EQUILIBRIUM EXIT IN STOCHASTICALLY DECLINING INDUSTRIES

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ABSTRACT

We study a complete information model of exit in which the stage payoffs are governed by a nonstationary Markov process that reflects the stochastic decline of the industry. For a monopolist, our model is an optimal stopping problem. For duopolists, we analyze the (perfect) stopping time equilibria of the exit game. There are multiple perfect equilibria of our exit game, in contrast to several papers in the literature. We explain how relaxing an arguably unrealistic assumption in those models will give multiple equilibria in those models also. Finally, we show an equivalence between stopping time equilibria, and perfect equilibria of complete-information, stochastic exit games.

1. INTRODUCTION

The decline of products and industries is a natural by-product of the growth and evolution of industrial economies. Economic decline puts pressure on firms' profits and on their ability to remain in the business. Most firms eventually depart if the decline is severe. However, those who manage to stay in after a shakeout can often earn significant profits.

In this paper, we analyze firms' decisions to exit an industry that is suffering from economic decline. In contrast to the literature on entry, the study of exit decisions by firms in oligopolistic industries is not extensive. Ghemawat and Nalebuff (1985) analyze a continuous time, perfect information model for an asymmetric duopoly in a deterministically declining industry, where the firms differ by their capacities and related fixed costs per period. Unless the high-fixed-cost firm has significantly lower operating costs than the low-fixed-cost firm, the unique perfect Nash equilibrium in their model is for the firm with the larger capacity and fixed costs to exit at the first time that its duopoly profits are nonpositive. The smaller firm then remains until its monopoly profits become negative.

Fudenberg and Tirole (1986) also analyze an asymmetric duopoly exit game in a continuous time model with deterministic demand paths. However, they assume that each firm does not know the fixed operating cost per period of its rival. Thus their model is a war of attrition where each firm continually revises downward its estimate of the other's costs, as long as the contest for the market lasts. With an assumption that each firm assesses positive probability that its rival will never find it optimal to exit, the model

yields a unique perfect Nash equilibrium where the higher cost firm exits the market before the firm with lower costs.

We also model an industry that begins as a duopoly. Like Ghemawat-Nalebuff and Section 5 of Fudenberg-Tirole, we assume that the industry declines over time so that the profits of the two firms shrink until one of them exits, leaving a monopoly for the other. In contrast to the results of Ghemawat and Nalebuff, and Fudenberg and Tirole (each of whom obtains a unique perfect equilibrium in a continuous time, deterministic model), there are multiple exit-time equilibria in our stochastic, discrete time framework. As a consequence of the multiplicity of the equilibria, the stronger firm, which has a stage-payoff advantage, may not always exit last.

The multiplicity of equilibria in our model is a consequence of the fact that the industry demand process can jump from a point where both firms are viable as duopolists to a point where neither firm is viable as a duopolist but each is viable as a monopolist. At such a point in time, exit by either could constitute an equilibrium in stopping times. Both Ghemawat and Nalebuff and Fudenberg and Tirole avoid this problem by assuming that demand falls continuously (and deterministically) over time. However, their models would also exhibit multiple equilibria as ours does if their demand process were allowed to have jumps. Thus, an assumption in the direction of more realism (a demand process with discrete jumps) destroys the equilibrium uniqueness in each of their models.

We characterize the generic form of any equilibrium in our model. These equilibria are obtained by solving a time-indexed sequence of fixed point

problems. Among the equilibria, two are particularly interesting because they provide the upper and lower bounds for any equilibrium exit times and because each player prefers one of the two equilibria to all other equilibria. In particular, each firm favors the equilibrium which makes his active period the longest. We provide a necessary and sufficient condition for the "natural" equilibrium (in which the stronger firm always outlasts its rival) to be the unique perfect equilibrium.

To model the exit game, we adopt a natural multiperson extension of the optimal stopping time concept (see, e.g., Breiman (1964), Shiryayev (1978), Dynkin (1969), and references in Monahan (1980)) which is the concept of stopping time equilibrium, a Nash equilibrium in stopping time strategies. (Our concept of stopping time equilibrium is not unlike that of Ghemawat and Nalebuff or Fudenberg and Tirole, except that in their models the single-firm optimal stopping problem is of little intrinsic interest because their models have deterministic demand paths.) In general, any stochastic dynamic game in which each player's strategy is a single dichotomous decision at each stage can be formulated as a stopping time problem. Further, in such games, the notion of equilibrium stopping time can be applied. For this type of game, we show that the stopping time equilibria correspond to the subgame perfect equilibria in the natural extensive form game.

One characteristic of the firms' optimal policies in our model is that one or both firms may earn negative profits in some period(s), but remain in the market. This can occur for either of two reasons. First, industry demand can suffer a stochastic negative shock, such that, demand will increase in the next period, in expectation. In such cases, firms will absorb the loss in

that period, knowing that demand is likely to recover. This result accords with observed practice. Most industries experience stochastic downturns that do not lead to mass exit from the industry. A firm may also choose to absorb losses and remain in the market in the expectation that its rival will exit the industry first, leaving a more profitable, monopolistic industry for the remaining firm. Although the former effect is unique to our model, the latter effect is also present in the Fudenberg-Tirole formulation.

For both the single-firm and two-firm cases, we solve examples with linear demand and linear costs. For the single-firm example, the optimal exit time is decreasing in the firm's unit costs, the slope of the demand curve, and the per unit fixed cost of being in the market. The optimal exit time increases in the firm's discount factor. In the two-firm example with Cournot competition, these results also hold for each firm. In addition, the high cost firm's optimal exit time increases in the low cost firm's unit costs.

The remainder of the paper is organized as follows: In Section 2, the single-firm exit problem is formulated as a stopping time problem and several basic results are established. In Section 3 the notion of stopping time equilibrium is presented, and the main results regarding equilibrium exit behavior in a duopoly are proved and discussed. Concluding remarks and possible extensions follow in Section 4.

2. THE SINGLE-FIRM MODEL

In this section, we formulate the dynamic programming model for the single-firm problem and provide some comparative statics on the optimal stopping rule. We assume that at time zero, a single firm is in the market earning nonnegative profits. At any later point in time, if the firm concludes, after observing demand, that remaining in the market is no longer profitable, it may exit the market. Reentry is assumed to be prohibitively costly.

We let industry demand be indexed by a Markov sequence $\{a_t\}$. For each period t, the demand a_t takes value in $Z \subseteq \mathbf{R}$ and has a conditional probability measure $P_t\{\cdot|a_{t-1}\}$ on Z. For convenience, we assume Z = R in the later discussion. In order to formulate downward drifting demand representing a shrinking market, we use the familiar notion of first-degree stochastic dominance (Hadar and Russell (1969)). That is, we assume that for any $z \in Z$, $P_t\{\cdot|z\}$ stochastically dominates $P_{t+1}\{\cdot|z\}$ for all $t \geq 0$.

