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TIME-DOMAIN SYNTHESIS BY MEANS OF TRIGONOMETRIC POLYNOMIAL APPROXIMATIONS

Michael Strieby

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Abstract

The area of time-domain synthesis considered here is the process of finding the system function of a lumped-element, linear, passive, bilateral network whose impulse response approximates a prescribed function of time.

Time-domain synthesis can be regarded as essentially equivalent to the design of a rotational delay-line approximant. Suppose e^{-sT*} to be such an approximant. $H_p(s)$ is the Laplace transform of a suitably chosen semiperiodic (periodic for t greater than zero, and zero for t less than zero) trigonometric polynominal approximation of period T to the desired network impulse response that is also of duration T. Then the function $H_p(s)(1 - e^{-sT*})$ is the desired rational system function.

The first method of synthesis bases the choice of a rational delay-line approximant upon the particular impulse response that is being synthesized. Care must be taken in the selection to insure realizability of the resultant network and proper convergence of the impulse response. However, the method is straightforward and gives good results: a fairly accurate error prediction can easily be made, so that no trial-and-error procedure is needed. Several examples show that a close approximation to the desired impulse response is attainable with relative economy of network elements.

The second of the many possible procedures based on the same underlying philosophy uses a rational delay-line approximant which is independent of the function that is being synthesized. One example seems to show that this approach also produces satisfactory results.

I. WHAT IS TIME-DOMAIN SYNTHESIS?

Let us start with a formal definition. Time-domain synthesis is the design of lumped-element, linear, passive, bilateral, electric networks, with the network design criterion that the response to a given excitation approximates a prescribed function of time. The word "approximates" is important here. In the usual case, we have to be content with an approximation to the desired response; and the problem of time-domain synthesis can be correctly described as an approximation problem. The reader will notice that, by implication, we are assuming that only one excitation and one response pertain to a given network. We are, therefore, limiting our definition to one- or two-terminal-pair configurations. We shall similarly limit our discussion.

The words "time domain" in the definition have a very specific meaning. They imply, of course, that we are more interested in the input and output time functions associated with a network than we are in other properties. But we can be more specific: we shall gauge the success or failure of the synthesis, in the time domain. Denoting by the word "error," the difference between the desired response and the actual network response, both as functions of time, we shall assess the error. A small error denotes a good synthesis, a large error denotes a less satisfactory one.

We do not mean to imply by our definition that we exclude the possibility of using functions of frequency in the synthesis process. These functions usually enter the process at one stage or another; indeed almost the entire synthesis can be carried out in the frequency domain. The name time domain refers to our objectives, not to our methods.

The synthesis process is often considered in three parts. In the first part, we start with an excitation function and a response function. The problem is to find the impulse response that corresponds to this excitation-response pair. In particular, given the excitation function, $f_e(t)$, and the desired response function, $f_r(t)$, a solution, $f_d(t)$, to the equation,

$$f_e(t) * f_d(t) = f_r(t) \tag{1}$$

is to be found, where the star denotes convolution. In practical situations, of course, the solution to Eq. 1 is apt to be an approximate one. The second part consists in obtaining a realizable network system function whose inverse transform approximates the desired impulse response. (By a realizable system function we mean a quotient of finite polynomials in which the denominator is a Hurwitz polynomial having at least the same degree as the numerator.) In the third part, a network realization of the system function is found.

In this report we shall consider only the second of the three steps outlined. We refer the reader who is interested in part 1 to references (1) and (2). Many treatments of part 3 have been published; for a particularly interesting discussion, see reference (3).

II. THE RELATION OF DELAY LINES TO TRANSIENT SYNTHESIS

There is a close connection, either explicit or implied, between delay line concepts and time-domain synthesis. We shall show how, by use of a delay line approximant, a system function that has as its inverse Laplace transform an arbitrary impulse response can be approximated by Fourier methods. We shall also provide the conceptual basis for the next section, which deals with actual methods of finding this system function.

It may, at first sight, seem that, in considering delay lines as a means to achieve time-domain synthesis, we are proposing to approach the study of a problem that is merely difficult by first attacking one that is impossible. Thus it is evident that any rational function can only approximate a delay line in a limited sense, because the phase of a delay line increases without limit as the applied frequency increases. However, let us pursue this apparently illogical course a little longer, for it will appear that the delay line is acceptable, even if it is far from perfect; and in any case, by considering its shortcomings we shall achieve a better insight into the mechanism of the synthesis process.

We are concerned with both functions of "s," which are system functions, and with their inverse Laplace transforms, which are functions of time. We shall not, for the moment, be concerned with networks corresponding to these system functions, or even with their potential realizability (these subjects are covered in Section 3). It may, nonetheless, be helpful in visualizing what is going on, to think of the system functions as though they had network realizations; we resort to this device when it seems expedient. We use the fact that the cascading of two networks corresponds to multiplication of their system functions, while the impulse response of the combination is the convolution of the inverse Laplace transforms of the system functions.

2.1 DERIVATION OF A SYSTEM FUNCTION

Let us denote by $f_d(t)$ the desired function, that is, the ideal impulse response that is to be approached by the final network. This function is, of course, zero for $t < 0$, and we suppose it to be limited in duration to T: $f_d(t) = 0$ for $t > T$. The restriction implied by this statement is practically negligible, as it will appear subsequently. Denote by $f_{dp}(t)$, a function constructed by repeating $f_d(t)$ periodically; for the present purpose, the period is conveniently chosen equal to T; it must be at least T, but could equally well be greater.

$$f_{dp}(t) = f_d(t) + f_d(t-T) + f_d(t+T) + f_d(t-2T) + f_d(t+2T) + \dots$$

If $f_{dp}(t)$ is now approximated by a finite trigonometric series, $f_t(t)$,

$$f_{dp}(t) \approx f_t(t)$$

$$f_t(t) = \sum_{k=-n}^{n} a_k e^{jk\omega t} \qquad \omega = 2\pi/T$$

where complex notation has been used for convenience in the subsequent manipulations.

Let

$$h_p(t) = f_t(t) \qquad t > 0$$
$$= 0 \qquad t \leq 0$$

$$H_p(s) = L[h_p(t)]$$
$$= \sum_{k=-n}^{n} \frac{a_k}{s - jk\omega}$$

Examples of $f_d(t)$ and $h_p(t)$ are shown in Fig. 1a and b.

Thus $h_p(t)$ is a repetition, for positive values of t, of a trigonometric approximation to $f_d(t)$, and is evidently Laplace transformable because it is zero for $t < 0$. For the same reason, it is not strictly periodic. Nonetheless, the notation was chosen because it was suggestive of the repetitive character of the function; the subscript p will be used consistently to denote either a periodic function or a periodic function which has been modified by the requirement that it be zero for negative values of time.

Evidently, H_p has only j-axis poles. A network realization would therefore be lossless, and its impulse response, $h_p(t)$, would never die out, which accords with the definition of $h_p(t)$ as an infinite train of pulses.

The next step is to obtain an impulse response that is one pulse alone, rather than the unending function $h_p(t)$. To this end, $H_p(s)$ is multiplied by the function G(s), defined as follows:

$$G(s) = 1 - e^{-sT}$$

Here we encounter the delay line mentioned above; the function e^{-sT} corresponds in the time domain to a perfect delay of T seconds. For the moment, we merely use this function without being concerned about the question of how it might be realized.

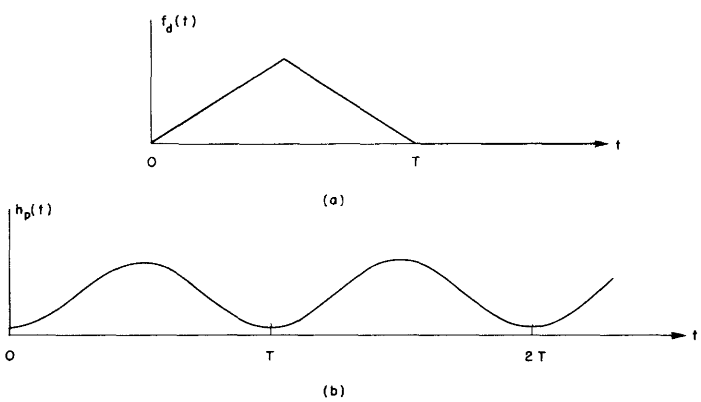


Fig. 1

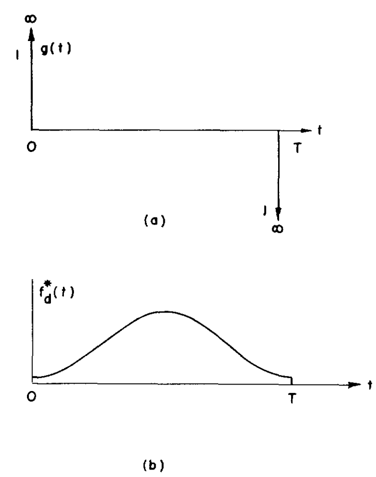


Fig. 2

The inverse transform of G(s) is:

$$g(t) = u_o(t) - u_o(t - T)$$

where the notation $u_o(t)$ denotes the impulse at t = 0; g(t) is shown in Fig. 2a.

Consider now the system function $H_p(s) \cdot G(s)$, and its inverse transform. The latter can be evaluated by convolving $h_p(t)$ with g(t); hence it is evidently just equal to one of the pulses whose repetition makes up $h_p(t)$. Since each of these pulses approximates $f_d(t)$, we have achieved an approximation to the desired impulse response. This we denote $f_d^*(t)$, using a star to indicate approximation.

$$f_d^*(t) = L^{-1}[F_d^*(s)] \tag{2}$$

$$F_d^*(s) = H_p(s) \cdot G(s) \tag{3}$$

$f_d^*(t)$ is shown in Fig. 2b.

It is both instructive and useful to obtain an alternative derivation of the results of the last paragraph, by making use of the frequency domain. From this point of view, $H_p(s)$ represents a lossless network which rings when excited. G(s) can be represented by a device that has two channels; one channel passes the input without distortion, while the other delays all important frequencies a time equal to one period of the fundamental. The response corresponding to G(s) is the difference of the outputs of the two channels. Suppose, now, that $H_p(s)$ and G(s) are cascaded and an impulse is applied; then, the response is evidently caused by $H_p(s)$ alone, for a single period, T; but, for $t > T$, there is no response at all. In other words, the impulse response is $f_d^*(t)$. Thus, consideration of the pertinent function of frequency leads, as it must, to an interpretation entirely consistent with that already obtained by using the convolution viewpoint.

2.2 PRACTICAL CONSIDERATIONS

So far, we have made no attempt to discuss realizability of the functions of "s" with which we have been concerned. Nonetheless, it is not difficult to appreciate from physical considerations that a time-domain synthesis method based on the foregoing discussion is feasible. A rigorous treatment will be given below and specific methods will be introduced. Here,we appropriately present some plausibility arguments to help motivate the subsequent material.

