

AN APPROXIMATION METHOD WITH RATIONAL FUNCTIONS

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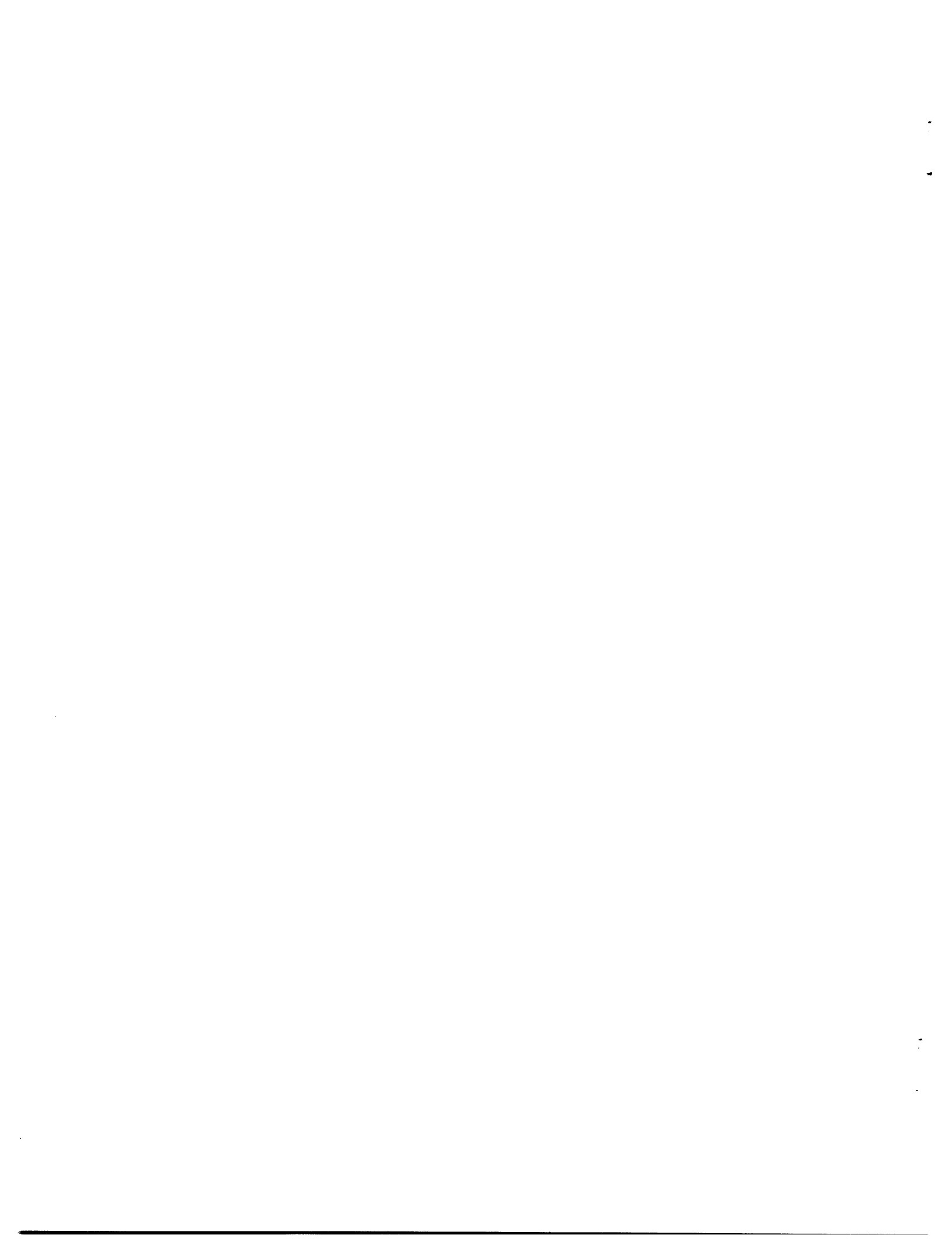
Nick De Claris

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Abstract

A general method is developed for finding functions of a single complex variable s , which approximates an assigned network characteristic, within the special class of functions realizable as networks of linear lumped parameter elements. The method is based upon an interpolation technique with a series of general rational functions on the unit circle of the z -plane. A number of transformations that map the interval of interest of the s -plane into the unit circle of the z -plane are discussed.

A great advantage of this method is that it allows one to preassign the pole location of the desired rational function anywhere in the left-half of the s -plane. Following a formal mathematical treatment, procedures are outlined for three cases of approximation in both the frequency and the time domain. A number of examples illustrate the wide range of applicability.



I. INTRODUCTION

Statement of the Problem. Effects of preassigning the pole location. Approximation problems in the time domain and the complex-frequency domain.

In using the synthesis techniques of lumped parameter network theory, one is dealing, essentially, with rational functions of a single complex variable s . The investigation described in this report was undertaken in an effort to establish a general method for determining transfer impedance or admittance functions with an arbitrary but finite number of preassigned poles anywhere in the left-half of the s -plane; that is, a desired functional characteristic will be effectively approximated by a rational function in s , the location of the poles being preassigned.

Preassigning the pole location of a transfer function plays a very important role in the final realization of the network. In particular, it determines: (a) the "character" of the network. That is, whether the network will contain two types of elements (RC or RL) or all three types (RLC); (b) the "general form" of the network; (c) the "Q" of the elements to be used in the network. Clearly an approximation procedure of this type enables one to design high-Q circuits with relatively low-Q elements.

This report provides separate sections for the mathematically-minded and for the applied engineer. Thus, Section I contains a formal mathematical treatment of the approximation problem leading to an analytic expansion by means of a complete set of rational functions orthogonal on the unit circle. The coefficients of the expansion are evaluated by interpolation; we use a recurrence formula to avoid complex integration. Section II is intended especially for the applied engineer. Here all of the mathematical developments of the previous section are summarized and presented in a case-study manner to maximize their use in the design of linear lumped parameter networks.

A very important feature of the approximation method described in this report is that it enables one to approximate on some contour of the complex plane the function itself, rather than its modulus. Indeed, in some applications of electric circuitry, one is interested in both the magnitude and phase characteristics of a network in a specified frequency interval.

Dr. M. V. Cerrillo has established a number of very important relationships concerning the parameter presentation of time- and frequency-domain functions. Of primary importance in his formulation, which unifies some problems of network synthesis in the time and frequency domains, is the so-called density distribution function $U(\gamma_0, \lambda)$ — that is, the real part of the complex transfer function on a contour characterized by γ_0, λ . The method presented in this report can most easily be used to approximate the $U(\gamma_0, \lambda)$ function. Since very little is included in this report about the $U(\gamma_0, \lambda)$ function, the reader interested in this topic is advised to acquaint himself with Dr. M. V. Cerrillo's "On Basic Existence Theorems in Network Synthesis."

