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ECONOMIC LOT SIZE DETERMINATION IN MULTI-STAGE ASSEMBLY SYSTEMS

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I. INTRODUCTION

In this paper we discuss an algorithm for computing lot sizes for multi-stage assembly systems. In a multi-stage assembly system each stage, or facility, requires inputs from a number of immediate predecessor stages, and supplies, in turn, one immediate successor. This structure includes the important special case of facilities arranged in series. We consider the case of constant demand over an infinite horizon, with instantaneous production.

Our objective is the choice of a lot size for each facility which minimizes average per period production and inventory holding costs, where

[1] This paper is a synthesis of papers submitted to Management Science in February, 1971 by Crowston and Wagner [4] and Williams [16]. Sections I - VIII represent a combination of material from both. The proof of integrality contained in the Appendix was unique to reference [4]. In addition, reference [4] presents results for finite horizon models of multi-stage assembly systems which are omitted here.

at each facility production costs consist of a setup cost and possibly a linear component, and inventory holding costs are linear. Under these and a few additional assumptions to be described in section II, we prove that an optimal set of lot sizes exists such that the lot size at each facility is a positive integer multiple of the lot size at its successor facility. This fact is used in the development of a dynamic programming algorithm for determining optimal lot sizes.

In recent years a number of new algorithms for production scheduling in multi-echelon systems have been developed. Both the finite horizon case, where decisions are made at discrete points in time, and the infinite horizon case have been discussed. Discrete dynamic programming models developed by Zangwill [17-20] , and Veinott [14], and Love [9] assume a finite horizon and a known but possibly varying demand. Production and holding costs in these models are assumed to be concave. Love shows that his approach may be extended to the infinite horizon case for the facilities in series structure. This allows periodic lot sizes at each stage, a possibility which we explicitly rule out in our development. Love's algorithm cannot be extended in any obvious fashion to the general assembly system structure. We remark in this connection that our reason for assuming that the lot size at a stage is constant over time is primarily computational, but that this restriction may often be reasonable in practice in light of scheduling and administrative considerations.

Discrete linear programming models have been developed by Manne [10], Von Lanzenauer [15], and Gorenstein [6,7]. These models schedule several different products and assume a finite horizon and known demand. However, Manne's model neglects holding cost. The objective function in his model

minimizes overtime requirements. In [15] Von Lanzenauer has integrated production scheduling and sequencing decisions for multi-stage, multi-product systems. However, successful implementation of [15] depends on the development of more efficient zero-one programming techniques. The assembly system structure in particular is considered by Gorenstein [7].

Published models for the infinite horizon case assume constant demand and time invariant cost parameters. For the facilities in series structure there have been two recent contributions in addition to the aforementioned work of Love. The model of Taha and Skeith [12] allows non-instantaneous production, delay between stages, and back-orders for the product of the final stage. They assume that in an optimal solution the lot size at a stage is an integer multiple of the lot size at the succeeding stage and suggest the problem be solved by examining all combinations of such integer values. Jensen and Khan [8] also allow non-instantaneous production but do not use the assumption of positive integers. Instead they have constructed a simulation model which evaluates the average inventory at a stage, given the lot size at that stage and at the succeeding stage, along with the production rate at both stages. A dynamic programming algorithm is then formulated in which the simulation model is used in evaluation of each functional equation. They note that high average inventories result if the integer multiple assumption is not followed and discuss a problem for which non-constant lot size is optimal.

For the multiple predecessor case Schussel [11] develops a simulation model and heuristic decision rule which again assumes that integer multiples are optimal. He adds a "learning curve" function so that unit

production cost decreases with lot size and allows costs to be discounted over time. Crowston, Wagner and Henshaw [3] tested four heuristic rules and compared them with a version of the dynamic programming algorithm developed in this paper.