In the derivation of $f_d^*(t)$ from the frequency functions, $H_p(s)$ and G(s), we observed that the role of the term e^{-sT}, in the expression $G(s) = 1 - e^{-sT}$, is that of delaying and inverting all important frequency components of $H_p(s)$. The key word here is "important." Evidently, for any practical $f_d(t)$, $|H_p(j\omega)|$ becomes small as ω increases indefinitely; in other words, the periodic extension of the desired function has a limited spectrum. Evidently, the behavior of G(s) need be controlled only over the same limited spectrum; hence it is reasonable to assume that a satisfactory rational approximation to G(s) can be made.

We do not consider the details of this approximation now, but some of the considerations involved may be mentioned. Since e^{-sT} is the troublesome quantity in the expression for G(s), we confine our attention to this term, denoting by e^{-sT*} a rational approximant to e^{-sT}.

In the first place, we cannot expect the relation

$$e^{-sT*} \approx e^{-sT} \qquad s = j\omega \tag{4}$$

to be valid for all ω values at which $H(j\omega)$ is non-zero; rather, the objective is to satisfy this relation well for those values of ω for which $|H_p(j\omega)|$ is largest. In the second place, $|H_p(j\omega)|$ is certainly negligibly small for all values of $|\omega|$ larger than some constant ω_0. Hence the approximation 4 need be valid only over a finite frequency band; accordingly, we expect that a finite network realization of e^{-sT*} can be found.

An error will occur in $f_d^*(t)$, as a result of the inaccuracy of relation 4, and it is collaterally interesting to notice that this could have been predicted from the nature of finite networks. The function $f_d(t)$, is of limited duration, as is one cycle of its trigonometric approximation. If $g^*(t)$ were the idealized function shown in Fig. 3a, which is also dead for $t > T$, then $f_d^*(t)$ would likewise possess this property. However, $f_d^*(t)$ cannot be of limited duration, because it is the impulse response of a finite lumped-element network and must therefore be equal to a finite sum of damped sinusoids. Hence we conclude that $f_d^*(t)$ is not identically zero for large t, the discrepancy being an error, and that the error must arise because $g^*(t)$ is not ideal. Thus the approximate nature of the delay line causes an imperfect $g^*(t)$, which, in turn, produces an infinite tail on the impulse response of the final network. Our approach produces an error of a kind that is unavoidable in this problem.

With regard to the question of choosing the rational delay-line approximant, no simple criterion for making a choice is given; but certain frequency functions are more convenient than others. The reader may have noticed, and worried about, the j-axis poles that seemed to occur in the final system function $G(s) \cdot H_p(s)$ as a result of their presence in the latter factor of this product. If they are not cancelled by corresponding zeros in G(s), they will cause undamped, hence undesirable, terms in the network impulse response $f_d^*(t)$. It is therefore expedient to choose e^{-sT*} so that the zeros of G(s) coincide with the poles of $H_p(s)$; this choice corresponds to placing the zeros of $1 - e^{-sT*}$ at the 2n zeros of $1 - e^{-sT}$ that are nearest the origin and is, therefore, not the cause of an unreasonable restriction. It is not essential to choose the zeros of $1 - e^{-sT*}$ in this manner because they will in any case almost cancel the poles of $H_p(s)$. This choice is computationally convenient, however. (See Section 3.3 and Fig. 21.)

III. DERIVATION AND APPLICATIONS OF A SYNTHESIS PROCEDURE

The notation used in this section is, in general, consistent with that introduced in Section II. We have, however, extended the definition of the function G(s) to include the possibility of using a rational delay-line approximant. Thus we make use of the relation,

$$G(s) = 1 - e^{-s\tau^*}$$

Hereafter, we shall frequently encounter functions that are periodic in the range $t > 0$, but are defined to be zero for $t < 0$. The function, $h_p(t)$, introduced previously, is an example of a function of this character. We refer to these functions as "semiperiodic." In addition, it will often be convenient to speak of the "frequencies" and the "periods" of semiperiodic functions. These terms are useful, and their meanings should be clear, but we should, strictly speaking, apply them only to truly periodic functions.

3.1 A RATIONAL DELAY LINE APPROXIMANT

In this section we accomplish the first step in deriving $F_d^*(s)$, the selection of a suitable rational delay-line approximant. We derive an appropriate expression, starting with the desired network impulse response, $f_d(t)$. This function is now supposed to have a duration of $\tau/2$ seconds and to be repeated over and over, for both positive and negative time, with a period τ. Referring as before to the periodic extension as $f_{dp}(t)$, we have

$$f_{dp}(t) = f_d(t) + f_d(t-\tau) + f_d(t+\tau) + f_d(t-2\tau) + f_d(t+2\tau) + \dots$$

It is important to notice that $f_{dp}(t)$ is not made up merely by repeating the desired function, but by repeating the combination of $f_d(t)$ followed by a dead space, the whole having a period τ, of which the duration of $f_d(t)$ is half and the duration of the dead space is likewise half. Thus our present $f_{dp}(t)$ differs from that of the last chapter in that the fundamental period is now twice the duration of $f_d(t)$, instead of equal to it, this change being necessary in order to achieve the present purpose of deriving a delay-line approximant. To avoid confusion, we have introduced the new notation, τ, for the fundamental period; the duration of $f_d(t)$ is, accordingly, $\tau/2$.

As before, the function $f_{dp}(t)$ is approximated by a trigonometric polynomial denoted $f_t(t)$, and a corresponding function, dead for $t \leq 0$, is defined:

$$\left.\begin{aligned} f_t(t) &= \sum_{k=-n}^{n} \alpha_k e^{jk\omega t} && \omega = 2\pi/\tau \\ f_p(t) &= f_t(t) && t > 0 \\ &= 0 && t \leq 0 \end{aligned}\right\} \qquad (5)$$

Let

$$f_1(t) = \sum_{\substack{k=-n \\ k \text{ odd}}}^{n} \alpha_k e^{jk\omega t} \qquad t > 0$$
$$= 0 \qquad t \leq 0 \qquad (6)$$

$$f_2(t) = \sum_{\substack{k=-n \\ k \text{ odd}}}^{n} \alpha_k e^{jk\omega t} \qquad t > 0$$
$$= 0 \qquad t \leq 0 \qquad (7)$$

It is interesting to examine some of the properties of these semiperiodic functions, examples of which are shown in Figs. 3 and 4. Since it contains only even harmonics, $f_2(t)$ has a "frequency" twice that of the fundamental. Formally,

$$f_2(t) = f_2(t + \tau/2) \qquad t > 0 \qquad (8)$$

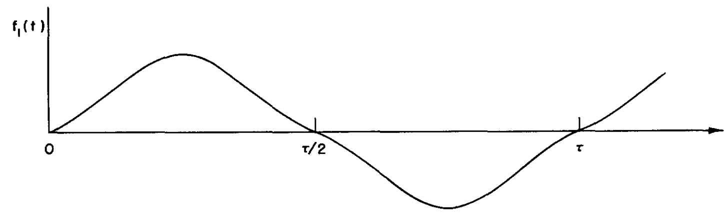


Fig. 3

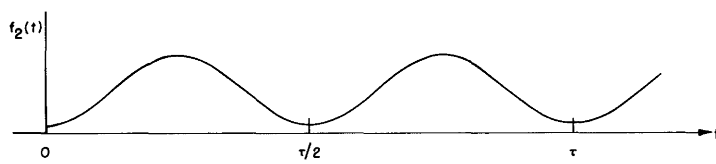


Fig. 4

Furthermore, any half-period of $f_2(t)$ is the Fourier approximation to $(1/2) \cdot f_d(t)$. This assertion follows from a consideration of the functions $f_d(t)$ and $(1/2)[f_d(t) + f_d(t - \tau/2)]$, together with their Fourier approximations. The fundamental period of each approximation is taken to be τ. Evidently, the coefficients of the even harmonic terms are the same for one approximation as they are for the other. Since the Fourier approximation to $(1/2)[f_d(t) + f_d(t - \tau/2)]$ contains only even-order terms, the sum of the even-order terms in the approximation to $f_d(t)$ has the period $\tau/2$, and the stated property follows.

Any half-period of $f_1(t)$ also approximates $(1/2)f_d(t)$ in the Fourier sense, but it has the opposite sign in alternate half-periods starting with the second. This property is deduced from Eq. 7 by reasoning entirely similar to that of the preceding paragraph. Since $f_1(t)$ contains only odd harmonics, the relation

$$f_1(t) = -f_1(t + \tau/2) \qquad t > 0 \qquad (9)$$

holds.

We are now in a position to consider two quantities which are the keys to our subsequent derivation: the sum and the difference of $f_2(t)$ and $f_1(t)$. (See Fig. 5.) By definition, $[f_2(t) + f_1(t)]$ is equal to $f_p(t)$, and its properties are, therefore, familiar (see Eq. 5). It is collaterally interesting, however, to elaborate upon these properties through use of those pertinent to the components $f_1(t)$ and $f_2(t)$ as described above. For values of t in the range from 0 to $\tau/2$, $f_1(t)$ and $f_2(t)$ are both Fourier approximations to

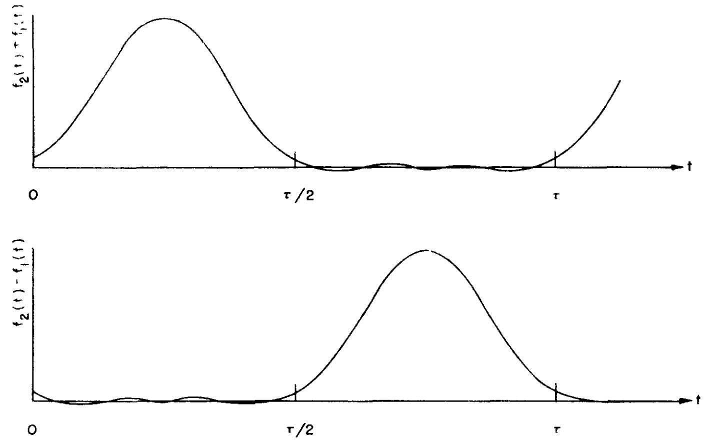


Fig. 5

$(1/2)f_d(t)$; accordingly, their sum is a Fourier approximation to $f_d(t)$ in this range. This conclusion is in accord with the definition in Eq. 5. However, for t-values greater than $\tau/2$ but less than τ, $f_2(t)$ and $f_1(t)$ are Fourier approximations to $(1/2) \cdot f_d(t - \tau/2)$ and $-(1/2) \cdot f_d(t - \tau/2)$, respectively. Hence, $[f_2(t) + f_1(t)]$ is a Fourier approximation to zero for these values of t, which again checks with our previous result.