II. MATHEMATICAL DEVELOPMENT

Functions of best approximation. Approximation through interpolation. Construction of rational functions of best approximation. Evaluation of the coefficients of interpolation.

2.1 Preliminary Definitions

In the present analysis we shall be concerned entirely with functions of a single complex variable. Frequent use will be made of the plane of the complex variable $z = x + jy$ or $s = \sigma + j\omega$; thus, knowledge of the elementary definitions of limit, continuity, and convergence is required.

An analytic function $F(z)$ is said to be approximated by a sequence of rational functions $R_n(z)$ on an arc or curve C if

$$F(z) = R_n(z) \quad \text{as } n \rightarrow \infty \quad (2.1)$$

where z is on C . The effectiveness of an approximation for a fixed index n is measured: (a) in the sense of least squares; (b) in the sense of Chebyshev.

Lemma I. Let the function $F(z)$ be analytic on a curve C . Then the function $R_n(z)$ is of best approximation to $F(z)$ on C in the sense of least squares if the integral

$$e_s = \int_C |F(z) - R_n(z)|^2 |dz| \quad z \text{ on } C \quad (2.2)$$

is minimum.

Lemma II. The function $R_n(z)$ is of best approximation to an analytic function $F(z)$ on C in the sense of Chebyshev if the value of the modulus

$$e_t = \max |F(z) - R_n(z)| \quad z \text{ on } C \quad (2.3)$$

is minimized.

The concept of best approximation is somewhat generalized by the introduction of a weight or norm function $N(z)$ in the following manner:

$$e_s = \int_C N(z) |F(z) - R_n(z)|^2 |dz| \quad z \text{ on } C \quad (2.4)$$

$$e_t = \max [N(z) |F(z) - R_n(z)|] \quad z \text{ on } C \quad (2.5)$$

For the purpose of this analysis, we shall always use $N(z) = 1$.

A further extension of best approximation in the sense of least squares is that, in the sense of least p^{th} powers as measured by the integral,

$$\int_C N(z) |F(z) - R_n(z)|^p |dz| \quad p > 0 \quad (2.4.1)$$

where z is on C .

2.2 Orthogonality and Best Approximation

Let a finite or infinite set of functions $D_0(z), D_1(z), \dots$, be integrable, in the sense of Lebesgue, on an arc or curve C of the complex plane z . The set of functions $D_i(z)$ is said to be orthogonal on C if we have

$$\int_C D_i(z) \overline{D_j(z)} |dz| = 0 \quad i \neq j \quad (2.6)$$

If a function $F(z)$ is equal on C to the series

$$F(z) = \sum_{i=0}^{\infty} A_i D_i(z) \quad (2.7)$$

then the coefficients A_i are uniquely determined by

$$A_i = \frac{\int_C F(z) \overline{D_i(z)} |dz|}{\int_C |D_i(z)|^2 |dz|} \quad (2.8)$$

and Eq. 2.7 is called the formal expansion of $F(z)$ on C in terms of the orthogonal functions $D_i(z)$.

Theorem 1. If a function $F(z)$ is analytic on a curve C and can be expressed by means of a formal expansion on C in terms of a set of orthogonal functions $D_i(z)$, then the linear combination

$$F_n(z) = A_0 + A_1 D_1(z) + A_2 D_2(z) + \dots + A_n D_n(z) \quad (2.9)$$

approximates $F(z)$ best on C in the sense of least squares.

Let a linear combination

$$\sum_{i=0}^n \lambda_i D_i(z) \quad (2.10)$$

be of best approximation to the function $F(z)$ in the sense of least squares. Then λ_i is chosen to minimize the integral

$$e_s = \int_C \left| F(z) - \sum_{i=0}^n \lambda_i D_i(z) \right|^2 |dz| \quad (2.11)$$

But

$$e_s = \int_C \left[F(z) - \sum_{i=0}^n \lambda_i D_i(z) \right] \left[F(z) - \sum_{i=0}^n \bar{\lambda}_i \overline{D_i(z)} \right] |dz|$$

or

$$\begin{aligned} e_s = & \int_C F(z) \overline{F(z)} |dz| - \sum_{i=0}^n |A_i|^2 \int_C |D_i(z)|^2 |dz| \\ & + \sum_{i=0}^n (A_i - \lambda_i) (\bar{A}_i - \bar{\lambda}_i) \int_C |D_i(z)|^2 |dz| \end{aligned} \quad (2.12)$$

Clearly, the foregoing expression is minimum when $\lambda_i = A_i$.

Corollary 1. In Theorem 1 the inequality known as Bessel's inequality is obvious.

$$\int_C |F(z)|^2 |dz| > \sum_{i=0}^n |A_i|^2 \int_C |D_i(z)|^2 |dz| \quad (2.13)$$

Corollary 2. In Theorem 1 the series

$$\sum_{i=0}^n |A_i|^2 \int_C |D_i(z)|^2 |dz| \quad (2.14)$$

is convergent.

Corollary 3. In Theorem 1 the difference $F(z) - F_n(z)$ is orthogonal to each of the functions $D_i(z)$; this property completely characterizes the function $F_n(z)$.

The proof follows from Corollary 1.

2.3 Approximation on the Unit Circle

It has been shown by a number of mathematicians that approximation is connected intimately with interpolation on a specified contour of the complex plane. Before we proceed in this direction, it is convenient to prove first:

Theorem 2. If a function $E(z)$ analytic on and within $C : |z| = 1$ vanishes in the point $z = 1/\bar{a}$, then $E(z)$ is orthogonal on C to the function

$$\frac{1}{z - a} \quad |a| > 1 \quad (2.15)$$

For z on $C : |z| = 1$ we have

$$z = e^{j\theta} \quad \text{and} \quad dz = je^{j\theta} d\theta$$

or

$$d\theta = -je^{-j\theta} dz$$

But

$$d\theta = |dz|$$

thus

$$|dz| = -j\bar{z} dz \tag{2.16}$$

To completely prove the theorem, it is sufficient from the definition of orthogonality that

$$\int_C E(z) \frac{|dz|}{\bar{z} - \bar{a}} = 0 \tag{2.17}$$

However, from this, since $z\bar{z}$ equals 1 it is obvious that

$$\frac{|dz|}{\bar{z} - \bar{a}} = \frac{j}{\bar{a}} \frac{dz}{z - 1/\bar{a}}$$

Thus

$$\int_C E(z) \frac{|dz|}{\bar{z} - \bar{a}} = \frac{j}{\bar{a}} \int_C E(z) \frac{dz}{z - 1/\bar{a}} \tag{2.18}$$

which vanishes by Cauchy's integral formula.

