WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

Generalized Picard-Lindelof Theory

Iris Marie Mack, Ph. D.

July 1991 WP 3317-91-MSA

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

Generalized Picard-Lindelof Theory

Iris Marie Mack, Ph. D.

July 1991 WP 3317-91-MSA

Forthcoming in *SIAM J. of Mathematical Analysis*

Generalized Picard-Lindelöf Theory

Abstract: In this paper, a constraint algorithm is developed for the purpose of obtaining conditions for the existence and uniqueness of solutions to systems of differential/algebraic equations. The underlying principle of our constraint theory is to conceptually reduce a system of differential/algebraic equations to a form so that we can prove the existence and uniqueness of its solution similar in manner to that done via utilizing the Picard-Lindelöf theory for ordinary differential equations.

Key Words: Picard-Lindelof theory, differential/algebraic equations, reduction technique, constraint, Lagrangian dynamics.

INTRODUCTION

The theory developed in this paper, which we will call our constraint theory, will be utilized to obtain conditions for the existence and uniqueness of the solution to a system of differential/algebraic equations as is done for ordinary differential equations. Recall that the Picard-Lindelöf theory (or the method of successive approximations) is used to investigate the conditions under which an initial valued ordinary differential equation has a solution and that that solution is unique. The underlying principle of our constraint theory is to conceptually reduce a system of differential/algebraic equations to a form so that we can prove existence and uniqueness of its solution in a manner similar to that done via utilizing the Picard-Lindelöf theory for ODE's

Our constraint algorithm can be summarized as follows. differentiate the system of differential/algebraic equations with respect to the independent variable x; study row spaces of certain Jacobian matrices; perform various algebraic manipulations to attempt to reduce the ill-posed system of differential/algebraic equations to a well-posed, overdetermined system; then invoke theorems such as The Fredholm Alternative Theorem, The Implicit Function Theorem, and The Picard-Lindelöf Theorem, to obtain conditions for the existence and uniqueness of the solution to the system of differential/algebraic equations.

Now we will briefly summarize what is to appear in the remainder of this paper Section one will contain a discussion of concepts from Lagrangian Dynamics, some of which will be utilized in our constraint theory In section two we will introduce our constraint theory and present several important theorems - which will be utilized in our analysis. Sections three and four will contain a derivation of our constraint algorithm for nonlinear systems of differential/algebraic equations. In section five we will present an application of our constraint algorithm. Finally, in section six, we will present a discussion of an extension of our local results to semi-local results.

1. Motivation from Lagrangian Dynamics

Our constraint theory is motivated by concepts from Lagrangian dynamics. For more details related to our short digression on Lagrangian dynamics, the interested reader is referred to [3] and [8]. The Lagrangian equations of motion for conservative mechanical systems can be written as,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial q'_i}\right] - \frac{\partial L}{\partial q_i} = 0 \qquad (1.1)$$

such that $L := L(t, \underline{q}(t), \underline{q}'(t))$ is the Lagrangian, and

$\underline{q} \in R^k$

contains the k dynamical coordinates, $q_i = q_i(t)$, for $1 \le i \le k$.

The second order system (1.1) represents a mechanical system of the ***standard case*** if it can be solved for the accelerations q''_i $(1 \le i \le k)$: that is, if q''_i can be expressed in terms of q_i and q'_i, for $1 \le i \le k$. In such a case, if the initial dynamical coordinates and velocities are prescribed, the accelerations can be obtained, and hence the motion of the mechanical system can be determined via integration. But if the mechanical system (1.1) is such that we are not able to solve for all k components of the acceleration q''_i $(1 \le i \le k)$, this is a ***nonstandard case*** and the mechanical system is said to be ***subject to constraints***. This means that the dynamical coordinates and velocities are not independent functions of time that is, a functional relationship exists between $\underline{q}$ and $\underline{q}'$

