.6, we will need a precise description of the $K_{\mathbb{C}}$-orbits on X for certain type A real forms. (This was exactly what we managed to avoid in Section 4 — now, unfortunately, we have no choice but to descend into the details.) The following result is (more or less) implicit in [MO], though it was undoubtedly part of the folk knowledge of experts. In the case of $U(p,q)$, Yamamoto [Ya] has given explicit proofs, which the interested reader can modify for the remaining cases.

Proposition 13.1. *Recall Notations 9.1 and 9.2, and fix B to be the upper triangular Borel in $GL(n,\mathbb{C})$ (or $GL(2n,\mathbb{C})$ in part (b) below).*

(a) *For $G = U(p,q)$, $K_{\mathbb{C}}\backslash X$ is parametrized by $\Sigma_{\pm}(p,q)$. The correspondence takes an involution with signed fixed points (σ,ϵ) to the $K_{\mathbb{C}}$ orbit through gB where $g \in GL(n,\mathbb{C})$ is defined as follows.*

(i) *If $\sigma(l) = l$ and $\epsilon_l = +$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, \epsilon_j = +\} \\ 0 & \text{else.} \end{cases}$$

(ii) *If $\sigma(l) = l$ and $\epsilon_l = -$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = p + \#\{j \mid j \leq l, \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(iii) *If $\sigma(l) > l$, then*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, \epsilon_j = +\} \\ 1 & \text{if } k = p + \#\{j \mid j \leq \sigma(l), \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(iv) *If $\sigma(l) < l$, then*

$$g_{kl} = \begin{cases} -1 & \text{if } k = \#\{j \mid j \leq \sigma(l), \epsilon_j = +\} \\ 1 & \text{if } k = p + \#\{j \mid j \leq l, \epsilon_j = -\} \\ 0 & \text{else.} \end{cases}$$

(b) *For $G = SU^*(2n)$, $K_{\mathbb{C}} \backslash X$ is parametrized by $\Sigma_0(2n)$. The correspondence takes a fixed-point free involution σ to the $K_{\mathbb{C}}$ orbit through gB where $g \in GL(2n, \mathbb{C})$ is defined by*

(i) *If $l < \sigma(l)$,*

$$g_{kl} = \begin{cases} 1 & \text{if } k = \#\{j \mid j \leq l, j < \sigma(j)\} \\ 0 & \text{else.} \end{cases}$$

(ii) *If $l > \sigma(l)$,*

$$g_{kl} = \begin{cases} 1 & \text{if } k = n + \#\{j \mid j \leq l, j > \sigma(j)\} \\ 0 & \text{else.} \end{cases}$$

(c) *For $G = GL(n, \mathbb{R})$, $K_{\mathbb{C}} \backslash X$ is parametrized by $\Sigma(n)$. The correspondence takes an involution σ to the orbit though gB with g defined by*

(i) *If $l = \sigma(l)$,*

$$g_{kl} = \begin{cases} 1 & \text{if } j = l \\ 0 & \text{else.} \end{cases}$$

(ii) *If $l \neq \sigma(l)$,*

$$g_{kl} = \begin{cases} \sqrt{\frac{1}{2i}} & \text{if } k = l \\ i\sqrt{\frac{1}{2i}} & \text{if } k = \sigma(l) \\ 0 & \text{else.} \end{cases}$$

(The factor $\sqrt{\frac{1}{2i}}$ is just a convenient normalization arranged so that for g as defined in part (c), g^2 is the permutation matrix corresponding to σ.)

Remark 13.2. The statement in part (a) remains valid if we replace $G = U(p,q)$ by $SU(p,q)$. When n is odd, part (c) remains unchanged if $GL(n, \mathbb{R})$ is replaced by $SL(n, \mathbb{R})$. For n even, the $O(n, \mathbb{C})$ orbit parametrized by $\sigma \in \Sigma_0(n) \subset \Sigma(n)$ is a union of two $SO(n, \mathbb{C})$ orbits, as can be see already in the $n = 2$ case.

Next we describe the $K_{\mathbb{C}}$ equivariant local systems on each $v \in K_{\mathbb{C}} \backslash X$. This amounts to computing the centralizer component group $A(v) = Z_{K_{\mathbb{C}}}(v)/Z^{\circ}_{K_{\mathbb{C}}}(v)$ (see [V5] for example).

Proposition 13.3. *Recall Proposition 13.1.*

(a) *For $G = U(p,q)$, each $A(v)$ is trivial.*

(b) *For $G = SU^*(2n)$, each $A(v)$ is trivial.*

(c) *For* $G = GL(n,\mathbb{R})$ *and* $v_\sigma \in K_{\mathbb{C}}\backslash X$ *parametrized by* $\sigma \in \Sigma(n)$, $A(v_\sigma) \simeq (\mathbb{Z}/2)^r$ *where* r *is the number of fixed points of* σ.

Corollary 13.4. *For* $G = GL(n,\mathbb{R})$, *the set of* $K_{\mathbb{C}}$ *equivariant local system on* $v_\sigma \in K_{\mathbb{C}}\backslash X$ *is in bijection with the set*

$$\{(\tau,\epsilon) \in \Sigma_{\pm}(n) \mid \tau = \sigma\}.$$

In particular, the set of pairs consisting of an orbit $v \in K_{\mathbb{C}}\backslash X$ *and a* $K_{\mathbb{C}}$ *equivariant local system on* v *is in bijection with* $\Sigma_{\pm}(n)$.