At each time t, the firm observes $(a_s; s \leq t)$, and then makes a decision about whether to stay in the market and get $\pi(a_t)$ in period t or to exit and get zero profit thereafter. We assume that $\pi(a_0) > 0$, and for each period t, we assume that present profits and future expected profits are increasing in period t demand, a_t. That is, $\pi(a_t)$ is increasing in a_t, and for $a_t \geq a_t'$, $P_{t+1}\{\cdot|a_t\}$ stochastically dominates $P_{t+1}\{\cdot|a_t'\}$ for all $t \geq 0$.

The firm's strategy is a random time $T:\Omega \to \mathbf{N} = \{0,1,2,...\}$, where Ω is the space of sample paths of $\{a_t, t \geq 0\}$. We require T to be <u>stopping time</u>. Formally,

let $\mathbf{F}_t = \sigma(a_s;\ s\leq t)$, the σ-field generated by $\{a_s; s\leq t\}$. We may think of $\mathbf{F}_t$ as the information available at time t obtained by observing $\{a_s;\ s \leq t\}$. A random time $T: \Omega \rightarrow \mathbf{N}$ is said to be a stopping time of $(\mathbf{F}_t)$ if $\{T\leq t\}\epsilon\mathbf{F}_t$ for every $t\epsilon\mathbf{N}$. In other words, information F_t available at time t is sufficient to tell whether the event $\{T\leq t\}$ has occurred.

Let t^* be the first time that demand has declined to the point where the industry is no longer profitable. That is, $t^* = \inf\ \{t:\pi(a_t) < 0, \text{a.s.}\}$. We assume that $t^* > 0$ to avoid a trivial case and that $t^* < \infty$ to guarantee that the firm will exit in finite time.

Let β be the firm's discount factor. The firm's problem is to choose stopping time T so as to maximize the expected discounted profit

$$E\ [\ \sum_{t=0}^{T-1} \beta^t\ \pi(a_t)\ \mid\ a_0]\ . \tag{1}$$

Proposition 1 below characterizes the optimal stopping times for this problem. It defines time indexed sets B_t such that the firm's optimal policy is to exit at the first time t such that $a_t \in B_t$. The sets B_t are defined by the recursively defined profit-to-go functions $u_t(a_t)$ and take the form $B_t = \{a_t \leq h_t\}$ where the h_t's are defined in terms of the $u_t(\cdot)$ functions. Proposition 1 is preceded by a technical lemma that orders the expectations of the positive parts of monotone ordered functions of random variables that are ordered by stochastic dominance.

<u>Lemma 1</u>. Suppose $f_1(\cdot)$, $f_2(\cdot)$ are increasing functions and $f_1(z) > f_2(z)$, for $z\varepsilon Z$. And suppose z_1, z_2 are random variables with probability measures μ_1, μ_2, respectively. Moreover, μ_1 stochastically dominates μ_2. Then $E[f_1(z_1)1_{(0,\infty)}(f_1(z_1)] \geq E[f_2(z_2)1_{(0,\infty)}(f_2(z_2))]$, and strict inequality holds if $E[1_{(0,\infty)}f_1(z_1)] > 0$.

<u>Proof</u>: Since $f_1(z) > f_2(z)$ implies $\{f_1(z) > 0\} \_ \{f_2(z) > 0\}$ and $\mu_1 > \mu_2$,

$$E[f_1(z_1)1_{(0,\infty)}(f_1(z_1))] = \int_{\{f_1(z)>0\}} f_1(z)d\mu_1(z)$$

$$\geq \int_{\{f_2(z)>0\}} f_1(z)d\mu_1(z) > \int_{\{f_2(z)>0\}} f_2(z)d\mu_1(z)$$

$$\geq \int_{\{f_2(z)>0\}} f_2(z)d\mu_2(z)$$

$$= E[f_2(z_2)\, 1_{(0,\infty)}(f_2(z_2))].$$ Q.E.D.

<u>Proposition 1:</u>

(i) A unique minimal optimal stopping time for the problem described in (1) can be characterized by $T = \inf\{t:a_t \varepsilon B_t\}$ where $B_t = \{u_t(a_t) \leq 0\}$, $u_t(a_t) = \pi(a_t) + \beta w_{t+1}(a_t)$, and

$$w_{t+1}(a_t) = E[u_{t+1}(a_{t+1})\, 1_{(0,\infty)}(u_{t+1}(a_{t+1}))|a_t],$$

with $w_t(\cdot) = 0$ for $t \geq t^*$.

(ii) For $t \geq 0$ and $z\varepsilon Z$, $w_t(z) \geq w_{t+1}(z)$ and $u_t(z) \geq u_{t+1}(z)$. Let $h_t \equiv \sup\{z : u_t(z) < 0\}$. Then h_t is increasing in t, $B_t = \{a_t \leq h_t\}$ and $B_t \quad B_{t+1}$ for $t \geq 0$.

Proof: Part (i) follows from the optimality principle of dynamic programming; cf. Shiryayev (1978, Chapter 2).

To prove (ii), note that

$$w_{t^*-1}(z) = E[\pi(a_{t^*-1})1_{(0,\infty)}(\pi(a_{t^*-1}))|z] \geq 0 = w_{t^*}(z).$$

Suppose $w_{t+1} - w_{t+2} \geq 0$ for $0 \leq t \leq t^*-2$. Note that $u_t(\cdot)$ is increasing since $\pi(\cdot)$ is increasing. Furthermore, $u_t(z) - u_{t+1}(z) = \beta(w_{t+1}(z) - w_{t+2}(z)) \geq 0$. Lemma 1 implies $w_t(z) \geq w_{t+1}(z)$ since $w_t(\cdot) = E[u_t(a_t)1_{(0,\infty)}(u_t(a_t))|\cdot]$.

Finally, we need to show that $u_t(\cdot)$ are monotone for $t \geq 0$ or equivalently, that $w_t(\cdot)$ are monotone for $t \geq 0$. Again, by induction, if $w_{t+1}(\cdot)$ is increasing, then $u_t(\cdot) = \pi_t(\cdot) + \beta w_{t+1}(\cdot)$ is increasing and $w_t(\cdot)$ $= E[u_t(a_t)1_{(0,\infty)}(u_t(a_t))|\cdot]$ is increasing by Lemma 1. Hence, B_t can be written as $\{a_t \leq h_t\}$ with $h_t \equiv \sup\{z:u_t(z) < 0\}$ for $t \geq 0$. Q.E.D.

Note that $u_t(a_t)$ is just the expected discounted profit from time t until exit under the optimal stopping policy. Therefore, the optimal policy is to stop the first time that a_t falls so low that $u_t(a_t) \leq 0$. This cutoff point for a_t is denoted by h_t, and the set of a_t values that dictate optimal exit, that is, the set of points less than or equal to h_t, are denoted by B_t.

