DETECTION THEORY AND PSYCHOPHYSICS

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Abstract

Part I: Traditional psychophysical models and Tanner and Swets' model are critically examined. Some of the weaknesses of these models are found to be eliminable by a more sophisticated analysis in terms of detection theory. Accordingly, psychophysical methods are re-examined, and the two-category forced-choice technique is found to be particularly advantageous on theoretical grounds. Application of detection theory to the problem of auditory masking with gaussian noise as measured by this forced-choice technique leads to the mathematical derivation of the theoretical ("ideal detector") psychophysical function for this situation.

Part II: Experiments using the forced-choice method with auditory signals masked by broadband gaussian noise are reported. The aim of these experiments is to determine the extent to which and the manner in which subjects differ from the "ideal detector" of detection theory. It is found that, except for being approximately 13 db less sensitive, subjects behave very much like the ideal detector – that is, in accordance with the mathematical predictions of Part I – when the signals are pure tones. Results with signals consisting of two-component tones require a somewhat enlarged model; such a model is developed.

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GLOSSARY OF SYMBOLS

β	A parameter determined by the payoff matrix and the *a priori* probability of a signal.
c	The criterion which must be exceeded by the effective stimulus magnitude for a perception (earlier model) or a response (later model) to occur.
$\underline{C}$	The momentary threshold.
C	(The event of) a correct response. Pr(C), the probability of a correct response, is called the detectability.
d'	The effective stimulus magnitude corresponding to a given signal.
E	The signal energy.
E( )	The expected value of .... $E(\underline{X})$ is the mean of the random variable $\underline{X}$.
exp[ ]	The number e raised to the power ....
f_N	The probability density of the receiver input when noise alone is present.
f_{SN}	The probability density of the receiver input when signal and noise are present.
F_N	The (complementary) distribution of the likelihood ratio of the receiver input when noise alone is present.
F_{SN}	The (complementary) distribution of the likelihood ratio of the receiver input when signal and noise are present. Thus $F_{SN}(\beta)$ is the probability that the likelihood ratio of a receiver input containing both signal and noise exceeds the number β.
L( )	The likelihood ratio. $L(\Psi) = f_{SN}(\Psi) / f_N(\Psi)$.
N_o	The noise power in a 1-cps band.
N[ ]	The normal distribution function. (See section on notation and formulas for the normal distribution).
P	(The event of) a perception. $\overline{P}$ is the complementary event.
Pr( )	The probability of .... (See section on notation for probabilities.)
Pr(C)	The probability of a correct response; the detectability.
Ψ	The set $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{2TW}$. Ψ may be thought of as an alternative representation of the receiver-input time-function.
Ψ_1	Analogously to Ψ, Ψ_1 is the receiver input in the first observation-interval of a two-category forced-choice trial.
Ψ_2	The receiver input in the second observation-interval of a forced-choice trial.
Q_i	The outcome of a detection trial. The event Q_1, Q_2, Q_3, or Q_4 occurs according as $S \cdot R$, $\overline{S} \cdot R$, $S \cdot \overline{R}$, or $\overline{S} \cdot \overline{R}$ occurs.
r	The signal-energy to noise-spectrum-level ratio expressed in decibels.
R	One of the two possible responses in a detection situation. R is the response that yields the higher payoff in case S obtains. The complementary response is $\overline{R}$. (These symbols are not used in the forced-choice situation.)

R_1	One of the two possible responses in a two-category forced-choice trial. R_1 is the response which yields the higher payoff in case $S{:}\bar{S}$ obtains. The complementary response is R_2.
S	One of the two possible states (of stimulation) in a detection situation. Specifically, in a yes-no situation in which noise occurs on every trial and a signal occurs on some trials, S is the event that the signal occurs. The complementary event is $\bar{S}$. (These symbols are not used in the forced-choice situation.)
$S{:}\bar{S}$	In a two-category forced-choice situation in which noise is presented in both observation-intervals and a signal is presented in only one, $S{:}\bar{S}$ is the event that the signal occurs in the first observation-interval. The complementary event is $\bar{S}{:}S$.
$S \cdot R$	The joint event S and R.
T	The duration of an observation-interval.
$\mathrm{Var}(\ )$	The variance of ....
v_i	The (payoff) value of the event Q_i.
W	The bandwidth of the noise.
$\underline{X}_i$	The value of the receiver-input time-function at the i^{th} sampling point; a random variable.

PART I. THEORY

1. TRADITIONAL PROBABILITY-MODELS IN PSYCHOPHYSICS AND THE CONCEPT OF DETECTION

We consider a given signal and a given condition under which this signal is presented to a subject. Our aim is to predict the probability that the subject will perceive the signal.

In order to make predictions we must have a model, and any psychophysical model must come to grips with at least two facts: (a) Subjects exhibit variability of a "random" sort: on some trials the signal is perceived (we shall say the event P occurs), on others the signal is not perceived (the event $\overline{P}$ occurs), and there is no way – at least no obvious way – of predicting the sequence of the events P and $\overline{P}$ in a group of trials. (b) The probability that the signal is perceived, Pr(P), increases as the signal magnitude increases, other things being equal.

The model to account for these facts is sometimes constructed as follows. Whether P or $\overline{P}$ occurs on a given trial, it is argued, depends on the value assumed on that trial by some variable quantity within the subject. Let us call this quantity "the effective stimulus magnitude" and designate it by d'. (This symbol is taken from Tanner and Swets (21) and introduced here in anticipation of our discussion of their model.) If on a given trial d' exceeds some cutoff value c, then P occurs; otherwise $\overline{P}$ occurs. That is, the signal must generate within the subject an effective stimulus magnitude greater than some minimum quantity in order to be perceived.

Next we must introduce some random variability and also bring the signal magnitude into the picture. Several approaches can be taken. We shall discuss two.

Threshold Approach. In this approach we argue that the quantity c is a random variable (which, in our notation, must be represented by $\underline{C}$ – with the exception of the random variable Ψ, all random variables will be designated by underlined capital letters). On a given trial, $\underline{C}$ assumes a certain value, which might be called the "true momentary threshold" for that trial. $\underline{C}$ will vary from trial to trial in some irregular fashion as a result, presumably, of some random variability in the subject.

The quantity d', on the other hand, is conceived of as a simple function of the signal magnitude. To each signal magnitude there corresponds an effective stimulus magnitude, the latter being presumably a nondecreasing function of the former.

It is frequently hypothesized [the "phi-gamma hypothesis" (7)] that the random variable $\underline{C}$ has a normal, or gaussian, distribution. In this case we have (see the sections on Notation for Probabilities and on Notation and Formulas for the Normal Distribution):

$$\Pr(\underline{C} < x) = N\,[x;\ E(\underline{C}),\ \mathrm{Var}(\underline{C})] \tag{1.1}$$

The basic statement of the model – that the quantity $\underline{C}$ must be exceeded by the effective stimulus magnitude for perception to occur – may be written

$$\Pr(P) = \Pr(\underline{C} < d') \tag{1.2}$$

From Eqs. 1.1 and 1.2 we get

$$\Pr(P) = N\,[d';\ E(\underline{C}),\ \mathrm{Var}(\underline{C})] \tag{1.3}$$

which is the "psychophysical function" according to the present model. That is, it is the function that relates the probability of perception to the signal magnitude, or, as in the present case, to a function of the signal magnitude.

Let us examine the significance of this result. We notice, first of all, that in order to calculate Pr(P) we need to specify two unknown parameters, $E(\underline{C})$, the expected true momentary threshold (referred to as "the threshold"), and $\mathrm{Var}(\underline{C})$, the variance of the true momentary threshold.

What is more unfortunate, however, is that even knowing these two parameters, we still cannot predict Pr(P) from a knowledge of the signal. This is because the independent variable d' of Eq. 1.3 is not a measure of the signal, but of the effective stimulus, and we do not know *a priori* what signals go with what effective stimuli. What we would have to do is to "scale" the signal magnitudes. That is, we would have to find the relation between some specific measure of signal magnitude (if the signals are tones, such a measure could be the sound pressure in dynes per square centimeter at the eardrum) and the corresponding effective stimulus magnitude. Having obtained this scale, we could then go from a particular signal to the corresponding effective stimulus, and, by means of the model, predict Pr(P).

Unless, however, this scale can be defined independently of the model, or unless the model is to be used for purposes other than predicting the psychophysical function, such a procedure would be trivial. For we would be using the model, in conjunction with the results of psychophysical experiments, to set up the scale, and then using this scale, in conjunction with the model, to predict the results of the psychophysical experiments.

A second way of deriving the psychophysical function might be called the criterion approach.

Criterion Approach. Here again we have a certain stimulus magnitude d' corresponding to a given signal. The effect of the subject's variability is, in the present case, to perturb, not the momentary threshold, but rather the quantity d' itself. If $\underline{H}$ is a random variable that assumes a value on each trial, and d' is the effective stimulus magnitude that would obtain in the absence of subject variability, then the total effective stimulus magnitude on a trial can be represented by $d' + \underline{H}$. The quantity $\underline{H}$, it is assumed, has a normal distribution, with the proviso, moreover, that $\underline{H}$ augments d' as often as it diminishes it; or, in other words, that $E(\underline{H}) = 0$. We have, therefore,

$$\Pr(\underline{H} < x) = N\,[x;\ 0,\ \mathrm{Var}(\underline{H})] \tag{1.4}$$

Again we argue that in order for P to occur on a given trial the (total) effective stimulus magnitude on that trial, $d' + \underline{H}$, must exceed a minimum quantity c. Here, this quantity c is not a random variable but a parameter of the situation, and is called "the subject's criterion." We have, therefore,

$$\Pr(P) = \Pr(d' + \underline{H} > c) \tag{1.5}$$

From Eqs. 1.4 and 1.5 we easily obtain the psychophysical function

$$\Pr(P) = N\,[d';\ c,\ \mathrm{Var}(\underline{H})] \tag{1.6}$$

Essentially, this result differs from that obtained with the threshold approach (Eq. 1.3) only in notation. Equations 1.3 and 1.6 both mean that the psychophysical function is a normal ogive when plotted on the scale of d', the scale of effective stimulus magnitudes. Since $E(\underline{C})$, $\mathrm{Var}(\underline{C})$, c, and $\mathrm{Var}(\underline{H})$ are unknown constants, we can equally well represent either equation simply by

$$\Pr(P) = N\,[d';\ \mu,\ \sigma^2]$$

The results of the threshold and criterion approaches are thus mathematically indistinguishable, and our earlier comments about the former apply equally well to the latter.

The criterion approach, however, is suggestive of further developments which would seem artificial in terms of the threshold approach. In the first place, we easily obtain an extension of the model for the case in which the signal is not presented on a trial. Our earlier result, Eq. 1.6, was

$$\Pr(P|S) = N\,[d';\ c,\ \mathrm{Var}(\underline{H})] \tag{1.7a}$$

in which we write $\Pr(P|S)$ to indicate that the function applies only to trials on which the signal is presented. [See the section on Notation for Probabilities.] The case in which no signal is presented is, one might argue, identical with the case given above, with the exception that d' now has a lower value. Since we measure d' on an unknown abstract scale, it is reasonable to let this lower value arbitrarily be zero. Then we write

$$\Pr(P|\bar{S}) = N\,[0;\ c,\ \mathrm{Var}(\underline{H})] \tag{1.7b}$$

for trials on which the signal is not presented.

A second advantage of the criterion approach lies in the names assigned to various quantities. In the threshold approach, Eq. 1.3, the "central-tendency" term, $E(\underline{C})$, was called "the threshold." We are tempted to think of a "physiological" quantity which is constant, as physiological quantities presumably are, under a wide range of possible psychophysical experiments. In the criterion approach, the central-tendency term, c, is called "the subject's criterion." This suggests a "psychological" quantity which may, as psychological quantities sometimes do, change its value from psychophysical experiment to experiment. It may even be that the quantity c is somehow correlated with the quantity d'. If this were true, the present model would be in serious trouble. The model is weak to begin with, but it can at least predict the shape of the psychophysical function plotted on the effective stimulus scale. If the correlation being considered existed, the model would not even be able to do that much without knowing the nature of the correlation. In the criterion approach we are forced to take this possibility more seriously.

Rather than turning to a discussion of this problem of the shifting criterion, we consider a more immediate objection. We have begun by considering the prediction of the event P, a perception of the signal. If, however, we have occasion to act as subject in a threshold experiment, we discover that the present "experience model" does not account for our actual experience. According to the experience model the event P occurs whenever the total effective stimulus magnitude exceeds the criterion c; we are tempted to think of a lamp inside the mind which either does or does not light up on a given trial. Our actual experience, however, is not like that; it does not come so neatly dichotomized. This fact, has, of course, long been realized, as is evidenced by the extensive discussion in the psychophysical literature of the problem of guessing (see refs. 1, 7, 21, and our Appendix A, in which the traditional "guessing-correction formula" is discussed). By "a guess" is presumably meant the act of responding "yes, I perceive the signal" when one does not perceive the signal, or at any rate when one is not certain whether or not he does. Now, if experience did come in the dichotomized fashion demanded by the experience model, a well-intentioned subject would obviously find no occasion for guessing. He would simply report whether or not P occurred. Theorists who believe that well-intentioned subjects guess thereby also believe that the experience model does not fit the facts.

There exist, fortunately, events in this situation which do come in the dichotomized fashion predicted by the model. These are the event R of responding "yes, I perceive the signal" and the complementary event $\overline{R}$. (Indeed, any two mutually exclusive events whose occurrence is under the subject's control would do.) We can, it will be seen, get out of our present predicament if we modify our model to predict the occurrence, not of P, but of R. Such a "response model," then, asserts that a response R is made whenever the total effective stimulus magnitude on a given trial exceeds the criterion, leaving questions of experience altogether out of the discussion. While this seems to take us away from traditional psychophysics, it does have the advantage of solving at this stage a further problem that we would sooner or later have had to face: the event P is, for scientific purposes, an unobservable event. To get into the realm of observables, we would have had to make additional assumptions to enable us to predict the occurrence of observable events, such as R, from the unobservable one, P. The response model does not need these additional assumptions. The questions relating to guessing which have bothered workers in this area need, moreover, no longer arise on the response model, since the experience of the subject is now irrelevant.

The response model, then, predicts directly from stimuli to responses, attaining thereby an objective character. There is, unfortunately, one further hitch. What task are we to assign the subject? Presumably we wish to instruct him to respond if and only if he perceives the signal. This is an unpleasant situation, though, for we now find that having deleted mention of experience from the model, it creeps back in the instructions. The model and the instructions under which the model applies thus bear no relation, at least on this point, to each other. If it were not for the fact that there is a way out of this difficulty – and one that will, moreover, prove helpful in handling the problem of the shifting criterion – we would be tempted here to revert to the experience model.

We extricate ourselves by the introduction into the experiment of catch-trials, i.e., trials on which the signal is not presented. Once such trials are introduced, a new mode of instruction becomes quite natural: we instruct the subject to try to be correct as often as possible; by "correct" we mean making a response R when there is a signal or making the complementary response $\overline{R}$ when there is no signal. Or, to make the situation even more general, we may assign values to the various outcomes of a trial. The subject's task is then to maximize the values which accrue to him as a result of such trials. (The term "value" is used in the sense in which it is said that gold has greater value than silver.)

The introduction of catch-trials is, of course, nothing new to psychophysics. What is relatively new, perhaps, is the realization that introducing them brings the psychophysical experiment under the rubric of detection, thereby rendering psychophysical problems amenable to analysis in terms of the concepts and results of detection theory. In order to understand this, let us examine in a general way what we mean by a detection situation.

Many different formulations are possible. In this report we shall use the concept exclusively in accordance with the following definition.

Definition of the Detection Situation

1. On each "trial" one or the other of two mutually exclusive states S or $\overline{S}$ obtains. The *a priori* probability Pr(S) of the occurrence of S is assumed to be a fixed given number.

2. On each trial the subject (the "detector") brings about the occurrence of one or the other of two mutually exclusive events ("responses") R or $\overline{R}$.

3. Event Q_1, Q_2, Q_3, or Q_4 occurs on a given trial according as the joint event $S \cdot R$, $\overline{S} \cdot R$, $S \cdot \overline{R}$, or $\overline{S} \cdot \overline{R}$ also occurs on that trial.

4. The occurrence of the event Q_i $(i = 1, \ldots, 4)$ has a specifiable value, v_i (the "payoff value" or "payoff") for the subject. ("Value" is used in the sense defined above.)

5. The following inequalities hold: $v_1 > v_3$; $v_4 > v_2$.

Example: A Discrimination Experiment. A rat is placed on a jumping-stand. Facing it across a gap are two cards, one bearing a circle, the other a triangle. An air blast forces the rat to jump at one of the two cards on each trial. The system is so arranged that if the rat jumps at the circle, the card folds back and the rat goes through to find food (the rat is "reinforced"). If it jumps at the triangle, it finds that it cannot go through, bumps its nose, and falls into a net (it is "punished"). On each trial, then, the rat jumps either to the right (R) or left ($\overline{R}$). The card bearing the circle is either on the right (S) or left ($\overline{S}$). If it is on the right and the rat jumps to the right ($S \cdot R$), the rat is reinforced on the right (Q_1). If the circle is on the left and the rat jumps to the right ($\bar{S}\cdot R$), the rat is punished on the right (Q_2). If the circle is on the right and the rat jumps to the left ($S\cdot\bar{R}$), the rat is punished on the left (Q_3). And, lastly, if the circle is on the left and the rat jumps to the left ($\bar{S}\cdot\bar{R}$), the rat is reinforced on the left (Q_4). We assume that being reinforced has a greater value for the rat than being punished, so that v_1, the value of the occurrence of Q_1, is greater than v_3, the value of the occurrence of Q_3; likewise v_4 is greater than v_2.

Remark 1. On occasion we shall refer to certain mechanical devices as "detectors." This is a loose way of speaking, since, presumably, the occurrence of events can be said to have value only to organisms. Strictly speaking, it is the person who uses such a mechanical device who is doing the detecting; it is for him that Q_i has value v_i.

Remark 2. We shall arrange the payoff values in a payoff matrix:

	S	$\bar{S}$
R	v_1	v_2
$\bar{R}$	v_3	v_4

Remark 3. Clause 5 of the definition, included primarily for subsequent ease of exposition, is not as restrictive as might at first appear. What it does is to rule out two classes of situations. (a) It rules out degenerate situations exemplified by the following payoff matrix:

	S	$\bar{S}$
R	1	2
$\bar{R}$	-3	1

Consider what happens in this situation: If S obtains, response R is preferable, since R yields a larger payoff than $\bar{R}$; if $\bar{S}$ obtains, response R is again preferable for the same reason. Optimal behavior would therefore consist in making response R on every trial without regard to whether S or $\bar{S}$ obtained. Such situations are clearly of little practical interest. (b) Clause 5 excludes situations that differ from the included ones only trivially in notation, as, for instance,

	S	$\bar{S}$
R	-1	1
$\bar{R}$	1	-1

If in this matrix we replace S by $\bar{S}$ and $\bar{S}$ by S, we obtain a matrix that satifies the inequalities of clause 5. Excluding such "mirror image" cases enables us to state very simply what we mean by the accuracy of detection.