The concept of echelon stock which we find useful for computing total system inventory holding cost was originally propounded by Clark and Scarf [1,2]. They define installation stock at any given facility as the stock which is stored between that facility and its immediate successor facility. The average installation stock depends both upon the lot size at the predecessor and successor stages. Echelon stock, on the other hand, is defined to be the number of units in the system which have passed through facility i but have not as yet been sold. The use of the echelon stock concept allows inventory holding cost to be regarded as a function of only the predecessor facility. Echelon n stock may often be considered to be the facility n value-added inventory, and the concept has broad implications for more general multi-echelon structures than the assembly system which we consider. The Clark-Scarf models allow stochastic demand and convex holding costs, but setup costs are assumed to be associated with no more than two facilities.

We present our results in the following order. In section II we describe the problem. We follow in section III with a statement of Theorem 1, which characterizes the form of the optimal solution. The result is used in the derivation of the total cost model provided in section IV. Simple extensions of the model are considered in section V. We then present the basic dynamic programming algorithm in Section VI. Some computational refinements are discussed in section VII. We conclude

in Section VIII with a summary of results and a brief discussion of implications of the model. Proof of Theorem 1 is provided in the Appendix.

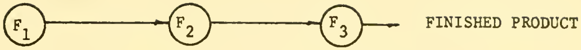
II. PROBLEM DESCRIPTION

In a multi-stage system, the manufacture of final product requires completion of a number of operations or stages. We use interchangeably the terms stage and facility. A stage might consist of an operation such as procurement of raw materials, fabrication of parts, or assembly. A fixed sequence of operations is assumed, so that output from one stage serves as input to an immediate successor stage. The final stage is an exception in that its output is a finished product used to service customer demand. Output from any stage may be stored until needed in that stage's installation inventory.

A multi-stage assembly system is characterized by the restriction that each stage has at most one immediate successor. We emphasize that, in general, a stage may have any number of immediate predecessors. Examples of multi-stage assembly systems are depicted in Figures 1 and 2, with Figure 1 illustrating the facilities in series case.

We shall denote a stage F_n, where n is an index ranging from 1 to N, and F_N is the final stage. Let a(n) be the index of the immediate successor of F_n, A(n) the set of indices of all successors (immediate or otherwise), b(n) the set of indices for all immediate predecessors and B(n) for all predecessors. In Figure 2 for example

$$a(6) = [7],\ b(6) = [4,5],\ A(6) = [7,17],\ B(7) = [1,2,3,4,5,6].$$



3 level, 1 stage per level system (facilities in series)

FIGURE 1

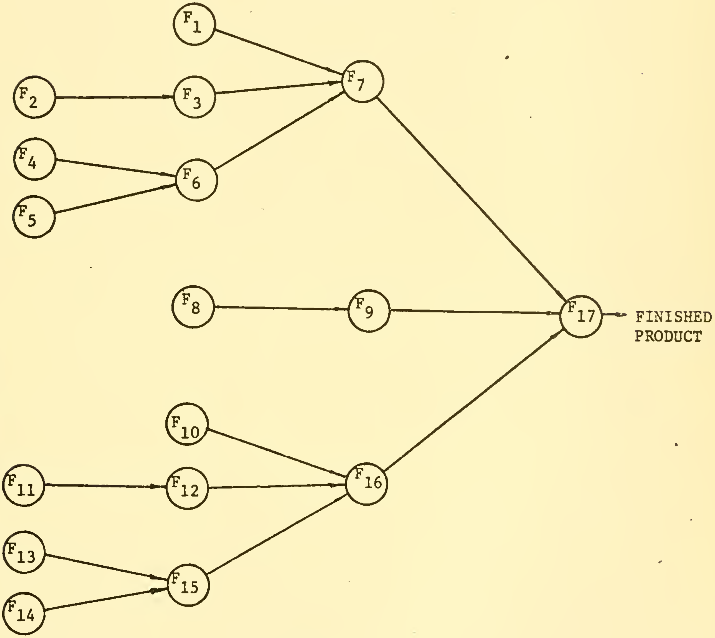


FIGURE 2

L_1 L_2 L_3 L_4

4 levels, multi-stage per level system

For expositional convenience, we introduce the notion of a level, where stages are assigned to levels according to: the final stage F_N is in level L_M, and F_n is in level L_m if its successor $F_{a(n)}$ is in L_{m+1}.