Now suppose we consider systems with k ***degrees of freedom*** (or independent generalized coordinates). Given the Lagrangian $L(t,\underline{q}(t),\underline{q}'(t))$, we define the following matrix of partial derivatives with respect to the velocities:

$$W(t,\underline{q}(t),\underline{q}'(t)) = [W_{ij}] = \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right] \tag{1.2}$$

such that $1 \le i,j \le k$

By expanding the first term of the Lagrangian equations of motion (1.1) and using the matrix defined by (1.2) we obtain equations of the form

$$\sum_{j=1}^{k} W_{ij}\ddot{q}_j = \alpha_i(t,\underline{q}(t),\underline{q}'(t)), \tag{1.3}$$

such that $1 \le i \le k$. The accelerations can be uniquely solved for; that is, the Lagrangian is standard, if and only if the matrix W is of maximal rank. But if W is rank-deficient, the Lagrangian equations of motion do not yield all the accelerations as functions of the dynamical coordinates and velocities

In [8] the authors examine in some detail the Lagrangian equations of motion for the nonstandard case to understand the nature of the solution to the equations of motion.

Let us now examine the case where W in (1.2) is rank-deficient that is, its rank R is less than k. Therefore, there exist k-R linearly independent null eigenvectors

$$\underline{v}^{(m)}(t,\underline{q}(t),\underline{q}'(t)) := [v_1^{(m)}(t,\underline{q}(t),\underline{q}'(t)), \ldots ,v_k^{(m)}(t,\underline{q}(t),\underline{q}'(t))]^T$$

for the matrix W, such that

$$\sum_{j=1}^{k} v_j^{(m)}(t,\underline{q}(t),\underline{q}'(t))\, W_{ij}(t,\underline{q}(t),\underline{q}'(t)) = 0, \qquad (1.4)$$

for $1 \le m \le k-R$ and $1 \le j \le k$. Equations (1.3) and (1.4) yield the following k-R relations involving the q_j and q'_j:

$$\sum_{j=1}^{k} v_i^{(m)}(t,\underline{q}(t),\underline{q}'(t))\, \alpha_i(t,\underline{q}(t),\underline{q}'(t)) = 0, \qquad (1.5)$$

for $1 \le m \le k-R$. The relations defined in (1.5) are called ***constraints*** in the Lagrangian sense. They are consequences of the equations of motion, and place restrictions on the choice of the initial values of the dynamical coordinates and velocities.

2. Introduction to Constraint Algorithm

Suppose we consider general nonlinear overdetermined systems of differential/algebraic equations of the form:

$$\underline{F}(x,\mathbf{y}(x),\mathbf{y}'(x)) = \underline{0}, \quad x \in [x_0,x_{fin}], \tag{2.1}$$

subject to the initial conditions

$$\mathbf{y}(x_0) = \mathbf{y}_0,$$

such that

$$\mathbf{y}:[x_0,x_{fin}] \rightarrow \mathbf{R}^n,$$

$$\underline{F}:[x_0,x_{fin}] \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^m,$$

and

$$m \geq n.$$

To utilize our constraint theory to obtain conditions for the existence and uniqueness of solutions to systems of the form (2.1), we will find it necessary to assume certain algebraic and smoothness properties for (2.1), which will be valid in some neighborhood around a specified point in the domain of our problem. Given these algebraic and smoothness assumptions, we will obtain a local solution to a certain class of systems of the form (2.1). These local results will then be extended to semi-local results over larger subdomains of the domain of our problem - for which our algebraic and smoothness assumptions also hold.

Given certain problem-dependent matching or interface conditions, we may also be able to extend our semi-local results to global results for systems of differential/algebraic equations with discontinuities by "piecewise patching" our semi-local results together at the discontinuous interfaces of our domain. If system (2.1) is such that $\partial \underline{F}/\partial \mathbf{y}'$ undergoes a rank change, it is a system with turning points (which is analogous to a system with discontinuities.) The extension of our semi-local results to handle systems of differential/algebraic equations with turning points will not be discussed in any detail, in part because our motivating problems are decoupled systems of differential/algebraic equations without turning points.

2.1 Theoretical Tools

Important tools, which we will utilize in our constraint analysis, are the Fredholm Alternative Theorem [7], the Implicit Function Theorem [5], and the Picard-Lindelof Theorem [1]. Recall The Fredholm Alternative Theorem is a tool used in describing the solution to rectangular systems of linear equations of the form

$$A\underline{x} = \underline{b}, \tag{2.2}$$

such that

$A \in \mathbf{R}^{m\times n}$,

$\underline{x} \in \mathbf{R}^{n}$,

and

$\underline{b} \in \mathbf{R}^{m}$.