Accuracy of Detection. To make response R when S obtains – this is sometimes called a "hit" – or to make response $\bar{R}$ when $\bar{S}$ obtains is, we shall say, to make a correct response. To make response R when $\bar{S}$ obtains – this is sometimes called a "false alarm" – or to make response $\bar{R}$ when S obtains – sometimes called a "miss" – is to make an incorrect response. The accuracy of detection in a given situation will, of course, be measured by the ratio of the number of correct responses to the total number of responses made in the situation.

If in a given situation the accuracy of detection is a_o, we shall say that the detectability of the signal (to be defined later) in that situation is a_o.

Expected Payoff. If we know the probability of a hit, $Pr(R|S)$, and the probability of a false alarm, $Pr(R|\bar{S})$, in a given situation, we can calculate the expected value of the payoff according to the following obvious formula:

$$\begin{aligned} E(\underline{V}) = {} & v_1 Pr(R|S)\, Pr(S) + v_2 Pr(R|\bar{S})\, Pr(\bar{S}) \\ & + v_3 Pr(\bar{R}|S)\, Pr(S) + v_4 Pr(\bar{R}|\bar{S})\, Pr(\bar{S}) \end{aligned} \tag{1.8}$$

2. TANNER AND SWETS' MODEL

In Tanner and Swets' model (19, 20, 21), the prototype of the psychophysical experiment is the detection situation defined in the previous section (with the event S being the occurrence of some signal). The mathematics of the model, however, is still for the most part that of traditional psychophysics. Indeed, the psychophysical function is assumed to be given by our Eqs. 1.7, with the additional statement that $\mathrm{Var}(\underline{H}) = 1$. The important assumption behind this statement is that the variance of $\underline{H}$ is independent of the various other parameters of the situation. Picking the value unity is of no particular significance, since the scale on which $\underline{H}$ is measured is still arbitrary: letting $\mathrm{Var}(\underline{H}) = 1$ amounts to picking a unit of measurement on this scale. Since we are dealing with a response model, we shall also change the equations to the extent of writing R where we previously wrote P. Equations 1.7a and 1.7b become

$$\Pr(R|S) = N[d'; c, 1] \tag{2.1a}$$

$$\Pr(R|\overline{S}) = N[0; c, 1] \tag{2.1b}$$

The reader will recall how these equations were obtained. On each trial there is assumed to exist an effective stimulus of total magnitude $d' + \underline{H}$, where $\underline{H}$ is a normally distributed random variable with mean zero. The quantity d' is assumed to be a (non-decreasing) function of the signal magnitude, so that $d' + \underline{H}$ will, on the average, be large or small, according as the signal is strong or weak. If no signal is presented, d' is assumed to have the value zero. Response R occurs on a given trial if $d' + \underline{H} > c$ on that trial, c being a criterion set by the subject.

From Eq. 2.1b we see that $\Pr(R|\overline{S})$, the probability of a false alarm, is a function solely of the criterion c. If the subject sets his criterion to a low value, i.e., one that will frequently be exceeded by $d' + \underline{H}$, the false alarm rate will be high; if he sets the criterion high, the false alarm rate will be low. Conversely, the false alarm rate determines the criterion; measuring the former in an experiment enables us by Eq. 2.1b to calculate the latter.

The experiment in which we measure the false alarm rate can also, of course, yield a measure of $\Pr(R|S)$, the probability of a hit. The measured probability of a hit, together with the calculated value of c, can then be introduced into Eq. 2.1a, and this equation solved for d'.

The whole process can then be repeated with a different signal magnitude. For each signal used we measure $\Pr(R|S)$ and $\Pr(R|\overline{S})$; these two quantities in turn yield a value of d'. We have thus constructed a scaling procedure, i.e., a procedure that enables us to determine d' for any given signal.

We could check on this procedure in the following manner. First we set up a detection situation s_1 by specifying a given signal, payoff matrix, and a priori probability Pr(S). In situation s_1 we measure $\Pr_1(R|S)$ and $\Pr_1(R|\overline{S})$. Then we set up situation s_2 with the same signal but with a different payoff matrix and/or Pr(S). Our psychophysical measurements now yield $Pr_2(R|S)$ and $Pr_2(R|\bar{S})$. Since s_1 and s_2 are different situations, we find, presumably, that $Pr_1(R|S) \neq Pr_2(R|S)$ and that $Pr_1(R|\bar{S}) \neq Pr_2(R|\bar{S})$. For a subject behaving optimally,* at any rate, these inequalities can be shown to hold. We now solve Eqs. 2.1, using $Pr_1(R|S)$ and $Pr_1(R|\bar{S})$, and obtain a value d'_1. Repeating the process, using $Pr_2(R|S)$ and $Pr_2(R|\bar{S})$, we obtain d'_2. Under ideal circumstances, i.e., if there is no experimental error, we should find that $d'_1 = d'_2$, since both of these numbers measure the effective stimulus magnitude of the same signal. Unfortunately, we would need further statistical analysis to determine what discrepancies we could tolerate, and such an analysis has not, to our knowledge, been developed.

Tanner and Swets' model enables us to relate our uncertainties one to the other.** We can, for example, write the probability of a hit as a function of d' and the probability of a false alarm, eliminating the criterion c. The predictive power per se of the model is not, however, increased thereby. If our interest lies in predicting the psychophysical function, i.e., in predicting the probabilities $Pr(R|S)$ and $Pr(R|\bar{S})$ corresponding to a given signal, we still have to know beforehand: (a) the magnitude of our signal on the effective stimulus scale; and (b) the value of the parameter c. From the point of view of experimental prediction we have improved on the venerable phi-gamma hypothesis only to the extent of writing Eq. 2.1b; that is, only to the extent of adopting the criterion approach of the last section.

We now want to gain some insight into how we can change the false alarm rate without changing the signal. It is essential that we be able to do so in order to have a check on the scaling procedure just described. Consider the following example.

Example. Let the payoff matrix be

	S	$\bar{S}$
R	100	0
$\bar{R}$	0	1

and let $Pr(S) = 0.99$. In such a situation a clever subject will almost always make the response R, since if he makes this response he stands a good chance – indeed a 99 to 1 chance – of being paid 100; but if he makes response $\bar{R}$, he can at most be paid 1 – and that much only once in a hundred times. The subject will therefore, presumably, set his criterion very low so that it will almost always be exceeded by $d' + \underline{H}$, and response

*This concept is defined below. In order for the inequalities to hold, it is sufficient (in optimal behavior) for the parameter β, defined by Eq. 2.2, to assume different values in situations s_1 and s_2.

**Tanner and Swets also show how predictions in the present kind of psychophysical experiment are related to predictions in the "forced-choice" experiment. This part of their model will not be discussed. The forced-choice situation in general will be discussed in sections 4 and 5.

R will be made on almost every trial. Only if the total effective stimulus magnitude is extremely low will he let himself be convinced that the signal was not presented (that $d' = 0$ on that trial) and make response $\bar{R}$.

If we change the situation by letting the payoff matrix be

	S	$\bar{S}$
R	1	0
$\bar{R}$	0	100

and by letting $Pr(S) = 0.01$, the reverse will presumably occur. The subject will now set his criterion very high and make response $\bar{R}$ on almost every trial. Only if the effective stimulus is extremely large, will he take it as evidence that a signal was presented (that $d' \neq 0$ on that trial) and make response R.

We see, therefore, that a clever subject will pick his criterion differently in different situations even though the signal remains the same. We might even suspect that there is an optimal way of picking the criterion so that the expected payoff to the subject is maximized in the given situation. A subject who sets his criterion to such an optimal value, c_{max}, would be said to be behaving optimally.

[A subject behaving optimally in the sense of the present discussion must not be confused with what we shall call an ideal detector. The latter maximizes the expected value of the payoff. The former maximizes the expected value of the payoff in a situation in which conditions (1.4) and (1.5) – the fundamental conditions of the model – hold. We may say that our optimally-behaving subject is an ideal detector for a situation in which the information about the signal and the variability ("noise") is available only in the form stipulated by Tanner and Swets' model.]

While Tanner and Swets themselves do not particularly stress this notion of optimal behavior, it might be of interest here to investigate it somewhat further. The mathematics of the situation is worked out in Appendix B. Here we merely state the result. First, a definition:

$$\text{Let } \beta = \frac{(v_4 - v_2)\, Pr(\bar{S})}{(v_1 - v_3)\, Pr(S)} \tag{2.2}$$

This quantity β incorporates the relevant information about the payoff matrix and the a priori probability of a signal. In terms of the (known) parameter β we find that

$$c_{max} = \frac{\ln \beta}{d'} + \frac{d'}{2} \tag{2.3}$$

Putting this result in Eqs. 2.1 and rearranging terms, we see that for a subject behaving optimally the psychophysical function is

$$\Pr(R|S) = 1 - N\left[\frac{\ln \beta}{d'} - \frac{d'}{2};\ 0,\ 1\right] \tag{2.4a}$$

$$\Pr(R|\bar{S}) = 1 - N\left[\frac{\ln \beta}{d'} + \frac{d'}{2};\ 0,\ 1\right] \tag{2.4b}$$

Note that these functions (plotting the probability versus d') are not, in general, in the shape of normal distributions (normal ogives are obtained only in case $\beta = 1$). This may be considered startling when we reflect that our basic equations (2.1) are, of course, normal ogives. The difference occurs because the subject behaving optimally changes his criterion not only with changes in the payoff matrix or the a priori probability but also with changes in the signal (changes in d'). Thus, the quantities c and d' appearing in Eqs. 2.1 are not independent of each other in the case of optimal behavior; insofar as our subjects approximate such optimal behavior the usefulness of these equations may therefore be seriously doubted even if the model that led to them is accepted.

In our discussion of the scaling procedure we saw that the two measured probabilities $\Pr(R|S)$ and $\Pr(R|\bar{S})$ corresponding to a given signal were necessary to yield an estimate of d'. We see from Eq. 2.4 that on the assumption of optimal behavior each of these two probabilities yields an estimate of d'. This provides us with a check for optimal behavior: The experimenter selects a value of β, that is, selects a payoff matrix and a Pr(S), and selects a particular signal to be used. The probability of a hit and of a false alarm are then measured. Each of these yields an estimate of d'. The two estimates ought to agree within the tolerance of experimental error if the subject is behaving optimally (some statistics would again be necessary to determine this tolerance). Unfortunately, the experiment performed by Tanner and Swets did not follow this straightforward procedure. Their experiment – which according to our definition is not a detection situation – was set up so that on each trial there was presented either no signal or one of four signals differing in magnitude, the payoff matrix being

	S_1	S_2	S_3	S_4	$\bar{S}$
R	v_1	v_1	v_1	v_1	v_2
$\bar{R}$	v_3	v_3	v_3	v_3	v_4

It is not clear to us how, from the results of such an experiment, one could determine whether or not the subject was behaving optimally.

3. DETECTION THEORY

[The material for this section is drawn from Peterson and Birdsall (15). The reader is referred to their monograph for further details, proofs, and a short bibliography of other writings on detection theory. A brief historical introduction to detection theory is given in our Appendix J.]

In discussing the models considered thus far we spoke of signals in a very general way. The specific nature of the signal – its duration, for instance – played no part in the theorizing. Likewise, the nature of the random variability was hardly analyzed: we arbitrarily assumed the existence of a single random variable $\underline{H}$ with normal distribution and fixed parameters.

A more sophisticated analysis is made in detection theory; it results, as one would expect, in a model of greater power, although, admittedly, also of greater mathematical complexity. It is the purpose of this section to present some of the fundamental ideas of this theory.

The Signal. By a signal, we shall understand a measurable quantity varying in a specified manner during a fixed observation-interval T seconds long. Letting time be measured from the beginning of the observation-interval, we may represent the signal by a time-function s(t) defined on the interval $0 < t < T$.

It will be assumed that s(t) is series-bandlimited to a bandwidth W_S. By this we mean that s(t) has a Fourier series expansion on the interval $0 < t < T$; that the series contains only a finite number of terms; and that the frequency of the highest harmonic is W_S.

The Noise. As in previous models, we shall assume that the signal is perturbed by random variability. Since the signal is now represented by a time-function, we must let this perturbation occur at each moment of time at which the function is defined. Mathematically we do this by adding to our signal-function s(t) a "random" time-function, i.e., a function whose value at each moment of time is specified only in a statistical sense. Such a function n(t), $0 < t < T$, will be said to describe the noise in the situation. It will be assumed that n(t) is series-bandlimited to a bandwidth $W \geqslant W_S$.

Receiver Inputs; the Events S and $\overline{S}$. Each observation-interval thus yields a time-function. Such a function – or its physical counterpart – is called a receiver input. It is designated by x(t).

By saying that the signal is presented during a given observation-interval, we mean that the receiver input for that interval is given by

$$x(t) = s(t) + n(t) \qquad 0 < t < T$$

By saying that the signal is not presented during a given observation-interval, we mean that s(t) = 0 for that interval, so that the receiver input is given simply by

$$x(t) = n(t) \qquad 0 < t < T$$

We say that the event S or $\overline{S}$ occurs according as the signal is or is not presented.

In either case the receiver input is given by a series-bandlimited time-function defined on the interval $0 < t < T$; this function is "random" since, in either case, it contains the "random" term $n(t)$.

The Sampling Theorem. We shall need one form of the so-called sampling theorem in the time-domain. [For a proof of this theorem see Appendix D of reference 15. Also compare references 5 and 16.] The theorem states that any fixed time-function defined on an interval $0 < t < T$ and series-bandlimited to a bandwidth W can be uniquely represented by a set of $2TW$ numbers, these numbers being the value of the function at "sampling points" $1/2W$ seconds apart on the time-axis.

Accordingly, if we knew the value of $x(t)$ at each of the $2TW$ sampling points, we could reconstruct $x(t)$. The function $x(t)$, however, is a "random" function; its value at the sampling points is determined only in a statistical sense. Thus, instead of representing this function by $2TW$ numbers, as we would in the case of an ordinary well-specified function, we must use, instead, a set of $2TW$ random variables, $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_{2TW}$.

On occasion it will be found convenient to denote this set by the single letter "Ψ". We have, thus, three equivalent ways of writing a receiver input: $x(t)$; $\underline{X}_1, \ldots, \underline{X}_{2TW}$; and Ψ.

Probability Density of the Receiver Input. We consider, next, the joint probability density of the random variables $\underline{X}_1, \ldots, \underline{X}_{2TW}$. (It is here assumed that the density exists, i.e., that the joint distribution function of the $\underline{X}_i$ has all derivatives.) This density will, by virtue of the sampling theorem, give us the complete statistical specification of the receiver input.

Actually, it will be found convenient to break up this density into two conditional densities, one for each of the two cases S and $\overline{S}$. The symbol "f_{SN}" will be used to denote the joint density (of $\underline{X}_1, \ldots, \underline{X}_{2TW}$) that obtains in case the receiver input contains both signal and noise; "f_N" will be used for the density in the alternative case of noise alone.

In formal notation we have

$$\Pr(\underline{X}_1 < a_1, \underline{X}_2 < a_2, \ldots, \underline{X}_{2TW} < a_{2TW} \mid S)$$

$$= \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} \cdots \int_{-\infty}^{a_{2TW}} f_{SN}(x_1, x_2, \ldots, x_{2TW})\, dx_1 dx_2 \ldots dx_{2TW}$$

$$\Pr(\underline{X}_1 < a_1, \underline{X}_2 < a_2, \ldots, \underline{X}_{2TW} < a_{2TW} \mid \overline{S})$$

$$= \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} \cdots \int_{-\infty}^{a_{2TW}} f_N(x_1, x_2, \ldots, x_{2TW})\, dx_1 dx_2 \ldots dx_{2TW}$$

Likelihood Ratio. Recall that the probability density of the receiver input is $f_{SN}(x_1, \ldots, x_{2TW})$ or $f_N(x_1, \ldots, x_{2TW})$ according as the signal is or is not presented.

We now define the likelihood of the receiver input to be $f_{SN}(\underline{X}_1, \ldots, \underline{X}_{2TW})$ or $f_N(\underline{X}_1, \ldots, \underline{X}_{2TW})$ according as the signal is or is not presented. In our summary notation, the likelihood is written $f_{SN}(\Psi)$ or $f_N(\Psi)$. The likelihood ratio $L(\Psi)$ of a receiver input is then defined as follows:

$$L(\Psi) = \frac{f_{SN}(\Psi)}{f_N(\Psi)} = \frac{f_{SN}(\underline{X}_1, \ldots, \underline{X}_{2TW})}{f_N(\underline{X}_1, \ldots, \underline{X}_{2TW})}$$

The likelihood ratio of a particular receiver input tells us, very roughly speaking, how likely it is that this input contains the signal; if the likelihood ratio is large, we may take it as evidence that a signal is present; the larger the likelihood ratio, the weightier the evidence. Since our receiver input is random, we can never, of course, be sure; some receiver inputs containing the signal and noise will yield low values of the likelihood ratio, and some containing only noise will yield high values.

The likelihood ratio must not be confused with the ratio

$$\frac{f_{SN}(x_1, \ldots, x_{2TW})}{f_N(x_1, \ldots, x_{2TW})}$$

of the densities. The variables occurring in the likelihood ratio, unlike those occurring in the ratio of the densities, are random variables. The likelihood ratio is thus a function of random variables, and, like any function of random variables, is itself a random variable. It is thus meaningful to speak of the distribution of the likelihood ratio, while it would be nonsensical to speak of the distribution of the ratio of densities, just as it would be nonsensical to speak of the distribution of a distribution.

As in the case of the receiver-input density, we break up the distribution of the likelihood ratio into two conditional distributions, one for each of the two cases S and $\overline{S}$. Denoting these distributions* by F_{SN} and F_N, we have

$$F_{SN}(x) = \Pr(L(\Psi) > x \mid S) \tag{3.1a}$$

$$F_N(x) = \Pr(L(\Psi) > x \mid \overline{S}) \tag{3.1b}$$

Thus, the expression $F_{SN}(\beta)$, to be used later, designates the probability that the likelihood ratio of a receiver input containing the signal exceeds the number β. The expression $F_N(\beta)$ designates the corresponding probability for a receiver input containing noise alone.

To fix our ideas, let us consider the following simple example.

Example: Likelihood Ratio in a Discrete Case. We are given a loaded coin, and we know, for one reason or another, that the odds are 10 to 1 in favor of a particular one of the two faces coming up when the coin is tossed, but we do not know which face

*Notice the direction of the inequality in Eqs. 3.1a and 3.1b. The functions are complementary distribution functions, since they give the probability that $L(\Psi)$ lies between x and $+\infty$, rather than between $-\infty$ and x, as is usual.

is favored. We decide to toss the coin once and count the number $\underline{K}$ (0 or 1) of "heads" that come up. $\underline{K}$ is our receiver input. We shall say that the event S or $\overline{S}$ obtains according as the true probability of heads is 1/11 or 10/11.