It is assumed throughout that demand is known with certainty. The objective is minimization of the cost of satisfying all demand with no backorders. Costs are assumed to depend upon the stage, F_n, there being a fixed charge for production setup, S_n (\$/setup), and a linear per unit installation inventory holding cost, H_n (\$/unit/time). One unit at a stage is the quantity required in one unit of final product.

We will find it convenient to refer to the n-echelon per unit holding cost, h_n, defined by: $h_n = H_n - \Sigma_{m \varepsilon b(n)} H_m$. The concept of an incremental holding cost is closely related to that of "value added" at a production stage. In fact, the holding cost in many situations might be: $H_n = V_n c$, where V_n = total dollar value of a completed stage n unit and c is a cost of carrying inventory per dollar of inventory and $h_n = v_n c$, where v_n = value per unit added by the stage n process. We note that a direct per unit production cost, P_n, can easily be added to the models discussed herein, but such a term has no effect upon the lot size decision and simply adds a constant to the total costs.

We now list our assumptions.

1. Stages are arranged in an assembly structure with each stage having at most one successor.
2. Inventory can be stored between facilities. Where some facilities do not allow this, we can redefine facilities so that a model facility corresponds to two or more actual facilities and storage between model facilities is allowed.

3. If there are delays in moving from one facility to the next, the delays are constant and thus not a function of the lot size.
4. There are no capacity constraints.
5. At each facility, production is instantaneous.
6. Final product demand is constant: D per period. We initially assume demand is discrete, but consider the continuous case in section VII.
7. Stockouts are not permitted.
8. At any given facility, marginal production costs are constant. Thus they may be ignored in the optimization.
9. There are setup costs or ordering costs at each facility. If a given facility has a setup cost of zero, total costs are minimized by producing, at that facility, the least possible amount at the latest possible time, subject to input requirements from the successor facility. Thus the lot size for a facility with a setup cost of zero should be the same as the lot size at its immediate successor facility. In the model, such a facility should be combined with its successor to form a single model-facility.
10. Holding cost per period on installation stock at a given facility is a linear function of the quantity of stock at that facility. Furthermore, at any given facility, installation stock holding cost per product unit per time period is never smaller than installation stock holding cost per product unit per time period at any predecessor facility. (Product units are defined such that a product unit at any given facility is a quantity which will eventually form one unit of final product.)

11. The lot size at a given facility is constant (rational number).

We shall refer to the problem of minimizing the average cost per unit time under assumptions 1 - 11 as the Basic Problem.

III. FORM OF THE OPTIMAL SOLUTION

We consider only solutions which can be characterized by a single lot size for each stage. Let $k_n = Q_n/Q_{a(n)}$ and $K_n = Q_n/Q_N$. A particular solution is given by $k^j = \{k_1^j, k_2^j, \ldots, k_{N-1}^j, 1\}$ and Q_N^j or by $K^j = \{K_1^j, K_2^j, \ldots, K_{N-1}^j, 1\}$ and Q_N^j. Then it can be shown that the ratio of lot sizes between successor and predecessor stages, k_n, must be a positive integer. The result is summarized in Theorem 1.

Theorem 1: Form of the Optimal Sclution. Of the set of all solutions to the Basic Problem which can be characterized by a set of rational lot size multiples, k^j, and final stage quantity Q_N^j, a minimum cost solution exists with Q_N^j and k^j all positive integers.

A detailed proof is given in the Appendix. An expression is derived for the costs associated with a lot size Q_n given $Q_{a(n)}$. This function is shown to be minimized with $k_n = Q_n/Q_{a(n)}$ a positive integer. Proof then follows by induction over the levels of the system.