Remark 13.5. When $G = SU(p,q)$, the groups $A(v)$ are either trivial or isomorphic to $\mathbb{Z}/2$. The latter case happens precisely when $p = q$ and v corresponds to (σ,ϵ) with $\sigma \in \Sigma_0(2p)$ being fixed-point free. Any orbit for $SL(n,\mathbb{R})$ parametrized by $\sigma \in \Sigma(n)$ with r fixed points has $A(v)$ equal to the subgroup $(\mathbb{Z}/2)^r$ consisting of those r tuples with an even number of nontrivial elements.

We now turn to representation theory. In [V6], Vogan defines a duality on irreducible Harish-Chandra modules of different real forms (of a fixed complex group) that behaves nicely with respect to composition series. The duality is not unique; nonetheless, we will write $L^\vee$ for any choice of the dual of L. The key formal property that we will need is that, up to tensoring with the sign representation, the duality intertwines the coherent continuation representation.

Proposition 13.6. *Recall Theorem 3.1 and suppose* G *is a real form of* $GL(n,\mathbb{C})$. *Then the tableau parametrizing* $Ann(L^\vee)$ *is the transpose of the one parametrizing* $Ann(L)$.

We now write down (a choice of) the duality on the level of orbits.

Proposition 13.7. *Fix* $\sigma \in \Sigma_0(2n)$; *then (by definition—see Notation 9.2) there is a unique* ϵ *with* $(\sigma,\epsilon) \in \Sigma_{\pm}(2n)$. *Let* v_σ *and* $\check{v} = v_{(\sigma,\epsilon)}$ *denote the corresponding orbits (Proposition 13.1). We have*

$$(L_{SU^*(2n)}(v_\sigma))^\vee = L_{SU(n,n)}(\check{v},\check{\phi}),$$

where $\check{\phi}$ *is the unique nontrivial local system on the indicated orbit (Remarks 13.2 and 13.5).*

In order to describe the duality for $GL(n,\mathbb{R})$ we need to give a combinatorial construction of an involution with signed fixed points $(\check{\sigma},\check{\epsilon}) \in \Sigma_{\pm}(n)$ from an involution $\sigma \in \Sigma(n)$. So fix $\sigma \in \Sigma(n)$, take $\check{\sigma} = \sigma$, and define

$$\epsilon_i = + \text{ if } i < \sigma(i),$$
$$\epsilon_i = - \text{ if } i > \sigma(i);$$

these definition are required by the normalization in the definition of $\Sigma_{\pm}(n)$. We assign the first fixed point of σ a $+$ sign, and then require the signs to alternate along the remaining fixed points; more precisely, list the fixed points of σ in increasing order as $r_1,\ldots,r_l$, and set $\epsilon_{r_i} = (-1)^{i+1}$. (We could just as well have chosen $\epsilon_{r_i} = (-1)^i$, reflecting the nonuniqueness of the duality.)

Example 13.8. Given $\sigma = (36)(49) \in \Sigma(9)$, we apply the algorithm as follows.

$$\sigma \in \Sigma(9) \qquad \bullet \quad \bullet \quad 3 \quad 4 \quad \bullet \quad 6 \quad \bullet \quad \bullet \quad 9$$

$$\rightsquigarrow$$

$$(\check{\sigma}, \check{\epsilon}) \in \Sigma_{\pm}(5,4) \qquad + \quad - \quad 3^{+} \quad 4^{+} \quad + \quad 6^{-} \quad - \quad + \quad 9^{-}$$

The picture means that $\check{\sigma} = (36)(49)$ while

$$\check{\epsilon} = (+, -, +, +, +, -, -, +, -).$$

Proposition 13.9. *Let $\sigma \in \Sigma(n)$ and let $(\check{\sigma}, \check{\epsilon}) \in \Sigma_{\pm}(p, q)$ be as defined in the previous paragraph, and let v and $\check{v}$ denote the corresponding orbits under Proposition 13.1. Then*

$$(L_{GL(n,\mathbb{R})}(v))^{\vee} = L_{U(p,q)}(\check{v}).$$

Remark 13.10. Using Corollary 13.4, we see that the full duality

$$\widehat{GL(n, \mathbb{R})}_{\rho} \longrightarrow \coprod_{p+q=n} \widehat{U(p,q)}_{\rho}$$

amounts to a bijection

$$\Sigma_{\pm}(n) \longrightarrow \coprod_{p+q=n} \Sigma_{\pm}(p,q).$$

The algorithm given before the statement of the proposition can be naturally extended to this larger domain giving an explicit formulation of the duality on all of $\widehat{GL(n, \mathbb{R})}_{\rho}$ — we leave the precise formulation to the reader. Below, however, we will make use of one qualitative feature of the answer: if ϕ is any $K_{\mathbb{C}}$ equivariant local system on an orbit v_{σ} for $GL(n, \mathbb{R})$, and

$$(L_{GL(n,\mathbb{R})}(v_{\sigma}, \phi))^{\vee} = L_{U(p,q)}(v_{(\check{\sigma}, \check{\epsilon})}),$$

then $\check{\sigma} = \sigma$. (In words: the full duality, like the algorithm given in Proposition 13.9, doesn't alter the non-fixed points of σ.)