Since we are dealing with discrete quantities, we shall write probability distributions rather than densities. Thus,

$$f_{SN}(n) = \Pr(\underline{K} = n \,|\, S) = (1/11)^n \, (10/11)^{1-n} \tag{3.2a}$$

$$f_N(n) = \Pr(\underline{K} = n \,|\, \overline{S}) = (10/11)^n \, (1/11)^{1-n} \tag{3.2b}$$

$$(n = 0 \text{ or } 1)$$

The likelihood ratio is

$$L(\underline{K}) = \frac{f_{SN}(\underline{K})}{f_N(\underline{K})} = \frac{(1/11)^{\underline{K}} \, (10/11)^{1-\underline{K}}}{(10/11)^{\underline{K}} \, (1/11)^{1-\underline{K}}} = 10^{1-2\underline{K}} \qquad (\underline{K} = 0 \text{ or } 1)$$

The likelihood ratio is thus seen to be a simple function of the random variable $\underline{K}$. It is itself a random variable and has a probability distribution. Let us calculate this distribution for the case in which S obtains; that is, let us calculate

$$\Pr(L(\underline{K}) = m \,|\, S) = \Pr(10^{1-2\underline{K}} = m \,|\, S) \qquad (m = 1/10 \text{ or } 10)$$

Replacing the expression $10^{1-2\underline{K}} = m$ by the equivalent expression $\underline{K} = (1 - \log m)/2$, we have

$$\Pr(L(\underline{K}) = m \,|\, S) = \Pr\left(\underline{K} = \frac{1 - \log m}{2} \;\Big|\; S\right)$$

Replacing n in Eq. 3.2a by $(1 - \log m)/2$ we obtain our result

$$\Pr(L(\underline{K}) = m \,|\, S) = (1/11)^{\frac{1-\log m}{2}} \, (10/11)^{\frac{1-\log m}{2}} \qquad (m = 1/10 \text{ or } 10)$$

By a similar argument we can obtain

$$\Pr(L(\underline{K}) = m \,|\, \overline{S}) = (10/11)^{\frac{1-\log m}{2}} \, (1/11)^{\frac{1-\log m}{2}}$$

We see, therefore, that the random variable $L(\underline{K})$ assumes the values 1/10 and 10, respectively, with probabilities 1/11 and 10/11 if S obtains, and with probabilities 10/11 and 1/11 if $\overline{S}$ obtains.

A Fundamental Theorem of Detection Theory. Consider a detection situation as defined in section 1, with given payoff matrix and Pr(S), and with the event S defined to be the occurrence of a given signal; and consider the following Decision Procedure: For any receiver input make response R if and only if $L(\Psi) > \beta$, where β is given by Eq. 2.2.

Theorem. A detector behaving in accordance with the decision procedure maximizes the expected value of the payoff in the given detection situation. Conversely, a detector maximizing the expected value of the payoff in the given detection situation makes the same responses as one behaving in accordance with the decision procedure.

[For proof, see Peterson and Birdsall's Theorems 1 and 2, section 2.4.2 (ref. 15).]

A detector behaving in accordance with the decision procedure will be called "an ideal detector". For an ideal detector we see that

$$\Pr(R|S) = \Pr(L(\Psi) > \beta|S) = F_{SN}(\beta) \tag{3.3a}$$

$$\Pr(R|\bar{S}) = \Pr(L(\Psi) > \beta|\bar{S}) = F_{N}(\beta) \tag{3.3b}$$

Continuation of the Preceding Example. Suppose that on the basis of our toss of the coin, i.e., on the basis of $\underline{K}$, we are required to choose between response R and $\bar{R}$. Suppose, moreover, that we are given that Pr(S), the probability that the coin is biased toward tails, is 1/12, and that the payoff matrix is

	S	$\bar{S}$
R	1	-1
$\bar{R}$	-1	1

With this payoff matrix and this value of Pr(S) we find that $\beta = 11$, and the fundamental theorem tells us to make response R if and only if $L(\underline{K}) > 11$. In the present example, however, the likelihood ratio never exceeds 10, as we have already seen. Thus, $\Pr(R|S) = \Pr(R|\bar{S}) = 0$. The expected value of the payoff in this case is $10/12 = 0.82$, as may be established by substitution into Eq. 1.8.

Suppose that we pick a decision procedure not in accordance with the fundamental theorem. Thus, for example, suppose that we decide to make response R if and only if $L(\underline{K}) > 1$. We have already calculated that the probability of this event is 10/11 in case S obtains, and 1/11 in case $\bar{S}$ obtains. Thus, $\Pr(R|\bar{S}) = 10/11$ and $\Pr(R|S) = 1/11$. Putting these values into Eq. 1.8 we find that the expected value of the payoff is now reduced to $49/66 = 0.74$.

We thus see that in ideal detection the critical quantities are the distribution functions F_{SN} and F_N of the likelihood ratio of the receiver input. Fortunately, Peterson and Birdsall (15) have calculated these functions for a wide variety of signals presented in uniform gaussian noise.* We can introduce their results directly into Eq. 3.3 to

*By a uniform gaussian noise we mean a time-function whose distribution of instantaneous amplitudes is gaussian (normal) and whose power-spectrum is uniform over a given range and zero outside that range.

obtain the psychophysical function for the ideal detector. If the signal is known exactly, as it would be with a pure tone of known amplitude, frequency, and phase, we obtain

$$\Pr(R|S) = 1 - N\left[\frac{\ln \beta}{d} - \frac{d}{2};\ 0,\ 1\right] \tag{3.4a}$$

$$\Pr(R|\bar{S}) = 1 - N\left[\frac{\ln \beta}{d} + \frac{d}{2};\ 0,\ 1\right] \tag{3.4b}$$

where

$$d^2 = \frac{2E}{N_o},$$

E being the signal energy (the signal power multiplied by T), and N_o being the noise power per unit bandwidth (cps).

We notice, incidentally, that if we write d' for d in the expressions above, we obtain Eqs. 2.4, which, it will be recalled, hold for the case of optimal behavior on Tanner and Swets' model. The difference is that in the present equations the variable d is measured in actual physical units, while in Tanner and Swets' situation the variable d' is measured on an unknown scale of effective stimulus magnitudes. It may be added, parenthetically, that if we had chosen a somewhat different situation – for example, one in which the signal had unknown phase – we would have obtained an entirely different set of functions for $\Pr(R|S)$ and $\Pr(R|\bar{S})$.

The two probabilities $\Pr(R|S)$ and $\Pr(R|\bar{S})$ may, of course, be combined into a single quantity, the probability $\Pr(C)$ of a correct response, or detectability, as defined in section 1. The formula for doing so is, obviously,

$$\Pr(C) = \Pr(R|S)\Pr(S) + \Pr(\bar{R}|\bar{S})\Pr(\bar{S}) \tag{3.5}$$

The detectability function for the case of the signal known exactly may be found by substituting the expressions for $\Pr(R|S)$ and $\Pr(R|\bar{S})$ from Eqs. 3.4 in Eq. 3.5. The resulting function with $\Pr(S)$ set to 1/2 is plotted in Fig. 1 for several values of β.

We have seen in a general way how the ideal-detector psychophysical function for a particular kind of signal and of noise can be obtained. Let us examine this result. It has already been remarked that, in the present case, unlike that of optimal behavior on Tanner and Swets' model, there is no uncertainty about scales. We notice, moreover, that the effect of changing the duration of the signal is immediately given by Eqs. 3.4. For instance, we see that increasing the signal duration and decreasing the signal power by a given factor leaves everything unchanged. This is the so-called I-T Law of psychophysics, which is found to hold within limits (11). The equations also predict constant detectability for constant signal-to-noise ratio, other things being equal; this, too is found to hold within limits in auditory experimentation (11). Furthermore, the equations show that for an ideal detector the detectability is independent of the noise bandwidth; auditory experiments reveal that this is the case for human subjects, again within limits (11).

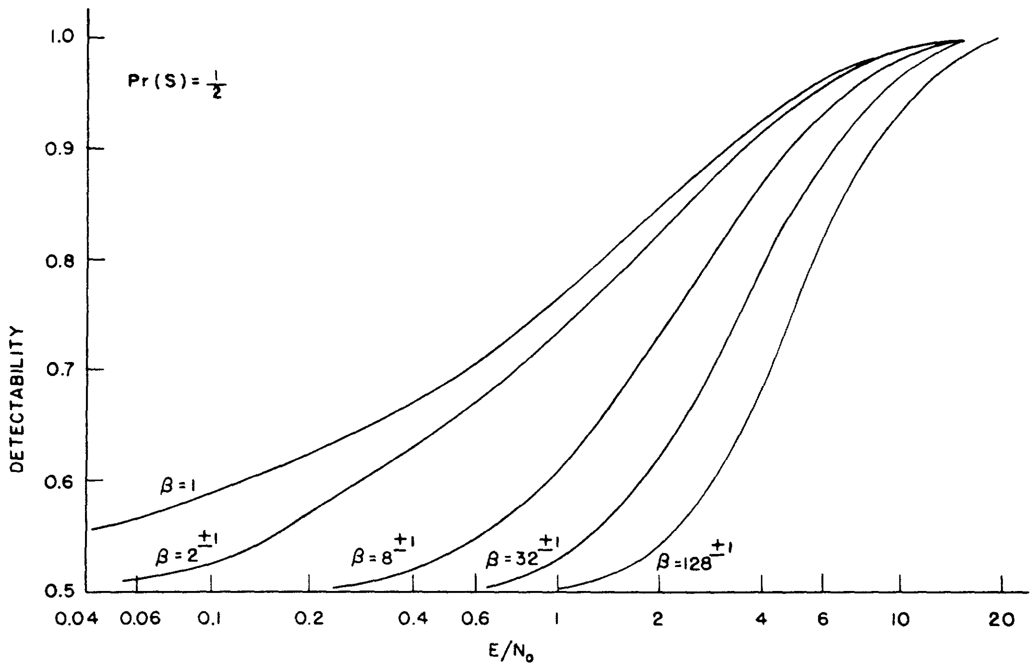

Fig. 1. Ideal-detector psychophysical functions. Signal known exactly; Pr(S) = 1/2. ("Yes-No" situation.)

Aside from these (perhaps accidental) facts, we might ask what relevance this kind of model has, in general, to empirical psychophysics, knowing, as we do in advance, that no subject is an ideal detector. First of all, the model allows us to place lower bounds on our threshold measurements. For example, we see from Fig. 1 that for a signal known exactly, a uniform gaussian noise, and Pr(S) = 1/2, the detectability cannot exceed 0.76 if the signal energy does not exceed the noise power per unit bandwidth. In similar fashion, lower bounds can be set for a wide variety of situations. Second, establishing the manner in which and the extent to which a subject's behavior differs from that of an ideal detector might lead to interesting insights into sensory psychology. If, moreover, the difference between subject and ideal detector is found to be expressible in terms that are sufficiently simple, we might be able to generate, by simple additions to the present model, a powerful tool for the prediction of actual behavior.

4. PROBLEMS OF EXPERIMENTAL TECHNIQUE AND THE FORCED-CHOICE METHOD

A basic problem concerning the payoff matrix has thus far been avoided. Recall that the v_i are the values, for the subject, of various events in the detection situation. How are we to determine these values? To be sure, the experimenter may try to assign the values by using a system of monetary rewards and fines. There is no guarantee, however, that the value for the subject corresponds to the number of cents involved; the value of two cents may be different from twice the value of one cent.

Moreover, the various events in the detection situation may have "private" values for the subject, prior to or independently of the assignment of values by the experimenter. Thus, in a particular experiment we might have:

Q_2: subject is fined \$1 upon making response R when $\bar{S}$ obtains (i.e., upon making a false alarm);

Q_3: subject is fined \$1 upon making response $\bar{R}$ when S obtains (i.e., upon missing the signal).

Even though the money involved in Q_2 and Q_3 is the same, the amount of displeasure experienced by the subject may be quite different in the two cases. If Q_3 occurs, the subject may reflect that he has made response $\bar{R}$ (which is appropriate to trials on which there is no signal, since in that case it earns \$1) on a trial on which there was a signal; but after all, no one is perfect, the signal was very weak, etc., ... On the other hand, if Q_2 occurs, he may reflect that he has made response R (appropriate to trials on which there is a signal) on a trial on which there was no signal; the experimenter may think he is having hallucinations, perhaps there is something wrong with his sense organs, etc., ... If the subject has such feelings, the values of Q_2 and Q_3 will be quite different.

It would be excellent, therefore, if we could get along without the v_i altogether. This we cannot do. We can, however, take a step in that direction with the help of the following result.

It is easy to show (see Appendix C) that the maximum expected payoff is independent of the v_i so long as

$$v_1 = v_4 \text{ and } v_2 = v_3 \tag{4.1}$$

A payoff matrix for which Eq. 4.1 holds will be called symmetrical.

Thus, if a subject is instructed to try to maximize the values that accrue to him,* his task, and therefore presumably his behavior, will be independent of the payoff matrix, so long as the latter is symmetrical. Instead of having to worry about four quantities, therefore, we need worry only about the symmetry of the payoff matrix.

*Strictly speaking, these instructions are unnecessary, since what we presumably mean by "value" is "something which people try to maximize."

As our discussion has just shown, however, the kind of experimental situation we have been considering is basically asymmetrical: Q_2 and Q_3 – or Q_1 and Q_4, for that matter – are psychologically different and may therefore have intrinsically different values for the subject. It is felt that condition (4.1), if not totally satisfied, can at least be approximated better in an experiment using the so-called (temporal) two-category forced-choice technique.

The experimental technique implicit in our discussion thus far has been the "yes-no" method. We are using this method whenever we set up our experiment as a detection situation in accordance with the definition of section 1, interpreting the event S of that definition to be the occurrence of a signal (as defined in section 3). Thus, on each trial of a yes-no experiment there is an observation-interval during which the signal may occur.

In the two-category forced-choice method there are on each trial two observation-intervals, during one and only one of which the signal occurs (whether on a given trial the signal occurs during the first or second observation-interval is determined, of course, at random); the noise (if any) occurs in both observation-intervals. Otherwise the forced-choice and the yes-no experiments are identical.

The forced-choice situation, it will be observed, fits our definition of a detection situation if we interpret the event S of that definition to be the occurrence of a signal in the first observation-interval (with the event $\bar{S}$ consequently to be interpreted as the occurrence of a signal in the second observation-interval).

In order for the notation to be indicative of the kind of experiment being discussed, the symbols S, $\bar{S}$, R, and $\bar{R}$, will be reserved for the yes-no situation, and the symbols $S{:}\bar{S}$ (signal in the first interval, no signal in the second), $\bar{S}{:}S$, R_1, and R_2 will be used for the corresponding events in the forced-choice situation.

The objectional kind of asymmetry found in the yes-no technique is not present to such an extent in the forced-choice technique. We may set up the following forced-choice situation, where, to emphasize the symmetry, we let the subject indicate his responses by pushing a switch to the left or right:

Q_1: subject is paid D dollars upon receiving the signal in the earlier observation-interval and pushing the switch to the left;

Q_2: subject is fined D' dollars upon receiving the signal in the later interval and pushing the switch to the left;

Q_3: subject is fined D' dollars upon receiving the signal in the earlier interval and pushing the switch to the right;

Q_4: subject is paid D dollars upon receiving the signal in the later interval and pushing the switch to the right.

Except for an interchange of "earlier" with "later" and of "left" with "right," our descriptions of Q_1 and Q_4, and of Q_2 and Q_3, are identical. In order for Eq. 4.1 to hold, therefore, we need only assume that such temporal and spatial differences do not affect the values that the subject places on the events.

If this assumption is true, i.e., if the value of Q_1 equals that of Q_4 and the value of Q_2 equals that of Q_3, then our previous result tells us that it is immaterial what the actual values are. For that matter, the monetary values D and D' assigned by the experimenter are also immaterial; we may replace the event of being paid D dollars by a flash of green light, for example, and the event of being fined D' dollars by a flash of red light, with the understanding that the green light is "correct" or "good," while the red light is "incorrect" or "bad." Our instructions to the subject may then simply be to try to maximize the occurrence of flashes of the green light.

In order to complete the symmetry of the situation we may want to set

$$\Pr(S{:}\bar{S}) = \Pr(\bar{S}{:}S) = 1/2 \tag{4.2}$$

A forced-choice situation in which both Eqs. 4.1 and 4.2 hold will be called a "symmetrical forced-choice situation."

5. IDEAL DETECTION IN A TWO-CATEGORY FORCED-CHOICE SITUATION

Let us consider a signal s(t) and a noise n(t), as defined in section 3. Since we are dealing with a forced-choice situation, we shall have two observation-intervals and, hence, two receiver inputs, $x_1(t)$ and $x_2(t)$. If $x_1(t) = s(t) + n(t)$ and $x_2(t) = n(t)$, we say that $S{:}\bar{S}$ obtains; if $x_1(t) = n(t)$ and $x_2(t) = s(t) + n(t)$, we say that $\bar{S}{:}S$ obtains. Either $S{:}\bar{S}$ or $\bar{S}{:}S$ must, of course, obtain on a given trial.

We have already seen how a receiver input may be considered to be a set Ψ of 2TW random variables. Two such sets, Ψ_1 and Ψ_2, will be necessary here, to correspond to the two receiver inputs $x_1(t)$ and $x_2(t)$. Notice that on any given trial one of the two quantities, Ψ_1, Ψ_2, has the density f_{SN} and the other the density f_N.

The payoff matrix for the forced-choice situation is written

	$S{:}\bar{S}$	$\bar{S}{:}S$
R_1	v_1	v_2
R_2	v_3	v_4

where $v_1 > v_3$, and $v_4 > v_2$. Corresponding to our earlier definition, we have

$$\beta = \frac{(v_4 - v_2)\ \Pr(\bar{S}{:}S)}{(v_1 - v_3)\ \Pr(S{:}\bar{S})} \tag{5.1}$$

We now need a decision procedure for ideal detection in the forced-choice situation to correspond to the one in the yes-no situation. (See section 3, Fundamental Theorem of Detection Theory.) It is shown in Appendix D that the desired rule is as follows.

Decision Procedure for Ideal Detection in the Forced-Choice Situation. Make response R_1 if and only if $L(\Psi_1) > \beta L(\Psi_2)$, where the likelihood ratio is defined as before.

This provides the general solution. From it we can calculate solutions for particular cases as was done in a previous section for the yes-no experiment. Ideal-detector psychophysical functions are derived in Appendix E for two kinds of signals presented in uniform gaussian noise: (a) signals known exactly; (b) signals known except for phase (the strict definition of the kind of signal considered here is given by Eq. E.14 of Appendix E).

Case of the Signal Known Exactly. From Appendix E, Eqs. E.12 and E.13, we have

$$\Pr(R_1 \mid S{:}\bar{S}) = 1 - N\left[\frac{\ln \beta}{h} - \frac{h}{2};\ 0,\ 1\right] \tag{5.2a}$$

$$\Pr(R_1 \mid \bar{S}{:}S) = 1 - N\left[\frac{\ln \beta}{h} + \frac{h}{2};\ 0,\ 1\right] \tag{5.2b}$$

where

$$h^2 = \frac{4E}{N_o}$$

It is interesting to compare these equations with the corresponding equations (3.4a and 3.4b), for the yes-no situation. It will be seen that if the signal-to-noise ratio, E/N_o, in the yes-no situation is made twice as great as in the forced-choice situation, the psychophysical functions become identical.

Equations 5.2a and 5.2b may be combined into a single detectability function by means of the following obvious formula:

$$Pr(C) = Pr(R_1 \mid S{:}\overline{S})\, Pr(S{:}\overline{S}) + Pr(R_2 \mid \overline{S}{:}S)\, Pr(\overline{S}{:}S) \tag{5.3}$$

where Pr(C) is the probability of a correct response.

If $\beta = 1$ and $Pr(S{:}\overline{S}) = Pr(\overline{S}{:}S) = 1/2$, the detectability function is

$$Pr(C) = N\left[\left(\frac{E}{N_o}\right)^{1/2};\ 0,\ 1\right] \tag{5.4}$$

This function is plotted in Fig. 2.