Corollary. If the function $E(z)$ analytic on and within $C : |z| = 1$ vanishes in the origin, then $E(z)$ is orthogonal on C to the function 1.

Clearly, this is the limiting case of Theorem 2 where a is infinite.

We are now in a position to consider the following:

Theorem 3. Let the function $F(z)$ be analytic on and within $C : |z| = 1$. Then a rational function $R_n(z)$ of the form

$$R_n(z) = \frac{B_n z^n + B_{n-1} z^{n-1} + \dots + B_0}{\prod_{i=1}^n (z - a_i)} \quad |a_i| > 1 \tag{2.19}$$

is unique and of best approximation to $F(z)$ on C in the sense of least squares if it interpolates $F(z)$ in the points $0, 1/\bar{a}_i$ where $i = 1, 2, \dots, n$.

If we let

$$F(z) - R_n(z) = E(z)$$

from Theorem 2, it is evident that if $R_n(z)$ interpolates $F(z)$ in the points $0, 1/\bar{a}_i$, then

Furthermore, it is essential for the set given above that

$$C_j(z) \Big|_{z=1/\bar{a}_i} = 0 \quad i = 0, 1, 2, \dots, k$$

for $j \geq k + 1$; that is, the rational function $C_j(z)$ vanishes at the points $0, 1/\bar{a}_1, 1/\bar{a}_2, \dots, 1/\bar{a}_{j-1}$. Thus

$$Q_j(z) = z \prod_{i=1}^{j-1} (1 - z\bar{a}_i) \quad (2.23)$$

We have therefore established:

Theorem 4. The sequence of general rational functions

$$\left. \begin{aligned} C_0(z) &= 1 \\ C_1(z) &= \frac{z}{z - \alpha_1} \\ C_2(z) &= \frac{z(1 - \bar{a}_1 z)}{(z - \alpha_1)(z - \alpha_2)} \\ &\dots \\ C_n(z) &= \frac{z(1 - \bar{a}_1 z)(1 - \bar{a}_2 z) \dots (1 - \bar{a}_{n-1} z)}{(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)} \end{aligned} \right\} \quad (2.24)$$

forms a complete set of interpolating functions of best approximation on $C : |z| = 1$ in the sense of least squares.

Corollary. The functions

$$1, \frac{z}{z - \alpha_1}, \frac{z(1 - \bar{a}_1 z)}{(z - \alpha_1)(z - \alpha_2)}, \dots$$

are orthogonal on $C : |z| = 1$.

The sum $S_n(s)$ of the first $n + 1$ terms of the sequence given above interpolates $F(z)$ in the points $0, 1/\bar{a}_1, 1/\bar{a}_2, \dots, 1/\bar{a}_n$; hence, by Corollary 3 of Theorem 1, the function $F(z) - S_n(s)$ is orthogonal to each of the functions given above.

Now we are in a position to state the following:

Theorem 5. Let the function $F(z)$ be analytic on and within $C : |z| = 1$. Then the series

$$R_n(z) = D_0 + \sum_{i=1}^n D_i \frac{z \prod_{j=1}^{i-1} (1 - \bar{a}_j z)}{\prod_{j=1}^i (z - \alpha_j)} \quad i > j > 0 \quad (2.25)$$

is of best approximation to $F(z)$ on C in the sense of least squares.

From the Corollary of Theorem 4, it is obvious that the series of Eq. 2.25 as $n \rightarrow \infty$ is identical with the formal expansion of $F(z)$ in a series of orthogonal functions. Furthermore, $R_n(z)$ in this form indeed interpolates $F(z)$ on C in the points $0, 1/\bar{a}_1, 1/\bar{a}_2, \dots, 1/\bar{a}_n$. Thus the truth of the statement of Theorem 5 is obvious.

Corollary. For the conditions of Theorem 5, we have

$$D_i = \frac{\alpha_i \bar{a}_i - 1}{2\pi j} \int_C F(z) \frac{\prod_{k=1}^{i-1} (z - a_k)}{z \prod_{k=1}^i (1 - \bar{a}_k z)} dz \quad (2.26)$$

When α_i equals ∞ , the factor $\alpha_i \bar{a}_i - 1$ is replaced by 1.

From the definition of approximation, it is evident that in the limit

$$F(z) = R_n(z) \quad \text{as } n \rightarrow \infty$$

for z on $C : |z| = 1$. Thus

$$F(z) = \sum_{i=0}^{\infty} D_i C_i(z) \quad z \text{ on } C \quad (2.27)$$

However, since $C_i(z)$ is a set of orthogonal functions,

$$\int_C F(z) \frac{\bar{z}(1 - \alpha_1 \bar{z}) \dots (1 - \alpha_{i-1} \bar{z})}{(\bar{z} - \bar{a}_1) \dots (\bar{z} - \bar{a}_i)} |dz| = D_i \int_C \frac{\bar{z}(1 - \alpha_1 \bar{z}) \dots (1 - \alpha_{i-1} \bar{z})}{(\bar{z} - \bar{a}_1) \dots (\bar{z} - \bar{a}_i)} |dz| \quad (2.28)$$

Furthermore, since the contour of integration is the unit circle,

$$z = \frac{1}{\bar{z}} \quad \text{and} \quad |dz| = -j\bar{z} dz \quad (2.29)$$

Evaluating the right-hand side of Eq. 2.28, we have

$$D_i \int_C \frac{\bar{z}(1 - \alpha_1 \bar{z}) \dots (1 - \alpha_{i-1} \bar{z})}{(\bar{z} - \bar{a}_1) \dots (\bar{z} - \bar{a}_i)} |dz| = D_i \frac{2\pi}{\alpha_i \bar{a}_i - 1} \quad (2.30)$$

With Eq. 2.29, it can be easily shown that

$$\int_C F(z) \frac{\bar{z}(1 - \alpha_1 \bar{z}) \dots (1 - \alpha_{i-1} \bar{z})}{(\bar{z} - \bar{a}_1) \dots (\bar{z} - \bar{a}_i)} |dz| = \frac{1}{j} \int_C F(z) \frac{(z - \alpha_1) \dots (z - \alpha_{i-1})}{z(1 - \bar{a}_1 z) \dots (1 - \bar{a}_i z)} dz \quad (2.31)$$

Therefore, from Eqs. 2.28, 2.30, and 2.31,

$$D_i \frac{2\pi}{\alpha_i \bar{a}_i - 1} = \frac{1}{j} \int_C F(z) \frac{(z - \alpha_1) \dots (z - \alpha_{i-1})}{z(1 - \bar{a}_1 z) \dots (1 - \bar{a}_i z)} dz$$

