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A FOUR-FLAGGED LEMMA

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Abstract of:

A Four-Flagged Lemma

Murat R. Sertel

The occasion for the lemma is the Gorman - Vind debate in RES (Review of Economic Studies), January, 1971, concerning whether or not Gorman's Lemma 1 (RES, 1968) can be strengthened by relaxing Gorman's assumption of arc-connectivity for the space of prospects to connectivity alone. A lemma is proved showing the mentioned relaxation feasible and furnishing proof for Gorman's Lemma 1. This supplies a missing foundation stone of Gorman's "Structure of Utility Functions" and generalizes the results therein.

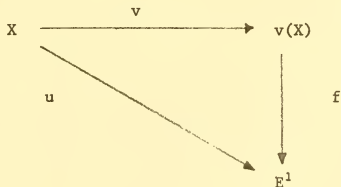
A FOUR-FLAGGED LEMMA¹

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The lemma stated below has been the subject of a recent summit debate [4,5,7,8]. As it turns out, the proof of the proposition takes up less space than did the proceedings of this debate, in which latter the participants addressed themselves to, among other such things, how easy the proposition was to prove, whether or not it had been proved, and whether, in fact, its hypothesis had to be strengthened (from assuming X connected to assuming X arc-connected) before we could trust it to be true. While the proof below admits Professor Vind to have been correct in his conjecture [7] that connectedness of X suffices, so that Professor Gorman's stronger assumption of arc-connectedness [3] is unwarranted, there is room to think that the sketch [8] which Professor Vind offers in persuasion may not so clearly constitute a proof to the reader accustomed to cautious step-by-step deduction. (Note that Professor Gorman [5], even at the end of the debate, fails to acknowledge that Professor Vind has proved what he claims - a failing which I feel I must confess to share with Professor Gorman.) Such caution might be especially justified, furthermore, in view of the fact that not even Professor Vind appears immune to oversights whereby a non-proof may appear to have earned a 'q.e.d.' at its end.² It appears only proper, therefore, to prove the following

LEMMA: Let $\{X_i \mid i \in M\}$ be a family of non-empty sets indexed by the finite set $M = \{0, 1, \dots, m\}$ of (w.l.g.) the first $m + 1$ non-negative integers, and denote $X = \prod_M X_i$, $X^i = \prod_{M \setminus \{i\}} X_j$, $M^0 = M \setminus \{0\}$. Equip X with a topology yielding X connected and with a complete preorder $\leq_M \subset X \times X$ such that (1) \leq_M is semiclosed, i.e., $\{x \in X \mid x \leq_M \bar{x}\}$ and $\{x \in X \mid \bar{x} \leq_M x\}$ are closed for each $\bar{x} \in X$, and (2) for each $i \in M^0$, if $(x_i, y^i) \leq_M (z_i, y^i)$ for some $x_i, z_i \in X_i$ and some $y^i \in X^i$, then $(x_i, x^i) \leq_M (z_i, x^i)$ for all $x^i \in X^i$, so that a (complete) preorder $\leq_i \subset X_i \times X_i$ is defined on X_i by setting $[x_i \leq_i z_i] \Leftrightarrow [(x_i, y^i) \leq_M (z_i, y^i) \text{ for some } y^i \in X^i]$. Given a real-valued function $u: X \rightarrow E^1$ preserving \leq_M with $u(X)$ connected, for each $i \in M^0$, let $v_i: X_i \rightarrow E^1$ preserve \leq_i , denoting the identity map of X_0 by $v_0: X_0 \rightarrow X_0$, and define a map $v: X \rightarrow X_0 \times E^m$ by $v(x) = \{v_i(x_i)\}_{i \in M}$, where $x_i = \pi_{X_i}(x)$.³ Then any function f which makes the diagram



commute is continuous.