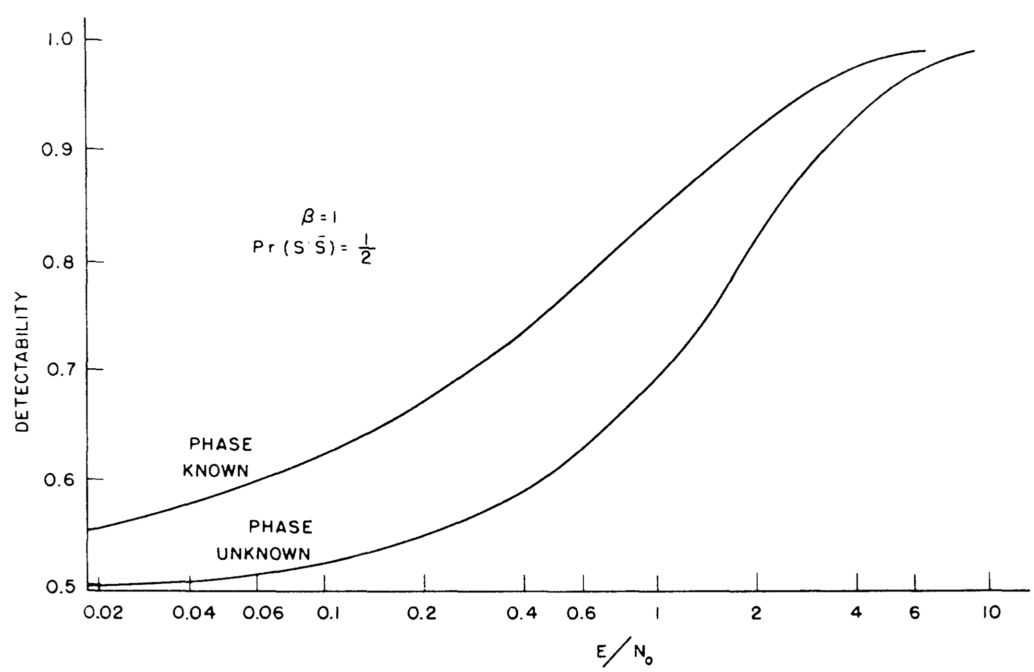

Fig. 2. Ideal-detector psychophysical functions for the two-category forced-choice situation. Signal known exactly and known except for phase; $\beta = 1$; $Pr(S{:}\overline{S}) = 1/2$.

Case of the Signal Known Except for Phase. This case is solved in Appendix E for $\beta = 1$ only. From Eqs. E. 29 and E. 27 we have

$$\Pr(R_1 \mid S:\bar{S}) = 1 - \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{5.5a}$$

$$\Pr(R_1 \mid \bar{S}:S) = \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{5.5b}$$

Combining these two equations into a single detectability function by formula 5.3 we have, for $\Pr(S:\bar{S}) = \Pr(\bar{S}:S) = 1/2$ (and $\beta = 1$),

$$\Pr(C) = 1 - \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{5.6}$$

This equation is plotted in Fig. 2.

We have already discussed the advantages of the detection-theory model as it applies to the yes-no situation (section 3). Our comments apply equally well to the present forced-choice situation.

PART II. EXPERIMENTAL RESULTS AND DISCUSSION

6. INTRODUCTION

The experiments discussed in this report were undertaken with the aim of exploring the manner in which and the extent to which a human subject differs from an ideal detector in the performance of an auditory detection task.

The two-category symmetrical forced-choice technique was used throughout. For a discussion of this technique and its advantages the reader is referred to Part I, section 4.

Two experiments were performed in order to measure with a considerable degree of accuracy the subjects' ability to detect auditory signals in the presence of broadband gaussian noise. Experiment 1 dealt with sinusoidal signals. Experiment 2 dealt with signals having two sinusoidal components. All listening was monaural.

7. DESCRIPTION OF THE EXPERIMENTS

7.1 DESIGN

Signals. The experiments consisted in the measurement of the detectability of auditory signals in the presence of noise. In Experiment 1 the signals were sinusoids of known level and frequency but unknown phase. Two frequencies were used: 500 cps and 1100 cps.* At 500 cps, 21 levels – 1 db apart – were selected in such a way as to cover roughly the range of detectability from 0.5 (chance) to 1.0 (perfect detectability). Similarly, at 1100 cps, 19 such levels were chosen.

In Experiment 2 each signal was composed of two sinusoidal components of the kind used in Experiment 1. Three pairs of frequencies were used: 500-540 cps, 1060-1100 cps, and 500-1100 cps. For each of the component frequencies three levels, labeled "soft", "medium", and "loud", were selected. The attempt was made to set these levels in such a way that soft components were poorly detectable, loud components were highly (though less than perfectly) detectable, medium components being somewhere in between. Since, however, these settings had to be made in advance of the experiment, an element of guesswork was involved. For each pair of frequencies the following five pairs of levels were used: loud-loud, medium-medium, soft-soft, loud-soft, soft-loud.

In addition to measuring the detectability of the two-component signals, it was also necessary to measure the detectability of each component taken by itself. For components of 500 cps and 1100 cps these measurements fall within the plan of Experiment 1. It remains, then, for Experiment 2 to measure the detectability of sinusoids of 540 cps and 1060 cps at each of three levels.

In Experiment 1, forty sinusoidal signals were used; in Experiment 2, six sinusoidal and fifteen complex signals were used. All in all, then, the detectability of each of 61 signals was measured.

To make results directly comparable, the two experiments were run concurrently, in the sense that the measurements in Experiment 1 were interspersed with those in Experiment 2. The specific manner in which this was done is explained in section 7.2.

The exact electrical specification of signal levels is given in section 7.4.

Noise. The masking noise was identical in level and power spectrum for all 61 signals. It was considered by the subjects to be "moderately loud". For exact specification see section 7.4.

Duration of Signal. The duration of each of the two observation-periods of the forced-choice trial was 1 second. During one or the other of these periods the signal was always present; the noise, of course, was present during both. Thus, on each trial

*For theoretical reasons discussed below it was necessary for the noise spectrum to be "reasonably flat" in the neighborhood of the signal frequencies. This entails staying in that part of the spectrum in which the earphone frequency-response is flat, i.e., in the lower part of the audio range.

the signal was presented for 1 second.

Subjects. All measurements were replicated on each of two subject (HF and PFL). Both were male students at the Massachusetts Institute of Technology. They were paid for their services.

7.2 PROCEDURE

The interest in the present experiments lay in measuring the detectability of signals whose characteristics, except for phase, could be considered known. To approximate this condition as closely as possible two devices were used. (a) The experiment was broken up into a succession of runs of 180 trials, with rest pauses between runs. Signals occurring during a given run differed from one another only in phase; the subject was thus able to become "set" for a given intensity and frequency. Furthermore the first 18 trials of a run were not recorded; these trials served the function of allowing the subject to become acquainted with the noise and with the particular signal used in the run. (b) At the beginning of each trial a short preview of the signal was presented in the absence of the masking noise. The subject was thus constantly reminded of the signal characteristics.

The duration of a trial was 3 1/3 seconds, a run of 180 trials thereby lasting 10 minutes. Each trial was divided into four consecutive periods. (Period 1): A period lasting 1/3 second during which the signal-preview was presented. (Periods 2, 3): Two consecutive observation-intervals, each lasting 1 second, during each of which masking noise was presented, and during one of which the signal was also presented. We speak of the occurrence of the event $S{:}\bar{S}$ or $\bar{S}{:}S$ on a given trial according as the signal on that trial occurred in period 2 or 3. Which of these two events occurred on a given trial was determined at random, with $Pr(S{:}\bar{S}) = Pr(\bar{S}{:}S)$, the events being independent from trial to trial. (Period 4): A response and payoff period lasting 1 second. During this period the subject indicated his response by pushing a lever switch left or right of center, causing either a green or a red indicator light near the lever switch to flash. We speak of the occurrence of the event R_1 or R_2 according as the switch was pushed to the left or right.

A response was considered correct if either $S{:}\bar{S}$ and R_1 or $\bar{S}{:}S$ and R_2 occurred; it was considered incorrect if either $S{:}\bar{S}$ and R_2 or $\bar{S}{:}S$ and R_1 occurred. The green or red light flashed according as the response was correct or incorrect.

The subject's task was to try to maximize the number of correct responses, or, equivalently, of flashes of the green light.

All measurements were repeated a number of times, i.e., each signal was used on a number of runs. Per subject there were (on the average)* six runs to each signal in

*Slight differences in the number of runs per signal were arranged with a view to a somewhat complicated statistical analysis of the data. This analysis proved more lengthy than rewarding and was abandoned.

Experiment 1 and eight runs to each signal in Experiment 2. The particular signal used on a given run was selected at random. In all, 810 runs were made.

During the entire duration of each run a low-intensity background noise was presented to mask possible switching transients.

7.3 APPARATUS

During a run the subject wore earphones (Permoflux PDR-8) in sponge-neoprene cushions (MX-41/AR). Since the listening was monaural, only one of the phones was connected.

Responses were made on a response box, held in the subject's lap. On the top of this box were mounted, in addition to the response switch and the two indicator lights already discussed, three period-indicator lights and the start button. Period-indicator 1, 2, or 3 was lit according as period 1, 2, or 3 of the trial was in progress. The start button started the run, which ended automatically after 180 trials.

Signals were generated by means of two oscillators (Krohn-Hite 430-A). Care was exercised in setting the two frequencies (in the case of two-component tones) to be the requisite number of cycles per second apart: a signal was generated whose frequency was equal to the frequency-difference of the two components; this signal was then compared by the Lissajous technique with the output of a third oscillator set to the desired difference.

Noise with flat power-spectrum (up to a cutoff at 20 kc) was provided by a noise generator (Grason-Stadler 455-A).

Appropriate audio amplifiers, linear adders, and attenuators were used for adding the signal components to one another and to the noise.

The timing of the various events was performed by a rotary mechanical timer of period 3 1/3 seconds, and associated relays. Runs were programmed in advance on a teletype tape, so that the sequence of events during a run was generated automatically.

Four electrical counters were used to count the frequency of occurrence during a run of the four joint events $S{:}\bar{S} \cdot R_1$, $\bar{S}{:}S \cdot R_1$, $S{:}\bar{S} \cdot R_2$, and $\bar{S}{:}S \cdot R_2$.

7.4 ELECTRICAL MEASUREMENTS

Levels (unless otherwise stated, all levels are given in decibels re 1 volt rms) were measured directly across the phone with a self-calibrating "True RMS" voltmeter (Ballantine Laboratories Model 320). These readings, which are accurate to approximately 0.1 db, will in many cases be sufficient for the statement and discussion of our results. In some instances, however, we shall need to know the signal to noise-spectrum-level ratio. Knowledge of this ratio is limited by the inaccuracy of our noise-spectrum-level measurements; the latter may be taken as correct only to approximately 0.8 db.

Signals. The twenty-one 500-cps signals of Experiment 1 ranged in 1-db steps from -64 db to -44 db. The nineteen 1100-cps signals ranged in 1-db steps from -61 db to -43 db.

It will be recalled that in Experiment 2 we dealt at each of four frequencies with sinusoids at three levels: loud, medium, and soft. For a given frequency these three levels were 3 db apart. The levels for the "loud" sinusoids were -49 db at 500 cps and -50 at 540 cps. For subject PFL, the level for the "loud" sinusoids at 1100 cps and 1060 cps was set at -46 db; for subject HF, at -48 db. Thus, for example, the signal designated by "loud 500 cps, soft 1100 cps" consisted, for subject HF, of the sum of two sinusoids, 500 cps at -49 db, and 1100 cps at -54 db.

Noise. The same masking noise, having a voltage-level across the phone of -19 db, was used on all runs. The background noise present throughout each run was set at -41 db.

Noise Spectrum-Level. The noise spectrum-level is given by $10 \log_{10} N_o$, where N_o is the noise power in a band 1 cps wide. This quantity may be calculated from the output of a calibrated narrow-band filter placed in parallel with the earphone. On the basis of nine such measurements, made with different filters and different passbands, the noise spectrum-level was found to be (to approximately 0.8 db) -66 db in the neighborhood of 500 cps and 1100 cps.

Sound Pressure-Level. From the phone calibration we find that, for sinusoids, the sound pressure-level in decibels re 0.0002 dyne/cm^2 can be obtained approximately from the voltage-level by adding 113 at 500 cps and 112 at 1100 cps.

8. EXPERIMENT 1: RESULTS

The results of Experiment 1 are given by the data points in Fig. 3.

In Part I, Eq. 5.6, the psychophysical function for an ideal detector operating under what are roughly the conditions of the present experiment was seen to be

$$\Pr(C) = 1 - \frac{1}{2} \exp\left[- \frac{E}{2N_o} \right] \tag{8.1}$$

where, it will be recalled, Pr(C) is the probability of a correct response, or detectability; E is the signal energy; and N_o is the noise power per unit bandwidth. Since we shall want to use the decibel notation, let us define the signal-energy to noise-spectrum-level ratio, r, to be $10 \log_{10} (E/N_o)$. Equation 8.1 can then be written as

$$\Pr(C) = 1 - \frac{1}{2} \exp\left[- \frac{10^{r/10}}{2} \right] \tag{8.2}$$

In order for this equation to hold, it is necessary, first of all, that $\beta = 1$ (see Eq. 5.1, section 5). Since we set $\Pr(S{:}\bar{S}) = \Pr(\bar{S}{:}S)$, this reduces to the requirement that the payoff matrix be symmetrical. The extent to which this requirement can be considered to be fulfilled has been discussed in section 4.

For the equation to apply it is necessary, moreover, that the noise be (a) gaussian, (b) series-bandlimited, (c) uniform in spectrum-level over the band, and (d) of known spectrum-level; and that the signal be (e) series-bandlimited, (f) of known energy, (g) of known frequency, but (h) of unknown phase.

Conditions (a), (b), and (e) were obviously met. Owing mostly to the earphone characteristic, condition (c) was not met. Suppose, however, that a narrow-band filter centered at the signal frequency had been inserted into the circuit ahead of the phone. It is known in psycho-acoustics (3, 4, 18) that the detectability of a signal in noise is independent of the noise bandwidth so long as the latter exceeds a so-called critical bandwidth of less than 50 cps (in the lower part of the audio range). From the characteristics of our circuit it is safe to assume that within such a narrow band the spectrum-level of our noise was very nearly uniform, i.e., the spectrum was very nearly flat. The insertion into the circuit of our hypothetical filter set to a bandwidth of 50 cps would therefore (a) leave our data unchanged, and (b) lead to behavior on the part of an ideal detector as specified by Eq. 8.2, provided the remaining requirements are met.

It seems intuitively evident that condition (h) is fulfilled: slight inaccuracies in the timer and frequency-drift in the oscillator contribute toward randomizing the phase. (Even without this, our signals ought probably to be considered as being of unknown phase from the point of view of a human subject.)

The only remaining requirements are that the noise level and the signal energy and frequency be known. These requirements were, it is hoped, approximated as closely

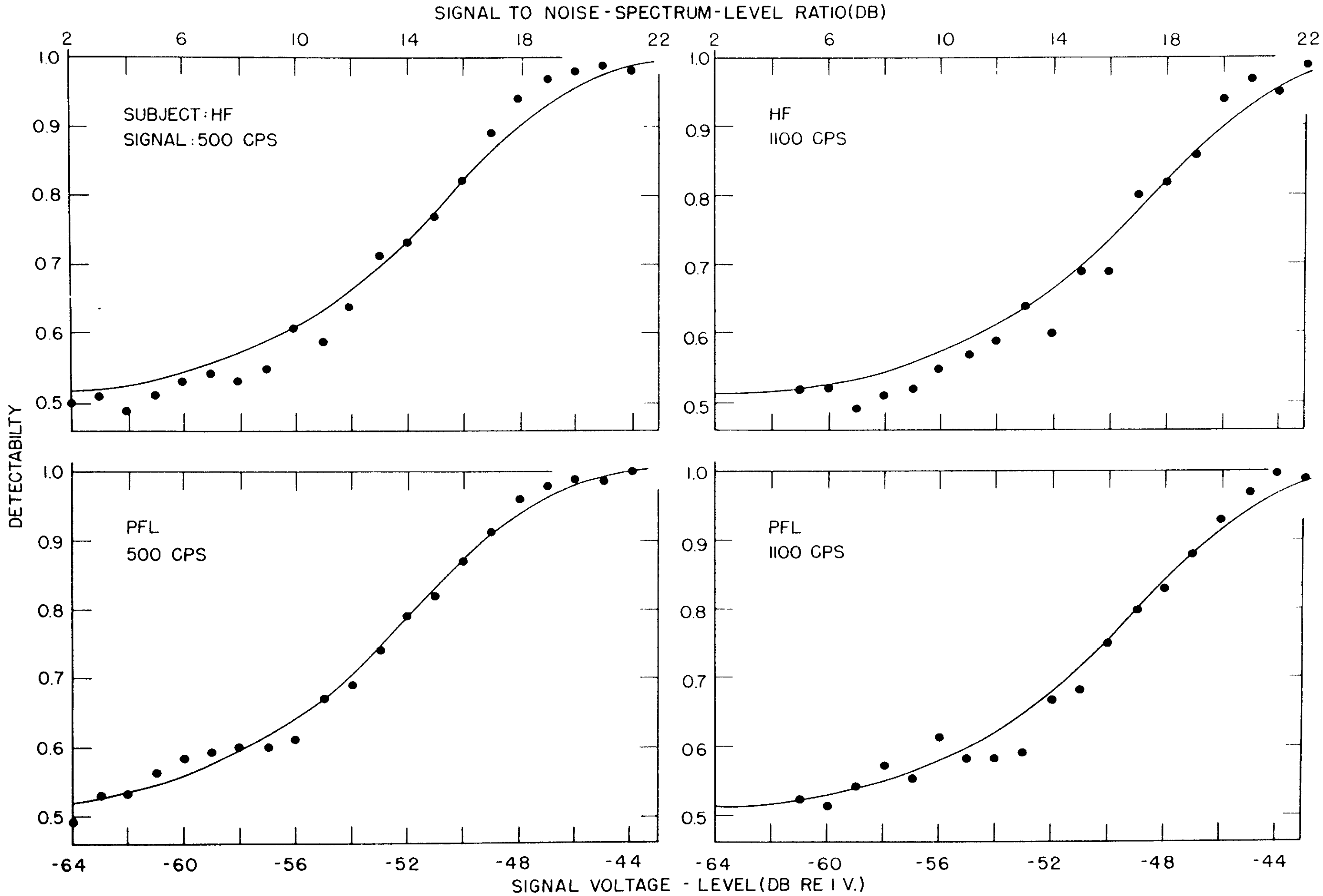

Fig. 3. Psychophysical functions obtained in Experiment 1. The points represent observed values. The solid curves are fitted theoretical functions, as discussed in the text.

as possible by our experimental conditions, as discussed in section 7.2.

We have established, then, that our ideal-detector function may reasonably be compared with our data.

As the first item of comparison we notice, and this comes as no surprise, that our subjects were not ideal detectors. The ideal detector operates with 75 per cent accuracy when r = 1.4. For our subjects to detect with the same accuracy we must have r approximately equal to 15; that is, we must increase the signal-to-noise ratio by approximately 13.5 db.