The most important property of the quantity $[f_2(t) - f_1(t)]$ is derived by taking the difference of Eqs. 8 and 9. We obtain, using the semiperiodicity of $f_1(t) + f_2(t)$,

$$f_2(t) - f_1(t) = f_2(t - \tau/2) + f_1(t - \tau/2) \qquad t > \tau/2 \tag{10}$$

which expresses the fact that $[f_2(t) - f_1(t)]$ is essentially a delayed version of $[f_2(t) + f_1(t)]$. The equation holds for $t > \tau/2$, and evidently holds also for $t \leq 0$, since both sides are zero for this range of t. It does not hold for intermediate t-values in the range $0 < t \leq \tau/2$, since we have

$$f_2(t - \tau/2) + f_1(t - \tau/2) = 0 \qquad 0 < t \leq \tau/2 \tag{11}$$

although

$$f_2(t) - f_1(t) = f_2(t + \tau/2) + f_1(t + \tau/2) \qquad 0 < t \tag{12}$$

where use is made of Eqs. 8 and 9 in deriving Eq. 12.

Let us define, for convenience,

$$\begin{aligned} e(t) &= f_2(t) - f_1(t) && 0 < t \leq \tau/2 \\ &= 0 && \text{otherwise} \end{aligned}$$

which gives

$$f_2(t) - f_1(t) = f_2(t - \tau/2) + f_1(t - \tau/2) + e(t) \tag{13}$$

This equation is valid for all values of t, according to the argument of the preceding paragraph. Let us examine the role played by the function e(t) in Eq. 13. From Eq. 12 and the definition of e(t), it follows that

$$\begin{aligned} e(t) &= f_2(t + \tau/2) + f_1(t + \tau/2) && 0 < t \leq \tau/2 \\ &= 0 && \text{otherwise} \end{aligned}$$

Now, if we use

$$f_p(t) = f_1(t) + f_2(t)$$

and recall that $f_p(t)$ is a Fourier approximation to zero in the range $\tau/2 < t \leq \tau$, we find that e(t) is small. Furthermore, as the degree of approximation implied by Eqs. 6 and 7 is increased, e(t) becomes still smaller; the trigonometric polynomial may be so chosen that the absolute value of e(t) is less than an arbitrary positive epsilon for all t. This result can be achieved by choosing a trigonometric approximation which converges uniformly, as does the Féjèr series, for instance. It is apt to be more expedient to use Fourier series, however, even at the expense of mathematical neatness. Hence, it is justifiable to write, for all t,

$$f_2(t) - f_1(t) \approx f_2(t-\tau/2) + f_1(t-\tau/2) \tag{14}$$

which is the basic equation leading to the delay line design. It will be necessary to keep in mind the approximation involved in relation 10, but we shall, for the moment, proceed to use this relation without worrying about the error that may result.

It is convenient to use the frequency-domain equivalent of relation 14,

$$[F_2(s) + F_1(s)]e^{-s\tau/2} \sim F_2(s) - F_1(s)$$

or

$$[F_2(s) + F_1(s)]e^{-s\tau/2*} = F_2(s) - F_1(s)$$

where a star, as usual, denotes an approximant. Formal manipulation immediately gives:

$$e^{-s\tau/2*} = \frac{F_2(s) - F_1(s)}{F_2(s) + F_1(s)} \tag{15}$$

$$e^{-s\tau*} = \left[\frac{F_2(s) - F_1(s)}{F_2(s) + F_1(s)}\right]^2 \tag{16}$$

The function on the right is rational, since $F_1(s)$ and $F_2(s)$ are rational; also, it approximates $e^{-s\tau}$. It is the desired delay-line approximant.

3.2 DERIVING THE SYSTEM FUNCTION

It is only necessary to follow the procedure outlined in the previous section to obtain a system function whose inverse Laplace transform approximates the desired network impulse response. In accordance with the discussion there, but using τ in place of T, and with the aid of Eq. 10, we obtain

$$\begin{aligned} G(s) &= 1 - e^{-s\tau*} \\ &= 1 - \left[\frac{F_2(s) - F_1(s)}{F_2(s) + F_1(s)}\right]^2 \\ &= \frac{4F_1(s) \cdot F_2(s)}{[F_1(s) + F_2(s)]^2} \end{aligned}$$

$$F_d^*(s) = F_p(s) \cdot G(s)$$

$$= [F_1(s) + F_2(s)] \cdot \frac{4F_1(s) \cdot F_2(s)}{[F_1(s) + F_2(s)]^2}$$

$$F_d^*(s) = \frac{4F_1(s) \cdot F_2(s)}{F_1(s) + F_2(s)} \qquad (17)$$

The last expression is the desired system function. Its inverse transform is the solution to our problem of finding an impulse response that approximates the desired function, $f_d(t)$. Formally,

$$f_d(t) \approx f_d^*(t) = L^{-1}[F_d^*(s)]$$

It may be mentioned that, if the use of Eq. 17 were entirely straightforward, our problem would be solved. There would be no reason to go further, except, perhaps, to consider some applications. However, as with so many things in life, there are problems. These problems will be discussed and resolved below. The reader should be aware that Eq. 17 is a fundamental result, but it will require some later interpretation and modification. Equation 17 was first derived by E. A. Guillemin (4) who used a method quite different from that given above. This work gave rise to subsequent further investigations (9) and (10) leading to a better understanding of the principles involved in this kind of an approach.

There is one feature of the derivation that may require some explanation, namely, the fact that the basic building block whose repetition makes up $f_p(t)$ is not an approximation to $f_d(t)$ alone; instead it approximates both $f_d(t)$ and a dead space of equal duration which follows it. This particular definition of $f_p(t)$ arose in the preceding section and was essential there, but for present purposes it may seem to be unnecessarily complicated. A more logical procedure would, perhaps, be to use an $f_p(t)$ that satisfies the relation:

$$f_p(t) \approx f_d(t) + f_d(t - \tau/2) + f_d(t - \tau) + \dots$$

We should, of course, make a corresponding change in the delay-line approximant to accommodate the new period $\tau/2$, instead of τ. Let us explore the consequences of this minor change of approach.

It is necessary, first, to obtain the modified version of $f_p(t)$, i.e., a semiperiodic function, any cycle of which approximates $f_d(t)$. This requirement is fulfilled, except for a multiplying constant, by $f_2(t)$, as can be seen by referring to the definition and subsequent discussion about this function. The multiplying constant is 2. Since the period of $2f_2(t)$ is $\tau/2$, the modified G(s) is

$$G(s) = 1 - e^{-s\tau/2*}$$

and the use of Eq. 15 yields:

$$G(s) = 1 - \left[\frac{F_2(s) - F_1(s)}{F_2(s) + F_1(s)}\right]$$

$$= \frac{2F_1(s)}{F_1(s) + F_2(s)}$$

Now, substitution in

$$F_d^*(s) = F_p(s) \cdot G(s)$$

gives

$$F_d^*(s) = \left[\frac{4F_1(s) \cdot F_2(s)}{F_1(s) + F_2(s)}\right]$$

which is the same as the expression derived originally.

We are now in a position to answer the question about the logic of the original method of derivation. We see that the approach of Section 3.1 is feasible, and so is that in the foregoing section; both give identical results. We have what amounts to two ways of looking at the same thing; neither is inherently more correct than the other. There are additional ways of deriving Eq. 17, but it would be an unnecessary digression to detail these various approaches here. The interested reader can find an alternative method treated in reference (4).

3.3 REALIZABILITY

In the derivation of Eq. 17, which gives the desired network system function, we were not concerned with any question of realizability. But our reasoning started with a more or less arbitrary time function, $f_d(t)$, and there is no assurance that every $f_d(t)$ that may be selected for approximation will lead to a system function without right-half-plane poles. In addition, for any particular choice of $f_d(t)$, any one of numerous degrees of approximation might be used – that is, in Eq. 5 any integer value of n is possible – and any of these approximations might or might not lead to a realizable result in Eq. 17. Accordingly, we shall discuss the question of determining the realizability of the system function when a given $f_d(t)$ is to be approximated.

$F_d^*(s)$ can be thought of as a product of two other functions of "s", $F_p(s)$ (or $H_p(s)$) and G(s), as shown in Eq. 3. It will be recalled that $F_p(s)$ possesses only j-axis poles and that these poles are in any case cancelled by zeros of G(s). Hence, any right-half-plane poles in $F_p(s)$ arise from G(s). The formula for G(s) is

$$G(s) = 1 - e^{-s\tau^*} \tag{18}$$

therefore any right-half-plane poles of G(s) are possessed by $e^{-s\tau^*}$ and also by $e^{-s\tau/2^*}$ (see Eqs. 15 and 16). Our problem is to find out why right-half-plane poles occur in $e^{-s\tau/2^*}$.

It is illuminating, in this connection, to consider the inverse transform of $e^{-s\tau/2*}$. Define $u_0^*(t - \tau/2)$ by

$$u_0^*(t - \tau/2) = L^{-1}[e^{-s\tau/2*}] \tag{19}$$

Now Eq. 14 can be written as

$$f_2(t) - f_1(t) = u_0^*(t - \tau/2) * f_2(t) + f_1(t) \tag{20}$$

by using Eqs. 15 and 19. The function $f_2(t) - f_1(t)$ is approximately a delayed version of $f_2(t) + f_1(t)$, as we see from Eq. 13. Hence we conclude that $u_0^*(t - \tau/2)$ is approximately a unit impulse occurring at $t = \tau/2$ seconds. (In this conclusion, there are some assumptions about the nature of the function $f_p(t) = f_2(t) + f_1(t)$. It is not obvious just what mathematical restrictions must be placed on $f_p(t)$, but it seems clear that they are satisfied by any function which is acceptable on physical grounds.) The word "approximately" refers, of course, to the convolution properties of $u_0^*(t-\tau/2)$, and does not imply that the difference between the two functions is small for all values of t.

It is important to consider the range of values of the independent variable, t, in the convolution product of Eq. 20. We mean, by "small" values of t, those values in the range from zero to several times the fundamental period, τ, this statement being intentionally imprecise. We want mainly to exclude large asymptotic values of time, and to include some values of t larger than $\tau/2$. With the aid of this definition, we can state specifically that $u_0^*(t - \tau/2)$ approximates $u_0(t - \tau/2)$ for "small" values of t. Furthermore, as the approximation in Eq. 5 is refined, $u_0^*(t - \tau/2)$ becomes a better and better approximation to $u_0(t - \tau/2)$, although it is still only for "small" values of t.

It is worth digressing to note that we have not stated any criterion of approximation. The degree of approximation needed in Eq. 5, that is, the value of "n" which must be selected, depends on several factors, of which the most important are the nature of $f_d(t)$, and the quality of approximation to $f_d(t)$ that is demanded. In a general treatment like this, with these factors left arbitrary, we have to be satisfied with what is essentially a plausibility argument, and rest our final proof on an empirical test. Basically, the reason for this attitude lies in the complexity of solutions to convolution equations. Some latitude must, therefore, be left in the choice of n, by any treatment that is not restricted to one particular function or class of functions.