By direct application of this notation, one can easily rewrite Eq. 2.34 in the form

$$\begin{aligned}
 D_1(\bar{a}_1) &= \left[F(1/\bar{a}_1) - D_0 \right] (1 - \alpha_1 \bar{a}_1) \\
 D_2(\bar{a}_2) &= \left[D_1(\bar{a}_2) - D_1(\bar{a}_1) \right] \frac{1 - \alpha_2 \bar{a}_2}{\bar{a}_2 - \bar{a}_1} \\
 D_3(\bar{a}_3) &= \left\{ \left[D_1(\bar{a}_3) - D_1(\bar{a}_1) \right] \frac{1 - \alpha_2 \bar{a}_3}{\bar{a}_3 - \bar{a}_1} - D_2(\bar{a}_2) \right\} \frac{1 - \alpha_3 \bar{a}_3}{\bar{a}_3 - \bar{a}_2} \quad (2.35) \\
 D_4(\bar{a}_4) &= \left(\left\{ \left[D_1(\bar{a}_4) - D_1(\bar{a}_1) \right] \frac{1 - \alpha_2 \bar{a}_4}{\bar{a}_4 - \bar{a}_1} - D_2(\bar{a}_2) \right\} \frac{1 - \alpha_3 \bar{a}_4}{\bar{a}_4 - \bar{a}_2} - D_3(\bar{a}_3) \right) \cdot \left(\frac{1 - \alpha_4 \bar{a}_4}{\bar{a}_4 - \bar{a}_3} \right) \\
 &\dots
 \end{aligned}$$

A study of these expressions will verify the following recurrence formula

$$D_{i+1}(\bar{a}_{i+1}) = \frac{1 - \alpha_{i+1} \bar{a}_{i+1}}{\bar{a}_{i+1} - \bar{a}_i} \left[D_i(\bar{a}_{i+1}) - D_i(\bar{a}_i) \right] \quad i = 1, 2, 3, \dots, n \quad (2.36)$$

2.6 Convergence

In representing or approximating a given function by means of a sequence of auxiliary special functions, it is necessary that one study the convergence of functions of best approximation. Of central importance in our study of convergence is a theorem of Blaschke's.

Theorem 6. If the points β_1, β_2, \dots lie interior to $C : |z| = 1$, and if the product $\prod_{i=1}^N |\beta_k|$ diverges, then the product

$$\prod_{i=1}^{\infty} \frac{\bar{\beta}_i}{|\beta_i|} \frac{z - \beta_i}{\bar{\beta}_i z - 1} \quad (2.37)$$

also diverges interior to C . On every closed set interior to C , we have uniformly

$$\lim_{N \rightarrow \infty} \prod_{i=1}^N \frac{\bar{\beta}_i}{|\beta_i|} \frac{z - \beta_i}{\bar{\beta}_i z - 1} = 0 \quad (2.38)$$

The product of Eq. 2.37 is called the Blaschke product corresponding to the numbers β_i . An infinite product, with only a number of its factors zero, is said to converge if and only if the product obtained by omitting those factors converges. An infinite product,

none of whose factors is zero, is said to converge if and only if the product of the first N factors approaches a finite limit different from zero as N becomes infinite. The convergence of an infinite product can be handled by the following corollary.

Corollary. A necessary and sufficient condition for the convergence of the product

$$\prod_{i=1}^{\infty} |\beta_i| \quad (2.39)$$

is the convergence of the series

$$\sum_{i=1}^{\infty} (1 - |\beta_i|) \quad (2.40)$$

We are now in position to investigate the convergence of the series of Eq. 2.25. Consider the n^{th} term

$$C_n(z) = \frac{z(1 - \bar{a}_1 z)(1 - \bar{a}_2 z) \dots}{(z - a_1)(z - a_2)(z - a_3) \dots} \quad |a_i| > 1$$

or

$$C_n(z) = \frac{z}{z - a_n} \prod_{i=1}^{n-1} \frac{1 - \bar{a}_i z}{z - a_i}$$

Now let β_i equal $1/\bar{a}_i$; thus

$$C_n(z) = \frac{z}{z - a_n} \prod_{i=1}^{n-1} \frac{\bar{\beta}_i}{\beta_i} \frac{\beta_i - z}{z\bar{\beta}_i - 1}$$

or

$$C_n(z) = \frac{z}{z - a_n} \prod_{i=1}^{n-1} \frac{\bar{\beta}_i^2}{|\beta_i|^2} \frac{\beta_i - z}{z\bar{\beta}_i - 1}$$

We identify the Blaschke product corresponding to the sequence β_i ; since, inside the unit circle, the function $z/z - a_n$ is bounded, it is evident that

$$\lim_{n \rightarrow \infty} C_n(z) = 0 \quad (2.41)$$

provided that the product

$$\prod_{i=1}^{n-1} \left| \frac{1}{\bar{a}_i} \right| \quad (2.42)$$

diverges. Consequently, it has been proved that the series

$$R_n(z) = D_0 + D_1 \frac{z}{z - \alpha_1} + \dots + D_n \frac{z(1 - \bar{\alpha}_1 z) \dots (1 - \bar{\alpha}_{n-1} z)}{(z - \alpha_1) \dots (z - \alpha_n)} \quad (2.25)$$

converges on the circumference of C if the points $1/\bar{\alpha}_i$ lie interior to $C : |z| = 1$, subject to the condition stated in Eq. 2.42.

The sequence of α_i is not to be arbitrary. Further study of the product

$$\prod_{i=1}^n \left| \frac{1}{\alpha_i} \right|$$

is required to establish the degree of convergence of the series in Eq. 2.25.

III. APPLICATION TO NETWORK SYNTHESIS

Necessary and sufficient conditions for rational network functions. Conformal mapping. Cases of approximation. Approximation on the real part. General procedure.

3.1 Realizability Conditions

In Section II, we developed a method of approximation on the unit circle with a set of general rational functions of a complex variable z . Before we apply our results to problems of network synthesis, we shall state a number of classical theorems and results of network theory. The proof of these theorems will be omitted. However, for further information the reader is referred to the excellent texts of Bode and Guillemin.

Given a four-terminal network, Fig. 3.1, the ratio of $V_1(s)$ to $I_1(s)$ is defined as the driving-point impedance $Z_{11}(s)$. Thus

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} \quad (3.1)$$

Similarly, the ratio of $V_2(s)$ to $I_1(s)$ is defined as the transfer impedance

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)} \quad (3.2)$$

Alternatively, the driving-point admittance is defined as

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} \quad (3.3)$$

and the transfer admittance as

$$Y_{12}(s) = \frac{I_2(s)}{V_1(s)} \quad (3.4)$$

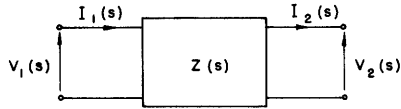


Fig. 3.1

Characterization of a network in the complex frequency domain.