Aside from the question of absolute sensitivity, how do our subjects compare with the ideal detector? Let us take the ideal-detector's psychophysical function (Eq. 8.2) and displace it in toto to the right along the r-axis until the curve best "fits" the data-points. In other words, let us compare the subjects' psychophysical function for our situation with the ideal-detector's function for a situation in which either the signal energy is reduced or the noise power increased. This procedure yields the solid curves of Fig. 3. The ideal-detector's function was displaced by the amounts given in column 2 of Table 1. (These values are, of course, only as accurate as our noise spectrum-level measurements. The significance of column 3 is explained in section 9.)

TABLE 1

	1	2	3
	Curve (cps)	Displacement (db re ideal detector)	Associated hypothetical bandwidth (cps)
PFL:	500	11.6	14
	1100	14.5	28
HF:	500	13	20
	1100	15	32

The fit of these curves may be considered quite satisfactory; the best-fitting curve (PFL:500 cps) accounts for 99.3 per cent of the variance in the graph; the worst-fitting curve (HF:1100 cps) accounts for 97.3 per cent. (The curves were fitted by eye. Had a least-squares method for fitting the curves been applied, the percentage of variance accounted-for could only have increased.)

The argument could be made that the obtained fit is not surprising; the observed psychophysical functions, like most such functions, are roughly in the shape of a normal ogive; any reasonable theory would predict curves approximately of this general shape; all such functions could be made to fit our data fairly well. Such an argument would, however, be misleading. It is true that our data, when plotted on a decibel scale, look very much like, let us say, normal ogives. On a different scale the data would have a different appearance; our theoretical curve would then, however, also change correspondingly; there is no ambiguity here concerning the scale on which to plot the data in order for them to fit the model. This fit, moreover, is accomplished by adjusting a single parameter, in contrast with the ordinary two-parameter normal-ogive fit. In the language of traditional psychophysics we may say that the present model predicts the scale and the slope of the psychophysical function. (A quick statistical test, worked in Appendix F, may even convince us that our theoretical curves fit better than normal ogives.)

9. EXPERIMENT 1: DISCUSSION

On the basis of our results thus far we can conclude that our subjects differed from the ideal detector only to the extent of being, on the average, 13.5 db less sensitive at the frequencies tested. How do we account for these 13.5 db?

We might consider the hypothesis that the 13.5 db difference is due to the internal noise of the subject. Assume, first, that the two kinds of noise, internal and external, are linearly additive; we should then expect the internal noise to become less and less important, and the difference between our subjects and the ideal detector to diminish rapidly, as the external noise level is raised. It is well known in psycho-acoustics, however, that subjects will detect signals with constant accuracy over a wide range of noise levels if the (external) signal-to-noise ratio remains constant (8); the same, of course, is true for the ideal detector at all noise levels, as is seen from Eq. 8.2. We know, therefore, that if we repeated our experiment with a noise of higher level, we should arrive again at a figure close to 13.5 db. This refutes the hypothesis of linearly additive internal noise.

If we still wish to explain the 13.5 db on the basis of internal noise, we must postulate that the addition is nonlinear. It is possible to speculate, for instance, about "automatic volume control" of the input to the auditory detector. While it is likely that such a hypothesis could be worked out in plausible fashion, we shall not try to examine it here. Instead, we turn to speculations that have the advantage of being considerably more familiar to workers in psycho-acoustics and also of being more useful in our discussion of the second experiment.

The fact of constant signal-to-noise ratio for a given detectability and the results of Experiment 1 lead us to conjecture about an auditory mechanism which is somehow like the ideal detector except for multiplying the noise power by a constant. For, consider that we install at the input of the ideal detector a mechanism which multiplies the noise power by 22. The noise-level reaching the detector would then be 13.5 db higher than the level of the presented noise, and the ideal detector would behave approximately as our subjects did; and the requirement of constant detectability for constant signal-to-noise ratio would be satisfied by virtue of Eq. 8.2.

Such a noise-multiplying mechanism is not hard to come by, if we are willing, for a moment, to be quite vague. Suppose that we could think of our ideal detector as being essentially a bandpass filter (the "ideal filter"). Our data could then be generated by a detector consisting of a filter like the ideal filter except for having a passband approximately 13.5 db wider.* For the output of this filter would be like that of the ideal filter except that the total noise-power output would be 13.5 db greater.

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*A band of frequencies f cps wide is said to have a bandwidth in decibels (re 1 cps) of $10 \log_{10} f$. The power in a band of uniform noise is given by the sum of the noise spectrum-level (i.e., the power in decibels in a 1-cps band) and the bandwidth in decibels.

Such an ideal filter is indeed specifiable. It is shown by Peterson and Birdsall (ref. 15, Part II, p. 20) that for the kind of signal and noise under consideration a filter having impulse response

$$h(t) = \begin{cases} \cos\left[\alpha(T-t)\right] & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{9.1}$$

where α is the signal frequency and T is the duration, has an output whose envelope at time T is related to the likelihood ratio $L(\Psi)$ of the input by a known monotonic function. The filter specified by Eq. 9.1 could therefore form the basis of an ideal detector.

When T is large compared with a period of the signal – and this is the case with the signals of Experiment 1 – the gain-versus-frequency characteristic of the ideal filter is approximately of the form shown in Fig. 4; the larger T, the more peaked and narrow the graph.

Assume that the ideal filter is centered at the frequency of the signal. If the filter bandwidth is increased, that part of the filter output caused by the signal remains unchanged. More noise, however, is now present in the output, leading to greater variability in the measurement of the output-envelope, hence in the estimate of $L(\Psi)$. This increase in the variability could equally well have been produced by increasing the input noise-level. From the point of view of a mechanism that measures only the envelope of the filter output at a given moment of time, these two ways of increasing the variability are indistinguishable. A filter with greater bandwidth, as in Fig. 5, could therefore be made to operate like the ideal detector, except for requiring a higher signal-to-noise ratio at its input. Hypothesizing the existence of such a filter in the auditory detection-mechanism would account for our 13.5 db.

The difference in sensitivity between our subjects and between detection at 500 cps

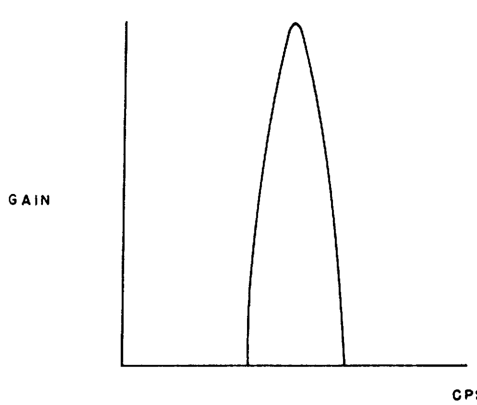

Fig. 4. Approximate shape of the gain-versus-frequency characteristic of the ideal filter (for the detection of long tones).

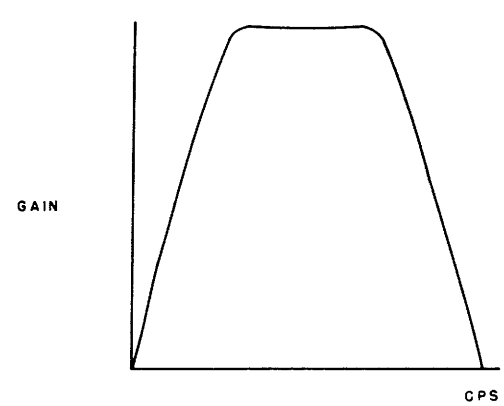

Fig. 5. Approximate shape of the gain-versus-frequency characteristic of a hypothetical "auditory filter".

and 1100 cps could be explained as a difference in the bandwidth of the auditory filtering mechanisms involved. Specifically, the bandwidth of these mechanisms would have to exceed the bandwidth of the ideal filter by just the number of decibels listed in column 2 of Table 1. [By bandwidth, in the present context, we shall understand "equivalent rectangular bandwidth," defined thus: The equivalent rectangular bandwidth of a bandpass filter F_1 whose gain-versus-frequency characteristic exhibits a maximum at frequency f_o shall be the difference in cycles per second between the upper and lower cutoff frequencies of a bandpass filter F_2 whose gain-versus-frequency characteristic is rectangular and whose output signal-to-noise ratio is equal to that of F_1 when the input is a flat noise and a sinusoid of frequency f_o.]

To obtain an estimate of the actual bandwidth of the "auditory filters" we need first to know the bandwidth of the ideal filter. This latter is shown in Appendix G to be $1/T$ cps; that is, under the condition of our experiments, 1 cps. Thus the bandwidth in decibels of the hypothetical auditory filters is given by column 2 of Table 1. The corresponding bandwidth in cycles per second is listed in column 3.

We have spoken of the ideal detector as being basically a bandpass filter. It must be emphasized that such an interpretation is not necessary; no mention of filters was made in the exposition of detection theory in Part I.

We might wonder how, except by a process of filtering, the behavior of an ideal detector can be independent of the noise bandwidth W, as demanded by Eq. 8.2. To clarify this we recall that the detector input can be considered to be a set of 2TW random variables $\underline{X}_i$. If, as in the present case, the noise is uniform and gaussian, these variables are statistically independent (24). If we increase the bandwidth of the noise, we increase the total noise power and, thereby, the variance of each $\underline{X}_i$; but we also increase the number of independent variables $\underline{X}_i$ with which we deal; and Eq. 8.2 tells us that this increase in the number of variables just makes up for the increase in variance. The situation is identical with the more familiar one of testing on the basis of a number of sample-values whether the population from which they were drawn has mean zero or $\mu \neq 0$. To make our decision with a given level of confidence, we need to take many or few sample-values, according as the variance is large or small.

In the case of the human subject, however, there is, clearly, a limit to the amount of information which can be handled in a short time; we cannot readily conceive of the subject computing functions of an indefinitely large number of independent variables. Nevertheless, for him, too, the bandwidth of the noise is irrelevant, as we have mentioned, so long as the bandwidth exceeds a given value, the critical bandwidth. The only explanation here does seem to be in terms of some filtering process that reduces the noise band to a constant width at the input to the system.

The auditory theory of the present section is, of course, essentially Fletcher's critical-band theory (3, 4). His theory rests (as does the present discussion) primarily on the evidence that widening the noise band beyond a critical value leaves detectability unchanged.

There do seem, however, to be certain advantages in our formulation. Consider, for instance, the fact of constant detectability for constant signal-to-noise ratio. While this fact is necessary to keep the critical-band theory within manageable bounds, it is inexplicable on that theory. On our formulation, however, it is explicable; the explanation rests in the derivation of Eq. 8.2, which predicts this fact.

We have estimated critical bandwidths, as does Fletcher, from detectability data. Our rationale, however, has differed from his. Fletcher's calculations are based on the assumption that "at threshold" the noise power in a critical band is equal to the signal power. This assumption is, it would seem, _ad hoc_. No such assumption was found necessary to perform our estimations.

10. EXPERIMENT 2: RESULTS

Experiment 2 deals with signals composed of two sinusoidal components known except for phase; the experimental results are given in column 6 of Tables 2, 3, and 4.

As discussed in Appendix H, Eq. 8.2 may be taken – for purposes of comparison – as representing the ideal-detector psychophysical function for Experiment 2 as well as for Experiment 1. It will be recalled that the only variable of Eq. 8.2 is r, the signal-energy to noise-spectrum-level ratio; this means that for the ideal detector it is irrelevant whether the signal energy is concentrated in a single sinusoid or spread among a number of components.

Suppose that the same were true of our subjects; the detectability of the two-component signals could then be predicted from the detectability of the components. Since the detectability of each relevant single sinusoid was measured, it is a simple matter of interpolation to find what the detectability would have been had all the energy in the two-component signals been concentrated in a single component.

Such predictions are given in column 5 of Tables 2, 3, and 4; the corresponding observed detectability is given in column 6. For signals having energy of 500 cps and 540 cps the predictions were made by considering the energy to be concentrated at 500 cps. Correspondingly, for signals having energy at 1060 cps and 1100 cps the predictions were made by considering the energy to be concentrated at 1100 cps. This procedure is legitimate, since there does not appear to be any difference in detectability between 500 cps and 540 cps nor between 1060 cps and 1100 cps.* There is a problem about predicting the detectability of signals having components of 500 cps and 1100 cps, since these components are not equally detectable. The results of Experiment 1 show, however, that 1100-cps tones are approximately as detectable as 500-cps tones x db lower in level, where x is 2.9 for subject PFL and 2 for subject HF. If we subtract x from the level of the 1100-cps component, this diminished level may then be combined with the level of the 500-cps component; the predicted detectability may then be taken to be the detectability of a 500-cps tone having this combined level. Column 5 of Table 4 was obtained in this manner.

Let us now examine these predictions. In Table 2 we find a very close agreement between the predicted and observed detectability; the predictions account for 96.5 per cent of the variance in column 6. In Table 3 the agreement is still appreciable, but less than in Table 2; here the predictions account for 73.6 per cent of the variance in column 6. In Table 4 the agreement between predicted and observed detectability is practically nil; the variance accounted for is only 25 per cent. We notice, in fact, that we do better in this case by letting the predicted detectability of the two-component signal be equal to the detectability of the more detectable of the two components, i.e., by

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*No difference in detectability between 500 cps and 540 cps nor between 1060 cps and 1100 cps could be found by the t-test technique. The difference between 500 cps and 1100 cps was found, however, to be highly significant.

TABLE 2

1 frequency of components (cps)	2 level of components (db re 1 v)	3 detectability of components	4 level of sum of components (db re 1 v)	5 predicted detectability of sum of components	6 observed detectability of sum of components
			Subject HF		
1100 1060	-48 -48	.82 .84	-45	.98	.96
1100 1060	-48 -54	.82 .59	-47.1	.85	.85
1100 1060	-54 -48	.59 .84	-47.1	.85	.86
1100 1060	-51 -51	.69 .68	-48	.83	.80
1100 1060	-54 -54	.59 .59	-51	.70	.66
			Subject PFL		
1100 1060	-46 -46	.93 .93	-43	.99	.99
1100 1060	-46 -52	.93 .70	-45.1	.96	.96
1100 1060	-52 -46	.67 .93	-45.1	.96	.97
1100 1060	-49 -49	.80 .81	-46	.93	.93
1100 1060	-52 -52	.67 .70	-49	.80	.78

TABLE 3

1 frequency of components (cps)	2 level of components (db re 1 v)	3 detectability of components	4 level of sum of components (db re 1 v)	5 predicted detectability of sum of components	6 observed detectability of sum of components
			Subject HF		
500 540	-49 -50	.89 .85	-46.5	.97	.94
500 540	-49 -56	.89 .61	-48.2	.93	.91
500 540	-55 -50	.62 .85	-48.8	.90	.86
500 540	-52 -53	.73 .70	-49.5	.86	.77
500 540	-55 -56	.62 .61	-52.45	.72	.65
			Subject PFL		
500 540	-49 -50	.91 .86	-46.5	.98	.97
500 540	-49 -56	.91 .61	-48.2	.94	.95
500 540	-55 -50	.67 .86	-48.8	.92	.89
500 540	-52 -53	.79 .73	-49.5	.89	.82
500 540	-55 -56	.67 .61	-52.5	.77	.68

TABLE 4

1	2	3	4	5	6
frequency of components (cps)	level of components (db re 1 v)	detectability of components	level of sum of components (db re 1 v)	predicted detectability of sum of components	observed detectability of sum of components
		Subject HF			
500 1100	-49 -48	.89 .82	-45.5	.97	.88
500 1100	-55 -48	.62 .82	-47.2	.92	.82
500 1100	-49 -54	.89 .59	-47.8	.94	.84
500 1100	-52 -51	.73 .69	-48.5	.87	.75
500 1100	-55 -54	.62 .59	-51.5	.71	.64
		Subject PFL			
500 1100	-49 -46	.91 .93	-44.3	.99	.96
500 1100	-55 -46	.67 .93	-45.5	.96	.91
500 1100	-49 -52	.91 .67	-47.3	.96	.90
500 1100	-52 -49	.79 .80	-47.3	.91	.84
500 1100	-55 -52	.67 .67	-50.3	.79	.67

ignoring the less detectable component altogether. If we do this, our predictions account for 93.5 per cent of the variance; a t-test fails to reveal any difference between the observed detectability and that of the more detectable component.

[Similar t-tests reveal that the difference in detectability between the two-component signal and the more detectable of the two components is highly significant in both Tables 2 and 3, indicating that in these cases the presence of the less detectable component increases the detectability of the total signal.]

In summary, then, we may say that the detectability of signals having components of 1060 cps and 1100 cps is governed by the total energy in the signal, as would be the case for the ideal detector; the detectability of signals having components of 500 cps and 1100 cps is governed by the detectability of the more detectable component alone; and the detectability of signals having components of 500 cps and 540 cps lies somewhere between the first two cases.

11. EXPERIMENT 2: DISCUSSION

Our discussion of the results of Experiment 1 led us to speculate about the presence in the auditory system of mechanisms similar in function to bandpass filters. The passband of the filter was considered to be centered at the frequency of the signal. In extending this point of view to the present case of multicomponent signals, it is necessary to visualize a number of such filters, the passband of any one of these overlapping, perhaps, with that of the filter nearest in frequency.

Case 1

Consider first that both components of the signal are close enough in frequency to fall within the passband of the same filter. More exactly, if the filter characteristics are as shown in Fig. 5, assume that the components fall within the flat part of the characteristic. The detectability of the signal will then depend only on the total signal energy, independently of the number of components, and on the noise power.

Thus to explain the fact that a signal with components at 1060 cps and 1100 cps is as detectable as a signal having the same energy but only one component at 1100 cps, we need only assume that 1060 cps and 1100 cps fall within the flat portion of the same filter characteristic.

Case 2

Next suppose that the two components are far removed in frequency; let us say that the component of lower frequency, f_1, falls in the middle of the passband of some filter A, while the component of higher frequency, f_2, falls in the middle of the passband of some filter B. Assume, moreover, that a negligible amount of power from f_1 is passed by B, and a negligible amount from f_2 by A; A and B are thus "nonoverlapping." If the same input containing gaussian noise is applied to both filters, the outputs of the two filters will be statistically independent. If these filters are now incorporated in signal-detection systems, the decisions of these two systems will also be statistically independent.

All detectors considered thus far have been visualized as devices that calculate, on each trial, a quantity $\underline{Z}$, and, in a symmetrical forced-choice situation, make one or the other of two responses according as $\underline{Z}$ is positive or negative. (If the detector is ideal, $\underline{Z} = L(\Psi_1) - L(\Psi_2)$. If the detector is not ideal, $\underline{Z}$ is a similar quantity having larger variance, in accordance with the discussion of sec. 9.) In the present case, however, we are visualizing two (independent) filters, and hence two detectors. Associate the random variable $\underline{Z}$ with the first detector and the (independent) variable $\underline{Z}'$ with the second. A simple extension of our earlier model would lead us to say that the decision as to which response is to be made in the two-detector case rests solely on the signs of $\underline{Z}$ and $\underline{Z}'$. That is to say, whether R_1 or R_2 is made depends on which of the four events

$(\underline{Z} > 0,\ \underline{Z}' > 0)$ $(\underline{Z} > 0,\ \underline{Z}' < 0)$

$(\underline{Z} < 0,\ \underline{Z}' > 0)$ $(\underline{Z} < 0,\ \underline{Z}' < 0)$

occurs.