14. Annihilators for $U(p,q)$, $GL(n,\mathbb{R})$, and $SU^{*}(2n)$

In this section we recall Garfinkle's algorithms to compute $\mathrm{Ann}(L(v))$ for $v \in K_{\mathbb{C}} \backslash X$. The first group we treat is $U(p,q)$. (The following holds verbatim for $SU(p,q)$.) Given $(\sigma, \epsilon) \in \Sigma_{\pm}(p,q)$, form a sequence of pairs of the form

$$(i, \epsilon_i) \text{ if } \sigma(i) = i; \text{ and}$$
$$(i, \sigma(i)) \text{ if } i < \sigma(i).$$

Arrange the pairs in order by their largest entry, with the convention that a sign has numerical size zero. Write $\pi_1, \ldots, \pi_r$ for the resulting ordered sequence. (For instance,

$$(1,+), (2,-), (5,+), (3,6), (7,+), (8,-), (4,9)$$

is the sequence corresponding to $(\breve{\sigma}, \breve{\epsilon})$ in Example 13.8).

We now give Garfinkle's algorithm describing a same-shape a pair of tableaux

$$(\Psi_1^a(v), \Psi_2^a(v)) \in \mathcal{T}_\pm(p+q) \times \mathcal{T}(p+q).$$

Each tableau is constructed by inductively adding the pairs π_j. So suppose that we have added $\pi_1, \ldots, \pi_{j-1}$ to get a (smaller) same-shape pair of tableau $(T_\pm, T)$. If $\pi_j = (k, \epsilon_k)$, then we first add the sign ϵ_k to the topmost row of (a signed tableau in the equivalence class of) $T_\pm$ so that the resulting tableau has signs alternating across rows. Then add the index j to T in the unique position so that the two new tableaux have the have the same shape. If $\pi_j = (k, \sigma(k))$ we first add k to T using the Robinson-Schensted bumping algorithm to get a a new tableau T', and then add a sign ε (either $+$ or $-$ as needed) to $T_\pm$ so that the result is a signed tableau $T'_\pm$ of the same shape as T'. We then add the pair $(\sigma(k), -\varepsilon)$ (by the recipe of the first case) to the first row strictly below the row to which ε was added. We continue inductively to get $(\Psi_1^a(v), \Psi_2^a(v)) \in \mathcal{T}_\pm(p+q) \times \mathcal{T}(p+q)$. (For a more formal definition, the reader is referred to [G].)

Theorem 14.1. *Let $G = U(p,q)$, take $(\sigma, \epsilon) \in \Sigma_\pm(p,q)$, and let $v \in K_{\mathbb{C}} \backslash X$ be the corresponding $K_{\mathbb{C}}$ orbit of Proposition 13.1. Then, given the tableau parametrizations of Theorem 3.1 and Lemma 12.1, $(\Psi_1^a(v), \Psi_2^a(v))$ is the associated variety and annihilator of $L_{U(p,q)}(v)$. The subsets*

$$\{L(v) \mid \Psi_1^a(v) = \mathcal{O}\}$$

of $\widehat{U(p,q)}_\rho$ exhaust the cells of Harish-Chandra modules as $\mathcal{O}$ ranges over the nilpotent $K_{\mathbb{C}}$ orbits in $\mathfrak{p}$.

Pf. We have arranged our parametrization of $K_{\mathbb{C}} \backslash X$ to coincide with the $\mathbb{Z}/2$ data (or, equivalently, Langlands parameters) that Garfinkle uses in [G]. So the annihilator part follows. The associated variety and cell computation are due to Barbasch and Vogan, and are discussed in Remark 4.8 and the beginning of Section 5. $\square$

Next we turn to $SU^*(2n)$. Given $\sigma \in \Sigma_0(2n)$, or the corresponding orbit $v \in K_{\mathbb{C}} \backslash X$, we describe an element $\Psi_2^b(v) \in \mathcal{T}_e^{tr}(n)$. As above σ gives rise to an ordered sequence of pairs of integers $\pi_1, \ldots, \pi_n$, by ordering the pairs

$$(i, \sigma(i)) \text{ for } i < \sigma(i)$$

by their maximal entry. We construct the *transpose* of $\Psi_2^b(v)$ inductively by adding the pairs π_j. So suppose the pairs $\pi_1, \ldots, \pi_{j-1}$ have been added to produce a tableau T. To add the jth pair $(k, \sigma(k))$, we first add k to T using the Robinson-Schensted procedure to get a tableau T', and then add $\sigma(k)$ to the end of the (unique) row of T' which is longer than the corresponding row of T. Inductively we obtain an element of $\mathcal{T}_e(2n)$ whose transpose (which lives in $\mathcal{T}_e^{tr}(2n)$) we define to be $\Psi_2^b(v)$. We then define $\Psi_1^b(v) \in \mathcal{D}_e^{tr}(2n)$ to be the shape of $\Psi_2^b(v)$. (Again this is redundant, but we elect to preserve the analogy.)