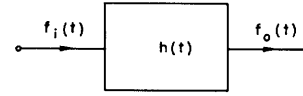


Fig. 3.2

Characterization of a network in the real time domain.

It should be noted that the foregoing expressions for current and voltage are functions of the complex frequency $s = \sigma + j\omega$. In the time domain (that is, one is using only functions of the real time variable t) input and output functions (Fig. 3.2) are related by the convolution integral

$$f_o(t) = \int_0^t f_i(t-\tau) h(\tau) d\tau \quad (3.5)$$

where $h(t)$ is the unit impulse response of the system. Since convolution in the real domain goes over into multiplication in the complex domain, taking the Laplace transform of Eq. 3.5

$$F_o(s) = F_i(s) \cdot H(s)$$

and if $F_i(s) = 1$, then $F_o(s) = H(s)$.

Theorem 1. The necessary and sufficient conditions that a rational function

$$Z_{11}(s) = \frac{p_0 + p_1s + \dots + p_ms^m}{q_0 + q_1s + \dots + q_ns^n} \quad (3.6)$$

be a driving-point impedance function are: (a) $Z(s)$ must be real for real s ; (b) $Z(s)$ must be analytic in the right-half of the s -plane; (c) $\text{Re}[Z(j\omega)] \geq 0$. Any j -axis poles must be simple with real and positive residues.

Theorem 2. The necessary and sufficient conditions that a rational function

$$Z_{12}(s) = C \frac{\prod_{i=1}^m (s - b_i)}{\prod_{i=1}^n (s - a_i)} \quad (3.7)$$

be a transfer-point impedance function are: (a) $\text{Re}[a_i] < 0$; (b) a_i and b_i , if not real, occur in complex conjugate pairs; (c) upper bound on $m \geq n + 1$.

The following theorem was proved by Cerrillo.

Theorem 3. Given a contour Γ_o on the complex plane of the type shown in Fig. 3.3, then, for a rational transfer function $F(s)$, necessarily

$$\left. \begin{aligned}
 U(\gamma_0, \lambda) &= \text{Real } F(s) \Big|_{s \in \Gamma_0} \text{ bounded} \\
 F(s) &= \frac{2(s - \gamma_0)}{\pi} \int_{\Gamma_0} \frac{U(\gamma_0, \lambda)}{(s - \gamma_0)^2 - \lambda^2} d\lambda \\
 f(t) &= \frac{2e^{\gamma_0 t}}{\pi} \int_{\Gamma_0} U(\gamma_0, \lambda) \cos \lambda t d\lambda
 \end{aligned} \right\} \quad (3.8)$$

where $\gamma_0 > c_0$, and c_0 is the abscissa of convergence of $F(s)$.

Consider the special case where γ_0 equals 0. Let

$$F(s) = \frac{p_m(s)}{q_n(s)} = \frac{p_m(s) q_n(\bar{s})}{|q_n(s)|^2}$$

and

$$|q_n(s)|^2 = Q_n(s^2)$$

$$p_m(s) q_n(\bar{s}) = M(s^2) + s N(s^2)$$

However, since Γ_0 coincides with the j -axis and s equals $j\lambda$

$$U(0, \lambda) = \frac{M(\lambda^2)}{Q(\lambda^2)}$$

From this expression it is evident that $U(0, \lambda)$ is an even function, which is true for any γ_0 . Thus, in general,

$$\text{Real } F(s) \Big|_{s \in \Gamma_0} = U(\gamma_0, \lambda^2) \quad (3.9)$$

3.2 Transformations to the Unit Circle

In network theory dealing with real time functions, one is concerned mainly with the real frequency variable ω . Thus, approximations should be based on the j -axis of the complex plane s , rather than on the unit circle used in Section II. It is evident that before one applies the results of Section II a conformal mapping should be made to relate the unit circle of the z -plane to the contour of interest in the s -plane. However, in using these types of transformation, care should be taken that the area inside the unit circle of the z -plane coincide with an analytic area of the s -plane. (See Fig. 3.31.)

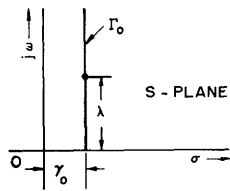


Fig. 3.3
The Γ_0 contour.

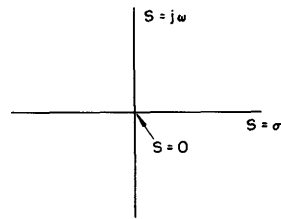


Fig. 3.31
The s-plane.

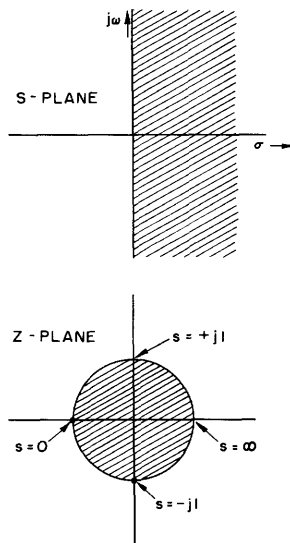


Fig. 3.4
A unit circle transformation (A).

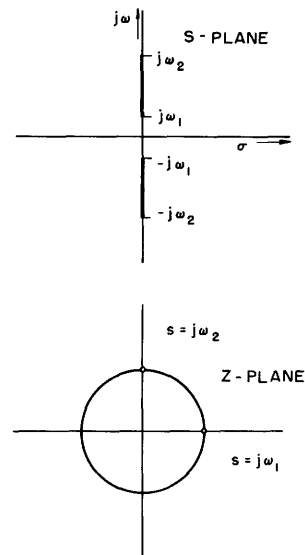


Fig. 3.5
A unit circle transformation (B).