Under such circumstances it can be shown (see Appendix I) that the probability of making the correct decision as to which response to make is no greater than it would have been had the decision been based on only one of the two filter outputs, provided we pick the one that gives us the better results. In other words, if responding on the basis of the sign of $\underline{Z}$ gives us better results than responding on the basis of the sign of $\underline{Z}'$, we can ignore $\underline{Z}'$ altogether without lowering the probability of making the correct response.

Thus, on our model, to explain the fact that a signal with components at 500 cps and 1100 cps is only as detectable as the more detectable of the two components, we need only assume that 500 cps and 1100 cps fall within the passbands of nonoverlapping filters.

Case 3

The third possible case occurs if the two components are not close enough in frequency to fall within the flat portion of the characteristic of one single filter or far enough apart to fall in nonoverlapping filters. In that case one of the components falls, it is assumed, in the flat portion of the characteristic of some filter, while the other falls on the sloping portion of the same characteristic in such a way as to be attenuated somewhat (but not so much as to give a negligible output). Here we should expect the detectability to fall somewhere between Case 1 and Case 2.

Thus, to explain the detectability of a signal with components at 500 cps and 540 cps we need only assume that these frequencies come within the scope of Case 3.

The nature of the difference in detectability between signals having components at 500 cps and 540 cps and those with components at 1060 cps and 1100 cps implies that the auditory filters in the neighborhood of 1100 cps have wider bandwidths than those in the neighborhood of 500 cps. By the argument of section 9 this, in turn, implies that detectability at 500 cps is better than at 1100 cps, which was found to be true in Experiment 1.

Acknowledgment

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Appendix A

Guessing-Correction Formula

The traditional guessing-correction formula is sometimes (ref. 1) given as:

$$p = \frac{p' - c}{1 - c} \tag{A.1}$$

where p is the corrected (or true) proportion of perceptions given that a signal was presented, p' is the observed proportion of responses given that a signal was presented, and c is the proportion of responses given that a signal was not presented. Putting this into probability notation, we have

$$\Pr(P|S) = \frac{\Pr(R|S) - \Pr(R|\overline{S})}{1 - \Pr(R|\overline{S})} \tag{A.2}$$

where S = a signal is presented, R = a response is made, and P = a perception occurs.

This formula purports to yield the probability of a perception from the probabilities of responses. We can find its underlying assumptions by writing the identity:

$$\Pr(P|S) \equiv \frac{\Pr(R|S) - \Pr(R \mid \overline{P} \cdot S)}{\Pr(R \mid P \cdot S) - \Pr(R \mid \overline{P} \cdot S)} \tag{A.3}$$

Clearly, then, if we assume

$$\Pr(R \mid \overline{P} \cdot S) = \Pr(R|\overline{S}) \tag{A.4}$$

and

$$\Pr(R \mid P \cdot S) = 1 \tag{A.5}$$

we have the desired correction formula (A.2).

It is unlikely that we shall ever be able to test the truth of these assumptions (A.4 and A.5) directly. If we wish to evaluate them, we must do so with respect to a given model. We would most likely use such a formula as Eq. A.2, involving the term P, only if we had in mind a model that asserts that information is transmitted from signals to responses through the intermediary state P only. Otherwise, it would not pay to introduce the state P into the model at all; we would rather talk of the transmission of information from the signal to the response directly. In other words we would presumably use Eq. A.2 only if we had in mind a model that implied, among other things, that

$$\Pr(R \mid \overline{P} \cdot S) = \Pr(R \mid \overline{P} \cdot \overline{S}) \tag{A.6}$$

Now, if we believe Eq. A.6 we can make the following deduction. Write the identity

$$\Pr(P|\overline{S}) \equiv \frac{\Pr(R|\overline{S}) - \Pr(R \mid \overline{P} \cdot \overline{S})}{\Pr(R \mid P \cdot \overline{S}) - \Pr(R \mid \overline{P} \cdot \overline{S})} \tag{A.7}$$

Use assumption (A.4) on identity (A.7) to get

$$\Pr(P|\bar{S}) = \frac{\Pr(R \mid \bar{P} \cdot S) - \Pr(R \mid \bar{P} \cdot \bar{S})}{\Pr(R \mid P \cdot \bar{S}) - \Pr(R \mid \bar{P} \cdot \bar{S})} \tag{A.8}$$

Now use assumption (A.6) on (A.8) to get

$$\Pr(P|\bar{S}) = \frac{\Pr(R \mid \bar{P} \cdot \bar{S}) - \Pr(R \mid \bar{P} \cdot \bar{S})}{\Pr(R \mid P \cdot \bar{S}) - \Pr(R \mid \bar{P} \cdot \bar{S})} = 0 \tag{A.9}$$

This last statement (Eq. A.9) is undoubtedly false, as anyone who has served as subject in a threshold experiment would testify. We conclude that the guessing-correction formula is inapplicable.

Appendix B

Calculation of the Criterion, c, for Optimal Behavior on Tanner and Swets' Model

We wish to find the value of c for which the expected value $E(\underline{V})$ of the payoff is maximal. This value of c will be denoted by c_{max}.

The expected value of the payoff is

$$E(\underline{V}) = v_1 \Pr(R|S) \Pr(S) + v_2 \Pr(R|\bar{S}) \Pr(\bar{S})$$

$$+ v_3 \Pr(\bar{R}|S) \Pr(S) + v_4 \Pr(\bar{R}|\bar{S}) \Pr(\bar{S})$$

$$= (v_1 - v_3) \Pr(R|S) \Pr(S) + (v_2 - v_4) \Pr(R|\bar{S}) \Pr(\bar{S})$$

$$+ v_3 \Pr(S) + v_4 \Pr(\bar{S})$$

$$\frac{E(\underline{V})}{(v_1 - v_3) \Pr(S)} = \Pr(R|S) - \frac{(v_4 - v_2) \Pr(\bar{S})}{(v_1 - v_3) \Pr(S)} \Pr(R|\bar{S}) + \frac{v_3 \Pr(S) + v_4 \Pr(\bar{S})}{(v_1 - v_3) \Pr(S)}$$

Notice that the denominator on the left side is a positive constant and that the last term on the right is a constant. Hence to require that $E(\underline{V})$ be a maximum is to require that the following expression be a maximum:

$$y = \Pr(R|S) - \beta \Pr(R|\bar{S}) \tag{B.1}$$

where

$$\beta = \frac{(v_4 - v_2) \Pr(\bar{S})}{(v_1 - v_3) \Pr(S)}$$

From Eqs. 2.1a and 2.1b we have

$$\Pr(R|S) = N[d'; c, 1]$$

$$\Pr(R|\bar{S}) = N[0; c, 1]$$

So that

$$y = N[d'; c, 1] - \beta N[0; c, 1]$$

$$= 1 - N[c; d', 1] - \beta \left(1 - N[c; 0, 1]\right)$$

$$= \beta N[c; 0, 1] - N[c; d', 1] + 1 - \beta$$

Differentiating with respect to c, and setting the derivative equal to zero, we obtain

$$\frac{\partial y}{\partial c} = \beta n[c; 0, 1] - n[c; d', 1] = 0$$

Hence,

$$\beta = \frac{n[c;\ d',\ 1]}{n[c;\ 0,\ 1]} = \exp\left[cd' - \frac{d'^2}{2}\right]$$

$$\ln \beta = cd' - \frac{d'^2}{2}$$

$$c = \frac{\ln \beta}{d'} + \frac{d'}{2} = c_{max}$$

Appendix C

Maximum Expected Payoff with Symmetrical Payoff Matrices

From Appendix B, Eq. B. 1, we have that the expected payoff is maximized if and only if the following quantity is also maximized:

$$\Pr(R|S) - \beta \Pr(R|\bar{S}) \tag{C.1}$$

where

$$\beta = \frac{(v_4 - v_2) \Pr(\bar{S})}{(v_1 - v_3) \Pr(S)}$$

We say that a payoff matrix is symmetrical if $v_1 = v_4$ and $v_2 = v_3$. Thus, for a symmetrical payoff matrix,

$$\beta = \frac{\Pr(\bar{S})}{\Pr(S)}$$

Hence, for a symmetrical payoff matrix the quantity (C. 1), and therefore the expected payoff, is maximized independently of the v_i.

Appendix D

Decision Procedure for Ideal Detection in a Two-Category Forced-Choice Situation

In a yes-no situation we deal with a receiver input, $\Psi = (\underline{X}_1, \ldots, \underline{X}_{2TW})$. This quantity Ψ has one or the other of two densities according as one or the other of two states obtains. We have called the two states S and $\bar{S}$ and the corresponding densities f_{SN} and f_N.

We define a payoff matrix

	S	$\bar{S}$
R	v_1	v_2
$\bar{R}$	v_3	v_4

specifying the value of the two responses R and $\bar{R}$ in case S or $\bar{S}$. The fundamental theorem (section 3) tells us that the ideal detector behaves according to the rule: Make response R if and only if $L(\Psi) > \beta$, where

$$L(\Psi) = \frac{f_{SN}(\Psi)}{f_N(\Psi)} = \frac{f_{SN}(\underline{X}_1, \ldots, \underline{X}_{2TW})}{f_N(\underline{X}_1, \ldots, \underline{X}_{2TW})}$$

and

$$\beta = \frac{(v_4 - v_2)\Pr(\bar{S})}{(v_1 - v_3)\Pr(S)}$$

The forced-choice situation at first glance looks dissimilar. We have two receiver inputs, Ψ_1 and Ψ_2, one for each of two observation-intervals, let us say 0 to T and T to 2T. One of these receiver inputs has the density f_{SN} and the other the density f_N, but it is not known how the pairing is to be made.

Otherwise, the situations are the same. We again have two possible states, which are here called $S{:}\bar{S}$ and $\bar{S}{:}S$. We say $S{:}\bar{S}$ obtains if Ψ_1 has density f_{SN} and Ψ_2 has density f_N; we say $\bar{S}{:}S$ obtains if Ψ_1 has density f_N and Ψ_2 has density f_{SN}. We again define a payoff matrix (which differs from the above only trivially in notation):

	$S{:}\bar{S}$	$\bar{S}{:}S$
R_1	v_1	v_2
R_2	v_3	v_4

Let us, however, consider the forced-choice situation to have, not two receiver inputs

$$x_1(t) \qquad 0 < t < T$$

and

$$x_2(t) \qquad T < t < 2T$$

but a single input x(t) defined by

$$x(t) = \begin{matrix} x_1(t) & \qquad 0 < t < T \\ x_2(t) & \qquad T < t < 2T \end{matrix}$$

This function x(t) can be represented by a set Ψ' of 4TW random variables:

$$\Psi' = (\underline{X}_{1,1}, \underline{X}_{1,2}, \ldots, \underline{X}_{1,2TW}, \underline{X}_{2,1}, \underline{X}_{2,2}, \ldots, \underline{X}_{2,2TW})$$

The values of x(t) at sampling points in the range 0 to T are represented by the variables $\underline{X}_{1,i}$; the values in the range T to 2T by the variables $X_{2,i}$. Now the quantity Ψ' has one or the other of two densities according as $S:\overline{S}$ or $\overline{S}:S$ obtains. Let us call the two densities $f_{S:\overline{S}}$ and $f_{\overline{S}:S}$.

In this formulation the forced-choice and yes-no situations do not differ at all as far as the fundamental theorem is concerned. This theorem then tells us that the ideal detector behaves according to the rule: Make response R_1 if and only if $L'(\Psi') > \beta$, where

$$L'(\Psi') = \frac{f_{S:\overline{S}}(\Psi')}{f_{\overline{S}:S}(\Psi')}$$

and

$$\beta = \frac{(v_4 - v_2)\ Pr(\overline{S}:S)}{(v_1 - v_3)\ Pr(S:\overline{S})}$$

Now consider the quantity

$$f_{S:\overline{S}}(\Psi') = f_{S:\overline{S}}(\underline{X}_{1,1}, \ldots, \underline{X}_{1,2TW}, \underline{X}_{2,1}, \ldots, \underline{X}_{2,2TW}) \tag{D.1}$$

This is the likelihood if $S:\overline{S}$ obtains, i.e., if signal and noise occur during the interval from 0 to T and noise alone occurs from T to 2T. Hence the first 2TW variables have the density f_{SN} and the last 2TW variables have the density f_N.

We now make the assumption that the samples of noise in the two intervals 0 to T and T to 2T are statistically independent. This would be true if the noise were caused by random disturbance in an electrical circuit or if it were drawn from a random-noise generator. On this assumption we then have

$$f_{S:\overline{S}}(\Psi') = f_{SN}(\underline{X}_{1,1}, \ldots, \underline{X}_{1,2TW})\ f_N(\underline{X}_{2,1}, \ldots, \underline{X}_{2,2TW})$$

$$= f_{SN}(\Psi_1)\ f_N(\Psi_2) \tag{D.2}$$

By a similar argument we get

$$f_{\bar{S}:S}(\Psi') = f_N(\Psi_1)\, f_{SN}(\Psi_2) \tag{D.3}$$

Hence, the likelihood ratio is

$$L'(\Psi') = \frac{f_{SN}(\Psi_1)\, f_N(\Psi_2)}{f_N(\Psi_1)\, f_{SN}(\Psi_2)} = \frac{L(\Psi_1)}{L(\Psi_2)}$$

Thus, if the noise is independent in the two observation-intervals of the forced-choice trial, the ideal detector behaves according to the Decision Procedure: Make response R_1 if and only if $L(\Psi_1) > \beta\, L(\Psi_2)$.

Appendix E*

Ideal Detection in a Two-Category Forced-Choice Situation with Uniform Gaussian Noise

The two receiver inputs are assumed to be independent in the sense of Eqs. D.1, D.2, and D.3. It will be convenient to work with the probability of making an incorrect response. In a two-category forced-choice situation the two conditional probabilities of an incorrect response are $\Pr(R_1 \mid \bar{S}{:}S)$ and $\Pr(R_2 \mid S{:}\bar{S})$.

From Appendix D we have, for an ideal detector,

$$\Pr(R_1 \mid \bar{S}{:}S) = \Pr(L(\Psi_1) > \beta\, L(\Psi_2) \mid \Psi_1 \in N,\ \Psi_2 \in SN)$$

$$\Pr(R_2 \mid S{:}\bar{S}) = \Pr(L(\Psi_1) < \beta\, L(\Psi_2) \mid \Psi_1 \in SN,\ \Psi_2 \in N)$$

where $\Psi \in SN$ (or $\Psi \in N$) means that Ψ has the density f_{SN} (or f_N); hence $\Psi_1 \in SN$, $\Psi_2 \in N$ means the same as $S{:}\bar{S}$, and $\Psi_1 \in N$, $\Psi_2 \in SN$ means the same as $\bar{S}{:}S$.

We can put these expressions in the following form (interchanging subscripts throughout the second equation):

$$\Pr(R_1 \mid \bar{S}{:}S) = \Pr\left(\frac{1}{\beta} L(\Psi_1) > L(\Psi_2) \mid \Psi_1 \in N,\ \Psi_2 \in SN\right) \tag{E.0a}$$

$$\Pr(R_2 \mid S{:}\bar{S}) = \Pr(\beta\, L(\Psi_1) > L(\Psi_2) \mid \Psi_1 \in N,\ \Psi_2 \in SN) \tag{E.0b}$$

We can find $\Pr(R_1 \mid \bar{S}{:}S)$ by computing the probability that $(1/\beta)\, L(\Psi_1)$ is greater than some number b and that $L(\Psi_2) = b$, and then integrating over all b:

$$\Pr(R_1 \mid \bar{S}{:}S) =$$

$$\int_0^\infty \Pr\left(\frac{1}{\beta} L(\Psi) > b \mid \Psi \in N\right) \Pr(L(\Psi) = b \mid \Psi \in SN) \tag{E.1a}$$

Likewise we can write

$$\Pr(R_2 \mid S{:}\bar{S}) =$$

$$\int_0^\infty \Pr(\beta\, L(\Psi) > b \mid \Psi \in N) \Pr(L(\Psi) = b \mid \Psi \in SN) \tag{E.1b}$$

By definition we have

$$\Pr(L(\Psi) > b \mid \Psi \in N) = F_N(b) \tag{E.2a}$$

$$\Pr(L(\Psi) > b \mid \Psi \in SN) = F_{SN}(b) \tag{E.2b}$$

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*The author is indebted to Dr. Bert F. Green, Jr., for several of the results of this appendix.

A probability element of Eq. 2.b can be written

$$\Pr(L(\Psi) = b \mid \Psi \in SN) = -d\, F_{SN}(b) \tag{E.3}$$

The unusual minus sign appears because of the unusual direction of the inequality in Eq. E.2b. See Peterson and Birdsall (ref. 15, Part I, p. 27).

Putting Eqs. E.2a and E.3 into Eqs. E.1a and E.1b we have

$$\Pr(R_1 \mid \bar{S}{:}S) = -\int_0^\infty F_N(\beta b)\, d\, F_{SN}(b) \tag{E.4a}$$

$$\Pr(R_2 \mid S{:}\bar{S}) = -\int_0^\infty F_N\left(\frac{b}{\beta}\right) d\, F_{SN}(b) \tag{E.4b}$$

Case of the Signal Known Exactly

From Peterson and Birdsall (ref. 15, Part II, p. 10) we have, for the case of the signal known exactly,

$$F_N(b) = \left(\frac{N_o}{4\pi E}\right)^{1/2} \int_a^\infty \exp\left[-\frac{1}{2}\frac{N_o}{2E} y^2\right] dy$$

$$d\, F_{SN}(b) = -\left(\frac{N_o}{4\pi E}\right)^{1/2} \exp\left[-\frac{N_o}{4E}\left(a - \frac{2E}{N_o}\right)^2\right] da$$

where

$$a = \frac{E}{N_o} + \ln b$$

Let

$$g = \frac{2E}{N_o}\text{; then } a = \frac{g}{2} + \ln b$$

and

$$F_N(b) = \left(\frac{1}{2\pi g}\right)^{1/2} \int_a^\infty \exp\left[-\frac{y^2}{2g}\right] dy \tag{E.5a}$$

$$d\, F_{SN}(b) = -\left(\frac{1}{2\pi g}\right)^{1/2} \exp\left[-\frac{1}{2g}(a - g)^2\right] da \tag{E.5b}$$

Substituting βb for b in Eq. E.5a, we have

$$F_N(\beta b) = \left(\frac{1}{2\pi g}\right)^{1/2} \int_{a'}^\infty \exp\left[-\frac{y^2}{2g}\right] dy \tag{E.6}$$

where

$$\alpha' = \frac{g}{2} + \ln \beta b = \frac{g}{2} + \ln b + \ln \beta$$
$$= \alpha + \ln \beta \tag{E.7}$$

Substituting b/β for b in Eq. E.5a, we have

$$F_N\left(\frac{b}{\beta}\right) = \left(\frac{1}{2\pi g}\right)^{1/2} \int_{\alpha''}^{\infty} \exp\left[-\frac{y^2}{2g}\right] dy \tag{E.8}$$

where

$$\alpha'' = \frac{g}{2} + \ln \frac{b}{\beta} = \frac{g}{2} + \ln b - \ln \beta$$
$$= \alpha - \ln \beta$$

We can now write an expression for $\Pr(R_1 \mid \bar{S}{:}S)$. Putting Eqs. E.5b and E.6 into Eqs. E.4a, we have

$$\Pr(R_1 \mid \bar{S}{:}S) = -\int_{-\infty}^{\infty} \left\{ \left(\frac{1}{2\pi g}\right)^{1/2} \int_{\alpha'}^{\infty} \exp\left[-\frac{y^2}{2g}\right] dy \right\} \left\{ -\left(\frac{1}{2\pi g}\right)^{1/2} \exp\left[-\frac{(\alpha - g)^2}{2g}\right] d\alpha \right\}$$

$$= \int_{-\infty}^{\infty} \int_{\alpha'}^{\infty} \frac{1}{2\pi g} \exp\left[-\frac{1}{2g}\left\{y^2 + (\alpha - g)^2\right\}\right] dy\, d\alpha \tag{E.9}$$

Notice that the range of integration in Eq. E.4a is from 0 to ∞, while the range on the outside integral here is from -∞ to ∞. The change occurs because in Eq. E.4a the integration is with respect to b, while here it is with respect to $\alpha = E/N_o + \ln b$; and $0 \leq b \leq \infty$ implies $-\infty \leq \alpha \leq \infty$.