Theorem 14.2. *Let $G = SU^*(2n)$, take $\sigma \in \Sigma_0(2n)$, and let v denote the corresponding $K_{\mathbb{C}}$-orbit (Proposition 13.1(b)). Then the pair $(\Psi_1^b(v), \Psi_2^b(v))$ is the the associated variety and annihilator of $L_{SU^*(2n)}(v)$. The fibers*

$$\{L(v) \mid \Psi_1^b(v) = \mathcal{O}\}$$

exhaust the cells of Harish-Chandra modules for $SU^*(2n)$ as $\mathcal{O}$ ranges over the nilpotent $K_{\mathbb{C}}$ orbits on $\mathfrak{p}$.

Pf. Again, we have arranged our parametrization of $K_{\mathbb{C}}\backslash X$ to coincide with the $\mathbb{Z}/2$ data Garfinkle uses in [G]. So the annihilator part follows. The associated variety part is trivial, since $\Psi_1^b(v)$ is the unique nilpotent orbit whose shape coincides with the shape of $\Psi_2^b(v)$. The cell computation is described in McGovern [Mc1]. $\square$

Finally we treat $GL(n,\mathbb{R})$. Given $v \in K_{\mathbb{C}}\backslash X$, let $\check{v}$ denote the orbit for $U(p,q)$ described in Proposition 13.9. We define $\Psi_2^c(v) \in \mathcal{T}(n)$ denote to be the transpose of $\Psi_2^a(\check{v})$, and let $\Psi_1^c(v) \in \mathcal{D}(n)$ (redundantly) denote its shape.

Theorem 14.3. *Let $G = GL(n,\mathbb{R})$, take $\sigma \in \Sigma(n)$, and let v denote the corresponding $K_{\mathbb{C}}$ orbit (Proposition 13.1(c)). The pair $(\Psi_1^c(v), \Psi_2^c(v))$ is the associated variety and annihilator of $L_{GL(n,\mathbb{R})}(v)$.*

Pf. The annihilator statement follows from definition of Ψ_2^c, together with Proposition 13.6 and the corresponding computation of annihilators for $U(p,q)$ (Theorem 14.1). The associated variety statement follows from same-shape considerations. $\square$

Remark 14.4. Except in special cases, the fibers of Ψ_1^c do *not* parametrize cells of Harish-Chandra modules for $GL(n,\mathbb{R})$.

We now can give some explicit examples of Corollary 12.10. If σ is an element of $\Sigma(n)$, write ${}^{\tau}\sigma$ for $w_o\sigma w_o$. Similarly if (σ,ϵ) is in $\Sigma_{\pm}(p,q)$ write $({}^{\tau}\sigma, {}^{\tau}\epsilon) \in \Sigma_{\pm}(p,q)$ for the pair

$$\begin{aligned} {}^{\tau}\sigma &= w_o w w_o; \\ {}^{\tau}\epsilon_i &= \epsilon_{w_o i w_o} \quad \text{if } {}^{\tau}\sigma(i) = i; \\ {}^{\tau}\epsilon_i &= -\epsilon_{w_o i w_o} \quad \text{if } {}^{\tau}\sigma(i) \neq i. \end{aligned}$$

(The different condition on the signs arises from the normalizations in the definition of $\Sigma_{\pm}(p,q)$.) Write v and ${}^{\tau}v$ for the corresponding orbits. Then one can verify that for $G = U(p,q)$, $SU(p,q)$, $SU^*(2n)$, $GL(n,\mathbb{R})$, or $SL(n,\mathbb{R})$,

$${}^{\tau}L_G(v) = L_G({}^{\tau}v).$$

The interested reader can then use the algorithms given in Theorems 14.1–14.3 for computing annihilators to explicitly verify Corollary 12.10.

15. Proof of Theorem 12.6

In this section we prove Theorem 12.6. We first treat the case of $U(p,q)$. In terms of the notation established in Corollary 12.2 and Theorems 14.1, we are to prove

$$(\Phi_1^a, \Phi_2^a) = (\Psi_1^a, \Psi_2^a).$$

We have essentially already dealt with half of this.

Lemma 15.1. $\Phi_1^a = \Psi_1^a$.

Pf. As discussed in Section 4, Yamamoto [Ya] has given an algorithm to compute $\Phi_1^a(v)$. So to prove the lemma, we have only to compare her algorithm with Garfinkle's. This is possible, but very complicated (mainly because Yamamoto's algorithm itself is complicated). We omit the details. (Note that when v parametrizes an $A_{\mathfrak{q}}(\lambda)$ module, we gave a detailed proof that $\Phi_1^a(v) = \Psi_1^a(v)$ in Section 4; see Remark 4.8 and Appendix 4.1.) $\square$

The next lemma, which follows directly from [St, Lemma 1.2], will be crucial for the other half of the theorem.

Lemma 15.2. *Let $F = (0 = F_0 \subset \cdots F_n = \mathbb{C}^n)$ be an n dimensional flag fixed by $1+N$. Let $U \simeq \mathbb{C}^{n-2}$ be an N-stable hyperplane in F_{n-1} and write $F' = F \cap U$ for the flag*

$$(F_0 \cap U) \subset \cdots \subset (F_n \cap U).$$

Then for some index k we can write F' as

$$0 = F_0 \subset \cdots F_{k-1} \subset (F_{k+1} \cap U) \subset \cdots \subset (F_{n-1} \cap U) = U.$$

Let N' denote restriction of N to U, so that F' is fixed by $1+N'$. Assume that the restriction $N'' = N|_{F_{n-1}}$ is a generic extension of N' to F_{n-1} in the sense that $\dim(G_{\mathbb{C}} \cdot N'')$ is maximal subject to the condition that

(a) *$N''|_U = N'$; and*

(b) *$1 + N''$ fixes $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$.*

Let T' denote the tableau obtained from $\gamma(N, F')$ by switching the entries from $1, \ldots, n-2$ to $1, \ldots, k-1, k+1, \ldots, n-1$. Then the first $n-1$ boxes of $\gamma(N, F)$ are obtained by adding k to T' using Robinson-Schensted insertion.