Theorem 2.1: Fredholm Alternative Theorem

For any A and $\underline{b}$, one and only one of the following systems has a solution:

(i) $A\underline{x} = \underline{b}$

(ii) $\underline{y}^{T}A = \underline{0}^{T}$, $\underline{y}^{T}\underline{b} \neq 0$. □

The general version of the Implicit Function Theorem can be stated as follows:

Theorem 2.2: Implicit Function Theorem

Let $A \subset \mathbf{R}^{q+r}$ be an open set. Consider a general nonlinear system of equations of the form

$$\underline{E}(\underline{w},\underline{z}) = \underline{E}(w_1, \ldots, w_q, z_1, \ldots, z_r) = \underline{0}, \qquad (2.3)$$

such that

$\underline{E}: A \rightarrow \mathbf{R}^r$,

$\underline{w} \in \mathbf{R}^q$,

and

$\underline{z} \in \mathbf{R}^r$

Let $\underline{E}$ be a function of class $C^p(A)$. Suppose $(\underline{w}_0,\underline{z}_0) \in A$ and $\underline{E}(\underline{w}_0,\underline{z}_0) = \underline{0}$ Suppose $\partial\underline{E}/\partial\underline{z}$ - evaluated at $(\underline{w}_0,\underline{z}_0)$ - is of maximal rank. Then there is a neighborhood $U \subset \mathbf{R}^q$ of $\underline{w}_0$, and a neighborhood $V \subset \mathbf{R}^r$ of $\underline{z}_0$, and a unique function $\underline{\Phi}: U \rightarrow V$, such that

$$\underline{E}(\underline{w},\underline{\Phi}(\underline{w})) = \underline{0}, \qquad (2.4)$$

for all $\underline{w} \in U$. Furthermore, $\underline{\Phi}$ is of class C^p.□

Once we have utilized the Implicit Function Theorem to reduce an implicit system of differential equations to an explicit system, we can invoke the Picard-Lindelöf Theorem to prove existence and uniqueness of a solution to the system of implicit differential equations. The following definition is needed to understand the Picard-Lindelöf Theorem.

Definition 2.1: Suppose $\underline{f}$ is defined in a domain $\mathfrak{R}'' \subset [x_0,x_{fin}] \times \mathbf{R}^n$. If there exists a constant $L > 0$ such that, for every $(x,\underline{\mathbf{y}}_1)$, $(x,\underline{\mathbf{y}}_2) \in \mathfrak{R}''$

$$\|f(x,\underline{\mathbf{y}}_1) - \underline{f}(x,\underline{\mathbf{y}}_2)\| \leq L\|\underline{\mathbf{y}}_1 - \underline{\mathbf{y}}_2\|, \qquad (2.5)$$

then $\underline{f}$ is said to satisfy a ***Lipschitz condition*** (with respect to $\underline{y}$) in $\mathcal{R}''$. This fact will be denoted by $\underline{f} \in$ **Lip in** $\mathcal{R}''$. The constant L is called the ***Lipschitz constant***. If, in addition $\underline{f}$ is of class $\mathcal{C}^0(\mathcal{R}'')$, we can write $\underline{f} \in (\mathcal{C}^0$**,Lip) in** $\mathcal{R}''$.□

If the domain $\mathcal{R}''$ is convex then an application of the Generalized Derivative Mean-Value Theorem [6] shows that the existence and boundedness of $\|\partial\underline{f}/\partial\underline{y}\|$ in $\mathcal{R}''$ are sufficient for $\underline{f} \in$ Lip in $\mathcal{R}''$. The Lipschitz constant L can be defined as

$$L := \sup_{(x,\underline{y}) \in \mathcal{R}''} \|\partial\underline{f}(x,\underline{y})/\partial\underline{y}\|.$$