Consider the following relation

$$z = \frac{s-1}{s+1} \quad \text{or} \quad s = \frac{1+z}{1-z} \quad (3.10)$$

To study the conformal maps of these functions, let s equal $j\omega$; then

$$z = \frac{j\omega - 1}{j\omega + 1} = - \frac{(j\omega - 1)^2}{1 + \omega^2}$$

or

$$z = - \frac{1 - \omega^2}{1 + \omega^2} + j 2 \frac{\omega}{1 + \omega^2}$$

which results in

$$|z|^2 = \frac{(1 - \omega^2)^2 + 4\omega^2}{(1 + \omega^2)^2} = 1$$

Furthermore, there exists the following set of corresponding points:

z-plane	s-plane
-1	0
0	1
1	∞

Therefore, under the transformation of Eq. 3.10, the right half-plane of the s -plane is mapped upon the inside of the unit circle of the z -plane (Fig. 3.4). Thus if a function $Z(s)$ possesses singularities in the points a_i of the s -plane, the transformed function

$$F(z) = Z\left(\frac{1+z}{1-z}\right)$$

possesses singularities in the points

$$a_i = \frac{a_i - 1}{a_i + 1} \quad (3.11)$$

Notice that the real axis of the s -plane corresponds to the real axis of the z -plane in the following segments:

$$\begin{array}{ll} -\infty \leq z \leq -1 & -1 \leq s \leq 0 \\ -1 \leq z \leq 1 & 0 \leq s \leq +\infty \\ 1 < z \leq +\infty & -\infty < s \leq -1 \end{array}$$

Also, since $\text{Re}[a_i]$ is less than 0, then $|a_i|$ is greater than 1 as required by Theorem 3 of Section II.

In some instances, one is concerned with approximating a given predescribed

behavior of a network in a given interval rather than in the entire real frequency (ω) range. With the following transformation from Darlington

$$s^2 = -\omega_1^2 + \frac{\omega_2^2 - \omega_1^2}{4} \left(z + \frac{1}{z} \right)^2 \quad (3.12)$$

the segment $\omega_1 < \omega < \omega_2$ of the j -axis of the s -plane is transformed into the unit circle of the z -plane (Fig. 3.5). Notice the following set of corresponding points.

s -plane	z -plane
$\pm j\omega_1$	$\pm j1$
$\pm j\omega_2$	± 1

These are only two of a number of possible transformations that enable one to map any region of the s -plane upon the unit circle of the z -plane.

3.3 Cases of Approximation

The problem of approximation for network synthesis could be defined as one of constructing a suitable function of a single complex variable $Z(s)$ to approximate a given prescribed characteristic $f(s)$. For lumped parameter circuits $Z(s)$ is required to be a rational function. Depending on the data given for $f(s)$, one encounters the following cases.

Case 1. $Z(s)$ is to be found from the prescribed behavior of either $|f(j\omega)|$ or $\text{Arg}[f(j\omega)]$ over the entire frequency range ω .

Case 2. $Z(s)$ is to approximate an explicitly given function $f(s)$ on the j -axis or on a specified interval of the j -axis.

For realizability conditions $f(s)$ is required to be analytic over the entire right-half of the s -plane. Clearly $f(s)$ could be in the form of a (a) rational function; (b) transcendental function.

Case 3. $Z(s)$ is to be found from a given prescribed unit impulse response $h(t)$ in the time domain.

Of course, some means of measuring the effectiveness of the approximation in any of the foregoing cases should be used. The question of measuring the approximation for network synthesis purposes is a rather controversial one and has been considered by many workers in the circuit theory field. For the present, and as long as no discontinuities are included, the "mean square error" provides a sufficient criterion.

It is readily understood that the form of $Z(s)$ will greatly affect the synthesis of the final network. Thus two main problems exist in each of Cases 1, 2, and 3:

- (a) Number of poles of $Z(s)$ given, but their location arbitrary;
- (b) Number of poles of $Z(s)$ given and their location preassigned.

Case 1 is the one which is met most often in network synthesis problems, in filters,

equalizers, and so on. In this type of problem $|f(j\omega)|$ is given as a graphical plot of an ideal magnitude.

However, in a number of applications the location of the poles is restricted in certain regions of the s -plane. Thus, if the desired network must contain only resistors and capacitors, the poles of the transfer function must be located on the negative real axis.

3.4 Procedures

Summarizing the results we have obtained thus far, we shall outline a procedure for each of the three preceding cases of approximation.

Case 1. From the theorems of Section II, it is evident that the approximation method presented here is applied to a function itself rather than to its modulus. However, if $|f(j\omega)|$ is given as a graphical plot, one proceeds in the following manner:

(a) First approximate $|f(j\omega)|$ with an analytic expression of the variable ω^2 . The variable ω^2 is used since $|f(j\omega)|$ is an even function of ω . The approximating function $f(\omega^2)$ need not be rational.

(b) Let ω^2 equal v .

(c) The approximation as outlined in Section II is to be used on a z -plane whose unit circle corresponds to the positive real axis of the v -plane.

(d) Care should be taken that the chosen poles, if complex, form conjugate pairs in the v -plane.

(e) Finally, let v equal $-s^2$. It should be recognized that the resulting function $r(s^2)$ is the modulus of the desired network function $Z(s)$.

Although this is a double approximation procedure, very good results can be achieved. Observe that there is little difficulty in obtaining the function $f(\omega^2)$. For finding $f(\omega^2)$ one can use almost any convenient function; for example, polynomials, a series of transcendental functions, and so on; or, Lagrange's method of interpolation could be used, with a sufficient number of equally spaced interpolating points. In any event, $f(j\omega)$ is easily approximated to any degree of accuracy.

Case 2. The results of Section II can be most easily applied in connection with this case. As stated in Section III, 3.3, the given function $f(s)$ can be transcendental or rational and must be approximated with a set of preassigned poles.

Let the points a_i be the given poles in the s -plane. Since the interval of interest is the entire j -axis, Eq. 3.10 will be used for approximations in the z -plane.

The points a_i correspond in the z -plane to the points a_i . From Section II the interpolating points are, in the z -plane,

$$\beta_i = 1/\bar{a}_i$$

which, in the s -plane, correspond to the points b_i . In computing the coefficients D_i , it is more convenient to use

$$F(1/\bar{a}_i) = f(b_i) \tag{3.13}$$

and to compute the interpolating coefficients in the following sequence with the recurrence formula of Eq. 2.36

$$\begin{aligned}
 &D_0 \\
 &D_1(\beta_1) D_1(\beta_2) D_1(\beta_3) \dots \\
 &\quad D_2(\beta_2) D_2(\beta_3) \dots \\
 &\quad\quad D_3(\beta_3) \dots
 \end{aligned}$$

Care should also be taken that if the a_i are complex, they exist in conjugate pairs.