(Of course the statement is really about $n-1$ dimensional flags. We have phrased it in this slightly confusing way in order to make the applications below a little more transparent.)

Before proceeding to the proof, we set aside several characteristics of Garfinkle's algorithm; the easy verification is left to the reader.

Lemma 15.3. *Fix $\sigma \in \Sigma_{\pm}(p,q)$, write v for the corresponding orbit, and let $\pi_1, \ldots, \pi_r$ be the sequence of pairs as described above. Let T denote the tableau constructed from the first $r-1$ pairs $\pi_1, \ldots, \pi_{r-1}$. Then either $\pi_r = (n, \epsilon_n)$ or $\pi_r = (k, n)$. If $\pi_r = (k, n)$, then the first $n-1$ boxes of $\Psi_2^{\mathfrak{a}}(v)$ are obtained by adding k to T using the Robinson-Schensted insertion procedure.*

We now give a detailed argument that $\Phi_2^{\mathfrak{a}}(v) = \Psi_2^{\mathfrak{a}}(v)$. Take $\sigma \in \Sigma_{\pm}(p,q)$, write v for the corresponding orbit (Proposition 13.1), and write

$$F = (F_0 \subset F_1 \subset \cdots \subset F_n)$$

for the representative given in the proposition. Write $\pi_1, \ldots \pi_r$ for the sequence of pairs attached to σ by the procedure given before Theorem 14.1.

First assume that $\pi_r = (n, \epsilon_n)$ and (without loss of generality) that $\epsilon_n = -$. Let U denote the $n-1$ dimensional subspace of V (as in Notation 9.1) spanned by $e_1, \ldots, e_{n-1}$. Let $G' \simeq U(p, q-1)$ denote the subgroup of $GL(U)$ preserving the form $\langle\ ,\ \rangle$ (of Notation 9.1) restricted to U. Then set

$$F' = (F_0 \subset F_1 \subset \cdots \subset F_{n-1}).$$

From Proposition 13.1, one can check that F' is the representative of the orbit v' attached to $\sigma' \in \Sigma_{\pm}(p, q-1)$ determined by the sequence of pairs $\pi_1, \ldots, \pi_{r-1}$ (with π_r omitted).

Now let N be a generic nilpotent in the moment map image $\mu(T_v^*(X)|_F)$. One may verify directly that

$$N|_U \text{ is generic in } \mu(T_{v'}^*(X')|_{F'}).$$

Hence, by Remark 12.3, the first $n-1$ boxes of $\Phi_2^{\mathfrak{a}}(v)$ coincide with $\Phi_2^{\mathfrak{a}}(v')$ which, by induction, we can assume coincides with the tableau T obtained by applying Garfinkle's algorithm to the pairs $\pi_1, \ldots, \pi_{r-1}$. But (from the definition of Garfinkle's algorithm) these are the first $n-1$ boxes of $\Psi_2^a(v)$. The last two sentences imply that the first $n-1$ boxes of $\Phi_2^a(v)$ coincide with those of $\Psi_2^a(v)$. Lemma 15.1 finishes the proof in this case.

To complete the proof, we must treat the case when $\pi_r = (k, \sigma(k) = n)$. In this case, let U be the subspace of V (as in Notation 9.1) spanned by

$$e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{n-1},$$

and let $G' \simeq U(p-1, q-1)$ denote the subgroup of $GL(U)$ preserving $\langle\ ,\ \rangle$ restricted to U. Write

$$F' = (F_0 \cap U) \subset \cdots \subset (F_n \cap U).$$

Explicitly from Proposition 13.1, one sees that F' equals

$$0 = F_0 \subset \cdots F_{k-1} \subset (F_{k+1} \cap U) \subset \cdots \subset (F_{n-1} \cap U) = U,$$

and that F' is a representative that the proposition gives for the orbit v' corresponding to $\sigma' \in \sigma_{\pm}(p-1, q-1)$ attached to $\pi_1, \ldots, \pi_{r-1}$ (with π_r omitted).

We will show that the first $n-1$ boxes of $\Phi_2^a(v)$ and $\Psi_2^a(v)$ coincide. Appealing to Lemma 15.1 then shows that $\Phi_2^a(v) = \Psi_2^a(v)$. So let N be a generic nilpotent in $\mu(T_v^*(X))$. Again, one may verify that

$$N' = N|_U \text{ is generic in } \mu(T_{v'}^*(X')|_{F'}).$$

Hence, by induction, we may assume that the tableau T', obtained by relabeling the boxes of $\Phi_2^a(v')$ by $1, \ldots, k-1, k+1, \ldots, n-1$ (instead of $1, \ldots, n-2$), is the tableau that Garfinkle's algorithm attaches to $\pi_1, \ldots, \pi_{r-1}$. Lemma 15.2 implies that the first $n-1$ boxes of $\Phi_2^a(v)$ are obtained by inserting k into T' using Robinson-Schensted. (Actually, there is something subtle to check here; see the discussion in the next paragraph.) In any event, by Lemma 15.3, we see that the first $n-1$ boxes of $\Phi_2^a(v)$ and $\Psi_2^a(v)$ coincide. This completes the proof for $U(p,q)$.