We wish to emphasize that the assumption of a time-variant lot size for each stage is quite strong. The possibility of cyclic lot sizes, for example, is thus eliminated. The restriction may be justified, in some cases, by the costs of administering changing lot sizes. In any event, Theorem 1 leads to computationally powerful algorithms for finding the optimum in a class of easily implemented solutions. Given the results

of Theorem 1, we now derive expressions for the total costs of a particular solution k^j, Q_N^j.

IV. THE COST STRUCTURE

If all lot sizes within the system were equal, then, given assumption 5, inventory would only be held at the final stage. However, if $Q_n \neq Q_{a(n)}$ then installation stock is created at F_n and the average level of such inventory is a complicated function of Q_n and $Q_{a(n)}$. In Figure 3 we show the installation stock at each stage of a 3-stage serial production process with $Q_1 = 6Q_3$ and $Q_2 = 2Q_3$. In Figure 4 the echelon stock for each stage of the system is shown to form the familiar saw-tooth pattern of the ordinary Wilson lot size formula. Given assumption 6 the average echelon inventory at stage F_n is $(Q_n - 1)/2$. Thus the total holding and setup cost for the echelon stock will be

$$f_n(Q_n) = DS_n/Q_n + (Q_n - 1)h_n/2 \tag{1}$$

and the total cost for the system, s, will be

$$T = \sum_{n=1}^{N} f_n(Q_n) \tag{2}$$

This may be rewritten as

$$T = \sum_{n=1}^{N} \{DS_n/K_nQ_N + (K_nQ_N - 1)h_n/2\} \tag{3}$$

Note that for a particular vector $\underline{K^j}$ the optimal value of Q_N would be