The statement that $u_0^*(t - \tau/2)$ is approximately equal to $u_0(t - \tau/2)$ for small values of t has important consequences. From it we conclude that our delay-line approximation is valid for "small" delays; and from this fact and the discussion in Section 2 it follows that $f_d^*(t)$ approximates $f_d(t)$ for small t. Let us consider the components of $f_d^*(t)$. At small values of t, the decaying sinusoids are evidently most important, although, for large values of t, any growing sinusoids contained in $f_d^*(t)$ predominate. Thus, clearly, the part of $f_d^*(t)$ attributable to the decaying sinusoids is the one of interest; this part approximates $f_d(t)$ for small t and dies out for large t. We shall define a new notation, $f_d^{**}(t)$, for these decaying sinusoidal components of $f_d^*(t)$. The growing exponentials contribute little to $f_d^*(t)$ for small t and are undesirable at large values of t because they blow up. Consequently, their removal has little effect on $f_d^*(t)$ for values of t of the order of magnitude of $\tau/2$ or less, but it improves the behavior of $f_d^*(t)$ for large t. At the same time, the removal of these terms guarantees the realizability of the transform of what is left, that is, of $L[f_d^{**}(t)]$. Thus, the solution to the realizability problem is merely to throw away troublesome terms. Let us see how this can be done.

Since $f_d^*(t)$ is a finite sum of exponentially changing sinusoids, the removal of certain terms is effected by subtracting these terms from the sum. The corresponding process in the frequency domain is equally simple. $F_d^*(s)$ is expanded in partial fractions and the terms involving right-half-plane poles are dropped, which gives the result $F_d^{**}(s)$. The justification for this procedure – that the residues of $F_d^*(s)$ in its right-half-plane poles are small relative to the residues in the left-half-plane poles – is the analog of, and follows from, our conclusion about $f_d^*(t)$, that the exponentially growing terms in the function are relatively small for small t. It is worth noticing that obtaining $F_d^{**}(s)$ from $F_d^*(s)$ in this way does not involve any great computational labor, since $F_d^*(s)$ must, in any case, be expanded in partial fractions in order to compute $f_d^*(t)$.

Another method of eliminating right-half-plane poles from $F_d^*(s)$ is the association of each pole with the zero of $F_d^*(s)$ that is nearest to it in the s-plane. All such pole-zero pairs are then simply removed from $F_d^*(s)$ as though they were perfectly self-cancelling. This procedure is based on the assumption that a pole and a zero which are cancelled as a pair are reasonably close together in the s-plane. This condition is likely to occur, because the residues in any right-half-plane poles are small: a small residue in the pole of a rational function usually means that a zero is nearby. Hence cancellation of poles and zeros in the way described is not illogical. It has, nonetheless, the disadvantage that we do not know physically what happens to the time function during the cancellation process. Owing to this drawback, we have ordinarily used the method given first. However, an example is shown (Fig. 21) in which the cancellation scheme is used. It is interesting to notice, in that case, that the two methods give results which are comparable in quality.

Our discussion of realizability is now complete, and we shall use the results henceforth. In every case involving the use of an $F_d^*(s)$ computed from Eq. 17, we shall tacitly assume that the right-half-plane poles have been removed. The method of removal, except when we note otherwise, is by subtraction of the terms involving these poles from the partial fraction expansion of $F_d^*(s)$. Since we shall always be dealing with an $F_d^*(s)$ which has been thus modified, that is, with $F_d^{**}(s)$, no confusion will occur if we drop the second star from the notation for $F_d^{**}(s)$. Accordingly, $F_d^*(s)$

denotes henceforth a system function that has been modified, if necessary, to make it realizable.

3.4 ERROR PREDICTION

Error, in synthesis, is the difference between the desired result and the result actually obtained. In our own case the error is

$$f_d(t) - f_d^*(t)$$

The sine qua non of a synthesis procedure is that the error be easily predictable. This is particularly true of time-domain synthesis for the following reason: It is obviously vital in any synthesis problem to know something about the error. Yet exact computation of it is usually a tedious process in time-domain synthesis: for the method described in this chapter, an error computation involves finding and inverse-transforming $F_d^*(s)$ – no easy job. If the computation reveals that the error is unacceptably large, then the whole process must be repeated, perhaps many times. It is evident that, if a method is to be practical, the error must be easily predicted.

It is important to state that we do not expect our error prediction to be exact. If it were exact, it would be an error computation. We expect it to be good enough for practical use, and easy to determine. This is the basis of our present discussion of error.

It is convenient to obtain a new expression for $f_d^*(t)$. To this end, recollect Eq. 13. It can be rewritten as

$$e(t) = f_2(t) - f_1(t) - f_2(t-\tau/2) - f_1(t-\tau/2) \tag{21}$$

If we define E(s) as the Laplace transform of e(t), we can write the transform of Eq. 21.

$$E(s) = F_2(s) - F_1(s) - [F_2(s) + F_1(s)]e^{-s\tau/2} \tag{22}$$

From Eq. 22 there follow:

$$2E(s)e^{-s\tau/2} = 2[F_2(s) - F_1(s)]e^{-s\tau/2} - 2[F_2(s) + F_1(s)]e^{-s\tau} \tag{23}$$

and

$$\frac{E^2(s)}{F_2(s) + F_1(s)} = \frac{[F_2(s) - F_1(s)]^2}{F_2(s) + F_1(s)} - 2[F_2(s) - F_1(s)]e^{-s\tau/2} + [F_2(s) + F_1(s)]e^{-s\tau} \tag{24}$$

If we add Eqs. 23 and 24, we obtain

$$\frac{E^2(s)}{F_2(s) + F_1(s)} + 2E(s)e^{-s\tau/2} = \frac{[F_2(s) - F_1(s)]^2}{F_2(s) + F_1(s)} - [F_2(s) + F_1(s)]e^{-s\tau} \tag{25}$$

This result, in turn, can be rewritten as

$$\frac{4F_2(s) \cdot F_1(s)}{F_2(s) + F_1(s)} = [F_2(s) + F_1(s)] \cdot [1 - e^{-s\tau}] - 2E(s)e^{-s\tau/2} - \frac{E^2(s)}{F_2(s) + F_1(s)} \tag{26}$$

An equation similar to Eq. 26 was first derived by Guillemin (4). It is useful to consider the inverse transform of this equation. By recalling that

$$\frac{4F_2(s) \cdot F_1(s)}{F_2(s) + F_1(s)} = F_d^*(s)$$

and

$$F_2(s) + F_1(s) = F_p(s)$$

we can, at the same time, simplify the result:

$$f_d^*(t) = f_p(t) * [u_o(t) - u_o(t-\tau)] - 2e(t-\tau/2) - e(t) * e(t) * L^{-1}\left[\frac{1}{f_p(s)}\right] \tag{27}$$

(In this equation, as in some others we write, a star has two possible meanings that can be distinguished by its position. A star written as a part of the notation for a function denotes an approximation, whereas a star between any two functions means convolution.)

It is now possible to write an expression for the error,

$$\text{Error} = f_d(t) - f_d^*(t)$$

$$\text{Error} = (f_d(t) - f_p(t) * [u_o(t) - u_o(t-\tau)]) + 2e(t-\tau/2) + e(t) * e(t) * L^{-1}\left[\frac{1}{F_p(s)}\right] \tag{28}$$

Equation 28 is a fundamental result which will be interpreted in the following paragraphs.

The effect of convolving the semiperiodic function $f_p(t)$ with $u_o(t) - u_o(t - \tau)$ is therefore to extract one period of $f_p(t)$. Accordingly, the time function in the first parenthesis on the right-hand side of Eq. 28 is the error in the Fourier approximation to $f_d(t)$. Of course the term in this parenthesis contributes nothing to the error for $t > \tau$.

The term $2e(t - \tau/2)$ is easily interpreted with the aid of an equation derived in Section 3.1.

$$\left.\begin{aligned} e(t) &= f_2(t+\tau/2) + f_1(t+\tau/2) && 0 < t \leq \tau/2 \\ &= 0 && \text{otherwise} \end{aligned}\right\} \tag{29}$$

Manipulation of this equation gives

$$\left.\begin{aligned} 2e(t-\tau/2) &= 2[f_2(t) + f_1(t)] && \tau/2 < t \leq \tau \\ &= 0 && \text{otherwise} \end{aligned}\right\} \tag{30}$$

Referring to the discussion in Section 3.1 for aid in interpreting Eq. 30, we see that the contribution of $2e(t - \tau/2)$ to Eq. 28 is twice the Fourier approximation to zero in the range $\tau/2 < t \leq \tau$, and is zero elsewhere.

The triple convolution product in Eq. 28 is harder to assess because the function $L^{-1}[1/F_p(s)]$ is not known. Furthermore, we do not wish to compute $L^{-1}[1/F_p(s)]$ accurately because the computation is too long. We might, of course, approximate this function, and we shall discuss presently the practicability of doing so. First, however, a simpler approach will be considered. The term $e(t) * e(t)$ occurs in the triple convolution product. Since $e(t)$ is small, $e(t) * e(t)$ is smaller still. Hence we expect $e(t) * e(t) * L^{-1}[1/F_p(s)]$ to be negligibly small, and it seems reasonable to drop this term entirely. We do so, and investigate what remains of Eq. 30.

One of the two remaining terms, $2e(t - \tau/2)$, is zero for $t < \tau/2$, that is, for the entire duration of the desired pulse, $f_d(t)$. For the range $0 < t \leq \tau/2$, which is of greatest interest, the term $f_d(t) - f_p(t) * [u_o(t) - u_o(t - \tau)]$, alone, approximates the error. Hence we write

$$\text{Error} \approx f_d(t) - f_p(t) * [u_o(t) - u_o(t - \tau)] \tag{31}$$

This relation is less accurate in the range $\tau/2 < t \leq \tau$ than if we had included the term $2e(t - \tau/2)$ on the right-hand side but it is also simpler. Since $2e(t - \tau/2)$ and $f_d(t) - f_p(t) * [u_o(t) - u_o(t - \tau)]$ are of the same order of magnitude (both being of the order of magnitude of the error in the Fourier approximation to $f_d(t)$), our expression is still in the right ball park, even though it omits one of these terms. We use Eq. 31 for error prediction.

The specific way in which Eq. 31 is useful is as follows. When a function $f_d(t)$ is to be approximated, we begin by making a trigonometric series approximation to the combination of this function followed by a dead space. (See Section 3.1.) The error predicted for the synthesis procedure is then just the error in the trigonometric approximation, in accordance with Eq. 31. The trigonometric approximation is adjusted, if necessary, by using methods which are more fully discussed in Section 6, until the predicted error is satisfactory. We then proceed with the synthesis.

It is interesting to see how this process works out in practice. We show, in Figs. 6, 7, and 8 comparisons of computed error with error predicted according to relation 31. The graphs of $f_d^*(t)$ to which these error curves refer are shown in Figs. 17, 22, and 23.