Figure 1 illustrates a sample path of $\{a_t\}$, h_t as a function of t, and the optimal stopping time T. The dashed horizontal line represents the zero profit line $z = \pi^{-1}(0)$. For $t \geq 0$, if a_t falls above (below) this line, then $\pi(a_t) > 0$ ($\pi(a_t) < 0$). The cutoff points h_t always lie below this line. Thus, the firm is sometimes willing to sustain a certain level of current loss in the hope of receiving future profits. In fact, even if $\pi(a_t) < 0$ and $E[\pi(a_{t+1})|a_t] < 0$, the firm's optimal policy may be to remain in the market. To see this, suppose $\pi(a_t) = -\varepsilon < 0$ and $E[\pi(a_{t+1})|a_t] < 0$, but $\pi(a_t) + \beta\, E[\pi(a_{t+1})1_{(0,\infty)}(\pi(a_{t+1}))|a_t] > 0$. Then the firm can pay ε in period

t for the option of staying in the market in the hope that demand will be higher than expected next period. Because the firm can choose to exit at no cost after seeing a_{t+1}, the option of being able to observe a_{t+1} and profit from it has positive value. The variable h_t represents how heavy a loss the firm will tolerate in period t in the hope of future profits.

The next proposition provides general comparative static results for the model. It says that the optimal stopping time is increasing in the one-stage payoffs and in the discount factor.

<u>Proposition 2</u>: If $\pi'(z) \leq \pi''(z)$ for $z \in \mathbf{Z}$ or $\beta' \leq \beta''$, then $w_t^{'} \leq w_t^{''}$, $u_t^{'} \leq u_t^{''}$, $h_t^{'} \geq h_t^{''}$ for $t \geq 0$ and $T' \leq T''$ almost surely.

<u>Proof</u>: Let $t' = \inf\{t : \pi'(a_t) \leq 0, \text{a.s.}\}$ and $t'' = \inf\{t : \pi''(a_t) \leq 0, \text{a.s.}\}$. Assume $t'' < \infty$. Obviously $t' \leq t''$. For $t \geq t''$, $w_t^{'} = w_t^{''} = 0$. Assume that $w_{t+1}^{'} \leq w_{t+1}^{''}$. We want to show $w_t^{'} \leq w_t^{''}$. This follows from the fact that $u_t^{'}(z) = \pi'(z) + \beta w_{t+1}^{'}(z) \leq \pi''(z) + \beta\, w_{t+1}^{''}(z) = u_t^{''}(z)$ and an application of Lemma 1. Then $h_t^{'} \geq h_t^{''}$ is obvious by noticing that $h_t = u_t^{-1}(0)$.
In turn, $h_t^{'} \geq h_t^{''}$ implies $B_t^{'} \supseteq B_t^{''}$ and $\bar{B}_t^{'} \subseteq \bar{B}_t^{''}$, for $t \geq 0$. Thus, $\{T' > t\} = \{\bigcap_{s=0}^{t} \bar{B}_t^{'}\} \subseteq \{\bigcap_{s=0}^{t} \bar{B}_t^{''}\} = \{T'' > t\}$ for every $t \geq 0$. So $T' \leq T''$ almost surely.

The results for $\beta' \leq \beta''$ follow from a similar argument. Q.E.D.

Proposition 2 provides a very convenient tool for dealing with the comparative statics analysis of the optimal stopping time. It says that looking at the stage payoff is sufficient.

For example, consider a monopolist who faces a stochastic linear demand function and produces a homogeneous good at constant cost c. The random

variable a_t is the intercept of the inverse demand and the constant b is the slope. The opportunity cost of staying in the market is k. (Alternately, the firm has to pay a fixed fee, k, each period in order to participate in the production activity.) Then, at each period t, the maximum profit that the monopolist can obtain is

$$\pi(a_t) = \begin{cases} \frac{(a_t - c)^2}{4b} - k & \text{if } a_t \geq c \\ -k & \text{otherwise.} \end{cases}$$

Clearly $\pi(\cdot)$ is an increasing function. Also $\pi(a_t) \equiv \pi(a_t, b, c, k)$ decreases as c, b, or k increases. This fact, together with an application of Proposition 2, implies that the optimal exit time T decreases almost surely as c,b,k increases or β decreases. That is, the monopolist optimally exits earlier when the marginal cost of production is higher, a better alternative opportunity exits (a higher k), the price is less sensitive to the quantity change (a higher b), or the firm is less patient (a smaller β).

3. THE DUOPOLY MODEL - EXIT AS EQUILIBRIUM STOPPING TIME

We next analyze optimal exit in a duopoly. We formulate the exit game and characterize the optimal stopping equilibria. For i,j = 1,2, denote by $\pi_{ij}(a_t)$ the payoff to firm i in period t, given there are j firms in the market and the demand is a_t. We assume that either firm will prefer high demand to low demand and will prefer having the market to itself to sharing the market. That is, $\pi_{ij}(a_t)$ are increasing in a_t for i,j = 1,2, and $\pi_{i2}(z) \leq \pi_{i1}(z)$ for i=1,2 and $z \in Z$. At each time t, the active firms observe $(a_s; s \leq t)$ and then simultaneously decide whether to stay in the market and earn $\pi_{ij}(a_t)$ (depending on the number of firms who stay in) or to exit and get zero profit thereafter. Firm i's

strategy is a stopping time $T_i: \Omega \to N$. The payoff (as a function of T_j) to firm i, $i \neq j$, at time t is completely determined by observing the history $(a_s; s \leq t)$, i.e.,

$$\pi_t^i(a_t, T_j) \equiv \pi_{i2}(a_t) 1_{\{T_j > t\}} + \pi_{i1}(a_t) 1_{\{T_j \leq t\}}.$$

Let β_i $(0 < \beta_i \leq 1)$ be the discount factor for firm i. Then firm i's problem is an optimal stopping problem if its opponent's exit time, T_j, is also a stopping time. Given T_j, $j \neq i$, a stopping time, let

$$v_0^i(a_0) = \sup_T E\{ \sum_{t=0}^{T-1} \beta_i^t \pi_t^i(a_t, T_j) \mid a_0 \},$$

for i = 1,2, where the supremum is over all possible stopping times T.

<u>Definition 1</u>: (T_1, T_2) is a <u>stopping time equilibrium</u> if T_1, T_2 are stopping times and for i=1,2, and $j \neq i$,

$$v_0^i(a_0) = E\{ \sum_{t=0}^{T_i-1} \beta_i^t \pi_t^i(a_t, T_j) \mid a_0 \}.$$

<u>Definition 2</u>: Firm 1 is <u>stronger</u> than Firm 2 if $\pi_{1j}(z) > \pi_{2j}(z)$ for j=1,2 and every z.