$$Q_N^j = \sqrt{\{2D\,\Sigma(S_n/K_n)\}/\Sigma h_nK_n} \tag{4}$$

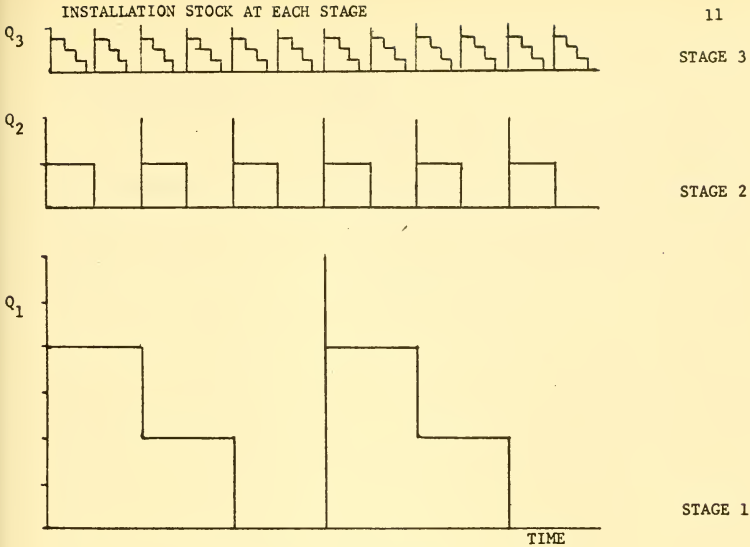


FIGURE 3

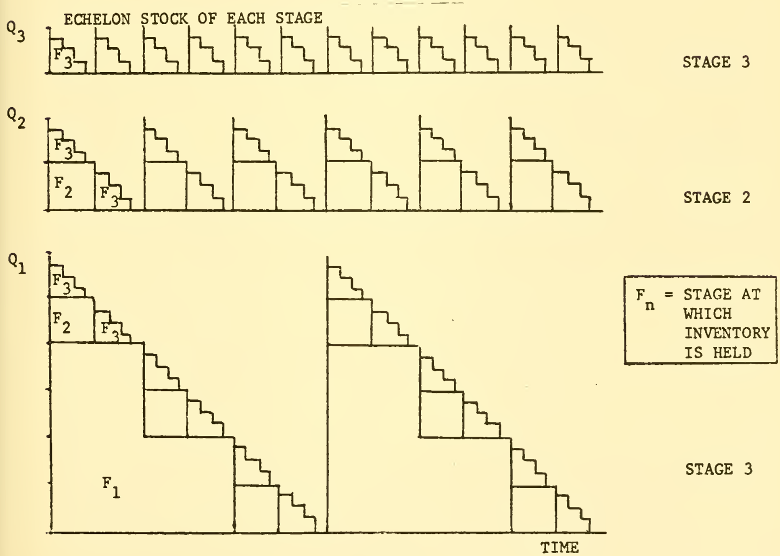


FIGURE 4

V. SIMPLE EXTENSIONS OF THE MODEL

In this section we briefly consider a special case of non-instantaneous production and the case of transfer delay between stages. If we assume production rate p_n at F_n and given $p_n \geq p_{a(n)}$ then the result of Theorem 1 applies. The cost function for the product of F_n will be

$$TC_n = DS_n/K_nQ_N + [(K_nQ_N - 1)/2][1 - D/p_n]\, h_n \quad . \tag{5}$$

Finally we observe that a transfer delay between stages simply adds a constant inventory term to either equation (3) or (5) and therefore does not affect the optimal solution. The fact that constant delays do not alter optimal policies is also mentioned in [12, 18].

VI. THE DYNAMIC PROGRAMMING ALGORITHM

The dynamic programming algorithm is written in terms of the simplest cost structure although it is clear that it could be modified to include the cost function for non-instantaneous production. Solution proceeds from the raw material stage to F_N with the recurrence relation defined as follows.

Let I denote the set of all positive integers, and let $T_n(Q_n)$ represent the optimal cost at F_n and all prior stages F_m, $m \varepsilon b(n)$, when Q_n is given. Then,

$$T_n(Q_n) = f_n(Q_n) + \Sigma_{m \varepsilon b(n)} \underset{\ell \varepsilon I}{\text{minimum}} \; T_m(\ell Q_n) \tag{6}$$

$$T^{*} = \text{optimal } T = \min_{Q_N} \; T_N(Q_N) \tag{7}$$

The computations proceed successively from the first level of the system to the last. At each stage n, $T_n(Q_n)$ is computed for all possible Q_n.

Optimal solutions for the system of Figure 2 have been obtained with this algorithm in approximately ten seconds of computation time

on a time-shared GE 645 system [3].

VII. COMPUTATIONAL REFINEMENTS

We will now develop both upper and lower bounds on Q_n so as to improve the computational efficiency of the dynamic programming algorithm.

If we assume a problem with the cost structure given in IV, then at F_n a lower bound, z_n, on the cost of system inventory of that stage will be

$$z_n = DS_n/\sqrt{2DS_n/h_n} + \sqrt{[(2DS_n/h_n) - 1]h_n/2} \tag{8}$$

This assumes no interdependency between successive stages. Then a lower bound for the complete system, L, will be

$$L = \sum_{n=1}^{N} z_n . \tag{9}$$

An upper bound on total cost, U, for the system may be derived from a feasible heuristic solution [3] to the problem, or from a modified dynamic programming solution using a coarse grid. This approach will be discussed below. With either method an upper bound on the cost of echelon stock for F_n may be determined:

$$U - (L - z_n).$$

Since $f_n(Q_n)$ is convex in Q_n, by setting $F_n(Q_n)$ equal to its cost upper bound, that is

$$DS_n/Q_n + (Q_n - 1)h_n/2 = z_n + U - L \tag{10}$$

we may solve directly for upper and lower bounds, Q_n^u and Q_n^ℓ on Q_n.

In addition, from Theorem 1, $Q_n \geq Q_{a(n)}$. Therefore better bounds on the optimal Q_n, such that $Q_n^{min} \leq$ optimal $Q_n \leq Q_n^{max}$, can be obtained as follows:

$$Q_n^{max} = \min \begin{cases} Q_n^u \\ \\ Q_m^u, \quad m \in B(n) \end{cases}$$

and

$$Q_n^{min} = \max \begin{cases} Q_n^\ell \\ \\ Q_m^\ell, \quad m \in A(n) \end{cases} .$$

Similar bounds may be calculated for the cost structure of equation (5).