We should mention the possibility of refining our error prediction by means of approximate evaluation of the triple convolution product appearing in Eq. 28. The essential problem in such evaluation is that of obtaining an approximant to $L^{-1}[1/F_p(s)]$, or, in other words, of solving equation

$$f_p(t) * x(t) = u_o(t) \tag{32}$$

for $x(t)$. Practical methods for approximate solution of equations like Eq. 32 are described in reference 2; therefore this approach seems feasible. The application of these methods to our problem has not been tried, largely because of the good approximating properties and simplicity of Eq. 31, but they might prove useful.

3.5 USE OF PREDICTED ERROR

In this section, we discuss the means of applying Eq. 31. This equation states that the total error is approximated by the error with which one period of a

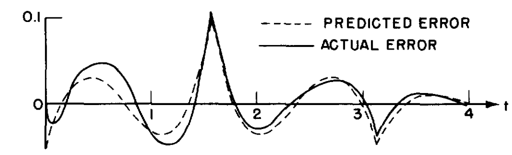


Fig. 6

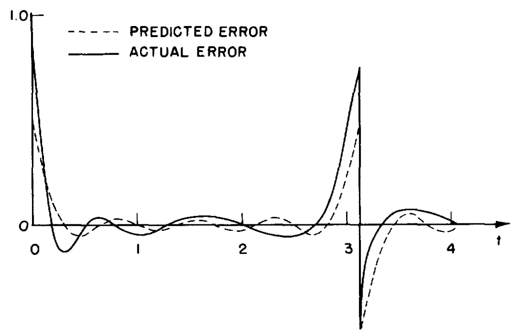


Fig. 7

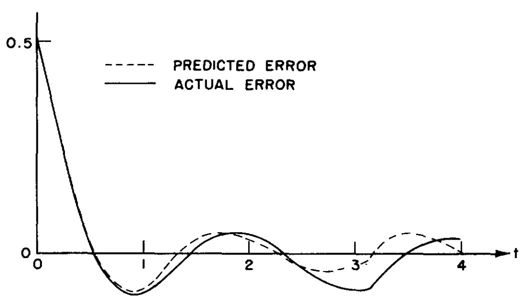


Fig. 8

trigonometric polynomial, $f_p(t)$, approximates the desired pulse. Accordingly, we shall touch on some ideas that are useful in deriving an acceptable trigonometric polynomial.

There are various series whose partial sums can be used as trigonometric approximating polynomials. Of these we shall discuss only the Fourier series, for two reasons. First, the Fourier series is easy to use, and the prospective network synthesizer is probably familiar with it. This familiarity is useful when cut-and-try enters into the selection of a series, as it sometimes does. Second, the Fourier series yields an approximation that is roughly equal-ripple in character, except at discontinuities, and this is generally desirable. It is true that at discontinuities of the approximated function, the Gibbs phenomenon causes a problem, but we shall mention presently how this can be handled.

It is helpful in thinking of the techniques we are discussing to have a specific example in mind, and we accordingly consider a function, $f_d(t)$, whose graph is a rectangle. This function is shown in Fig. 9. At the outset of the synthesis of this function it is evident that we cannot achieve an $f_d^*(t)$ which exactly equals $f_d(t)$, because of the discontinuity at $t = \tau/2$. We have to accept an approximating function whose slope is always finite, which has rounded "corners" and various other imperfections. To minimize the imperfections, we desire that the trigonometric polynomial be a very good approximation to $f_d(t)$. But this leads to a difficulty, since the better the approximation, the more poles there are in the system function; hence the more elements in the network realization that is ultimately obtained. Evidently, a compromise must be made in the choice of the trigonometric polynomial.

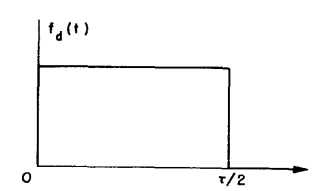


Fig. 9

The most obvious method of arriving at this compromise would be finding the Fourier series of the rectangular pulse, and computing several partial sums. By examining each partial sum in the light of the number of poles in the system function, we could select one that fits the requirements of the problem at hand. This method is straightforward and relatively simple. However, it can be varied in a way which is often useful.

We start with the observation made above, that we cannot expect perfection in our approximation. Accordingly, instead of using the Fourier partial sums derived from the rectangle function, we modify the function before finding its Fourier approximation. In effect, we choose a new $f_d(t)$ which is more realistic in that it is more easily approximated by a trigonometric polynomial.

A simple example of this process is shown in Fig. 10, in which the vertical sides of the rectangle have been replaced with sloping sides in order to make a symmetrical trapezoid. The dimension δ in this figure should be noticed. It is not a constant, but a parameter which can be varied; the Fourier coefficients relating to the modified $f_d(t)$ are functions of δ. We make a compromise, again, between the precision of approximation of the Fourier series and the allowable number of poles in $F_d^*(s)$. Now, however, there is a new parameter in the problem, δ, which can be used to advantage in any of several ways. 1. Perhaps a reasonable value of δ can be chosen by virtue of the requirements of the problem. That is, perhaps an acceptable slope of the sides of the rectangle can be decided upon in advance. If so, for any positive value of δ, the resulting Fourier series converges faster than the old one did. (See ref. 5 for the Fourier series for this and other modifications of the rectangle function.) Hence, fewer terms may be required in the partial sum. 2. Suppose that allowable network complexity limits the partial sum to a fixed number, n, of terms. In this case, the value of δ can be adjusted to suit the circumstances. There is a gain in flexibility over the original method, in which only a fixed, zero-value of δ, was considered. 3. The choice of a positive value of δ eliminates discontinuities from $f_d(t)$. Hence the Fourier series of $f_d(t)$ can be expected to converge to the function, and the Gibbs jump is avoided. It should be clear, in this connection, that the elimination of the Gibbs jump from the sum of a Fourier series does not automatically eliminate overshoots from the partial sums. Nonetheless, a partial sum can be chosen to make the overshoots as small as desired, provided that we are willing to include enough terms.

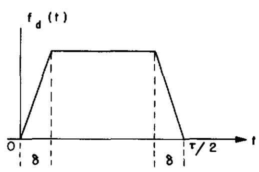


Fig. 10

As a practical matter, we might easily choose a more involved modification of the rectangular pulse than the one shown in Fig. 10. Nevertheless, the principle illustrated above of varying the modification in accordance with the number of terms in the Fourier series is still valid. Figure 22 illustrates a network impulse response for a case in which $f_d(t)$ is a modified rectangle. The exact modification that we used is not relevant here; it is given in Section 3.7.

One useful way of modifying $f_d(t)$ is to decrease its duration with respect to a full period of $f_p(t)$. That is, we propose to shrink the pulse uniformly by a scale change in its time coordinate. Insofar as $f_d(t)$ is concerned, this change is trivial, but the trigonometric polynomial approximant, $f_t(t)$, is affected drastically. It is evident that there might be an advantage, in some cases, in changing $f_t(t)$ in this way; an instance in which this is true will be given.

3.6 CONTROL OF INITIAL VALUE

A special situation arises near the origin, in that our method, as we have described it so far, sometimes fails to yield a good approximation to $f_d(0+)$ by the value $f_d^*(0+)$. Here we have used the symbol "0+" to denote a vanishingly small positive value of t. This is true even though the approximation is very good at larger t-values. We shall show, first, why this difficulty would be expected to occur, and second, how the remedy is found.

We start with Eq. 28. With the aid of a definition,

$$x(t) = L^{-1}\left[\frac{1}{F_p(s)}\right] \tag{33}$$

Equation 28 can be written,

$$\text{Error} = f_d(t) - f_p(t) * [u_o(t) - u_o(t-\tau)] + 2e(t-\tau/2) + e(t) * e(t) * x(t) \tag{34}$$

Let us consider the last term in this equation, the triple convolution product. The only unknown term in this product is x(t), a function which, from Eq. 33, satisfies the equation

$$f_p(y) * x(y) = u_o(t) \tag{35}$$

(We use "y" as a dummy variable to avoid the confusion that might result if there were two different uses of the symbol "t".)

In order to investigate x(t), it is convenient to state a known result about convolution equations. We consider the equation

$$f_p(y) * x(y) = z(t) \tag{36}$$

where, in the interest of generality, we have substituted z(t) for $u_o(t)$ on the right-hand side. Assume that the first n derivatives of $f_p(y)$ are continuous at the point $y = t_o$, but that the $(n+1)^{th}$ derivative is discontinuous there. We include, among the derivatives in this statement, the zeroth derivative, that is, $f_p(y)$ itself. Furthermore, let us assume that the first m derivatives of x(y) are continuous at t_o, but that the $(m+1)^{th}$ is discontinuous at this point. Our conclusion is that $d^{n+m+1}z(y)/dy^{n+m+1}$ is continuous at t_o, but $d^{n+m+2}z(y)/dy^{n+m+2}$ is not continuous at this point. It may be helpful to interpret graphically the convolution equation,

$$z(t) = \int_0^t x(y) f_p(t-y)\,dy \tag{37}$$

which is, of course, the equivalent of Eq. 36.

We are now in a position to deduce certain properties of x(y). At y = 0, $f_p(y)$ may be discontinuous, but at all positive values of y, $f_p(y)$ and all its derivatives are continuous, since $f_p(y)$ is a finite sum of sines and cosines. From Eq. 35 and the result of the last paragraph, if $f_p(y)$ has a discontinuity at y = 0, then x(y) must contain a doublet at this point, since we have reasoned that, if x(y) has merely a simple discontinuity or an impulse at y = 0, the right-hand-side of Eq. 35 is at most discontinuous, and cannot be an impulse. Similarly, if $f_p(y)$ is continuous, but has a discontinuous first derivative at y = 0, x(y) must contain a triplet at y = 0. Moreover, the statement of the previous paragraph can also be used to show that x(y) is continuous for $t > 0$. If this were not true, we could deduce, contrary to fact, that $u_0(t)$ or one of its derivatives is discontinuous at some finite, positive value of t.

From these results about x(t), we come to some interesting conclusions about the triple convolution product in Eq. 34: e(t) may have discontinuities, but it contains no singularity functions, so that e(y) * e(y) is surely continuous. Hence, e(y) * e(y) * x(y) is continuous for $t > 0$, but at the point t = 0, e(y) * e(y) * x(y) may fail to be continuous. Furthermore, the discontinuity at this point might be serious enough to make the error (Eq. 34) very large at small values of t. We shall investigate the discontinuity, and how its size can be controlled.