To aid the game-theoretic analysis that follows, we first solve four single-firm problems, two for each firm. That is, by applying the results in the previous section, we derive the optimal exit time for each firm i <u>as if</u> there were j firms in the market throughout i's stay in the market. Each single-firm problem is indexed by i and j and takes $\pi_{ij}(\cdot)$ as the stage payoff. We emphasize that the functions calculated below and used in Facts 1-4 <u>do not</u> represent equilibrium behavior. Rather, they provide useful manipulations of the two-firm profit function data, $\pi_{ij}(\cdot)$, that will

be used in the equilibrium analysis to follow. Following the notation in Section 2, define recursively for i=1,2, j=1,2,

$$u_t^{ij}(a_t) = \pi_{ij}(a_t) + \beta_i \, w_{t+1}^{ij}(a_t) ,$$

$$w_{t+1}^{ij}(a_t) = E\,[u_{t+1}^{ij}(a_{t+1})\, 1_{(0,\infty)}(u_{t+1}^{ij}(a_{t+1}))\,|\,a_t],$$

with $w_t^{ij} = 0$ for $t \geq t_{ij}^*$, where $t_{ij}^* = \inf\{t : \pi_{ij}(a_t) \leq 0, \text{ a.s.}\}$. By Propositions 1 and 2 we have the following results:

<u>Fact 1</u>: $T_{ij} = \inf\{t : a_t \leq h_t^{ij}\}$ is the optimal stopping time where $h_t^{ij} = \sup\{z : u_t^{ij}(z) < 0\}$.

<u>Fact 2:</u> w_t^{ij} are decreasing in t and h_t^{ij} are increasing in t for all i,j.

<u>Fact 3</u>: For i=1,2, and for all a_t, $w_{t+1}^{i1}(a_t) \geq w_{t+1}^{i2}(a_t)$, $h_t^{i1} \leq h_t^{i2}$ for $t \geq 0$, and $T_{i1} \geq T_{i2}$ almost surely.

<u>Fact 4</u>: If Firm 1 is stronger than Firm 2, then for j=1,2 and for all a_t, $w_{t+1}^{1j}(a_t) \geq w_{t+1}^{2j}(a_t)$, $h_t^{1j} \leq h_t^{2j}$ for $t \geq 0$, and $T_{1j} \geq T_{2j}$ almost surely.

Figure 2 illustrates h_t^{ij}, t_{ij}^*, a realization of the $\{a_t\}$ process, and stopping times, T_{ij} for the case when Firm 1 is stronger than Firm 2.

For i=1,2, and j≠i, we use the following notation to denote the equilibrium functions analogous to the single-firm functions described above. Suppose (T_1, T_2) is a stopping-time equilibrium. For i=1,2, and for each t, let B_{it} denote a subset of Z that represents the exit set for firm i in the equilibrium (T_1, T_2); and let $\bar{B}_{it}$ denote the complement of this set. That is, $T_i = \inf\{t : a_t \in B_{it}\}$

and $\bar{B}_{it} = Z \setminus B_{it}$. Whereas B_t is always an interval of the form $B_t = \{a_t \leq h_t\}$ in the single-firm problem, B_{it} is, in general, not an interval in the multiperson exit game.

For $t \geq t^*_{i1}$, define $w^i_t(\cdot) = 0$. Then define recursively for $t \leq t^*_{i1}-1$,

$$u^i_t(a_t) = [\pi_{i2}(a_t) + \beta_i\, w^i_{t+1}(a_t)]\, 1_{\bar{B}_{jt}}(a_t),$$

$$+ [(\pi_{i1}(a_t) + \beta_i w^{i1}_{t+1}(a_t)]\, 1_{B_{jt}}(a_t),$$

and

$$w^i_t(a_{t-1}) = E\,[u^i_t(a_t)\, 1_{(0,\infty)} u^i_t(a_t) | a_{t-1}].$$

Finally, define $h^i_t = \sup\{z : \pi_{i2}(z) + \beta_i w^i_{t+1}(z) < 0\}$. The h^i_t variable plays a somewhat different role in the multiperson game from the role of h_t in the single-firm model where we had $B_t = \{a_t \leq h_t\}$. Here, h^i_t does not completely describe the equilibrium strategy of firm i. Rather, it provides an upper bound for the exit set B_{it} of firm i.

Our first result is that the optimal single-firm stopping times, T_{i1} and T_{i2} impose upper and lower bounds for Firm i's equilibrium stopping time. That is, for any equilibrium, (T_1, T_2), $T_{i2} \leq T_i \leq T_{i1}$ almost surely, for i=1,2. The intuitive argument goes as follows: At any time t, it is not i's best response to exit as long as a_t is above h^{i2}_t. This follows because w^{i2}_{t+1} is the expected profit from t+1 on obtained by acting optimally in the situation when other firm will be in the market throughout the game. Therefore, w^{i2}_{t+1} is the minimum possible future gain that firm i can guarantee for itself. If w^i_{t+1} is the expected <u>equilibrium</u> profit from t+1 on, then $w^i_{t+1}(a_t) \geq w^{i2}_{t+1}(a_t)$ and $\pi_{ij}(a_t) + \beta_i w^i_{t+1}(a_t) \geq \pi_{i2}(a_t) + \beta_i w^{i2}_{t+1}(a_t) > 0$ for j=1,2. Hence, Firm i will stay active at time t. By a similar argument, Firm i will exit whenever a_t falls below h^{i1}_t since w^{i1}_{t+1} is the highest possible expected future gain

from t+1 on in the most optimistic situation, i.e., when he will be in the market alone from t+1 on. This completes the arguments required to prove

Proposition 3. Suppose (T_1,T_2) is an exit time equilibrium and $T_i = \inf\{t : a_t \in B_{it}\}$. Then $B_t^{i1} \subseteq B_{it} \subseteq B_t^{i2}$ for $t \geq 0$ and $T_{i2} \leq T_i \leq T_{i1}$, almost surely, for $i=1,2$.

We now turn to an asymmetric situation in which Firm 1 is stronger than Firm 2, that is, $\pi_{1j}(\cdot) > \pi_{2j}(\cdot)$. The natural equilibrium for this situation has $B_{1t} = B_t^{11}$ and $B_{2t} = B_t^{22}$; the weak firm exits the first time that its expected present-plus-future duopoly profits are negative, and the strong firm then remains as a monopolist until its expected monopoly profits turn negative. Although this is the unique euqilibrium that obtains in the models of Fudenberg-Tirole and Ghemawat-Nalebuff, it is only one of many equilibria here.

To built intuition for understanding the complete characterization of the equilibria in our model, consider the situation where $a_t \epsilon (h_t^{21}, h_t^{12}]$. In this case, either firm will find it profitable to stay in the market if his opponent exits, but both firms have negative expected profits if neither exits. Exit by either the strong firm or the weak firm could constitute an equilibrium.

In fact, any equilibrium in this game can be characterized by h_t^1, h_t^2, and a set $A_t \subseteq (h_t^{21}, h_t^1] \subseteq (h_t^{21}, h_t^{12}]$: If both firms are in the market at time t and $a_t \epsilon B_{1t} \equiv (-\infty, h_t^{11}] \cup A_t$, then Firm 1 exits at t and Firm 2 stays. If both firms are in at t and $a_t \epsilon B_t^2 = (-\infty, h_t^2] \cap \bar{A}_t$, then only Firm 2 exits.