Proof: Let W be a subbasic open set in E^1 . It suffices to show that $f^{-1}(W)$ is open. Assume, without loss of generality, that $W = \{w \in E^1 \mid w > w^*\}$. If $w^* \notin u(X)$, then the connectedness of $u(X) \subset E^1$ implies that either (i) $w^* < u(x)$ for all $x \in X$ or (ii) $w^* > u(x)$ for all $x \in X$. If (i) holds, then $u^{-1}(W) = X$, so that $v(u^{-1}(W)) = v(X) = f^{-1}(W)$ is open. If (ii) holds, then $u^{-1}(W) = \emptyset = v(u^{-1}(W)) = f^{-1}(W)$ is open. So, assume $w^* \in u(X)$, i.e., that $w^* = u(x^*)$ for some $x^* \in X$. Then, as u preserves the complete preorder \leq_M , $u^{-1}(W) = \{x \in X \mid x > x^*\}$, which is open (showing u to be continuous) by the semiclosedness of \leq_M and connected by the connectedness of X . Hence, the projections $P_i = \pi_{X_i}(u^{-1}(W)) \subset X_i$ are open and connected for each $i \in M$. Now $v_o(P_o) = P_o$ is open trivially. On the other hand, if $i \in M^o$, then the fact that v_i preserves \leq_i clearly implies that $v_i(P_i) = \{w \in v_i(X_i) \mid w > v_i(x_i^*)\}$, where $x_i^* = \pi_{X_i}(x^*)$, so that $v_i(P_i)$ is again open. Thus $v(u^{-1}(W)) = \prod_M v_i(P_i)$ is open, and, from the commutativity $v \circ u^{-1} = f^{-1}$, we conclude that $f^{-1}(W)$ is open, as to be shown.

N.B. The result, using no free sector $\{0\}$, which Debreu set out to prove in [2, p. 22, lines 5-18] is obviously a corollary to the above.

Note also that, although u and the v_i 's for $i \in M^o$ were not assumed continuous, they are easily shown to be so as a consequence of the hypothesis. In fact, a real-valued function u on a connected space X preserving a complete preorder on X is

continuous iff the preorder on X is semiclosed and $u(X)$ is connected: "only if" is obvious and "if" is noted, in passing, above in the proof of the lemma. Furthermore, the completeness of \leq_M and the commutation $u = f \circ v$ easily imply f to be increasing in each of its arguments v_i , since u preserves \leq_M .

It is clear, up to a possible need for rewording and the insertion of some obvious facts, that Professor Gorman's Lemma 1 [3, p. 387] is now proved, in fact strengthened so as not to require arc-connectivity but only connectivity for X . As a result, that "half" of his results relying on this lemma are given a foundation and strengthened to apply more generally.

Elsewhere [6] it is shown, by generalizing the first of Debreu's [1] two famous representation theorems, that the assumption of (topological) separability for X made throughout by Gorman [3] is also unnecessary, being so for his Lemma 1. This adds to the generality of Gorman's results. Given the insightful and essential nature of his work, the present sort of exercise becomes worthwhile and, indeed, a pleasure.

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FOOTNOTES

1. The first three flags, those of Debreu, Gorman and Vind, I found when I got to Clontarf [5], and I intend what is below as a truce banner.

This research was undertaken with the support of the MIPC (Management Information for Planning and Control) Group, Sloan School of Management, M.I.T. I should like to thank Paul Kleindorfer for a discussion (in which, incidentally, he suggested calling this lemma "Nameless Things").

2. In stating that Professor Debreu's proof [2, p. 22, lines 5-18] "is shorter and simpler than the proof in" Professor Gorman's [3], Professor Vind [7] evidently misses the fact that, while easy to mend, Professor Gorman's attempt actually fails to be a proof. For Professor Gorman concludes f to be continuous from his established fact that, for each sequence γ in $v(X)$, if γ converges to r^* , then $f(\gamma)$ converges to $f(r^*)$. What he established would have been sufficient to support his conclusion had Professor Gorman assumed $v(X)$ or - equivalently in this case - X_0 first countable or, without even invoking this assumption, simply had he established a generalized version of what he did, using nets rather than sequences.

I should, however, hasten to join Professor Vind in praising the rest of Professor Gorman's results in [3], especially his characterization of the case of an additively

separable utility function.

3. E^k denotes k -dimensional Euclidean space ($k = 1, 2, \dots$),
and $\pi_{X_i} : X \rightarrow X_i$ denotes the projection of X onto X_i ($i \in M$).

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