We now make the transformation

$$u = \alpha - y - g$$
$$v = y$$
$$du\, dv = dy\, d\alpha$$

Recalling that $\alpha' = \alpha + \ln \beta$, the old range of integration in Eq. E.9 was

$$\alpha + \ln \beta \leq y \leq \infty$$
$$-\infty \leq \alpha \leq \infty$$

The corresponding new range of integration, in terms of u and v, will be

$$-\infty \leq u \leq -\ln \beta - g$$
$$-\infty \leq v \leq \infty$$

so that Eq. E.9 becomes

$$\Pr(R_1 \mid \bar{S}{:}S) = \frac{1}{2\pi g}\int_{-\infty}^{\infty}\int_{-\infty}^{-\ln\beta - g} \exp\left[-\frac{1}{2g}\left\{v^2 + (u+v)^2\right\}\right] du\, dv$$

$$= \frac{1}{2\pi g}\int_{-\infty}^{-\ln\beta - g}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2g}\left(u^2 + 2uv + 2v^2\right)\right] dv\, du$$

$$= \frac{1}{2\pi g}\int_{-\infty}^{-\ln\beta - g} \exp\left[-\frac{u^2}{2g}\right]\left\{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2g}\left(2uv + 2v^2\right)\right] dv\right\} du \tag{E.10}$$

Let us call the integral within the curved brackets "I_2". Then

$$I_2 = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{g}\left(v + \frac{u}{2}\right)^2 + \frac{u^2}{4g}\right] dv$$

$$= \exp\left[\frac{u^2}{4g}\right]\int_{-\infty}^{\infty} \exp\left[-\frac{1}{g}\left(v + \frac{u}{2}\right)^2\right] dv$$

The integral in this expression is just $(\pi g)^{1/2}$ multiplied by the integral of the normal density function with mean $-u/2$ and variance $g/2$; hence it is equal to $(\pi g)^{1/2}$. Thus

$$I_2 = (\pi g)^{1/2} \exp\left[\frac{u^2}{4g}\right]$$

Hence, going back to Eq. E.10,

$$\Pr(R_1 \mid \bar{S}{:}S) = \frac{1}{2\pi g}\int_{-\infty}^{-\ln\beta - g} \exp\left[-\frac{u^2}{2g}\right](\pi g)^{1/2} \exp\left[\frac{u^2}{4g}\right] du$$

$$= \frac{1}{2(\pi g)^{1/2}}\int_{-\infty}^{-\ln\beta - g} \exp\left[-\frac{u^2}{4g}\right] du$$

Thus the last expression is just the normal density with mean zero and variance $2g$, integrated between $-\infty$ and $-\ln\beta - g$. In our notation,

$$\Pr(R_1 \mid \bar{S}{:}S) = N[-\ln\beta - g;\ 0,\ 2g]$$

$$= 1 - N\left[\frac{\ln\beta}{(2g)^{1/2}} + \frac{(2g)^{1/2}}{2};\ 0,\ 1\right] \tag{E.11}$$

Recall that $g = 2E/N_o$. To make Eq. E.11 directly comparable to Eq. 3.4b, we define a new symbol h to be $h = (2g)^{1/2}$. We then have

$$\Pr(R_1 \mid \bar{S}{:}S) = 1 - N\left[\frac{\ln \beta}{h} + \frac{h}{2};\ 0,\ 1\right] \tag{E.12}$$

where $h^2 = 4E/N_o$.

The development for $\Pr(R_2 \mid S{:}\bar{S})$ differs from the above only in that Eq. E.8 is used instead of Eq. E.6. The net effect of this change is only to change the sign of $\ln \beta$ in Eq. E.7. This sign is carried throughout. Hence,

$$\Pr(R_2 \mid S{:}\bar{S}) = 1 - N\left[- \frac{\ln \beta}{h} + \frac{h}{2};\ 0,\ 1\right]$$

Or, writing the probability of being correct when $S{:}\bar{S}$ occurs, that is, $\Pr(R_1 \mid S{:}\bar{S}) = 1 - \Pr(R_2 \mid S{:}\bar{S})$, we have

$$\Pr(R_1 \mid S{:}\bar{S}) = 1 - N\left[\frac{\ln \beta}{h} - \frac{h}{2};\ 0,\ 1\right] \tag{E.13}$$

where $h^2 = 4E/N_o$.

Case of the Signal Known except for Carrier Phase

This case will be treated only for the special condition that $\beta = 1$.

We shall deal with signals of the form

$$s(t) = f(t) \sin\left[\gamma t + \phi(t) - \underline{P}\right] \tag{E.14}$$

where $\underline{P}$ is a random variable with rectilinear distribution.

For this case Eqs. E.17-E.20 hold, as is shown by Peterson and Birdsall in reference 15. (See their Eqs. 4.19, 4.22, 4.25, and 4.26.)

First we define the quantity α by the following equation.

$$b = \exp\left[-\frac{E}{N_o}\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} \alpha\right) \tag{E.15}$$

where $I_o(x)$ is the zero-order modified Bessel function of the first kind; $I_o(x)$ is a monotonically increasing function having the properties

$$I_o(x) \geqslant 1;\ I_o(0) = 1 \tag{E.16}$$

In terms of these definitions we have

$$L(\Psi) = \exp\left[-\frac{E}{N_o}\right] I_o\left(\frac{r}{N}\right) \tag{E.17}$$

$$F_N(b) = \exp\left[-\frac{\alpha^2}{2}\right] \tag{E.18}$$

$$d\,F_{SN}(b) = -\exp\left[-\frac{E}{N_o}\right] \alpha \exp\left[-\frac{\alpha^2}{2}\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} \alpha\right) d\alpha \tag{E.19}$$

$$F_{SN}(b) = \exp\left[-\frac{E}{N_o}\right] \int_{\alpha}^{\infty} x \exp\left[-\frac{x^2}{2}\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} x\right) dx \tag{E.20}$$

From Eqs. E.16 and E.17 we see that in the present case the likelihood ratio, oddly enough, is never smaller than $b_o = \exp[-E/N_o]$. Since $F_{SN}(b)$ (Eq. E.20) is the probability that the likelihood ratio exceeds b, we have $F_{SN}(b_o) = 1$. From Eqs. E.15 and E.16 we see that when $b = b_o$, $\alpha = 0$. Hence

$$F_{SN}(b_o) = \exp\left[-\frac{E}{N_o}\right] \int_0^{\infty} x \exp\left[-\frac{x^2}{2}\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} x\right) dx = 1 \tag{E.21}$$

We have thus established the identity

$$\int_0^{\infty} x \exp\left[-\frac{x^2}{2}\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} x\right) dx = \exp\left[\frac{E}{N_o}\right] \tag{E.22}$$

Let us substitute E/2 for E in this identity; we obtain

$$\int_0^{\infty} x \exp\left[-\frac{x^2}{2}\right] I_o\left(\left(\frac{E}{N_o}\right)^{1/2} x\right) dx = \exp\left[\frac{E}{2N_o}\right] \tag{E.23}$$

Now express the identity in terms of the variable $y = x/\sqrt{2}$. The limits of integration are unchanged by this transformation. We get

$$\int_0^{\infty} y \exp\left[-y^2\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} y\right) dy = \frac{1}{2} \exp\left[\frac{E}{2N_o}\right] \tag{E.24}$$

We write Eq. E.4a for $\beta = 1$, recalling that in the present case the range of b is from b_o to ∞:

$$\Pr(R_1 \mid \bar{S}:S) = -\int_{b_o}^{\infty} F_N(b)\, d\,F_{SN}(b) \tag{E.25}$$

Introducing Eqs. E.18 and E.19 into Eq. E.25 and changing the range of integration accordingly, we have

$$\Pr(R_1 \mid \bar{S}:S) = \exp\left[-\frac{E}{N_o}\right] \int_0^{\infty} \alpha \exp\left[-\alpha^2\right] I_o\left(\left(\frac{2E}{N_o}\right)^{1/2} \alpha\right) d\alpha \tag{E.26}$$

The integral in Eq. E.26 is identical with Eq. E.24. Hence

$$\Pr(R_1 \mid \bar{S}{:}S) = \exp\left[-\frac{E}{N_o}\right] \frac{1}{2} \exp\left[\frac{E}{2N_o}\right] = \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{E.27}$$

The parallel development for $\Pr(R_2 \mid S{:}\bar{S})$ yields, as might be expected, the same function:

$$\Pr(R_2 \mid S{:}\bar{S}) = \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{E.28}$$

Thus the probability of being correct when the signal occurs in the first observation-interval is

$$\Pr(R_1 \mid S{:}\bar{S}) = 1 - \frac{1}{2} \exp\left[-\frac{E}{2N_o}\right] \tag{E.29}$$

Appendix F

Fitting Normal Ogives to the Obtained Psychophysical Data: A Quick Statistical Test

We have fitted function (8.2) to the data of Experiment 1 by allowing this function to be displaced to the right or left along the horizontal axis. Thus the fitted curves are given by

$$y = 1 - \frac{1}{2}\exp\left[-\frac{10^{(r-a)/10}}{2}\right] \tag{F.1}$$

where $\underline{a}$ is the constant to be determined by the fitting procedure.

As an alternative we can try fitting a normal ogive. This curve, as ordinarily written, $y = N[r;\ \mu,\ \sigma^2]$, is asymptotic to $y = 0$, and thus is not comparable to Eq. F.1, which is asymptotic to $y = 0.5$. We should like a function that has the properties of the normal ogive and is also asymptotic to $y = 0.5$. The desired function is

$$y = \frac{1}{2} + \frac{1}{2}N[r;\ \mu,\ \sigma^2] \tag{F.2}$$

This function is asymptotic to $y = 0.5$ and $y = 1$, and has an inflection point at $y = 0.75$ (where $r = \mu$).

Function (F.2) is easily shown to be symmetric about its inflection point, in the sense that if, for given values of μ and σ^2, a point (r, y) is on the curve, then the image (r', y') of this point (reflected through the inflection point of the curve) is also on the curve; the image being defined by

$$r' = 2\mu - r$$

$$y' = 2(0.75) - y$$

Pictorially we can think of the curve (F.2) as coming in two sections that are pivoted at the point p_c at which $y = 0.75$. Rotating the lower section about p_c by 180° superimposes it on the upper section.

Function (F.1), on the other hand is not symmetrical about the point p_c at which $y = 0.75$. If we pivot the lower part of this curve about p_c by 180°, we find it falling below the upper part. The distance between points on the upper part of the curve and corresponding points on the pivoted lower part is zero when $y = 0.75$, and becomes very small as y assumes values very close to unity; for $0.90 < y < 0.99$, however, this distance is of the order of 10 per cent of the range of y.

We wish to test the null hypothesis H_o that the data-points fall randomly about a true underlying ogive (F.2). To perform the test, each point below the line $y = 0.75$ is projected through p_c to form an image above this line. On H_o each image has an equal chance of falling above or below its corresponding point on the upper part of the curve. On the alternative hypothesis H_1 that the true underlying curve is given by the detection-theoretic function (F.1), an image has greater probability of falling below its corresponding point.

Our test consists in counting the number g_o of images which fall below their corresponding points. If this number is greater than could reasonably be expected on the basis of even chance, we feel justified in rejecting H_o.

In practice, three difficulties may arise in performing the test. (a) The location of p_c must be estimated; if there is doubt concerning the proper value to assume, various estimates must be made and the test repeated with each estimate. This procedure, of course, must then be taken into account in the statistical handling of g_o. (b) If the data-points are equally spaced along the abscissa, p_c must fall either above or below a data-point or half-way between the abscissas of two data-points; otherwise the image of a point would not fall directly either above or below its corresponding point. If p_c cannot be so located, the test must be performed for each of the nearest appropriate values of p_c. (c) The data-points may not be equally spaced along the abscissa. In this case appropriate makeshift repairs may suggest themselves, or the test may be considered inapplicable.

The test was applied to the four sets of data-points of Experiment 1 (Fig. 3). Fortunately, none of the difficulties mentioned above manifested itself.

Only points on the upper part of the curve for which $0.90 < y < 0.99$ were used in the test. In all, 17 such points were compared with corresponding images of points from the lower portion of the curve. All 17 differences were in the direction predicted on H_1, leading to a significant rejection of the null hypothesis.

Changing our estimate of p_c by $\pm$ 1 db does not effect the result of the test.

While the test rejects the normal-ogive hypothesis, it does not, of course, prove H_1. The results of the test are, however, compatible with the latter hypothesis.

Appendix G

Bandwidth of the Ideal Filter

As mentioned in section 9, the ideal filter for the detection of sinusoids of known frequency and amplitude but unknown phase has an impulse response

$$h(t) = \begin{cases} \cos[\alpha(T - t)] & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{G.1}$$

where α is the signal frequency in radians per second, and T is the signal duration. To simplify matters, let us set α or T so that $\alpha T = 2\pi n$ (n is an integer). The impulse response is then

$$h(t) = \begin{cases} \cos \alpha t & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{G.2}$$

The desired power-gain-versus-frequency characteristic of the filter is the power spectrum of the impulse response. In performing the Fourier analysis we use the notation and formulas given by Goldman (6). The Fourier integral is written

$$h(t) = 1/\pi \int_0^\infty S(\omega) \cos[\omega t + \mu(t)]\, d\omega$$

where $S^2(\omega)$, the power spectrum, is given by

$$S^2(\omega) = \left[\int_{-\infty}^{\infty} h(t) \cos \omega t\, dt\right]^2 + \left[\int_{-\infty}^{\infty} h(t) \sin \omega t\, dt\right]^2 \tag{G.3}$$

We shall also use the Fourier-integral energy theorem

$$1/\pi \int_0^\infty S^2(\omega)\, d\omega = \int_{-\infty}^{\infty} h^2(t)\, dt \tag{G.4}$$

Applying Eq. G.2 to Eq. G.4, we have

$$1/\pi \int_0^\infty S^2(\omega)\, d\omega = \int_0^T \cos^2 \alpha t\, dt = \frac{T}{2} + \frac{\sin 2\alpha T}{4\alpha} = T/2 \tag{G.5}$$

since $\alpha T = 2\pi n$.

Applying Eq. G.2 to Eq. G.3, we find $S^2(\alpha)$:

$$\int_{-\infty}^{\infty} h(t) \cos \alpha t\, dt = \int_0^T \cos^2 \alpha t\, dt = T/2$$

$$\int_{-\infty}^{\infty} h(t) \sin \alpha t \, dt = \int_{0}^{T} \cos \alpha t \sin \alpha t \, dt$$

$$= -\frac{1}{4\alpha} (\cos 2\alpha T - 1) = 0$$

so that $S^2(\alpha) = T^2/4$.

We now apply to the filter input a sinusoidal signal of frequency α and power p_o and a flat noise of power m_o per radian/second. That part of the output caused by the signal will have power $K\, p_o\, S^2(\alpha) = 1/4\, K\, p_o\, T^2$, where K is a proportionality constant, and that part of the output caused by the noise will have an over-all power of

$$\int_{0}^{\infty} K\, m_o\, S^2(\omega)\, d\omega = 1/2\, \pi\, K\, m_o\, T$$

where the equality holds by virtue of Eq. G.5. The output signal-to-noise ratio will therefore be, in power units,

$$\frac{p_o T}{2\pi m_o}$$

Consider applying the same signal and noise to a filter having rectangular gain-versus-frequency characteristic over a band $\Delta\omega$ wide (see section 9). The output caused by the signal will have power $C\, p_o$, and the output caused by the noise will have power $C\, m_o \Delta\omega$. The output signal-to-noise ratio will therefore be $p_o/(m_o \cdot \Delta\omega)$. Letting $\Delta\omega = 2\pi/T$, we have a filter whose output signal-to-noise ratio is equal to that of the filter originally considered. We say then that the equivalent rectangular bandwidth of the original filter is $\Delta\omega = 2\pi/T$, or, in cycles per second, $\Delta f = 1/T$.

Appendix H

The Sum of Two Sinusoids Expressed as a Frequency- and Amplitude-Modulated Sinusoidal Carrier

Certain results obtained by Peterson and Birdsall (15) were used in the derivation of Eq. 8.2. These results apply to situations in which the signal can be expressed as

$$s(t) = f(t) \sin[\alpha t + \phi(t) - \underline{P}] \tag{H. 1}$$

where $\underline{P}$ is a random variable with rectilinear distribution.

The signals used in Experiment 2 can be written as

$$s_1(t) = A_1 \sin(\omega_1 t + \underline{P}_1) + A_2 \sin(\omega_2 t + \underline{P}_2) \tag{H. 2}$$

In order for Eq. 8.2 to be applicable to Experiment 2, it is necessary to show that Eq. H. 2 can be put in the form of Eq. H. 1.

Let

$$l_1 = \omega_1 t + \underline{P}_1$$

$$l_2 = \omega_2 t + \underline{P}_2$$

$$l_3 = l_2 - l_1$$

Equation H. 2 can then be written

$$s_1(t) = A_1 \sin l_1 + A_2 \sin(l_1 + l_3)$$

$$= A_1 \sin l_1 + A_2 (\sin l_1 \cos l_3 + \cos l_1 \sin l_3)$$

$$s_1(t) = (A_1 + A_2 \cos l_3) \sin l_1 + (A_2 \sin l_3) \cos l_1 \tag{H. 3}$$

Let

$$u = \frac{A_2 \sin l_3}{A_1 + A_2 \cos l_3}$$

and

$$\phi = \tan^{-1} u$$

We then have

$$\cos \phi = 1/(u^2 + 1)^{1/2}$$

$$\sin \phi = u/(u^2 + 1)^{1/2}$$

We can then write

$$\sin(l_1 + \phi) = \frac{\sin l_1}{(u^2 + 1)^{1/2}} + \frac{u \cos l_1}{(u^2 + 1)^{1/2}} \tag{H.4}$$

and, replacing several of the u in Eq. H.4 by the definition of u, we have

$$\sin(l_1 + \phi) = \frac{(A_1 + A_2 \cos l_3) \sin l_1 + (A_2 \sin l_3) \cos l_1}{(A_1 + A_2 \cos l_3)(u^2 + 1)^{1/2}}$$

$$= \frac{s_1(t)}{(A_1 + A_2 \cos l_3)(u^2 + 1)^{1/2}}$$

replacing l_1 by its definition, letting the denominator in the above equation be $f_1(t)$, and writing $\phi = \phi_1(t)$, we have, finally,

$$s_1(t) = f_1(t) \sin[\omega_1 t + \phi_1(t) + \underline{P}_1]$$

which is in the desired form of Eq. H. 1. The amplitude modulation is thus given by

$$f_1(t) = (A_1 + A_2 \cos l_3)(u^2 + 1)^{1/2}$$

$$= \left(A_2^2 + A_1^2 + 2 A_1 A_2 \cos[(\omega_2 - \omega_1)t + (\underline{P}_2 - \underline{P}_1)]\right)^{1/2}$$

and the frequency modulation is given by

$$\phi_1(t) = \tan^{-1} \frac{A_2 \sin[(\omega_2 - \omega_1)t + (\underline{P}_2 - \underline{P}_1)]}{A_1 + A_2 \cos[(\omega_2 - \omega_1)t + (\underline{P}_2 - \underline{P}_1)]}$$

For $s_1(t)$ to fulfill the requirements of randomness associated with Eq. H. 1 it is necessary (a) that $\underline{P}_1$ be a random variable having rectilinear distribution, and (b) that $\underline{P}_2 - \underline{P}_1$ be constant, since f(t) and $\phi(t)$ are required to be ordinary (nonrandom) functions. Condition (a) is very likely approximated in Experiment 2 (as already discussed in connection with Experiment 1). Condition (b) is not met.