Proposition 4 formally states and proves this result.

Proposition 4 Suppose Firm 1 is stronger than Firm 2. Any exit time equilibrium (T_1,T_2) can be characterized as $T_i = \inf\{t: a_t \epsilon B_{it}\}$ for i=1,2. The exit sets B_{it}, i=1,2, $t \geq 0$ are defined recursively as follows:

$$B_{1t} = (-\infty, h_t^{11}] \quad A_t, \; B_{2t} = (-\infty, h_t^2] \cap \bar{A}_t,$$

A_t is a Borel set contained in the interval $(h_t^{21}, h_t^1]$, $\bar{A}_t = Z \setminus A_t$,

$$h_t^i = \sup\{a_t : \pi_{i2}(a_t) + \beta_i w_{t+1}^i(a_t) < 0\},$$

$$w_{t+1}^i(a_t) = 0, \text{ for } t \geq t_{11}^*, \text{ and}$$

$$w_{t+1}^i(a_t) = E[u_{t+1}^i(a_{t+1}) \, 1_{(0,\infty)}(u_{t+1}^i(a_{t+1})) \mid a_t],$$

$$\text{for } 0 \leq t < t_{11}^*,$$

where

$$u_t^i(a_t) = (\pi_{i2}(a_t) + \beta_i w_{t+1}^i(a_t) 1_{\bar{B}_{jt}}(a_t)$$

$$+ (\pi_{i1}(a_t) + \beta_i w_{t+1}^{i1}(a_t)) 1_{B_{jt}}(a_t), \quad j \neq i.$$

Proof: Backward induction.

For $t \geq t_{11}^*$, $P\{\pi_{ij}(a_t) > 0\} = 0$, i,j = 1,2. So $P\{a_t \, \epsilon \, \bar{B}_{it}\} \leq P\{\pi_{11}(a_t) > 0\} = 0$ and $P\{a_t \, \epsilon \, B_{it}\} = 1$ for i=1,2. Also $w_t^{i2}(a_t) = w_t^i(a_t) = w_t^{i1}(a_t) = 0$ for i=1,2. The proposition is trivially true.

Now assume that it is true for $s \geq t+1$. Then by the definitions of h_t^i and $u_t^i(a_t)$, $w_{t+1}^{i2}(a_t) \leq w_{t+1}^i(a_t) \leq w_{t+1}^{i1}(a_t)$ implies $h_t^{i1} \leq h_t^i \leq h_t^{i2}$

for i=1,2. Suppose both firms are in the market at time t. Given any strategy $(B_{2t}, \bar{B}_{2t})$ adopted by Firm 2, Firm 1 will be a monopolist from t on if $a_t \in B_{2t}$, and he will be involved in a game which gives expected payoff w^1_{t+1} if $a_t \in \bar{B}_{2t}$ and equilibrium strategies follow from t+1 on. Remaining active at t, Firm 1 expects to get

$$u^1_t(a_t) = (\pi_{12}(a_t) + \beta_1 w^1_{t+1}(a_t))\, 1_{\bar{B}_{2t}}(a_t) + (\pi_{11}(a_t) + \beta_1 w^{11}_{t+1}(a_t))\, 1_{B_{2t}}(a_t).$$

Firm 1's best response then is to exit if $u^1_t(a_t) \leq 0$ and stay if $u^1_t(a_t) > 0$.

Let $B_{1t} = \{u^1_t(a_t) \leq 0\}$ and $\bar{B}_{1t} = \{u^1_t(a_t) > 0\}$. Firm 1 will exit if Firm 2 exits and $a_t \leq h^{11}_t$ or if Firm 2 stays and $a_t \leq h^1_t$. That is,

$$B_{1t} = \{B_{2t} \cap (-\infty, h^{11}_t]\} \cup \{\bar{B}_{2t} \cap (-\infty, h^1_t]\}. \tag{3a}$$

Similarly,

$$\bar{B}_{1t} = \{B_{2t} \cap (h^{11}_t, \infty)\} \cup \{\bar{B}_{2t} \cap (h^1_t, \infty)\}. \tag{3b}$$

Given the strategy of Firm 1, if Firm 2 does not exit at t, he will expect

$$u^2_t(a_t) = (\pi_{22}(a_t) + \beta_2 w^1_{t+1}(a_t))\, 1_{\bar{B}_{1t}}(a_t)$$

$$+ (\pi_{21}(a_t) + \beta_2 w^{21}_{t+1}(a_t))\, 1_{B_{1t}}(a_t).$$

and 2's best response should satisfy

$$B_{2t} = \{B_{1t} \cap (-\infty, h^{21}_t]\} \cup \{\bar{B}_{1t} \cap (-\infty, h^2_t]\} \tag{4a}$$

$$\bar{B}_{2t} = \{B_{1t} \cap (h^{21}_t, \infty)\} \cup \{\bar{B}_{1t} \cap (h^2_t, \infty)\}. \tag{4b}$$

But, using (3a) and (3b) to expand the sets in (4b), we get

$$B_{1t} \cap (h^{21}_t, \infty) = \{\{B_{2t} \cap (-\infty, h^{11}_t]\} \cup \{\bar{B}_{2t} \cap (-\infty, h^1_t]\}\} \cap (h^{21}_t, \infty)$$

$$= \bar{B}_{2t} \cap (h^{21}_t, h^1_t] \quad \text{since } h^{11}_t \leq h^{21}_t.$$

Similarly,

$$\bar{B}_{1t} \cap (h_t^2, \infty) = \{\{B_{2t} \cap (h_t^{11}, \infty)\} \cup \{\bar{B}_{2t} \cap (h_t^1, \infty)\}\} \cap (h_t^2, \infty)$$

$$= (h_t^2, \infty) \setminus \{\bar{B}_{2t} \cap (h_t^2, h_t^1]\} \quad \text{since } h_t^{11} \leq h_t^{21} \leq h_t^2.$$

Using (4) and the above characterizations,

$$\bar{B}_{2t} = \{\bar{B}_{2t} \cap (h_t^{21}, h_t^1]\} \cup \{(h_t^2, \infty) \setminus \{\bar{B}_{2t} \cap (h_t^2, h_t^1]\}\}$$

$$= \{\bar{B}_{2t} \cap (h_t^{21}, h_t^1]\} \cup (h_t^2, \infty) \quad \text{since } h_t^{21} \leq h_t^2 .$$

Thus $\bar{B}_{2t}$ is part of the equilibrium if and only if $\bar{B}_{2t}$ satisfies

$$\bar{B}_{2t} = \{\bar{B}_{2t} \cap (h_t^{21}, h_t^1]\} \cup (h_t^2, \infty)\}$$

The only possible solution is of the form

$$\bar{B}_{2t} = A_t \cup (h_t^2, \infty), \text{ where } A_t \text{ is a Borel subset of } (h_t^{21}, h_t^1].$$

Furthermore, B_{2t} is just the complement of $\bar{B}_{2t}$, i.e.,

$$B_{2t} = \bar{A}_t \cap (-\infty, h_t^2], \quad \text{where } \bar{A}_t = Z \setminus A_t.$$

If $f_{2t}(B) \equiv \{B \cap (h_t^{21}, h_t^1]\} \cup (h_t^2, \infty)$, a mapping from a Borel set in Z to a Borel set in Z, then B_{2t} is an equilibrium strategy if and only if $\bar{B}_{2t}$ is a fixed point of $f_{2t}(\cdot)$, i.e., $f_{2t}(\bar{B}_{2t}) = \bar{B}_{2t}$.