The results of the Picard-Lindelöf Theorem can be deduced for all $(x,\underline{y})$ in a region $\mathcal{R}' \subset \mathbf{R}^{n+1}$, which is defined as follows:

$$\mathcal{R}': \quad |x - x_0| \le a' \qquad \|\underline{y} - \underline{y}_0\| \le b'$$

where $a' > 0$, $b' > 0$. But we will only be interested in the application of the Picard-Lindelöf Theorem when applied to an antisymmetric interval on the x-axis: that is, the results of applying the Picard-Lindelöf Theorem will be deduced for all $(x,\underline{y})$ in a region $\mathcal{R} \subset [x_0,x_{fin}] \times \mathbf{R}^n$, which is defined as follows:

$$\mathcal{R}: \quad [x_0,x_0+a] \qquad \|\underline{y} - \underline{y}_0\| \le b$$

where $a > 0$, $b > 0$. If $\underline{f}$ is of class $\mathcal{C}^0(\mathcal{R})$, then $\underline{f}$ is bounded there. Let

$$M := \sup_{(x,\underline{y}) \in \mathcal{R}} \|\underline{f}\|,$$

[redacted]

and

$\epsilon := \min(a, b/M)$.

We are now prepared to state the Picard-Lindelöf theorem for an antisymmetrical interval $[x_0, x_0+\epsilon]$

Theorem 2.3: Picard-Lindelöf Theorem

Suppose $\underline{f} \in (\mathcal{C}^0, \mathrm{Lip})$ on $\mathfrak{R}$. Consider the continuous successive approximations $\underline{\Phi}_k$ which exist on $[x_0, x_0+\epsilon]$, and are defined as

$$\underline{\Phi}_0(x) = \underline{y}_0, \tag{2.6}$$

$$\underline{\Phi}_{k+1}(x) = \underline{y}_0 + \int_{x_0}^{x} \underline{f}(s, \underline{\Phi}_k(s))ds, \tag{2.7}$$

for

$x \in [x_0, x_0+\epsilon]$,

$k \in \{0\} \cup \mathbf{N}$.

The successive approximations $\underline{\Phi}_k$ converge uniformly on $[x_0, x_0+\epsilon]$ to the unique solution $\underline{\Phi}$, such that $\underline{\Phi}(x_0) = \underline{y}_0$.$\square$

Thus if $\underline{f} \in \mathcal{C}^0(\mathfrak{R})$ and satisfies a Lipschitz condition with respect to $\underline{y}$ on $\mathfrak{R}$, the Picard-Lindelöf Theorem asserts that there exists a unique local solution to (2.1) .

3. Derivation of Constraint Algorithm

Our constraint theory may be applied to variable and constant coefficient regular systems of differential/algebraic equations. However, we will only present our constraint theory for rectangular nonlinear systems of the form (2.1). But for a derivation of our algorithm applied to constant and variable coefficient systems, see [4].

Suppose we consider a nonlinear function $\underline{F}$, which is defined on an open set $D_0 \subset \mathbf{R}^{2n+1}$, and takes on values in $\mathbf{R}^m$; that is, a function of the form

$$\underline{F}(x,\underline{y},\underline{z}), \tag{3.1}$$

such that

$(x,\underline{y},\underline{z}) \in D_0$,

$\underline{F}: D_0 \rightarrow \mathbf{R}^m$,

and

$m \geq n$.

The open set D_0 contains the convex set

$D := [x_0,x_{fin}] \times \mathbf{R}^n \times \mathbf{R}^n$.

Now suppose we temporarily assume that there exists a smooth function, $\underline{y}$, such that

$\underline{z} := \underline{y}'$,

and the following is true:

$$\underline{F}(x,\underline{y}(x),\underline{z}(x)) = \underline{0}, \tag{3.2}$$

subject to the initial conditions

$$\underline{z}(x_0) = \underline{z}_0$$

such that

$$x \in [x_0, x_{fin}],$$

$$\underline{y}, \underline{z}: [x_0, x_{fin}] \rightarrow \mathbf{R}^n,$$

and

$$m \geq n$$

We are interested in nonlinear systems of the form (3.2), such that one of the following three situations is true:

(i) $m = n$, and the Jacobian $[\partial \underline{F}/\partial \underline{z}]$ is rank-deficient and of constant rank on $\mathfrak{M} \subset D$, such that

$\mathfrak{M} := W_x \times W_{\underline{y}} \times W_{\underline{z}}$,

$W_{\underline{y}} \subset \mathbf{R}^n$ is a neighborhood of $\underline{y}_0$, and $W_{\underline{z}} \subset \mathbf{R}^n$ is a neighborhood of $\underline{z}_0$.