As we mentioned above, we must be a little careful about applying Lemma 15.2. The hypothesis of the lemma requires that $N'' = N|_{F_{n-1}}$ be a generic extension of $N' = F|_U$. This would seem to follow immediately from the generic assumption on N, but it is more subtle than that. The generic extension hypothesis of the lemma requires the $GL(F_{n-1})$ orbit through N'' to be maximal, but here we are dealing with $K_{\mathbb{C}}$ orbits. More precisely, the shape of a generic extension of N' to F_{n-1} is obtained by adding some specified corner to the shape of N'; the point is that that the resulting shape may not, a priori, be a subshape of the shape of N (since there are alternating sign conditions to worry about). This never causes problems in our setting because we have two dimensions of freedom: the shape of N' (which is the shape of a signature $(p-1, q-1)$ tableau) plus any corner is a subshape of the shape of a signature (p,q) tableau. So Lemma 15.2 applies, and the argument is complete.

Next we consider the case of $SU^*(2n)$. First we need to record some results analogous to those of Lemma 15.3 and Lemma 15.2.

Lemma 15.4. *Fix $\sigma \in \Sigma_0(2n)$, write $k = \sigma(2n)$, and let v denote the corresponding $K_{\mathbb{C}}$ orbit. Write $\sigma' \in \Sigma_0(2n-2)$ for the involution obtained by viewing σ as a permutation of the letters $1, \ldots, k-1, k+1, \ldots, 2n-1$, and write v' for the corresponding orbit. Let T' denote the tableau obtained by switching the entries of $\Phi_2^b(v')$ from $1, \ldots, 2n-2$ to $1, \ldots, k-1, k+1, \ldots, 2n-1$. Then the first $2n-1$ boxes of $\Phi_2^b(v)$ are obtained by adding k to the transpose of T' using Robinson-Schensted insertion, and then taking the transpose of the resulting diagram.*

The next lemma (especially its proof) explains why the transpose of Robinson-Schensted insertion is appearing.

Lemma 15.5. *Let $F = (0 = F_0 \subset \cdots F_n = \mathbb{C}^n)$ be an n dimensional flag fixed by $1 + N$. Let $U \simeq \mathbb{C}^2$ be an N-stable plane in F_n not contained in F_{n-1}, and write $F' = F/U$ for the flag*

$$0 = F_0/(F_0 \cap U) \subset \cdots \subset F_n/(F_n \cap U).$$

Then for some index k we can write F' as

$$\begin{aligned} 0 =& F_0 \subset \cdots \subset F_{k-1} \subset \\ & F_{k+1}/(F_{k+1} \cap U) \subset \cdots \subset F_{n-1}/(F_{n-1} \cap U). \end{aligned}$$

Let N' denote map induced by N on $F_{n-1}/(F_{n-1} \cap U)$ so that F' is fixed by $1 + N'$. Assume that $N|_{F_{n-1}}$ is a generic lift of N' (in the sense analogous to the condition in Lemma 15.2). Write T' for the tableau obtained from $\gamma(N, F')$ by changing the entries from $1, \dots, n-2$ to $1, \dots, k-1, k+1, \dots, n-1$. Then the first $n-1$ boxes of $\gamma(N, F)$ are obtained by first adding k to the transpose of T' using Robinson-Schensted insertion, and then taking the transpose of the resulting tableau.

Sketch. As in the discussion preceding Proposition 12.9, given any flag $F = (F_i)$, we can form a dual flag $F^\vee = (F_i^\vee)$ defined by $F_i^\vee = (F_n/F_{n-i})^*$. Note that the dual of the flag F/U is of the form $F^\vee \cap V^*$, where V^* is the vector space dual of an n–2 dimensional complement to U. Using this observation, together with the explicit form of Proposition 12.9 and the standard interpretation of transpose in terms of the Robinson-Schensted algorithm, one can then deduce the present lemma from Lemma 15.2. We omit the details. □

Now we prove $\Phi_2^b = \Psi_2^b$, thus completing the proof of Theorem 12.6(b) for $G = SU^*(2n)$. Fix $\sigma \in \Sigma_0$, and write $k = \sigma(n)$. Consider the subspace U of $\mathbb{C}^n$ spanned by the $2n-2$ vectors $e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_{2n-1}$, and let $G' = GL(U)$. As above, we can form the flag $F' = F \cap U$. Then F' is the representative that Proposition 13.1 gives for the orbit v' attached to the involution $\sigma' \in \Sigma_0(2n-2)$ obtained by viewing σ as an involution of the $2n-2$ letters $1, \dots, k-1, k+1, \dots, 2n-1$. But now a problem arises: if N is generic in $\mu(T_v^*(X))$, then (except in very special cases) $N|_U$ will not even fix the flag F', let alone be in the moment map image $\mu(T_{v'}^*(X'))$.