An expression for the initial value of $f_d^*(t)$ is needed, and this we derive as follows:

$$F_1(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots}{b_{n+1} s^{n+1} + b_{n-1} s^{n-1} + \dots} \tag{38}$$

$$F_2(s) = \frac{c_m s^m + c_{m-1} s^{m-1} + \dots}{d_{m+1} s^{m+1} + d_{m-1} s^{m-1} + \dots} \tag{39}$$

From the initial value theorem of Laplace transforms,

$$f_1(0+) = \frac{a_n}{b_{n+1}} \tag{40}$$

$$f_2(0+) = \frac{c_m}{d_{m+1}} \tag{41}$$

Now

$$F_d^*(s) = \frac{4 \cdot F_1(s) \cdot F_2(s)}{F_1(s) + F_2(s)}$$

which can be rewritten, with the aid of Eqs. 37 and 38, as

$$F_d^*(s) = 4 \cdot \frac{a_n c_m s^{n+m} + (a_{n-1} c_m + a_n c_{m-1}) s^{n+m-1} + \dots}{(a_n d_{m+1} + c_m b_{n+1}) s^{n+m+1} + \dots}$$

Hence, again, from the initial value theorem,

$$f_d^*(0+) = \frac{4 \cdot a_n c_m}{a_n d_{m+1} + c_m b_{n+1}}$$

and, using Eqs. 40 and 41, we find:

$$f_d^*(0+) = \frac{4 \cdot f_1(0+) \cdot f_2(0+)}{f_1(0+) + f_2(0+)} \tag{42}$$

This is the desired relation between the initial value of the impulse response and the initial values of $f_1(t)$ and $f_2(t)$.

Equation 42 is interesting because it confirms the result of our previous discussion. If $f_1(0+)$ and $f_2(0+)$ are nearly the same in magnitude but have opposite signs, $f_d^*(0+)$ may be very large and highly dependent on a small change in either $f_1(0+)$ or $f_2(0+)$. Of course, a small change in either of these functions reflects only a small change in $f_d(0+)$. Hence, in these circumstances, it is evident that $f_d^*(0+)$ may differ a great deal from $f_d(0+)$.

In addition, Eq. 42 points the way toward correction of the difficulty discussed above. We must choose $f_1(0+)$ and $f_2(0+)$ in such a way that $f_d^*(0+)$ has an acceptable value, such as the value $f_d(0+)$. We shall consider how we might proceed in order to choose $f_1(t)$ and $f_2(t)$.

We shall explain a process with reference to a specific example, for which $f_d(t)$ is shown in Fig. 11. In the light of the discussion in Section 3.5, we modify $f_d(t)$ to the form shown in Fig. 12, with "a" equal to 0.9π. Next, $f_1(t)$ and $f_2(t)$ are computed, and $f_d^*(0+)$ is determined by use of Eq. 42. (The reader interested in the computation will find expressions for $f_1(t)$ and $f_2(t)$ in Appendix I.) In this case, the value of $f_d^*(0+)$ is found to be unacceptably large for any "n"; therefore, further modification of $f_1(t)$ and $f_2(t)$ is indicated, in accordance with the suggestion of the last paragraph.

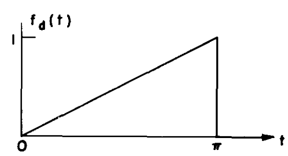


Fig. 11

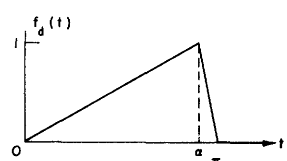


Fig. 12

We observe, from Eq. 42, that either $f_1(0+) = 0$ or $f_2(0+) = 0$ ensures that $f_d^*(0+) = f_d(0+) = 0$. Our object, then, is to modify, say, $f_1(t)$ so that $f_1(0+)$ is zero. We proceed by scaling the time coordinate of $f_d(t)$. It is helpful, in this regard, to look at a plot of $f_1(t)$. (See Fig. 13.) The value of n pertinent to Fig. 13 is 6, and only half a period is shown, since the second half-period is the same as the first, except for a change in sign. From Fig. 13 we can guess that the duration of $f_d(t)$ should be reduced by about 5 per cent in order to ensure that $f_1(0+)$ is approximately zero. The corresponding modified versions of $f_d(t)$ and $f_1(t)$ are shown in Fig. 14. In Fig. 14, the dimensions a and b correspond to the notation used in the expression for $f_p(t)$ (see Appendix II). The choice of 0.945π for the value of b is an informed guess based on the figure and the equation in Appendix II. It is apparent from Fig. 14 that $f_1(0+) \approx 0$. In fact, $f_1(0+)$ is so close to zero that a slight, and

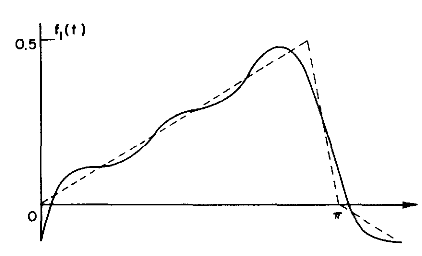


Fig. 13

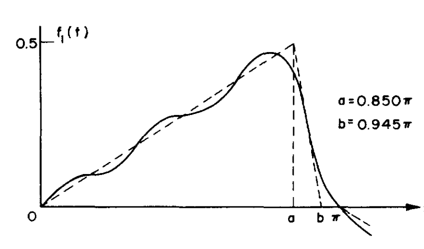


Fig. 14

otherwise completely trivial, change in any one of the odd coefficients in the Fourier series, $f_t(t)$, would suffice to make $f_1(t) = 0$. Actually, using Eq. 42, we obtain (0.004) for the value of $f_d^*(0+)$ pertaining to our last modification of $f_d(t)$; hence, no such change is necessary. The rest of the computation is straightforward, and the approximating impulse response is shown in Fig. 23.

There is a further technique which may sometimes be useful in the event that $f_d(0+)$ is not equal to zero. Suppose that

$$f_d(0+) = \alpha \neq 0$$

In this case, it is possible to subtract a decaying exponential term, such as $\alpha e^{-\beta t}$, from $f_d(t)$ before starting the synthesis procedure. We obtain, by so doing, some added ease of manipulation that comes from working with the value of $f_d(0+) = 0$. When the synthesis is done, a pole representing the removed exponential term must be added to the network. In some cases, this may not be too high a price to pay for added convenience.

It is possible to extend the techniques we have discussed, and to fix the first and higher initial derivatives of $f_d^*(t)$, just as we have fixed $f_d^*(0+)$. Formulas for these initial derivatives can be derived (the formula for the first derivative is given in Appendix III). However, we feel that, except in special instances, use of such formulas is computationally too involved to be worth while.

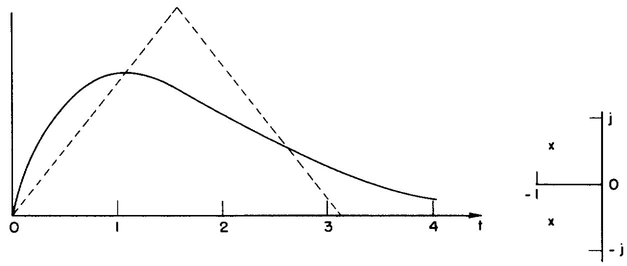
3.7 EXAMPLES

In this section we present some examples of applications of the method described above. We present examples in some detail because we wish to provide a basis for judging the usefulness of the method. An evaluation must obviously be performed, if an intelligent choice is to be made, in any particular case, between the various synthesis techniques that are available. A method is to be judged by two criteria: its ease of application and its results. Accordingly we give our results, in Figs. 15 through 23.

It is worth pointing out that specific examples, which have been derived by a given method, are particularly useful in time-domain synthesis. This is true because, as we pointed out in Section I, it is difficult to state an error criterion which is both accurate and generally applicable to time-domain synthesis problems. Consequently, we cannot usually know in advance exactly what result to expect from a given design. Under these circumstances, even the decision whether or not to use time-domain synthesis at all may be difficult, in the absence of previous results from which an assessment of quality can be made.

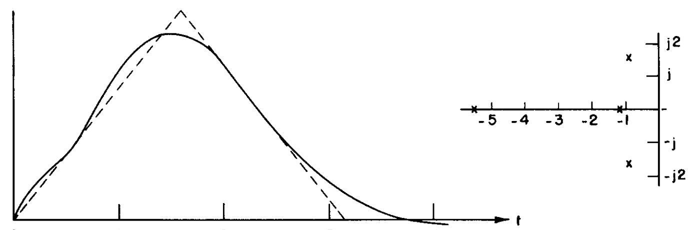
We have given various functions in Figs. 15-23. The literal expression for $f_d^*(t)$, valid for $t > 0$, is included in each case, so that the reader can check the plot against it. Any quantity labeled "term neglected" is the inverse transform of a term subtracted from $F_d^*(s)$ when that function is modified in accordance with the discussion

Fig. 15



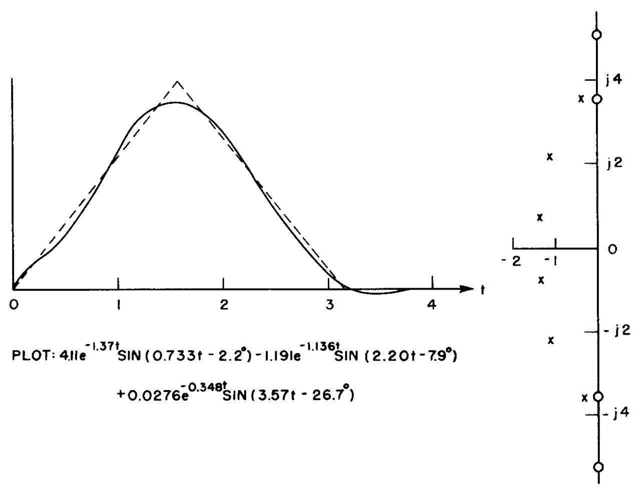
PLOT: $1.710e^{-0.810t}\cos(0.585t - \frac{\pi}{2})$

Fig. 16



PLOT: $1.647e^{-1.13t} - 0.400e^{-5.55t} + 1.332e^{-0.938t}\cos(1.570t + 159.6°)$