It may be possible to improve these bounds as the dynamic programming algorithm is in progress. For example, if the facilities are in series,

$$W_{n-I} + f_n(Q_n) + \sum_{k=n+1}^{N} z_n \leq U \tag{11}$$

where

$$W_{n-1} = \begin{cases} 0 \text{ if } n = 1 \\ \min\limits_{Q_{n-1}} \quad T_{n-1}(Q_{n-1}) \text{ if } n > 1 \end{cases} .$$

Thus updated bounds for Q_n can be computed. Similar inequalities provide bounds in more general multi-stage systems. In all cases the difference between the upper and lower bounds is an increasing function of U. Therefore, it is advantageous to begin with a reasonably good feasible solution.

One method for obtaining feasible solutions would be to allow the dynamic programming algorithm to search on a coarse grid. Let r be the choice of units (i.e., grid size) in terms of which the heuristic solution is to be obtained. Then, as before,

$$T_n(Q_n) = f_n(Q_n) + \sum_{m \varepsilon b(n)} \underset{\ell \varepsilon I}{\text{minimum}} \; T_m(\ell Q_n) \tag{12}$$

and

$$U = \underset{Q_N = r, 2r, \ldots, mr}{\text{minimum}} \; T_N(Q_N) \; . \tag{13}$$

Thus only quantities Q_n which are integer multiples of r need be considered. For a given value of r a vector $\underline{K^j}$ and a value Q_N^j will be obtained. This solution may be improved by solving equation (4) for an optimal Q_N^{j*}, given $\underline{K^j}$, but the result, which allows noninteger values for Q_N, is not admissible as an upper bound. Finally we note that if we had allowed continuous withdrawal, rather than discrete withdrawal, the formulation of equations (12), (13) would have been required for solution with $Q_n/2$ replacing $(Q_{n-1})/2$ as the expression for average inventory in $f_n(Q_n)$.

VIII. SUMMARY AND MODEL IMPLICATIONS

In this paper we have presented a dynamic programming algorithm which exploits the concept of echelon stock to obtain optimal constant lot sizes in a multi-stage assembly system.

A variety of heuristic rules have been suggested for problems having a structure similar to that of the multi-stage assembly system [3,11]. In addition, in industrial applications heuristics such as "constant lot size" at all stages, where the lot size is taken to be $Q_n = \sqrt{2DS_N/H_N}$, or "independent determination of lot size" at each stage are used. For the cost structure of (2) the optimal "constant lot size"

would be $\sqrt{2D\Sigma S_n/\Sigma h_n}$ although experimentation shows that this is a poor decision rule [3]. If "independent determination of lot size" is used, a common model is $Q_n = \sqrt{2DS_n/H_n}$. This implies the carrying cost of a unit of in-process inventory of F_n is a function of the total value of its components. Our model indicates that this results in double-counting. Finally, we would suggest that if heuristic decision rules are constructed for the more complicated case of multiple successors, incremental holding costs are again appropriate.

APPENDIX

Theorem 1: Form of the Optimal Solution

Of the set of all solutions to the Basic Problem which can be characterized by a set of rational lot size multiples k^j and Q_N^j, a minimum cost solution exists with Q_N^j and k^j all positive integers.

Proof: We will use Proposition 1, Proposition 2, and Lemma 2.

Proposition 1: An optimal solution to the Basic Problem with rational lot size multipliers k^j is in phase; that is, for each stage n, there is some point in time at which production occurs simultaneously with production at the successor stage a(n).