By (3), our characterizations of $\bar{B}_{2t}$ and B_{2t} give

$$B_{1t} = (-\infty, h_t^{11}] \cup A_t, \text{ and}$$

$$\bar{B}_{1t} = (h_t^{11}, \infty) \cap \bar{A}_t .$$

Finally, note that since $w^{i2}_{t+1}(a_t) \leq w^{i}_{t+1}(a_t) \leq w^{i1}_{t+1}(a_t)$, $u^{i2}_t(a_t) \leq u^{i}_t(a_t) \leq u^{i1}_t(a_t)$, which implies $w^{i2}_t(a_t) \leq w^{i}_t(a_t) \leq w^{i1}_t(a_t)$ by Lemma 1. We conclude the induction. Q.E.D.

The importance of Proposition 4 lies in the fact that it characterizes all possible equilibria. In particular, it points out that we can obtain all equilibria by varying A_t, a Borel subset contained in the interval $(h^{21}_t, h^1_t]$, for every t. Among these are two extreme equilibria which give upper and lower bounds on any equilibrium exit times (T_1, T_2). The first of these, $(\bar{T}_1, \underline{T}_2)$ is obtained by letting $A_t = \emptyset$ for all t; then $B'_{1t} = (-\infty, h^{11}_t]$, $B'_{2t} = (-\infty, h^{22}_t]$, almost surely for every t. In fact, $(\bar{T}_1, \underline{T}_2) = (T_{11}, T_{22})$, analogous to the equilibrium of Fudenberg-Tirole and Ghemawat-Nalebuff. This equilibrium is the one most preferred by Firm 1, the stronger firm, because, as will be shown in Proposition 5, it gives Firm 1 the latest possible exit time.

The second extreme equilibrium, $(\underline{T}_1, \bar{T}_2)$ gives Firm 2, the weaker firm, the latest possible exit time. It is obtained by setting $\underline{h}^1_{t-1} = h^{12}_{t-1}$ for $t = t^*_{11}$, and then defining recursively for $t \leq t^*_{11} - 1$, $A_t = (h^{21}_t, \underline{h}^1_t]$, and $\underline{h}^1_t, \bar{h}^2_t, \underline{w}^1_{t+1}(a_t)$, $\bar{w}^2_{t+1}(a_t)$, $\underline{u}^1_t(a_t)$, $\bar{u}^2_t(a_t)$ as in Proposition 4, where $B''_{1t} = (-\infty, h^{11}_t] \cup (h^{21}_t, \underline{h}^1_t]$ and $\bar{B}''_{2t} = (h^{21}_t, \underline{h}^1_t] \cup (\bar{h}^2_t, \infty)$. The next proposition shows that this equilibrium gives Firm 2 the longest possible time in the market.

<u>Proposition 5:</u> Let (T_1, T_2) be an equilibrium. Then $\underline{T}_1 \leq T_1 \leq \bar{T}_1$ and $\underline{T}_2 \leq T_2 \leq \bar{T}_2$, almost surely.

<u>Proof:</u> Let B_{1t}, $w_t^1(a_t)$, $u_t^1(a_t)$, $u_t^2(a_t)$, h_t^1, h_t^2, and $A_t \subsetneq (h_t^{21}, h_t^1]$ correspond to the equilibrium (T_1, T_2) in the usual way.

Note that for $t \geq t_{11}^*$, $w_t^{11} = w_t^1 = \underline{w}_t^1 = 0$ and $\bar{w}_t^2 = w_t^2 = w_t^{22} = 0$.

Assume $w_{t+1}^{11} \geq w_{t+1}^1 \geq \underline{w}_{t+1}^1$ and $\bar{w}_{t+1}^2 \geq w_{t+1}^2 \geq w_{t+1}^{22}$. Then, $h_t^{11} \leq h_t^1 \leq \underline{h}_t^1$.

Note that $B_{1t}' = (-\infty, h_t^{11}] \subseteq (-\infty, h_t^{11}] \cup A_t = B_{1t}$,

and $B_{1t} \subseteq (-\infty, h_t^{11}] \cup (h_t^{21}, h_t^1] \subseteq (-\infty, h_t^{11}] \cup (h_t^{21}, \underline{h}_t^1] = B_t''$.

Also, $\bar{B}_{2t}' = (h_t^{22}, \infty) \subseteq (h_t^2, \infty) \subseteq A_t \cup (h_t^2, \infty) = \bar{B}_{2t}$, and

$$\bar{B}_{2t} \subseteq (h_t^{21}, h_t^1] \cup (h_t^2, \infty) \subseteq (h_t^{21}, \underline{h}_t^1] \cup (\bar{h}_t^2, \infty) = \bar{B}_{2t}''.$$

It follows that $u_t^{11}(a_t) \geq u_t^1(a_t) \geq \underline{u}_t^1(a_t)$ and $\bar{u}_t^2(a_t) \geq u_t^2(a_t) \geq u_t^{21}(a_t)$ for every a_t. We just show $u_t^1(a_t) \geq \underline{u}_t^1(a_t)$ as an example.

$$\begin{aligned} u_t^1(a_t) &= (\pi_{12}(a_t) + \beta_1 w_{t+1}^1(a_t))\, 1_{\bar{B}_{2t}}(a_t) \\ &\quad + (\pi_{11}(a_t) + \beta_1 w_{t+1}^{11}(a_t))\, 1_{B_{2t}}(a_t) \\ &\geq (\pi_{12}(a_t) + \beta_1 \underline{w}_{t+1}^1(a_t))\, 1_{\bar{B}_{2t}''}(a_t) \\ &\quad + (\pi_{11}(a_t) + \beta_1 w_{t+1}^{11}(a_t))\, 1_{B_{2t}''}(a_t) = \underline{u}_t^1(a_t). \end{aligned}$$

Since $w^{1}_{t+1}(a_t) \geq \underline{w}^{1}_{t+1}(a_t)$, $\bar{B}_{2t} \subseteq \bar{B}''_{2t}$ and $B_{2t} \supseteq B''_{2t}$. Applying Lemma 1 again, we have $w^{11}_t(a_t) \geq w^{1}_t(a_t) \geq \underline{w}^{1}_t(a_t)$ and $\bar{w}^{2}_t(a_t) \geq w^{2}_t(a_t) \geq w^{22}_t(a_t)$. The induction is completed. As a result, $B''_{1t} \supseteq B_{1t} \supseteq B'_{1t}$ and $B'_{2t} \supseteq B_{2t} \supseteq B''_{2t}$ imply that $\underline{T}_1 \leq T_1 \leq \bar{T}_1$ and $\underline{T}_2 \leq T_2 \leq \bar{T}_2$. Q.E.D.