(ii) $m > n$, and the Jacobian $[\partial \underline{F}/\partial \underline{z}]$ is of maximal rank on $\mathfrak{M}$.

(iii) $m > n$, and the Jacobian $[\partial \underline{F}/\partial \underline{z}]$ is rank-deficient and of constant rank on $\mathfrak{M}$.

3.1 Definition of Constraints

The following definition is made for notational convenience.

Definition 3.1: The ***zeroth order constraints*** for the system (3.2) are defined as

$$\underline{F}^{[0]}(x,\underline{y},\underline{z}) := \underline{F}(x,\underline{y},\underline{z}) = \underline{0}, \tag{3.3}$$

such that

$$(x,\underline{y},\underline{z}) \in D,$$

and

$$\underline{F}^{[0]}: D \longrightarrow \mathbf{R}^m.$$

Differentiating system (3.3) with respect to x yields

$$d\underline{F}^{[0]}/dx := [\partial\underline{F}^{[0]}/\partial\underline{z}]\underline{z}' + [\partial\underline{F}^{[0]}/\partial\underline{y}]\underline{z} + [\partial\underline{F}^{[0]}/\partial x] = \underline{0}. \tag{3.4}$$

The system (3.4), along with the substitution $\underline{z} := \underline{y}'$, can be written as

$$\begin{bmatrix} I & 0 \\ 0 & \partial\underline{F}^{[0]}/\partial\underline{z} \end{bmatrix} \begin{bmatrix} \underline{y}' \\ \underline{z}' \end{bmatrix} = \begin{bmatrix} \underline{z} \\ \underline{\alpha}^{[0]} \end{bmatrix}, \tag{3.5}$$

such that

$$\underline{\alpha}^{[0]} := \underline{\alpha}^{[0]}(x,\underline{y},\underline{z}) := -([\partial\underline{F}^{[0]}/\partial\underline{y}]\underline{z} + [\partial\underline{F}^{[0]}/\partial x]). \tag{3.6}$$

we will denote the rank and nullity of $[\partial \underline{F}^{[0]}/\partial \underline{z}]$ on $\mathbf{M}$ as the constants

$$\mathcal{R}([\partial \underline{F}^{[0]}/\partial \underline{z}]) = R_{2,0}$$

and

$$\mathcal{N}([\partial \underline{F}^{[0]}/\partial \underline{z}]) = N_{2,0} = n - R_{2,0},$$

respectively. Also, let us denote the nullity of $[\partial \underline{F}^{[0]}/\partial \underline{z}]^T$ on $\mathbf{M}$ as the constant

$$\mathcal{N}([\partial \underline{F}^{[0]}/\partial \underline{z}]^T) = \mathbf{N}_{2,0} = m[0] - R_{2,0},$$

where $m[0] := m$.

Because the matrix $[\partial \underline{F}^{[0]}/\partial \underline{z}]^T$ is rank-deficient on $\mathbf{M}$ and/or $m > n$, $\mathbf{N}_{2,0} > 0$ on $\mathbf{M}$. The matrix $[\partial \underline{F}^{[0]}/\partial \underline{z}]^T$ has $\mathbf{N}_{2,0}$ linearly independent null vectors of the form

$$\underline{v}_0^{(i)} := \underline{v}_0^{(i)}(x,\underline{y},\underline{z})$$

$$= [v_{0,1}^{(i)}(x,\underline{y},\underline{z}), \ldots, v_{0,m}^{(i)}(x,\underline{y},\underline{z})]^T, \quad (3.7)$$

such that

$$[\underline{v}_0^{(i)}]^T[\partial \underline{F}^{[0]}/\partial \underline{z}] = \underline{0}^T, \quad (3.8)$$

for $1 \le i \le \mathbf{N}_{2,0}$, and $(x,\underline{y},\underline{z}) \in \mathbf{M}$.□

Definition 3.2: The ***first order constraints*** of the nonlinear system (3.2) are defined as

$$F_{m+i} := F_{m+i}(x,\underline{y},\underline{z}) := [\underline{v}_0^{(i)}]^T \underline{\boldsymbol{\alpha}}^{[0]} = 0, \quad (3.9)$$

for $1 \le i \le \mathbf{N}_{2,0}$, and $(x,\underline{y},\underline{z}) \in \mathbf{M}$.