Proof: Since Q_n and $Q_{a(n)}$ are rational, the quantity levels of installation stock at stage n cycle with period P where $P = q_n Q_n/D = q_{a(n)} Q_{a(n)}/D$ and q_n, $q_{a(n)}$ are relatively prime integers. Let Δt be the smallest interval of time between production at stage n and subsequent production at stage n+1 during the cycle. If $\Delta t \neq 0$, then all production at stage n (and stage n's predecessors B(n)) can be transferred to the future by the amount Δt with no increase in setup costs and reduced inventory costs.

Proposition 2: In a two-stage system with the successor stage lot size held constant at some level R (R is a positive rational number), and with the system in phase in accordance with Proposition 1, the total cost/unit time associated with lot-size Q_1 is given by

$$Z(Q_1) = S_1D/Q_1 + h_1(Q_1 - 1)/2 + RH_1(q_2 - 1)/q_2$$

where q_2 is defined by $q_1/q_2 = Q_1/R$ and q_1, q_2 are relatively prime integers.

Proof: There are three components of cost to consider:

1. The set-up cost— S_1D/Q_1.

2. The familiar inventory holding cost which arises from periodic addition to the entire system of the amount Q_1, and the intermittent flow out of the system of D units-- $h_1(Q_1 - 1)/2$. Note that the per unit holding cost is taken to be h_1, the echelon cost, even though the physical product does not remain in Stage 1 inventory.

3. The permanent Stage 1 installation stock that must be maintained to ensure that product is always available when required. Since Q_1 and R are assumed to be rational, we can find a cycle. The permanent component of installation stock is the amount which must be on hand at the beginning of the cycle to insure that Stage 1 installation stock remains non-negative. This amount can be found assuming that Stage 1 installation stock is zero at the start of the cycle, and finding the minimum (most negative) level which is attained during the cycle.

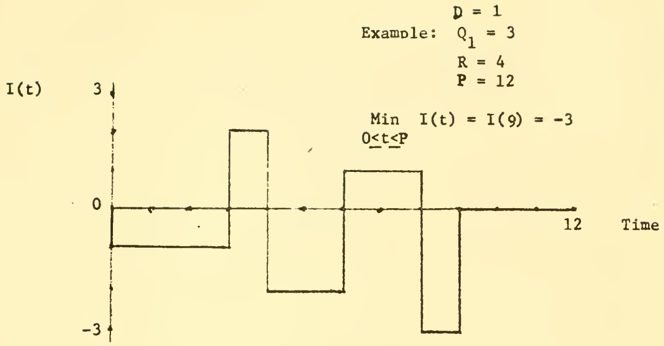


FIGURE 5

Let I(t) denote the quantity of installation stock on hand at any time $t \geq 0$ measured from the beginning of the cycle. Thus,

$$I(t) = [tD/Q_1 + 1]Q_1 - [tD/R + 1]R$$

where [ ] denotes "integer part". Then

$$I(t) = ([tD/Q_1] - R/Q_1[tD/R])Q_1 + Q_1 - R \ .$$

I(t) is clearly minimized for some t such that tD/R is integral, that is, immediately following a withdrawal to satisfy demand. Thus, $t = \ell R/D$, where ℓ is an integer, and

$$\min I(t) = \min_{\ell \text{ integer}} ([\ell R/Q_1] - R/Q_1[\ell R/R])Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} ([\ell R/Q_1] - \ell R/Q_1)Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} -(\ell q_2/q_1 - [\ell q_2/q_1])Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} -(\ell q_2 \bmod q_1)\ Q_1/q_1 + Q_1 - R.$$

Since q_2, q_1 are relatively prime, $\ell q_2 \bmod q_1$ takes on all the values $1,2,\ldots,q_1-1$. In particular, for some ℓ, $\ell q_2 \bmod q_1 = q_1 - 1$. Therefore,

$$\min I(t) = Q_1(1 - q_1)/q_1 + Q_1 - R = R(1 - q_1 + q_1 - q_2)/q_2$$

$$= - R(q_2 - 1)/q_2\ .\ ^*$$

Thus, $R(q_2 - 1)/q_2$ units of installation stock must be kept on hand permanently, at a cost of $RH_1(q_2 - 1)/q_2$.