Since Firm 1 is assumed to be stronger than Firm 2, the first extreme case (T_{11}, T_{22}) is, in some sense, the more appealing equilibrium. This equilibrium is also simple in structure, i.e., a cut-off equilibrium. Ghemawat and Nalebuff (1985) show that this is the unique subgame-perfect equilibrium in their deterministic, continuous time model. Their result depends on their assumption that industry demand declines continuously over time. An analogous assumption, for our model, would be that, with probability one, a_t falls into the interval $(\underline{h}^1_t, \bar{h}^2_t]$ before falling into $(h^{21}_t, \underline{h}^1_t)$. In this case, the unique equilibrium has the weaker firm exiting first. (This is stated and proved formally in Proposition 6.) Conversely, if Ghemawat and Nalebuff (or Fudenberg and Tirole) were to allow jumps of sufficient magnitude into their declining demand paths, they would lose uniqueness as our model does.

<u>Proposition 6</u>: Suppose Firm 1 is stronger than Firm 2. Then (T_{11}, T_{22}) is the unique equilibrium if and only if $P\{a_t \in (h^{21}_t, \underline{h}^1_t]\} = 0$ for $1 \leq t \leq \hat{t}_2 - 1$ where $\hat{t}_2 = \inf\{t: P\{\bigcap_{s=0}^{t} \{a_s \in \bar{B}''_{2t}\}\} = 0\}$.

<u>Proof</u>: For $t = \hat{t}_2 - 1$, $P\{a_t \in \bar{B}''_{2t}\} = 0$, or $P\{a_t \in (h^{21}_t, \underline{h}^1_t] \cup (\bar{h}^2_t, \infty)\} = 0$. This implies that $\bar{B}'_{2t} = \bar{B}''_{2t}$ a.s. and consequently $B'_{1t} = B''_{1t}$. Assume

$B'_{it+1} = B''_{it+1}$ a.s. It then follows that $\underline{h}_t^i = \bar{h}_t^{-i}$ which together with the condition $P\{a_t \in (h_t^{21}, \underline{h}_t^1]\} = 0$ imply that $B'_{it} = B''_{it}$ a.s. for i=1,2. The sufficiency follows.

To prove the necessary conditions, let

$$t = \sup \{s: \ 1 \leq s \leq \hat{t}_2 - 1, \ P\{a_s \in (h_s^{21}, \underline{h}_s^1]\} > 0\}.$$

Then,

$$P\{\bar{T}_2 > \underline{T}_2\} \geq P\{\bigcap_{s=0}^{t-1} \{a_s \in \bar{B}''_{2s}\} \cap \{a_t \in (h_t^{21}, \underline{h}_t^1]\}\}$$

$$= P\{a_t \in (h_t^{21}, \underline{h}_t^1] \mid a_{t-1} \in \bar{B}''_{2t-1}\} \cdot P\{\bigcap_{s=0}^{t-1} \{a_s \in \bar{B}''_{2s}\}\} > 0$$

since $\bigcap_{s=0}^{t-1} \bar{B}'_{2s} \subseteq \bigcap_{s=0}^{t-1} \bar{B}''_{2s}$ and $(h_t^{21}, \underline{h}_t^1] = \bar{B}''_{2t} \cap B'_{2t}$. Q.E.D.

The following example illustrates some comparative static results that can be obtained with linear cost and demand functions. A Cournot duopoly faces a stochastic linear inverse demand, $p_t = a_t - bQ$, where a_t are independent and decreasing in t, and produces a homogeneous good at constant costs c_i, i = 1,2. Assume that $c_1 \leq c_2$ so that Firm 1 is stronger than Firm 2. Firm i's opportunity cost of staying in the market is k_i. If both firms are in the market, the equilibrium stage payoffs are

$$\pi_{12}(a_t) = \begin{cases} \dfrac{(a_t - 2c_1 + c_2)^2}{9b} - k_1 & \text{if } a_t > 2c_2 - c_1 \\[2ex] \dfrac{(a_t - c_1)^2}{4b} - k_1 & \text{if } c_1 \leq a_t < 2c_2 - c_1 \\[2ex] - k_1 & \text{otherwise.} \end{cases}$$

$$\pi_{22}(a_t) = \begin{cases} \dfrac{(a_t - 2c_2 + c_1)^2}{9b} - k_2 & \text{if } a_t \geq 2c_2 - c_1 \\ - k_2 & \text{otherwise.} \end{cases}$$

If one firm is in the market alone, the monopoly stage profits are as derived in Example 1, for $i = 1,2$,

$$\pi_{i1}(a_t) = \begin{cases} \dfrac{(a_t - c_i)^2}{4b} - k_i & \text{if } a_t \geq c_i \\ - k_i & \text{otherwise.} \end{cases}$$

It is easy to see that $\pi_{i1}(a_t) \geq \pi_{i2}(a_t)$ for $i = 1,2$ and every a_t and if $c_1 \leq c_2$, $k_1 = k_2$ or $c_1 = c_2$, $k_1 \leq k_2$, then Firm 1 is stronger, i.e., $\pi_{1j}(a_t) \geq \pi_{2j}(a_t)$ for $j = 1,2$ and every a_t.

From the general results obtained earlier, we know that (T_{11}, T_{22}) is an exit equilibrium. The functions h_t^{11}, h_t^{22} can be calculated as follows:

$$h_t^{11} = \begin{cases} c_1 + \sqrt{4b(k_1 - \beta_1 w_{t+1}^{11})} & \text{if } k_1 \geq \beta_1 w_{t+1}^{11} \\ c_1 & \text{otherwise.} \end{cases}$$

$$h_t^{22} = \begin{cases} 2c_2 - c_1 + \sqrt{9b(k_2 - \beta_2 w_{t+1}^{22})} & \text{if } k_2 \geq \beta_2 w_{t+1}^{22}, \\ 2c_2 - c_1 & \text{otherwise.} \end{cases}$$

And h_t^{11} increases as c_1, b, k_1 increases or β_1 decreases, while h_t^{22} increases as c_2, b, k_2 increases or c_1, β_2 decreases. Thus T_{11} increases almost surely as c_1, b, k_1 decreases or β_1 increases and T_{22} increases almost surely as c_2, b, k_2 decreases or c_1, β_2 increases.