Instead we need to define $U = \mathbb{C}e_k \oplus \mathbb{C}e_{2n}$ and form the flag $F' = F/U$ described in Lemma 15.5. Define $G' = GL(F_n/U)$. Then one can check that for this G', we have that F' is again the representative that Proposition 13.1 gives for the orbit v' attached to the involution $\sigma' \in \Sigma_0(2n-2)$ described in the previous paragraph. Moreover, one can check directly that if N is generic in $\mu(T_v^*(X))$, then the projection of N on the quotient F_n/U is indeed generic in $\mu(T_{v'}^*(X'))$. Now the proof proceeds exactly as in the second case of the argument for $U(p,q)$, except that we instead use Lemma 15.4 and Lemma 15.5. (The same parenthetical caveat applies to the application of Lemma 15.5.) We conclude that the first $2n$–1 boxes of $\Phi_2^b(v)$ and $\Psi_2^b(v)$ coincide. Since there is a unique shape in $\mathcal{D}_e^{tr}(2n)$ containing the shape of the first $2n-1$ boxes of these tableaux, we conclude that $\Phi_2^b(v) = \Psi_2^b(v)$. The $G = SU^*(2n)$ case is complete.

Finally we consider $G = GL(n, \mathbb{R})$. We will deduce Theorem 12.6(c) from the following calculation. In its statement, we let $RS(\sigma)$ denote the standard Young tableau attached to $\sigma \in \Sigma(n)$ by the Robinson-Schensted algorithm.

Theorem 15.6. *Let $G = GL(n, \mathbb{R})$ and fix $v \in K_{\mathbb{C}} \backslash X$ corresponding (under Proposition 13.1) to $\sigma \in \Sigma(n)$. Then the generalized Robinson-Schensted algorithm for G coincides with the transpose of RS,*

$$\Phi_2^c(v) = RS(\sigma)^{tr}.$$

Theorem 12.6(c) now follows from explicit computation. In a little more detail, first fix $\sigma \in \Sigma(n)$, assume σ has no fixed points, and write (σ, ϵ) for the corresponding element of $\Sigma_{\pm}(n)$. Write v_σ and $v_{(\sigma,\epsilon)}$ for the orbits described in Proposition 13.1. By Proposition 13.3, the only $K_{\mathbb{C}}$ equivariant local system on v_σ is the trivial one. From Proposition 13.9 and Proposition 13.6, we have

$$\Phi_1^c(v_\sigma)^{tr} = \Phi_1^a(v_{(\sigma,\epsilon)}).$$

Directly from the definitions, one can verify that $\Phi_1^a(v_{(\sigma,\epsilon)}) = RS(\sigma)$, thus completing the proof of Theorem 12.6(c) in the fixed-point free case. On the other hand, assume σ has a least one fixed point. Then in view of Remark 13.10 and the definition of Ψ_2^c, Theorem 12.6(c) now follows from the following observation: given $\sigma \in \Sigma(n)$, there are exactly two elements of the form $(\sigma, \epsilon), (\sigma, \epsilon') \in \Sigma_{\pm}(n)$ with

$$\Phi_2^c(v_{(\sigma,\epsilon)}) = RS(\sigma) = \Phi_2^c(v_{(\sigma,\epsilon')}).$$

We leave the (easy) verification of these facts to the reader.

Now we turn to the proof of Theorem 15.6. We will begin by establishing that the first $n-1$ boxes of $\Phi_2^c(v_\sigma)$ and $RS(\sigma)^{tr}$ coincide.

So let $G = GL(n, \mathbb{R})$ and take $\sigma \in \Sigma(n)$. Write v for the corresponding orbit and $F = (F_i)$ for the representative given in Proposition 13.1. There are again two cases to consider. First assume that $\sigma(n) = n$. Write $\sigma' \in \Sigma(n-1)$ for the involution obtained by viewing σ as a permutation of n–1 letters $1, \ldots, n$–1. Let v' denote the corresponding orbit for $GL(n$–$1, \mathbb{R})$, and let F' denote the representative given in Proposition 13.1. If N is a generic nilpotent in $\mu(T_v^*(X)|_F)$, then once again one can verify directly that $N|_{F_{n-1}}$ is generic in $\mu(T_{v'}^*(X'))$. Hence, by Remark 12.3, we see that the first $n-1$ boxes of $\Phi_2^c(v)$ coincide with $\Phi_2^c(v')$. By induction we can assume that $\Phi_2^c(v') = \Psi_2^c(v')$. From the definition of Ψ_2^c, we see that the first $n-1$ boxes of $\Psi_2^c(v')$ coincide with $\Psi_2^c(v)$. Putting the last three sentences together, we conclude that the first $n-1$ boxes of $\Phi_2^c(v)$ coincide with those of $\Psi_2^c(v)$.

On the other hand, we can make the same conclusion in the case that $\sigma(n) = k \neq n$. The proof proceeds exactly as in the case of $SU^*(2n)$, once we notice that the obvious analog of Lemma 15.4 clearly holds for RS^{tr}. We omit the details.

Hence we conclude that the first n–1 boxes of $\Phi_2^c(v)$ and $RS(\sigma)^{tr}$ agree. To finish the proof of Theorem 15.6, it is enough to show that the shape of $\Phi_1^c(v_\sigma)$ matches the shape of $RS(\sigma)^{tr}$. One can prove this by duplicating Yamamoto's [Ya] moment-map image computations for $GL(n, \mathbb{R})$ and then verifying (as we did in the $U(p,q)$ case) that the shapes coincide. This is elementary, but extremely complicated. With a little sleight of hand, however, we can deduce it from Steinberg's calculation, part of which appears in the following lemma. (See Remark 12.4 for more details.)