Definition 3.3: We will let $\underline{F}^{[1]}$ denote the overdetermined system of differential/algebraic equations which consists of the zeroth order constraints (3.3), and the first order constraints (3.9): that is, an overdetermined system of the form

$$\underline{F}^{[1]} := \underline{F}^{[1]}(x,\underline{y},\underline{z}) := [\underline{F}^{[0]T}, F_{m+1}, \ldots, F_{m[1]}]^T = \underline{0}, \qquad (3.10)$$

such that

$$\underline{F}^{[1]} : \mathcal{M} \to R^{m[1]},$$

and

$$m[1] := m[0] + \mathcal{N}_{2,0}. \square$$

To further classify our first order constraints, we will need to define the following Jacobian matrices:

$$J^{1,0} := J^{1,0}(x,\underline{y},\underline{z}) := [\partial \underline{F}^{[0]}/\partial \underline{y}] \qquad (3.11a)$$

$$J^{2,0} := J^{2,0}(x,\underline{y},\underline{z}) := [\partial \underline{F}^{[0]}/\partial \underline{z}] \qquad (3.11b)$$

and

$$J^{3,0} := J^{3,0}(x,\underline{y},\underline{z}) := [J^{1,0}|J^{2,0}] \qquad (3.11c)$$

for $(x,\underline{y},\underline{z}) \in \mathcal{M}$. We will also assume that $R_{1,0} := \mathcal{R}(J^{1,0})$ and $R_{3,0} := \mathcal{R}(J^{3,0})$ are constants on $\mathcal{M}$.

We will now define the three types of first order constraints of the nonlinear system (3.2). Because the matrices defined in (3.11) are functions of $(x,\underline{y},\underline{z})$, our definitions of the three types of first order constraints will be pointwise: that is, will be for each $(x,\underline{y},\underline{z}) \in \mathcal{M}$.

Definition 3.4: ***First Order Constraints of Type A***

If $\partial F_{m+1}/\partial \underline{z}$ is not in the row space of $J^{2,0}$, F_{m+1} is a first order constraint of type A.□

Definition 3.5: ***First Order Constraints of Type B***

If $\partial F_{m+1}/\partial \underline{z}$ is in the row space of $J^{2,0}$, but $[\partial F_{m+1}/\partial \underline{y} | \partial F_{m+1}/\partial \underline{z}]$ is not in the row space of $J^{3,0}$, F_{m+1} is a first order constraint of type B.□

Definition 3.6: ***First Order Constraints of Type C***

If $[\partial F_{m+1}/\partial \underline{y} | \partial F_{m+1}/\partial \underline{z}]$ is in the row space of $J^{3,0}$, F_{m+1} is a first order constraint of type C.□

Let there be

$m_1^{[A]} := m_1^{[A]}(x,\underline{y},\underline{z})$

first order constraints of type A,

$m_1^{[B]} := m_1^{[B]}(x,\underline{y},\underline{z})$

of type B, and

$m_1^{[C]} := m_1^{[C]}(x,\underline{y},\underline{z})$

of type C, for all $(x,\underline{y},\underline{z}) \in \mathcal{M}$. We will assume that the function defined by (3.1) is such that $m_1^{[A]}$, $m_1^{[B]}$, and $m_1^{[C]}$ are constants on $\mathcal{M}$, with

$m_1^{[A]} + m_1^{[B]} + m_1^{[C]} = \mathcal{N}_{2,0}$.

Although we have $m_1^{[L]}$ first order constraints of type L (for L = A,B) on $\mathcal{M}$, they may not be linearly independent. Also, some of the first order constraints of type L may be linearly dependent on the zeroth order constraints. Therefore, let there be $n_1^{[\mathbf{L}]} \le \mathfrak{m}_1^{[\mathbf{L}]}$ linearly independent first order constraints of type $\mathbf{L}$ on $\mathfrak{M}$, which are also linearly independent of the zeroth order constraints, for $\mathbf{L}$ = A and B. Also, let $\mathfrak{m}_1^{[C]} \le \mathfrak{m}_1^{[C]}$ denote the number of linearly independent first order constraints of type C. These first order constraints are such that

$$n_1^{[A]} + n_1^{[B]} + n_1^{[C]} \le \mathfrak{N}_{2,0},$$

and $n_1^{[\mathbf{L}]}$ ($\mathbf{L}$ = A,B,C) are constants on $\mathfrak{M}$.