Case 3. It has been stated previously, that given a desired function in the time domain, $f(t)$, one can immediately compute the real part of the corresponding function in the complex domain by means of the integral

$$U(\gamma_0, \lambda) = \int_0^{\infty} e^{-\gamma_0 t} f(t) \cos \lambda t dt$$

Let $f(t)$ be a nonsymmetric pulse of length 2δ and mean delay t_0 as shown in Fig. 3.6. The area of the pulse is A . Then

$$U(\gamma_0, \lambda) = \int_{t_0 - \delta}^{t_0 + \delta} e^{-\gamma_0 t} f(t) \cos \lambda t dt \quad (3.14)$$

In the special case where $f(t)$ is symmetric about t_0 , after routine algebraic operations, one obtains

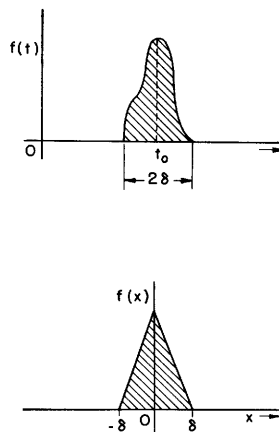


Fig. 3.6

A nonsymmetric and a symmetric pulse.

$$U(\gamma_0, \lambda) = e^{-\gamma_0 t_0} \left[M(\gamma_0, \lambda) \cos t_0 \lambda + N(\gamma_0, \lambda) \sin t_0 \lambda \right] \quad (3.15)$$

where

$$\left. \begin{aligned} M(\gamma_0, \lambda) &= 2 \int_0^\delta f(x) \cos h(\gamma_0 x) \cos \lambda x \, dx \\ N(\gamma_0, \lambda) &= 2 \int_0^\delta f(x) \sin h(\gamma_0 x) \sin \lambda x \, dx \end{aligned} \right\} \quad (3.16)$$

and

$$x = t - t_0$$

In general, $U(\gamma_0, \lambda)$ will be a bounded nonrational function of λ . Thus the problem is to approximate $U(\gamma_0, \lambda)$ by a rational function. It should be recognized that in many instances it may be more convenient to obtain a graphical plot of the function $U(\gamma_0, \lambda)$ instead of an analytic expression. In either case, one can easily locate the zeros of $U(\gamma_0, \lambda)$, if any, from a graphical plot of $U(\gamma_0, \lambda)$. Let these zeros be λ_i^2 . Since $U(\gamma_0, \lambda)$ is an even function one can instantly write

$$V(\gamma_0, \lambda^2) = \frac{1}{U(\gamma_0, \lambda^2)} = K_0 + \sum_{i=1}^n \frac{K_i}{\lambda^2 - \lambda_i^2} \quad (3.17)$$

If n equals ∞ , a sufficient upper bound must be decided. Let

$$\lambda^2 = v \quad \text{and} \quad \lambda_i^2 = v_i$$

Thus

$$V(\gamma_0, v) = K_0 + \sum_{i=1}^n \frac{K_i}{v - v_i}$$

For an appropriate transformation of the interval of interest into the unit circle of the z -plane, the reader is referred to the excellent dictionary of conformal representations by Kober. Having solved the problem of approximation as outlined in Section II, one can write

$$U(\gamma_0, \lambda^2) = B_0 + \sum_{i=1}^n \frac{B_i}{\lambda^2 - \mu_i} \quad (3.18)$$

and

$$F(s) = B_0 + \sum_{i=1}^m \frac{-B_i/\mu_i}{s - \gamma_0 - \mu_i} \quad (3.19)$$

IV. EXAMPLES

The examples presented in this section sketch a general picture of the procedures developed in the preceding sections. Of course, each case involves additional details that are treated individually. However, the main pattern of the approximation procedure remains the same.

4.1 Approximation of a Driving-Point Impedance

Find a five-pole approximation of the driving-point impedance of a short-circuited transmission line whose constants for the total length of the line are R , L , G , and C .

The impedance of a short-circuited line (Fig. 4.1) is

$$Z = Z_0 \tanh \Gamma \quad (4.1)$$

where

$$Z_0^2 = \frac{R + sL}{G + sC} \quad (4.2)$$

and

$$\Gamma^2 = (R + sL)(G + sC) \quad (4.3)$$

To decide the location of the poles in the s -plane, consider the admittance function

$$Y = \frac{1}{Z} = Y_0 \operatorname{ctn} h \Gamma \quad (4.4)$$

whose singularities are the zeros of Eq. 4.1 and occur at

$$\Gamma = j\pi k \quad k = 0, \pm 1, \pm 2, \dots$$

Therefore

$$\Gamma^2 = (R + sL)(G + sC) = -\pi^2 k^2$$

whose roots are

$$\left. \begin{aligned} a_k &= \bar{a}_{-k} = -\sigma_k + j\omega_k \\ \sigma_k &= \frac{G}{2C} + \frac{R}{2L} \\ \omega_k &= \left[\frac{\pi^2 k^2}{LC} - \left(\frac{G}{2C} - \frac{R}{2L} \right)^2 \right]^{1/2} \end{aligned} \right\} \quad (4.5)$$

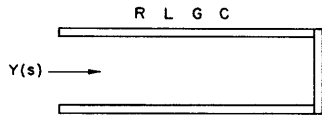


Fig. 4.1

A short-circuited transmission line.

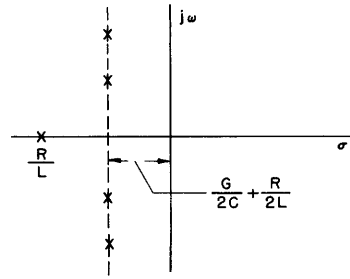


Fig. 4.2

Poles of $Y(s)$ in the s -plane.

as shown in Fig. 4.2, with an additional pole at

$$a_0 = -\frac{R}{L}$$

The necessity of dealing with actual numbers in the approximation procedure is readily understood from the previous section. Let, for convenience and without loss of generality,

$$R = L = G = C = 1$$

We, therefore, decide to perform the approximation with the poles

$$a_0 = -1$$

$$a_1 = \bar{a}_{-1} = -1 + j\pi$$

$$a_2 = \bar{a}_{-2} = -1 + j2\pi$$

To follow the notation of the previous sections, let

$$Z(s) = \text{ctn } h(1+s) \tag{4.6}$$

It should be noted that the approximation does not guarantee that the resulting function $Y(s)$ will be realizable as a driving-point admittance.