It is highly likely, however, that, from the point of view of the auditory mechanism, it is irrelevant whether requirement (b) is met or not. A change in the difference between phases is equivalent to a change in the time at which the observation-interval begins, i.e., equivalent to a change in the location of the observation-interval along an infinite signal waveform. So long as the duration of the observation-interval is large compared to a period of the waveform (as is the case in Experiment 2), such changes cannot matter to the ear. Had the condition of constant phase-difference been met, our data and our comparisons between ideal-detector behavior and actual behavior would have remained unchanged.

Appendix I

Detection with Two Independent Detectors in a Symmetrical Two-Category Forced-Choice Situation

Ideal Detection

To show: Detection with two independent detectors is no better than detection with only the better one of the two.

The symmetrical two-category forced-choice procedure is characterized by $\beta = 1$ and $\Pr(S{:}\bar{S}) = \Pr(\bar{S}{:}S)$. We wish to maximize the quantity $\Pr(R_1 \mid S{:}\bar{S}) + \Pr(R_2 \mid \bar{S}{:}S)$.

From Appendix D we have that the ideal decision procedure is to make response R_1 if and only if $L(\Psi_1) > L(\Psi_2)$. Let us define

$$\underline{Z} = L(\Psi_1) - L(\Psi_2)$$

The procedure is then to make response R_1 if and only is $\underline{Z} > 0$.

Now consider a second ideal detector operating simultaneously; this second detector calculates a quantity $\underline{Z}'$, which is independent of $\underline{Z}$ in the sense that

$$\Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid S{:}\bar{S}) = \Pr(\underline{Z} > 0 \mid S{:}\bar{S}) \Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \tag{I.1a}$$

$$\Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid \bar{S}{:}S) = \Pr(\underline{Z} > 0 \mid \bar{S}{:}S) \Pr(\underline{Z}' > 0 \mid \bar{S}{:}S) \tag{I.1b}$$

For ideal detectors we have, obviously, (for either $\underline{Z}$ or $\underline{Z}'$),

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) > \Pr(\underline{Z} > 0 \mid \bar{S}{:}S) \tag{I.2}$$

A decision procedure for two detectors consists of a plan that assigns one of the two responses to each of the four possibilities

$(\underline{Z} > 0,\ \underline{Z}' > 0)$ $\quad$ $(\underline{Z} < 0,\ \underline{Z}' > 0)$

$(\underline{Z} > 0,\ \underline{Z}' < 0)$ $\quad$ $(\underline{Z} < 0,\ \underline{Z}' < 0)$

There are, in all, sixteen possible decision procedures. These procedures may be represented abstractly by the matrix

	$\underline{Z} > 0$	$\underline{Z} < 0$
$\underline{Z}' > 0$	A_1	A_2
$\underline{Z}' < 0$	A_3	A_4

To obtain any particular procedure we replace each A_i by either R_1 or R_2.

a. The Two Detectors Agree. It is easy to decide which response to make if the two detectors agree: A_1 is R_1 and A_4 is R_2. That is to say, if the two detectors agree, make the response upon which they agree. This is shown as follows.

We wish to maximize the quantity $\Pr(R_1 \mid S{:}\bar{S}) + \Pr(R_2 \mid \bar{S}{:}S)$. This quantity may be written

$$\Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid U_1) + \Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid U_2)$$

$$+ \Pr(\underline{Z} > 0,\ \underline{Z}' < 0 \mid U_3) + \Pr(\underline{Z} < 0,\ \underline{Z}' < 0 \mid U_4) \qquad \text{(I. 3)}$$

where U_i is $S{:}\bar{S}$ or $\bar{S}{:}S$ according as A_i is R_1 or R_2. We maximize (I. 3) by making each term as large as possible. If we let A_1 be R_1 then the first term of (I. 3) — the only term affected — is

$$\Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid S{:}\bar{S}) = \Pr(\underline{Z} > 0 \mid S{:}\bar{S})\,\Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \qquad \text{(I. 4)}$$

If we let A_1 be R_2, the first term of Eq. I. 3 is

$$\Pr(\underline{Z} > 0,\ \underline{Z}' > 0 \mid \bar{S}{:}S) = \Pr(\underline{Z} > 0 \mid \bar{S}{:}S)\,\Pr(\underline{Z}' > 0 \mid \bar{S}{:}S) \qquad \text{(I. 5)}$$

Referring to (I. 2) we see that (I. 4) > (I. 5). Thus we want to let A_1 be R_1. By a similar argument we show that we want to let A_4 be R_2.

We are therefore left with only four decision procedures to investigate:

	$\underline{Z} > 0$	$\underline{Z} < 0$
$\underline{Z}' > 0$	R_1	R_1
$\underline{Z}' < 0$	R_1	R_2

Procedure 1

	$\underline{Z} > 0$	$\underline{Z} < 0$
$\underline{Z}' > 0$	R_1	R_2
$\underline{Z}' < 0$	R_2	R_2

Procedure 2

	$\underline{Z} > 0$	$\underline{Z} < 0$
$\underline{Z}' > 0$	R_1	R_2
$\underline{Z}' < 0$	R_1	R_2

Procedure 3

	$\underline{Z} > 0$	$\underline{Z} < 0$
$\underline{Z}' > 0$	R_1	R_1
$\underline{Z}' < 0$	R_2	R_2

Procedure 4

b. The Remaining Four Procedures. It is easy to show that the ideal detector has the same probability of making the correct response in case $S{:}\bar{S}$ as in case $\bar{S}{:}S$. (See Eq. E.0, Appendix E, letting $\beta = 1$.) Thus (for either $\underline{Z}$ or $\underline{Z}'$)

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) = \Pr(\underline{Z} < 0 \mid \bar{S}{:}S) \tag{I.6}$$

Assume first that the $\underline{Z}$-detector is better than the $\underline{Z}'$-detector, that is, assume that

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) > \Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \tag{I.7}$$

By writing out expression (I. 3) explicitly for each of the four decision procedures, and then applying (I. 2), (I. 6), and (I. 7), it is easy to convince oneself that (I. 3) is maximal for procedure 3. But procedure 3 is equivalent to paying attention only to the $\underline{Z}$-detector, as is evident from the matrix. By (I. 7) the $\underline{Z}$-detector is the better of the two detectors.

Assume next that the $\underline{Z}'$-detector is the better of the two. That is, instead of (I. 7), assume

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) < \Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \tag{I.8}$$

In this case by writing out (I. 3) explicitly for the four procedures, one can convince himself that procedure 4 is optimal. But this procedure is equivalent to paying attention only to the $\underline{Z}'$-detector, which is, by (I. 8), the better of the two.

Lastly, assume that the two detectors are equally good, that is, instead of (I. 7) or (I. 8), assume

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) = \Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \tag{I.9}$$

In this case we can convince ourselves, by writing out (I. 3) explicitly for all four cases, that all four decision procedures are equally good. Thus we can arbitrarily choose to consider only one of the two detectors. This completes the proof.

Nonideal Detection

When dealing with nonideal detection, we assume once again that there exist quantities $\underline{Z}$ and $\underline{Z}'$ that are similar to those quantities in the case of ideal detection, at least to the extent that Eq. I. 2 holds. In that case the proof given above holds through section a; that is we need to investigate only four decision procedures. On the other hand the part of the proof in section b may no longer hold, through failure of (I. 6) to hold.

We can however make an empirical investigation. To calculate the expected detectability on each of the four decision procedures we need to know

$$\Pr(\underline{Z} > 0 \mid S{:}\bar{S}) \qquad \text{and} \qquad \Pr(\underline{Z} < 0 \mid \bar{S}{:}S) \tag{I.9}$$

$$\Pr(\underline{Z}' > 0 \mid S{:}\bar{S}) \qquad \text{and} \qquad \Pr(\underline{Z}' < 0 \mid \bar{S}{:}S) \tag{I.10}$$

But by our model these probabilities are equal to

$$\Pr(R_1 \mid S{:}\bar{S}) \qquad \text{and} \qquad \Pr(R_2 \mid \bar{S}{:}S)$$

which have been measured in our experiment. Thus we may take the observed

$Pr(R_1 \mid S{:}\bar{S})$ and $Pr(R_2 \mid \bar{S}{:}S)$ for the case of a 500-cps signal, letting these be the probabilities in (I.9). Then take the observed $Pr(R_1 \mid S{:}\bar{S})$ and $Pr(R_2 \mid \bar{S}{:}S)$ for the case of an 1100-cps signal, letting these be the probabilities in (I.10). On the basis of these values we may then calculate for each of the four decision procedures what the observed detectability for the combination of these two tones would have been had detection been performed by two independent detectors, one listening to each tone.

These calculations were carried out five times, once for each of the five conditions of Table 4 for subject HF. In none of these five cases was the predicted combined detectability on any of the four decision procedures higher than the detectability of the more highly detectable component. Thus, in our experiment, the theorem holds for nonideal detection.

Appendix J

Some Background Notes on Detection Theory

The theory of detection is generally recognized to be a special case of the statistical theory of hypothesis-testing. And, as there are two general approaches to the theory of hypothesis-testing, we find two corresponding approaches to the theory of detection.

The older approach to hypothesis-testing consists in the calculation by means of Bayes's theorem of the *a posteriori* probability of the hypothesis under test, as in the following example.

Example. Consider the following two hypotheses S and $\overline{S}$ about a given coin (we assume that these are the only admissible hypotheses).

S: The probability of the coin landing heads is 1/11.

$\overline{S}$: The probability of the coin landing heads is 10/11.

Suppose moreover, that for one reason or another, we are able to assign specific *a priori* probabilities to the two hypotheses. Thus, suppose that

$$\Pr(S) = \Pr(\overline{S}) = 1/2$$

We decide to evaluate our hypotheses on the basis of $\underline{K}$, the number of heads that occur in two independent tosses of the coin. First, let us write the probability distribution of $\underline{K}$ on each of the two hypotheses. If S obtains, we have

$$\Pr(\underline{K} = k|S) = \frac{2}{k!\ (2-k)!}\ (1/11)^k\ (10/11)^{2-k}$$

If $\overline{S}$ obtains, we have

$$\Pr(\underline{K} = k|\overline{S}) = \frac{2}{k!\ (2-k)!}\ (10/11)^k\ (1/11)^{2-k}$$

Now write Bayes's theorem,

$$\Pr(S \mid \underline{K} = k) = \frac{\Pr(K = k \mid S)\Pr(S)}{\Pr(\underline{K} = k \mid S)\Pr(S) + \Pr(\underline{K} = k \mid \overline{S})\Pr(\overline{S})}$$

Introducing our *a priori* probabilities and probability distributions into Bayes's theorem, we have

$$\Pr(S \mid \underline{K} = k) = \frac{1}{1 + (10)^{2k - 2}}$$

Thus, for instance, if we observe that $\underline{K} = 2$, that is, if we obtain two heads, we find that the *a posteriori* probability of the hypothesis S is

$$\Pr(S \mid \underline{K} = 2) = 1/101$$

Proponents of the Bayes's theorem approach argue that obtaining a low value for the a posteriori probability of a hypothesis is evidence against that hypothesis. Thus, in our example, we would be said to have found evidence that hypothesis S is (probably) false.

The application of this theory to the problem of detection is immediate: The hypothesis to be tested is that the "input to the detector" (see section 3) contains the signal in addition to the noise; due to the presence of the random noise, this input is a random variable whose distribution on the hypotheses may be calculated once the characteristics of the signal and of the noise are specified; the a priori probability of the hypothesis is assumed known.

Detection theory along the lines just discussed has been developed primarily by Woodward and Davies (22, 23, 24, 25). In support of their approach these authors show that a detector that calculates the a posteriori probability extracts all the possible relevant information from its input; if we are required to specify how certain we are that a signal was present, we cannot give a better answer than the a posteriori probability.

In a series of joint papers beginning in 1928 Neyman and Pearson proposed an alternative approach to hypothesis-testing. Their theory – which forms the basis for a good portion of modern statistics – and their objections to the earlier Bayes' s theorem approach are contained in the following quotation (14):

> Now what is the precise meaning of the words "an efficient test of a hypothesis?" There may be several meanings. For example, we may consider some specified hypothesis , as that concerning the group of stars, and look for a method which we should hope to tell us, with regard to a particular group of stars, whether they form a system, or are grouped "by chance," their mutual distances apart being enormous and their relative movements unrelated.
>
> If this were what is required of an "an efficient test," we should agree with Bertrand in his pessimistic view [that no statistical test could give reliable results]. For however small be the probability that a particular grouping of a number of stars is due to "chance," does this in itself provide any evidence of another "cause" for this grouping but "chance?" "Comment définir, d'ailleurs, la singularité dont on juge le hasard incapable?" Indeed, if x is a continuous variable – as for example is the angular distance between two stars – then any value of x is a singularity of relative probability equal to zero. We are inclined to think that as far as a particular hypothesis is concerned, no test based upon the theory of probability can by itself provide any valuable evidence of the truth or falsehood of that hypothesis.
>
> But we may look at the purpose of tests from another viewpoint. Without hoping to know whether each separate hypothesis is true or false, we may search for rules to govern our behaviour with regard to them, in following which we insure that, in the long run of experience, we shall not be too often wrong. Here, for example, would be such a "rule of behaviour": to decide whether a hypothesis, H, of a given type be rejected or not, calculate a specified character, x, of the observed facts; if $x > x_o$ reject H, if $x < x_o$ accept H. Such a rule tells us nothing as to whether in a particular case H is true when $x < x_o$ or false when $x > x_o$. But it may often be proved that if we behave according to such a rule, then in the long run we shall reject H when it is true not more, say, than once in a hundred times, and in addition we may have evidence that we shall reject H sufficiently often when it is false.

The decision-procedure approach of Neyman and Pearson is illustrated in the following example.

Example. Let the situation be the same as in the preceding example, and consider the following decision procedure: Reject the hypothesis S if and only if two heads come up in the two tosses of the coin.

We now calculate the probability of being in error if we follow this procedure. If S is true, the probability of rejecting S is

$$\Pr(\underline{K} = 2 \mid S) = (1/11)^2 = 0.008$$

If S is false (i.e., if $\bar{S}$ holds), the probability of accepting S is

$$\Pr(\underline{K} \neq 2 \mid \bar{S}) = 1 - \Pr(\underline{K} = 2 \mid \bar{S}) = 1 - (10/11)^2 = 0.17$$

These probabilities of error may, of course, be reduced by devising different decision procedures based on a larger number of throws of the coin.

Notice, incidentally, that if we are satisfied with calculating these conditional probabilities of error, we do not need to know the *a priori* probabilities associated with the hypotheses. Only if we wish to calculate the absolute (unconditional) probability of error need the *a priori* probabilities enter the picture.

Neyman and Pearson suggest the following criterion for an optimal decision procedure to test hypothesis H: Pick the procedure so that (a) the probability of rejecting H when it is true is a small given number, and (b) the probability of accepting H when it is false is simultaneously minimized. They show that the calculation of such optimal procedures rests primarily on the calculation of the likelihood ratio (see section 3) of the random variable in question. Their theory thus leads to the modern "likelihood ratio tests."

The Neyman-Pearson theory was applied to signal-detection by Lawson and Uhlenbeck in 1950 (ref. 10, chap. 7). These authors, however, do not follow Neyman and Pearson's suggested criterion for the optimal decision procedure. Instead, they use as criterion the minimization of the absolute probability of error. As already mentioned, this requires the specification of the *a priori* probability of the relevant hypotheses. Despite this change of criterion, Lawson and Uhlenbeck's theory remains – to quote the authors themselves – "practically identical with the Neyman-Pearson theory" (ref. 10, p. 168).

Both approaches to detection theory are reviewed in the comprehensive monograph by Peterson and Birdsall (15). These authors suggest a somewhat more general criterion of optimal procedures than that used by Lawson and Uhlenbeck: As their criterion they use the maximization of the "payoff values" associated with the two kinds of correct and the two kinds of incorrect decisions that may be made in a detection situation (see sections 1 2, and 3). This again, however, does not affect the basic theory materially. In addition to exhibiting extensive proofs of many of the major results in detection theory, Peterson and Birdsall solve the decision-procedure model explicitly for a wide variety of interesting specific situations.

Among other workers who have contributed to detection theory may be mentioned Kaplan and Fall (9), Marcum (12), Middleton (13), and Reich and Swerling (17).

NOTATION FOR PROBABILITIES

"Pr(A)" designates the probability of the occurrence of the event A. "Pr(A|B)" designates the probability of the occurrence of the event A, given that the event B occurs.

Superscribed bars always indicate complementary events. Thus, for any events A and B,

$$\Pr(A) + \Pr(\bar{A}) = 1$$

$$\Pr(A|B) + \Pr(\bar{A}|B) = 1$$

Distribution Functions and Densities

If $\underline{Y}$ is a random variable and if

$$\Pr(\underline{Y} < y) = F(y)$$

then F(y) is called the distribution function (or, briefly, the distribution) of $\underline{Y}$.

If F(y) has a derivative

$$\frac{d\,F(y)}{dy} = f(y)$$

then f(y) is called the probability density of $\underline{Y}$. Thus

$$\int_{-\infty}^{y} f(x)\,dx = \Pr(\underline{Y} < y) = F(y)$$

NOTATION AND FORMULAS FOR THE NORMAL DISTRIBUTION

The normal, or gaussian, distribution function, i.e.,

$$\int_{-\infty}^{x} \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$

is represented by $N[x;\ \mu, \sigma^2]$. Thus, to say that the random variable $\underline{Y}$ is normally distributed with mean $E(\underline{Y})$ and variance $Var(\underline{Y})$ is to say that

$$Pr(\underline{Y} < y) = N[y;\ E(\underline{Y}),\ Var(\underline{Y})]$$

The normal density is written: $n[x;\ \mu, \sigma^2]$.
Thus

$$\frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] = n[x;\ \mu, \sigma^2]$$

The reader is assumed to be familiar with the following identities:

$$N[x;\ \mu, \sigma^2] \equiv N[x-\mu;\ 0, \sigma^2] \equiv N[\tfrac{x-\mu}{\sigma};\ 0, 1]$$

$$N[x;\ 0, \sigma^2] \equiv 1 - N[-x;\ 0, \sigma^2]$$

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