We will not discard any of our redundant or linearly dependent first order constraints because our classification is pointwise, becuase it may be difficult to numerically detect linear dependence, and because there is a notational advantage in perserving the size of $\underline{F}^{[1]}$.

Now we will define higher order constraints, and formalize our nonlinear constraint theory. Consider the overdetermined system

$$\underline{F}^{[\wp]} := \underline{F}^{[\wp]}(x,\underline{y},\underline{z}) = \underline{0}, \tag{3.12}$$

subject to the initial conditions

$$\underline{y}(x_0) = \underline{y}_0,$$

such that

$$\underline{F}^{[\wp]}:D \to \mathbf{R}^{m[\wp]},$$

$$\underline{y},\ \underline{z}:[x_0,x_{fin}] \to \mathbf{R}^n,$$

$$J^{1,\wp} := J^{1,\wp}(x,\underline{y},\underline{z}) := [J_{rs}^{1,\wp}] := \left[\frac{\partial \underline{F}^{[\wp]}}{\partial \underline{y}}\right] \tag{3.13a}$$

$$J^{2,\wp} := J^{2,\wp}(x,\underline{y},\underline{z}) := [J_{rs}^{2,\wp}] := \left[\frac{\partial \underline{F}^{[\wp]}}{\partial \underline{z}}\right] \tag{3.13b}$$

$$R_{1,\wp} := R_{1,\wp}(x,\underline{y},\underline{z}) := \mathcal{R}(J^{1,\wp}),$$

$$R_{2,\wp} := R_{2,\wp}(x,\underline{y},\underline{z}) := \mathcal{R}(J^{2,\wp}),$$

$$N_{2,\wp} := N_{2,\wp}(x,\underline{y},\underline{z}) := \mathcal{N}(J^{2,\wp}) = n - R_{2,\wp},$$

$$\mathbf{N}_{2,\wp} := \mathbf{N}_{2,\wp}(x,\underline{y},\underline{z}) := \mathcal{N}([J^{2,\wp}]^{+})$$

$$= m[\wp] - R_{2,\wp},$$

$$1 \le s \le n,$$

$$\wp \in \{0\} \cup \mathbf{N},$$

$$1 \le r \le m[\wp],$$

$$m[\wp] = \begin{cases} m & \text{for } \wp = 0 \\ m[\wp-1] + \mathbf{N}_{2\,\wp-1} & \text{for } \wp \geq 1, \end{cases}$$

and

$(x,\underline{y},\underline{z}) \in D$.

Also, for notational convenience, we define the following Jacobian matrix:

$$J^{3\wp} := J^{3\wp}(x,\underline{y},\underline{z}) := [J_{rs}{}^{3\wp}] := [\, J^{1\wp} \mid J^{2\wp} \,], \tag{3.13c}$$

such that

$R_{3\,\wp} := R_{3\,\wp}(x,\underline{y},\underline{z}) := \mathcal{R}(J^{3\wp})$,

$1 \leq s \leq n$,

$1 \leq r \leq m[\wp]$,

and

$(x,\underline{y},\underline{z}) \in D$.

We will also assume that $R_{1\,\wp}$, $R_{2\,\wp}$, $N_{2\,\wp}$, $\mathbf{N}_{2\,\wp}$, $R_{3\,\wp}$, etc., are constants on $\mathcal{M}$.