We conclude this section by pointing out that the stopping time equilibria and the subgame perfect equilibria of the exit game are equivalent. By way of constructing a stopping time equilibrium in Proposition 4, it is clear that the stopping time $T_i = \inf\{s: s \geq t, a_s \in B_{is}\}$ constitute an equilibrium for the remaining subgame, given the firms surviving until t. Conversely, if we analyze the exit problem in an extensive form game assuming players have perfect recall of the historic information but not the future, etc., then any subgame perfect equilibrium is a stopping time equilibrium.

Proposition 7. The stopping time equilibria are subgame perfect.

4. CONCLUSION

This paper illustrates how a stochastic dynamic game like exit can be formulated as a stopping time problem, how a stopping time equilibrium can be found by solving a sequence of fixed point problems, and how to derive the properties that the stopping time equilibria possess. Furthermore it shows how demand processes that are not continuous can give rise to multiple exit time equilibria.

There are several interesting extensions or variations that might be explored in the continuation of this work. A straightforward, but notationally burdensome, generalization is the exit game for an n-firm oligopoly. The equilibrium in which the stronger firms exit later always exists if the firms are ordered by strength. A second direction for generalizing the model is to incorporate entry decisions into the model to study the firm's behavior over the life cycle of an industry. Though we only

discuss the exit problem for oligopoly here, the approach employed can be applied to entry problems as well. Finally, other interesting extensions may include stopping time equilibria in stochastic exit games with incomplete information (as in Fudenberg and Tirole) and stochastic exit games in continous time.

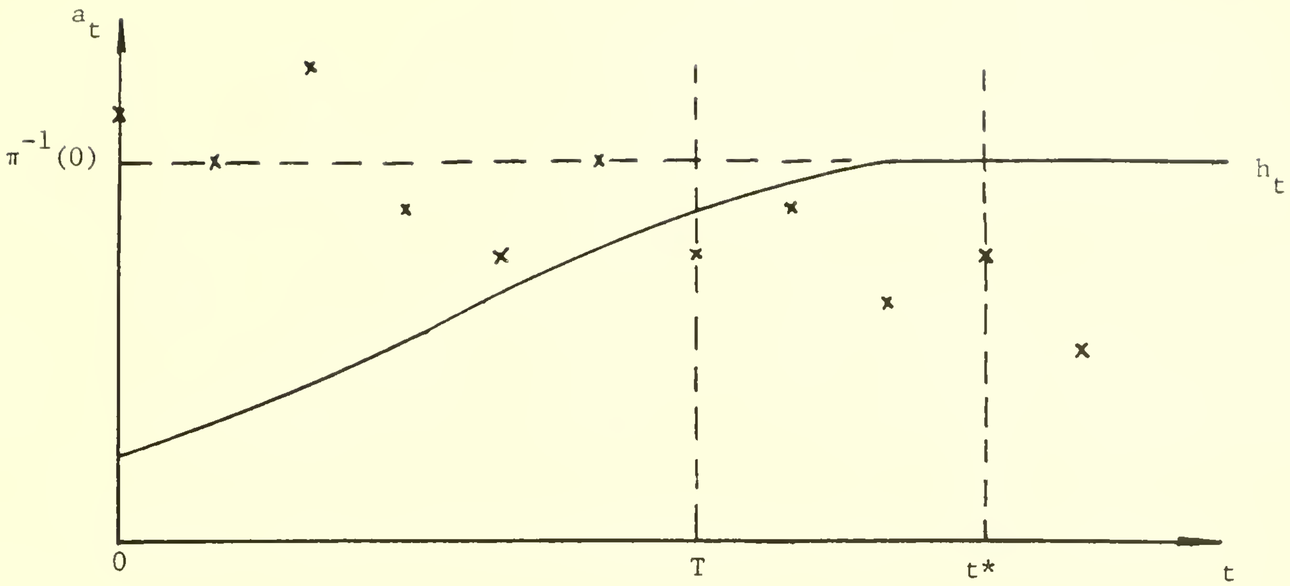

Figure 1. Single-firm stopping problem in the monopoly model

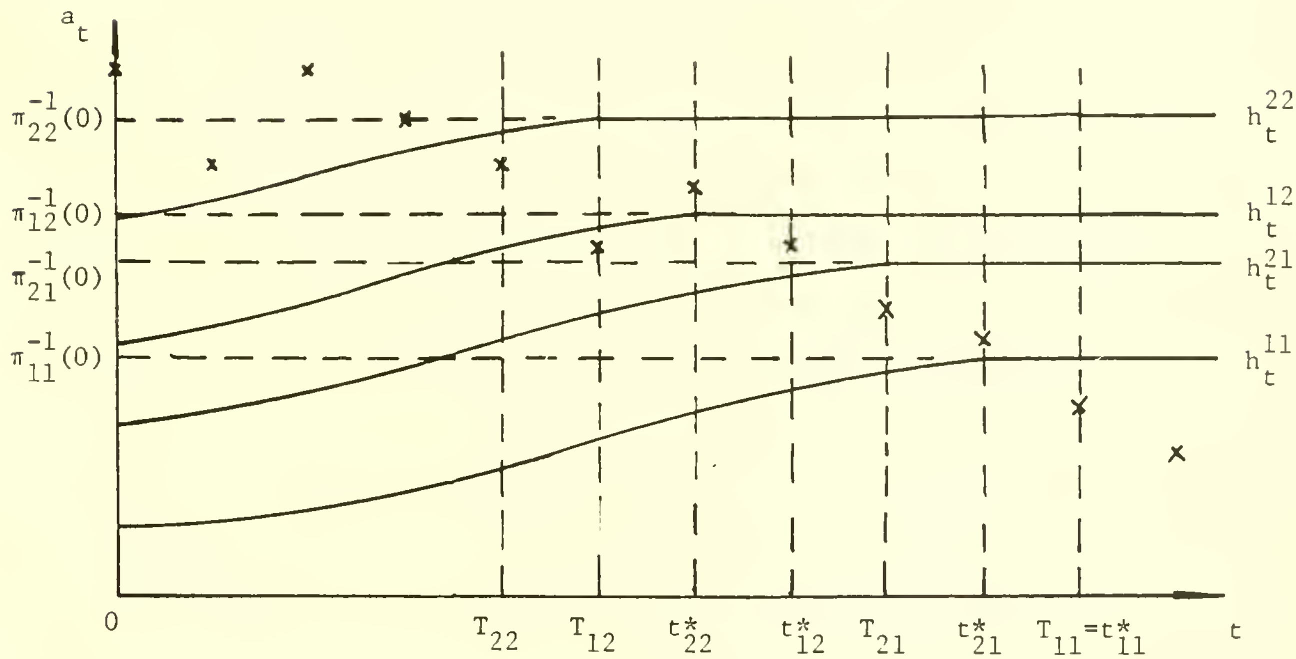

Figure 2. Four single-firm stopping problems in the duopoly model.

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