Lemma 2: A function

$$Z(Q) = C_1/Q + C_2(Q - 1)/2 + Q_2C_2(q_2 - 1)/q_2$$

*This result was suggested by William M. Hawkins, Sloan School of Management.

where C_1, C_2, Q_2 are constants and q_2 defined as in Proposition 2 is minimized for $q_2 = 1$, that is, with Q/Q_2 an integer.

Proof: Suppose Q_1^* minimizes Z and Q_1^*/Q_2 not integer. Define Q_1 by $Q_1^* = Q_1 + \Delta Q_2$ with Q_1/Q_2 an integer and $0 < \Delta Q_2 \leq Q_2$. This can be done because Q_1^* is clearly not zero. Then

$$Z(Q_1^*) = C_1/(Q_1 + \Delta Q_2) + C_2(Q_1 + \Delta Q_2)/2 + Q_2C_2(q_2 - 1)/q_2$$

Since Q_1^* not integer, $(q_2 - 1)/q_2 \geq 1/2$

$$Z(Q_1^*) \geq C_1/(Q_1 + \Delta Q_2) + C_2(Q_2 + \Delta Q_2)/2 + C_2Q_2/2$$

$$\geq C_1/(Q_1 + Q_2) + C_2(Q_1 + \Delta Q_2)/2 + C_2Q_2/2 \qquad \text{since } Q_2 \geq \Delta Q_2.$$

$$\geq C_1/(Q_1 + Q_2) + C_2(Q_1 + Q_2)/2 = Z(Q_1 + Q_2).$$

Thus $Z(Q_1^*) \geq Z(Q_1 + Q_2)$. Since Q_1/Q_2 is an integer by construction, $(Q_1 + Q_2)/Q_2$ is an integer.

Proof of Theorem 2 follows by induction over the levels of the multi-stage system. We assume we have an optimal solution Q^* and show that it must be integer. Let $Z_n(Q_n)$ be the cost for stage n and all predecessor stages, given that Q_n is produced at stage n and all predecessors produce optimally given Q_n. Consider the stages belonging to the first level, L_1. If $n \in L_1$, then $h_n = H_n$. Substituting $Q^*_{a(n)}$ for R in Proposition 2,

$$Z_n(Q_n) = S_n/Q_n + h_n(Q_n - 1)/2 + H_n Q^*_{a(n)}(q_{a(n)}-1)/q_{a(n)}.$$

Lemma 2 applies implying k_n is a positive integer.

Now suppose k_i is integer for all stages F_i, $i \in L_1 \quad L_2 \quad ,\ldots, \quad L_{j-1}$.

Let $n \in L_j$. Then the total cost associated with the choice of lot size Q_n is evidently

$$Z_n(Q_n) = DS_n/Q_n + h_n(Q_n - 1)/2 + Q^*_{a(n)}H_n(1 - q_{a(n)})/q_{a(n)}$$

$$+ \Sigma_{i\varepsilon b(n)} Z_i(k_iQ_n).$$

Noting that $(1 - q_{a(i)})/q_{a(i)} = 0$ if k_i is integral, then for each predecessor $i \in b(n)$,

$$Z_i(k_iQ_n) = DS_i/k_iQ_n + h_i(k_iQ_n - 1)/2 + \Sigma_{\ell\varepsilon b(i)}Z_\ell(k_\ell k_i Q_n)$$

thus, $Z_n(Q_n) = D/Q_n \cdot \Sigma_{i\varepsilon B(n)}S_i + Q_n/2\ \Sigma_{i\varepsilon B(n)}h_i + H_nQ^*_{a(n)}\ (1 - q_{a(n)})/q_{a(n)}$.

Since, by definition, $H_n = \Sigma_{i\varepsilon B(n)}\ h_i$, Lemma 2 applies directly and k_n must be an integer. The induction argument proves the theorem for all stages including the final stage if $Q^*_{a(N)}$ is taken to be D.

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