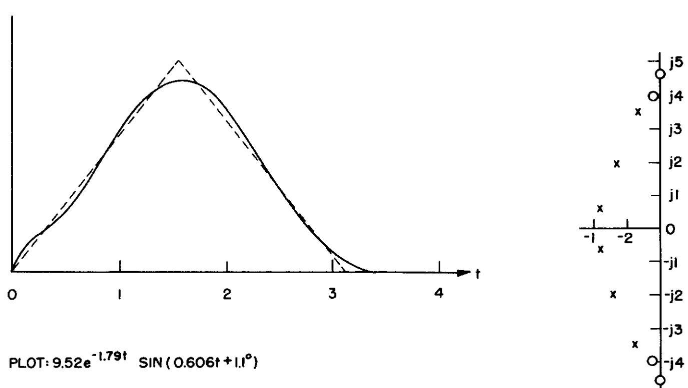
Fig. 17



PLOT: $4.11e^{-1.37t}\sin(0.733t - 2.2°) - 1.191e^{-1.136t}\sin(2.20t - 7.9°)$

$+0.0276e^{-0.348t}\sin(3.57t - 26.7°)$

Fig. 18



PLOT: $9.52e^{-1.79t}\sin(0.606t + 1.1°)$

$-2.68e^{-1.40t}\sin(1.97t + 3.35°)$

$+0.0845e^{0.726t}\sin(3.485t - 14.28°)$

TERM NEGLECTED: $0.0014e^{0.205t}\sin(4.24t - 49.47°)$

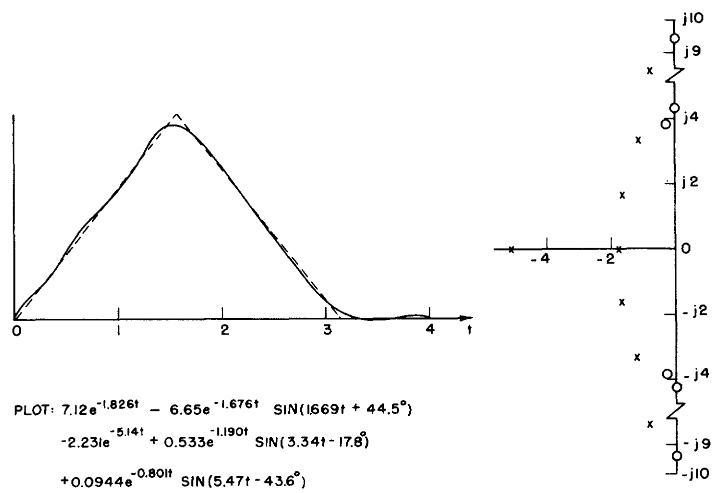


Fig. 19

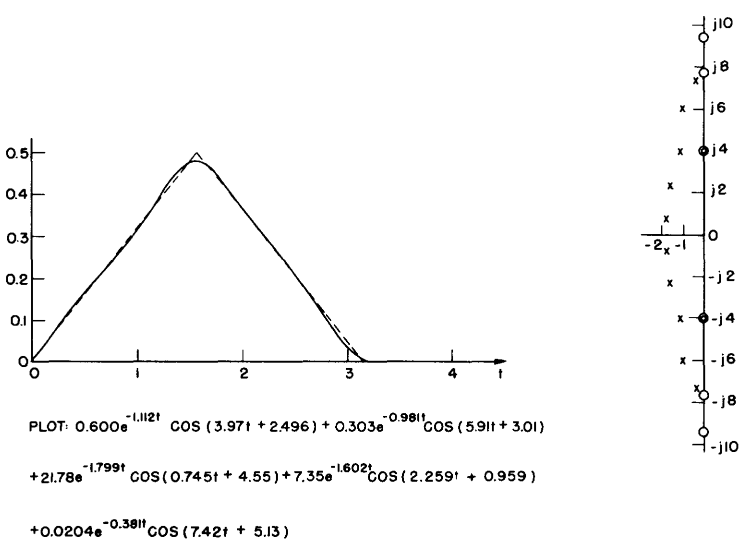


Fig. 20

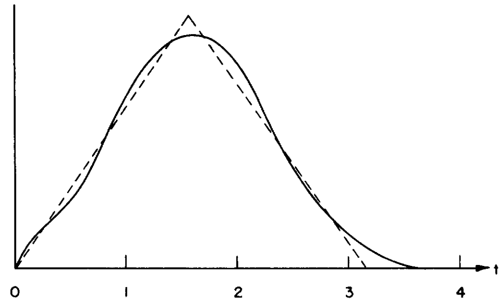
in Section 3.3. The "factor neglected" is a factor removed from $F_d^*(s)$ under the same circumstances.

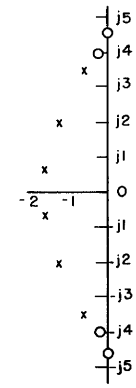
Some explanation of the symbolism in these figures may be helpful. In the plots, a dashed line denotes the graph of $f_d(t)$, a solid line denotes $f_d^*(t)$; π seconds was chosen for the pulse duration, for the sake of convenience. The pole plots and pole-and-zero plots refer to $F_d^*(s)$; for cases in which this function has been modified, the plots apply to the modified version of $F_d^*(s)$.

We shall now discuss special features of some of the functions we have synthesized. An appendix reference will be given to an intermediate computation that is of more than routine interest.

EXAMPLE 1. The Triangle Function. We refer to the function whose graph is the dashed curve in Fig. 15 as the triangle function. We have studied this function

Fig. 21





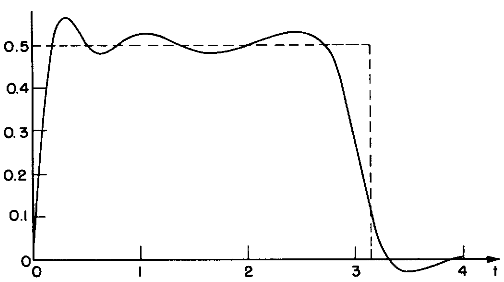
$F_d^*(s) = (0.703)\dfrac{(s^2+21.12)(s^2+0.460s+15.88)}{(s^2+3.58s+3.57)(s^2+2.80s+5.84)(s^2+1.451s+12.68)}$

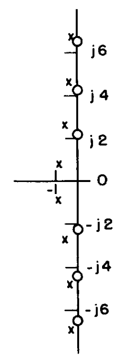
FACTOR NEGLECTED: $\dfrac{s^2-0.460s+15.88}{s^2-0.411s+17.96}$

PLOT: $10.50e^{-1.79t}\,\mathrm{SIN}\,(0.606t+1.8°) - 2.99e^{-1.40t}\,\mathrm{SIN}\,(1.97t+6.25°)$

$+0.971e^{-0.726t}\,\mathrm{SIN}\,(3.49t-3.17°)$

Fig. 22

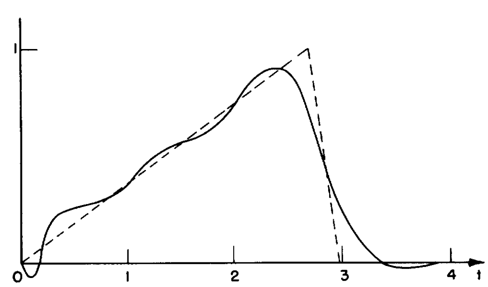


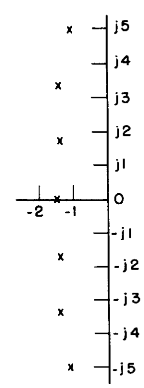


PLOT: $3.301e^{-0.8836}\,\mathrm{COS}\,(0.799t+4.44) + 0.704e^{-0.602t}\,\mathrm{COS}(2.679t+5.633)$

$+0.2205e^{-0.4114t}\,\mathrm{COS}\,(4.716t+5.980) + 0.0879e^{-0.2710t}\,\mathrm{COS}(6.78t+6.11)$

Fig. 23





ONE ADDITIONAL POLE AT s = -137

PLOT: $6.86e^{-1.411t}\,\mathrm{COS}(1.731t+116.6°)$

$-3.33e^{-1.449t}\,\mathrm{COS}(3.361t+38.3°)$

$+7.59e^{-1.080t}\,\mathrm{COS}(5.011t-38.0°)$

$+5.103e^{-1.501t} - 0.010e^{-136.7t}$

more thoroughly than any other, and it is worth pointing out the reason for this emphasis. First, we are anxious to test our synthesis method, and the triangle function is particularly informative in this regard. The function cannot be synthesized well by inspection, therefore its use is not trivial, but because the triangle function is continuous, a good approximation should be possible. (In contrast, we do not expect to be able to make a good approximation in the vicinity of a discontinuity (except at $t = 0$) of a discontinuous function.) Second, the triangle function is used as an example in references 1 and 6, and it is always helpful in judging a method to be able to compare alternative treatments with the one at hand.

Figures 15 through 20 form a group in that the various functions, $f_d^*(t)$, are all derived in the same way. A Fourier approximation was made to the unmodified triangle function, and $F_d^*(s)$ was computed according to Eq. 17. $F_d^*(s)$ was then expanded in partial fractions, and modified by the removal of all terms involving right-half-plane poles. In Figs. 15 through 17, no right-half-plane poles occurred in $F_d^*(s)$; consequently none were removed. In Figs. 18 through 20 one conjugate pair of right-half-plane poles occured in each case. These poles were deleted from $F_d^*(s)$, but the time functions corresponding to the removed poles are given in the figures. It is interesting to see how small these time functions are, for small values of t.

The example illustrated in Fig. 21 was computed exactly like the examples illustrated in Figs. 15-20, except that a different method was used in removing the right-half-plane poles from $F_d^*(s)$. $F_d^*(s)$ was written as a quotient of polynomials, with both numerator and denominator in factored form. The pair of right-half-plane poles and a pair of zeros were then cancelled out of the expression, as explained in Section 3.3. It should be noted that the cancellation process may affect the amplitude scale factor of $f_d^*(t)$; accordingly the scale factor in our example was adjusted after the poles and zeros were cancelled.

The triangle function is also synthesized in Section IV by an approach which is described there. Figure 24, which is comparable to Figs. 17, 18, and 21, is of interest in this connection.

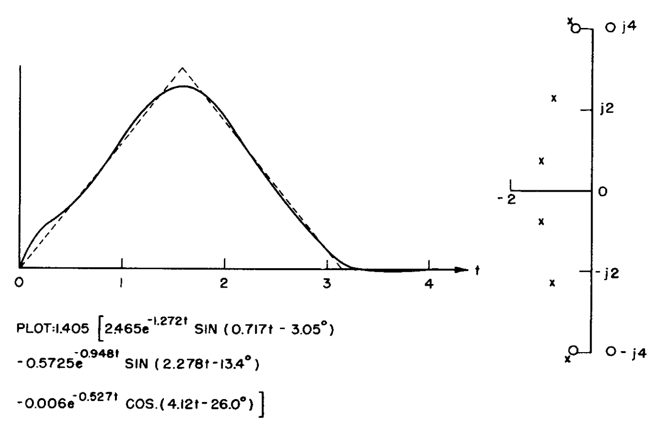


Fig. 24

EXAMPLE 2. The Rectangle Function. The rectangle function is shown by the dashed curve in Fig. 22. We have chosen to synthesize this function for two reasons. First, it is of interest to try out our method on a function that contains a discontinuity, and, second, various examples exist that furnish informative comparisons of the synthesis of the rectangle function[7, 8]. In this synthesis we start from a modified version of $f_d(t)$. (See Appendix IV for the pertinent expression.)

EXAMPLE 3. The Reversed Ramp Function. The reversed ramp function and its approximant are shown in Fig. 23. This function is of especial interest for our purposes, in the light of the discussion in Section 3.6. We mentioned the problem about the initial value of $f_d^*(t)$, which occurs in this case, and how the solution is obtained. In addition, right-half-plane poles occur in the function, $F_d^*(s)$, which is computed on the basis of Fig. 14, hence the methods of Section 3.3 have to be used. The small dip and the overshoot following it, found in the graph of $f_d^*(t)$, are there because the initial derivative, $f_d'^*(0+)$, was not controlled in this synthesis process. A synthesis which is better in this respect could be achieved by using the methods discussed in Section 3.6 and Appendix III.