We shall perform the approximation on the unit circle of the z -plane as defined by the transformation

$$z = \frac{1-s}{1+s}$$

The singularities a_i correspond in the z -plane to the points α_i as follows:

$$a_0 = -1$$

$$\alpha_0 = \infty$$

$$a_1 = -1 + j\pi$$

$$\alpha_1 = 1 + j\frac{2}{\pi}$$

$$a_{-1} = -1 - j\pi$$

$$\alpha_{-1} = 1 - j\frac{2}{\pi}$$

$$a_2 = -1 + j2\pi$$

$$\alpha_2 = 1 + j\frac{1}{\pi}$$

$$a_{-2} = -1 - j2\pi$$

$$\alpha_{-2} = 1 - j\frac{1}{\pi}$$

We shall therefore interpolate at the points $\beta_i = 1/\bar{a}_i$, where

$$\beta_0 = 0$$

$$\beta_1 = \frac{\pi^2}{\pi^2 + 4} \left(1 - j \frac{2}{\pi}\right) = \bar{\beta}_{-1}$$

$$\beta_2 = \frac{\pi^2}{\pi^2 + 2} \left(1 - j \frac{1}{\pi}\right) = \bar{\beta}_{-2}$$

Under the transformation, the function $Z(s)$ becomes

$$F(z) = \text{ctn h}\left(\frac{z}{1-z}\right) \quad (4.7)$$

It is apparent that, since both a_i and β_i occur in conjugate pairs, the following series of interpolation should be used:

$$R_n(z) = D_0 z + \frac{D_1}{(z - a_1)(z - a_{-1})} + \frac{D_2(\beta_1 - z)(\beta_{-1} - z)}{(z - a_1)(z - a_{-1})(z - a_2)(z - a_{-2})} \quad (4.8)$$

It was found that a more appropriate expression, computationally, is

$$R_n(z) = (1-z) \left[N_0 + \frac{N_1}{(z - a_1)(z - a_{-1})} + \frac{N_2(\beta_1 - z)(\beta_{-1} - z)}{(z - a_1)(z - a_{-1})(z - a_2)(z - a_{-2})} \right]$$

Proceeding in order we find

$$N_0 = \frac{1}{2}$$

$$N_1 = \left(\frac{2}{\pi}\right)^2$$

$$N_2 = \left(\frac{1}{\pi}\right)^2$$

and, finally, the driving-point admittance $Y(s)$ is written in partial fraction expansion form:

$$Y(s) = \frac{1}{1+s} + \frac{2(1+s)}{(1+s)^2 + \pi^2} + \frac{2(1+s)}{(1+s)^2 + (2\pi)^2} \quad (4.9)$$

Rearranging terms, we obtain (Fig. 4.3)

$$Y(s) = \frac{1}{1+s} + \frac{1}{1/2 + 1/2s + \left[1/(2/\pi^2) + (2/\pi^2)s\right]} + \frac{1}{1/2 + 1/2s + \left[1/(1/2\pi^2) + (1/2\pi^2)s\right]} \quad (4.10)$$

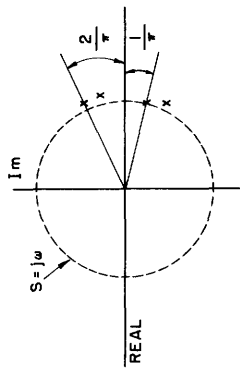


Fig. 4.3

Poles of $R_n(z)$ in the z -plane.

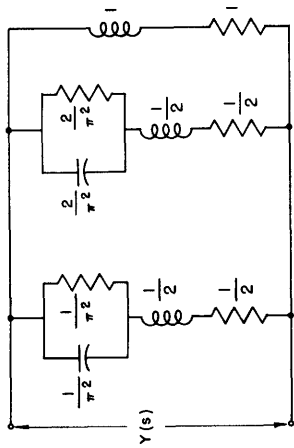


Fig. 4.4

Four-pole approximation of a short-circuited transmission line.

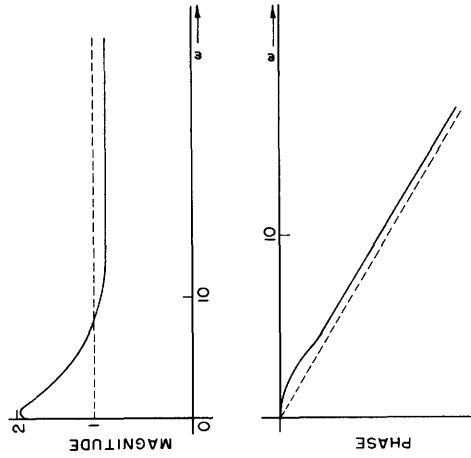


Fig. 4.5

Magnitude and phase characteristics of $r(j\omega)$.

The equivalent network is shown in Fig. 4.4.

4.2 Approximation of a Transcendental Function

Consider the transfer function of a unit delay network

$$Z(s) = e^{-s} \quad (4.11)$$

We will attempt to approximate the $Z(s)$ on the j -axis with a two-pole RC network. Let the poles be located in the points $s = -1/2$ and $s = -2$. Under the transformation

$$s = \frac{1+z}{1-z}$$

these points correspond in the z -plane to

s-plane	z-plane
$Z(s)$	$F(z)$
$a_1 = -1/2$	$a_1 = -3$
$a_2 = -2$	$a_2 = 3$

Thus, the interpolating points $\beta_i = 1/\bar{a}_i$ are

z-plane	s-plane
$\beta_1 = -1/3$	$b_1 = 1/2$
$\beta_2 = 1/3$	$b_2 = 2$

In these points we have

$$F(-1/3) = Z(1/2) = 0.606$$

$$F(1/3) = Z(2) = 0.1353$$

The D_i coefficients are computed by the recurrence formula in Eq. 2.36

$$D_0 = 1$$

$$D_1(-3) = 3.15$$

$$D_1(3) = 6.9$$

$$D_2(3) = 5.0$$

From Eq. 2.25

$$R_n(z) = 1 + 3.15 \frac{z}{z+3} + 5.0 \frac{z(1+3z)}{(z+3)(z-3)} \quad (4.12)$$

or

$$R_n(z) = \frac{19.15 z^2 - 4.45z - 9}{(z+3)(z-3)} \quad (4.13)$$

The corresponding function in the s-plane is

$$r(s) = \frac{5.70s^2 - 46.30s - 14.45}{(4s + 2)(2s + 4)} \quad (4.14)$$

The results are plotted in Fig. 4.5. To improve the approximation, one simply adds sufficient terms in Eq. 4.12.

V. CONCLUDING REMARKS

As it is true of any kind of research problem, the work on this topic is by no means complete. On the contrary, a number of rather interesting questions must arise in the mind of the reader.

It has been shown that a given analytic function can be effectively approximated with a rational function of preassigned poles. In some cases of approximation, the best choice of poles is evident. However, a study of the best possible set of poles to approximate a given function in general is still needed. It is suggested that the convergence of the series in Eq. 2.40 be investigated further. Also, we feel that a relation between the upper bound of the error in the approximation procedure and the magnitude of the preassigned poles can, in general, be found.

It is important that the reader realize the two general principles upon which this method of approximation is based: (a) approximation with directly rational functions and (b) approximation of the function rather than its modulus. Indeed, anyone who has worked in the field of network synthesis has at some time or other, felt the need of such an approximation procedure.

ACKNOWLEDGMENT

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