Lemma 15.7. *Fix $w \in S_n$ and let $A(w) \in GL(n, \mathbb{C})$ denote the corresponding permutation matrix. Let $\mathfrak{n}$ denote the upper-triangular nilradical of $\mathfrak{b}$, and suppose that N is generic in $Ad(A(w))\mathfrak{n} \cap \mathfrak{n}$. Then the shape of N coincides with the shape of $RS(w)$, where RS denotes the Robinson-Schensted algorithm.*

Now we prove that $\Phi_1^c(v_\sigma)$ coincides with the shape of $RS(\sigma)^{tr}$. Let $g_\sigma \in GL(n, \mathbb{R})$ denote the element attached to σ by Proposition 13.1. Using an invariant bilinear form to identify the fiber at eB of $T^*(X)$ with $\mathfrak{n}$, we get

$$\mu(T_v^*(X)) = Ad(g_\sigma)\mathfrak{n} \cap \mathfrak{p}.$$

Let N denote a generic element of the image. Since $\mathfrak{p}$ is the set of symmetric complex matrices, we can find $M \in \mathfrak{n}$ so that

$$(*) \qquad N = g_\sigma M g_\sigma^{-1} = (g_\sigma M g_\sigma^{-1})^{tr}.$$

Clearly N has the same shape as M, so we are to prove that the shape of M is the shape of $RS(\sigma)^{tr}$.

Now one may verify directly that $g_\sigma^2 = A(\sigma)$, the permutation matrix attached to σ. Combined with the fact that g_σ and g_σ^{-1} are symmetric, $(*)$ becomes

$$Ad(A(\sigma))M = M^{tr}.$$

Conjugating by $A(w_o)$, we get

$$Ad(A(w_o\sigma))M = M^{atr},$$

where M^{atr} denotes the anti-transpose of M, i.e. the reflection of M about its antidiagonal. Since $M \in \mathfrak{n}$, so is M^{atr}, and we can apply Lemma 15.7 to conclude that the shape of M coincides with the shape of $RS(w_o\sigma)$. By Proposition 12.9, this is the shape of $RS(\sigma w_o)$. Of course it is well-known that $RS(\tau w_o) = RS(\tau)^{tr}$ for any $\tau \in S_n$. Hence we conclude that the shape of M, and hence of N, coincides with the shape of $RS(\sigma)^{tr}$. The proof is complete.

Remark 15.8. We conclude by noting that, in some cases, we can give a completely self-contained computation of associated varieties. For any type A group considered above, let Φ_1 denote the map taking $K_\mathbb{C}\backslash X$ to $K_\mathbb{C}\backslash\mathcal{N}_\theta$. Of course we always have $\Phi_1(v) \subset \mathrm{AV}(L_G(v, \phi))$; see Propositions 2.6 and 2.8 in [BoBr], for example. Now Proposition 13.6 implies

$$\mathrm{shape}(\mathrm{AV}(L)) = \mathrm{shape}(\mathrm{AV}(L^\vee))^{tr},$$

and so we conclude

$$\mathrm{shape}(\Phi_1(v)) \subset \mathrm{shape}(\mathrm{AV}(L_G(v, \phi))) = \mathrm{shape}(\mathrm{AV}(L_{G^\vee}(\check{v}, \check{\phi}))^{tr} \subset \mathrm{shape}(\Phi_1(\check{v}))^{tr}.$$

We have given explicit formulas for Φ_1 and $\check{v}$, and one can check that in some cases the left and right ends of above chain of inclusions coincide. In these cases, we deduce the shape of $\mathrm{AV}(L_G(v, \phi))$; if $G = GL(n, \mathbb{R})$ or $SU^*(2n)$, this is of course $\mathrm{AV}(L_G(v, \phi))$. When $G = U(p, q)$, we can immediately conclude that $\Phi_1(v)$ is an irreducible component of $\mathrm{AV}(L_G(v))$. If there is only one signature (p, q) tableau of the relevant shape, then we can conclude that indeed $\Phi_1(v) = \mathrm{AV}(L_G(v))$. We can avoid the restrictions on the tableau if we are willing to admit the (relatively elementary) Barbasch-Vogan [BV4] result stating that $\mathrm{AV}(L_G(v))$ is irreducible.

The above method has obvious limitations. For instance, it is already inconclusive for the trivial representation of $U(p, q)$ when $|p - q| \geq 2$ and $\min(p, q) \geq 1$. Even so, the method does lead to some nontrivial computations. For instance, the method computes all associated varieties of the four modules $L_{GL(3,\mathbb{R})}(v_\sigma)$. (When $\sigma = (12)$ or (23), $L_{GL(3,\mathbb{R})}(v_\sigma)$ is not an $A_\mathfrak{q}(\lambda)$ module, so Proposition 4.4 does not apply.) For $GL(4, \mathbb{R})$ it computes the associated varieties of nine of the ten modules $L_{GL(4,\mathbb{R})}(v_\sigma)$, three of which are $A_\mathfrak{q}(\lambda)$'s; it is inconclusive when $\sigma = (23)$.

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