IV. AN ALTERNATIVE SYNTHESIS METHOD

We recognize that many variations are possible within the framework of the basic philosophy of synthesis presented in Section II. Just as there are many delay line designs, correspondingly many synthesis formulas can be developed. In this section, we derive one such formula for its collateral interest. The application that we give is a synthesis of the triangle function, and Fig. 24 is comparable with Figs. 17, 18, and 21 of the last chapter.

The Laplace transform of the impulse response approximation can be written

$$F_d^*(s) = 2F_1(s) \cdot G(s) \tag{43}$$

In Section 3.2 we derived the relation

A. $F_d^*(s) = 2F_2(s) \cdot (1 - e^{-s\tau/2*})$

In order to save space we did not derive

B. $F_d^*(s) = 2F_1(s) \cdot (1 + e^{-s\tau/2*})$

but we could equally well have done so. The function, G(s), corresponding to (B) is $G(s) = 1 + e^{-s\tau/2*}$. (A) and (B) serve equally well as bases for the development of explicit synthesis formulas. In Eq. 43, $F_1(s)$ is the transform of the function $f_1(t)$ introduced in Eq. 7. Use of the multiplying constant, 2, conforms to the notation in Section 3.2. Since $f_1(t)$ has a period of π, for G(s), we have

$$G(s) = 1 + e^{-s\tau/2*} = 1 + e^{-s\pi*} \tag{44}$$

We next derive the rational approximant, $e^{-s\pi*}$.

Let us denote by M(s) a polynomial which contains only even powers of s, and by N(s) a polynomial which contains only odd powers of s. Let us choose an $e^{-s\pi*}$ in the form

$$e^{-s\pi*} = \frac{M(s) - N(s)}{M(s) + N(s)} \tag{45}$$

By this choice, we obtain an approximation that is equal in magnitude to $e^{-s\pi}$ along the j-axis:

$$|e^{-s\pi*}|_{s=j\omega} = |e^{-s\pi}|_{s=j\omega}$$

To achieve the end of selecting M(s), we rewrite Eq. 43 as

$$F_d^*(s) = 2F_1(s) \cdot \left[1 + \frac{M(s) - N(s)}{M(s) + N(s)}\right] \tag{46}$$

or

$$F_d^*(s) = \frac{4F_1(s) \cdot M(s)}{M(s) + N(s)} \tag{47}$$

$F_1(s)$ has j-axis poles at $\pm j1$, $\pm j3$, ... These poles must be effectively cancelled by zeros of M(s), since otherwise $F_d^*(s)$ would contain them, and $f_d^*(t)$ would then possess an undesired periodic component. Hence we choose the zeros of M(s) at $s = \pm j1, \pm j3$, and so forth.

In choosing the zeros of N(s) we consider the imaginary parts of $e^{-s\pi^*}$ and $e^{-s\pi}$ on the $j\omega$ axis.

$$\mathrm{Im}\left[\frac{M(s) - N(s)}{M(s) + N(s)}\right]_{s=j\omega} = \left[\frac{-2M(s)\,N(s)}{M^2(s) - N^2(s)}\right]_{s=j\omega} \tag{48}$$

$$\mathrm{Im}\,(e^{-s\pi})_{s=j\omega} = -\sin(\omega\pi) \tag{49}$$

The right-hand side of Eq. 49 is zero for $\omega = 0, \pm 1, \pm 2, \ldots$. We should like to choose these same zero locations for the right-hand side of Eq. 48, in view of the desired relationships,

$$e^{-s\pi^*} \approx e^{-s\pi} \tag{50}$$

$$\mathrm{Im}(e^{-s\pi^*})_{s=j\omega} \approx \mathrm{Im}\,(e^{-s\pi})_{s=j\omega} \tag{51}$$

Equation 51 can be rewritten as

$$\left[\frac{-2M(s)\cdot N(s)}{M^2(s) - N^2(s)}\right]_{s=j\omega} \approx -\sin(\omega\pi) \tag{52}$$

Both sides of this equation are zero for $\omega = \pm 1, \pm 3, \ldots$ by virtue of our choice of M(s). We choose the zeros of N(s) at $s = 0, \pm j2, \pm j4$, and so forth, in accordance with Eq. 51, thereby ensuring that Eq. 52 is also an equality at these additional values of s. The reason for choosing those j-axis zeros of $\mathrm{Im}(e^{-s\pi})$ which are nearest the origin to be zeros of N(s), instead of zeros which occur at odd-integer multiples of 1 (but possibly at other than the smallest odd-integer multiples) is as follows. The largest values of $|F_1(j\omega)|$ usually occur at small values of ω; it is, therefore, at these values that the approximate relations 50 and 51 must be most accurate. Accordingly, we are naturally led to match zeros in Eqs. 48 and 49 at the lowest possible frequencies.

It is collaterally interesting to notice that we have chosen the zeros of M(s) and N(s) so that they interlace on the j-axis. We have, therefore, ensured that M(s) + N(s) is a Hurwitz polynomial; then, in accordance with Eq. 47, $F_d^*(s)$ is surely realizable.

It remains to choose the constant multipliers associated with M(s) and N(s). One of these is arbitrary, and we choose the constant multiplier of M(s) to be 1. The constant multiplier of N(s) can be chosen by referring again to Eq. 52. A minimum of $-\sin(\omega\pi)$ occurs for $\omega = (1/2)$, and we select the constant, k, in N(s), so that Eq. 52 holds exactly at this minimum:

$$-\sin(\pi/2) = \left[\frac{-2M(s)\cdot N(s)}{M^2(s) - N^2(s)}\right]_{s=j/2} = -1 \tag{53}$$

As might be expected, this choice of k incidentally leads to a phase characteristic of $e^{-j\omega\pi^*}$ that has good approximating properties at small values of ω.

We can now write M(s) and N(s) as follows:

$$M(s) = (s^2 + 1)(s^2 + 9) \ldots (s^2 + n^2) \tag{54}$$

$$N(s) = ks(s^2 + 4)(s^2 + 16) \ldots (s^2 + (n-1)^2) \tag{55}$$

In these equations, k is unspecified because its value is dependent upon the choice of n. We chose n equal to or greater than the "n" that appears in Eq. 6.

It is interesting to notice the value of $f_d^*(0+)$ in the synthesis procedure that we have described. We compute $f_d^*(0+)$ as follows. Regroup Eq. 47 as

$$F_d^*(s) = 4F_1(s) \cdot \frac{M(s)}{N(s) + M(s)} \tag{56}$$

In this equation, M(s) is of higher degree than N(s), so that the coefficients of the highest powers of s occurring in the numerator and denominator of the fraction on the right-hand side of the equation are the same. Consequently, if we apply the initial value theorem to Eq. 56, we obtain

$$f_d^*(0+) = 2 \cdot [2f_1(0+)] \tag{57}$$

If both sides of this equation are zero, we have, alternatively,

$$f_d'^*(0+) = 2 \cdot [2f_1'^*(0+)] \tag{58}$$

In other words, the initial value of the impulse response, $f_d^*(t)$, is twice that of the Fourier approximation to $f_d(t)$. In case Eq. 58 holds, the same remark can be made about the initial derivatives. It is, of course, possible to reduce or remove this discrepancy by the methods discussed in Sections 3.5 and 3.6, if it is necessary to do so.

In Fig. 24 we show an example of the synthesis of the triangle function by means of the method discussed above. The triangle function was used in unmodified form to obtain the function, $f_2(t)$, in the synthesis. The pertinent value of n is 5, and of k is 5.5.

Acknowledgment

The author is happy to acknowledge the inspiration of Professor Ernst A. Guillemin in the work represented by this report. Many thanks are due Dr. Robert A. Pucel and Professors William K. Linvill and George B. Thomas, Jr., for their helpful suggestions.

APPENDIX I

$$f_1(t) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \frac{1}{a i^2 (\pi - a)} \cdot \left[\left(\cos(ia) - 1 + \frac{2a}{\pi}\right)\cos(it) + \sin(ia)\sin(it)\right] \qquad t > 0$$

$$= 0 \qquad \text{otherwise}$$

$$f_2(t) = \sum_{\substack{i=1 \\ i \text{ even}}}^{n} \frac{1}{a i^2 (\pi - a)} \cdot [(\cos(ia) - 1)\cos(it) + \sin(ia)\sin(it)] + 1/4 \qquad t > 0$$

$$= 0 \qquad \text{otherwise}$$

APPENDIX II

The equation for $f_p(t)$ corresponding to $f_d(t)$, as shown in Fig. 14,is:

$$
\begin{aligned}
f_p(t) &= \frac{b}{4\pi} + \frac{1}{\pi(b-a)} \sum_{i=1}^{n} (1/i^2) \{[(b/a)\cos(ia) - \cos(ib) + 1 - b/a]\cos(it) \\
&\quad + [(b/a)\sin(ia) - \sin(ib)]\sin(it)\} \qquad t > 0 \\
&= 0 \qquad \text{otherwise}
\end{aligned}
$$

APPENDIX III

Define:

$$f_1(0+) = a_1$$

$$f_1'(0+) = a_2$$

$$f_2(0+) = b_1$$

$$f_2'(0+) = b_2$$

where d/dt is denoted by a prime.

Then,

$$f_d^*(0+) = \frac{4a_1 b_1}{a_1 + b_1}$$

$$f_d^{*\prime}(0+) = 2\,\frac{a_1^2 b_2 + a_2 b_1^2}{(a_1 + b_1)^2}$$

The most convenient way to derive expressions for higher initial derivatives of $f_d^*(t)$ is by using the Laplace transforms of $f_1(t)$, $f_2(t)$, and $f_d^*(t)$. Each of these transforms is a quotient of polynomials, the coefficients of whose Laurent expansion about the origin are the initial derivatives of the related time function. By substituting the pertinent Laurent expansions in Eq. 17, and then finding the Laurent expansion of $F_d^*(s)$, we arrive straightforwardly at pertinent expressions for the higher initial derivatives.

APPENDIX IV

The modification of $f_d(t)$ used in the computation of $f_d^*(t)$ in Fig. 22, with $\rho = \pi/10$, is:

$$
\begin{aligned}
f_d(t) &= 0 && t < 0 \\
&= \frac{t}{2\rho} + \frac{1}{2\pi}\sin(\pi t/\rho) && 0 \leq t \leq \rho \\
&= 1/2 && \rho < t < \pi - \rho \\
&= 0 && t > \pi
\end{aligned}
$$

The graph of the function is symmetrical about the line $t = \pi/2$, which completes the definition in the range $(\pi - \rho) \leq t \leq \pi$.

The pertinent function, $f_p(t)$, is:

$$
f_p(t) = 1/2 + \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} (2/k\pi) \cdot \left[\frac{\sin(k\rho/2)}{k\rho/2}\right] \cdot \left[\frac{\cos(k\rho/2)}{1 - k^2\rho^2/\pi^2}\right] \sin(kt)
$$

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