We see that the overdetermined system (3.12) is obtained from the zeroth to $(\wp-1)$st order constraints - that is, from $\underline{F}^{[\wp-1]}$ - and from the $\wp$th order constraints $F_{m[\wp-1]+i}$ $(1 \leq i \leq \mathbf{N}_{2\,\wp})$. Differentiating the nonlinear system (3.12) with respect to x yields

$$d\underline{F}^{[\wp]}/dx := J^{2\wp}\,\underline{z}' + J^{1\wp}\,\underline{z} + \partial\underline{F}^{[\wp]}/\partial x = \underline{0}. \tag{3.14}$$

The system (3.14), along with the substitution $\underline{z} = \underline{y}'$, can be written as

$$\begin{bmatrix} I & 0 \\ 0 & J^{2,\wp} \end{bmatrix} \begin{bmatrix} \underline{y}' \\ \underline{z}' \end{bmatrix} = \begin{bmatrix} \underline{z} \\ \underline{\alpha}^{[\wp]} \end{bmatrix}, \tag{3.15}$$

such that

$$\underline{\alpha}^{[\wp]} := \underline{\alpha}^{[\wp]}(x,\underline{y},\underline{z}) := -J^{1,\wp}\underline{z} - \partial \underline{F}^{[\wp]}/\partial x. \tag{3.16}$$

If $[J^{2,\wp}]^T$ is rank-deficient and/or $m[\wp] > n$ on $\mathcal{M}$, $[J^{2,\wp}]^T$ has $\mathcal{N}_{2,\wp} > 0$ linearly independent null vectors of the form

$$\underline{v}_\wp^{(i)} := \underline{v}_\wp^{(i)}(x,\underline{y},\underline{z})$$
$$:= [v_{\wp,1}^{(i)}(x,\underline{y},\underline{z}), \ldots, v_{\wp,m[\wp]}^{(i)}(x,\underline{y},\underline{z})]^T, \tag{3.17}$$

for $1 \le i \le \mathcal{N}_{2,\wp}$, $\wp \ge 1$, and $(x,\underline{y},\underline{z}) \in \mathcal{M}$. The first $\mathcal{N}_{2,\wp-1}$ null basis vectors in (3.17) can be chosen to be isomorphic to the vectors

$$\underline{v}_\wp^{(j)}(x,\underline{y},\underline{z}) := [v_{\wp-1,1}^{(j)}(x,\underline{y},\underline{z}), \ldots, v_{\wp-1,m[\wp-1]}^{(j)}(x,\underline{y},\underline{z}), 0, \ldots, 0]^T, \tag{3.18}$$

for $1 \le j \le \mathcal{N}_{2,\wp-1}$, $\wp \ge 1$, and $(x,\underline{y},\underline{z}) \in \mathcal{M}$. By definition of (3.17), we have

$$[\underline{v}_\wp^{(i)}]^T J^{2,\wp} = \underline{0}^T, \tag{3.19}$$

for $1 \le i \le \mathcal{N}_{2,\wp}$, and $(x,\underline{y},\underline{z}) \in \mathcal{M}$. From systems (3.15) and (3.19) we obtain the following $\mathcal{N}_{2,\wp}$ expressions:

$$F_{m[\wp]+i} := F_{m[\wp]+i}(x,\underline{y},\underline{z}) := [\underline{v}_\wp^{(i)}]^T \underline{\alpha}^{[\wp]} = 0, \tag{3.20}$$

for $1 \le i \le \mathcal{N}_{2,\wp}$, and $(x,\underline{y},\underline{z}) \in \mathcal{M}$.

Now we are prepared to pointwise define the three types of higher order constraints for rectangular nonlinear systems of differential/algebraic equations of the form (3.2): that is, $(\rho+1)$st order constraints of types A, B and C for (3.2), such that $(x,\underline{y},\underline{z}) \in \mathfrak{M}$.

Definition 3.7: ***$(\rho+1)$st Order Constraints of Type A***

If $\partial F_{m[\rho]+i}/\partial \underline{z}$ is not in the row space of $J^2\rho$, $F_{m[\rho]+i}$ is a $(\rho+1)$st order constraint of type A.□

Definition 3.8: ***$(\rho+1)$st Order Constraints of Type B***

If $\partial F_{m[\rho]+i}/\partial \underline{z}$ is in the row space of $J^2\rho$, but $[\partial F_{m[\rho]+i}/\partial \underline{y} \mid \partial F_{m[\rho]+i}/\partial \underline{z}]$ is not in the row space of $J^3\rho$, $F_{m[\rho]+i}$ is a $(\rho+1)$st order constraint of type B.□

Definition 3.9: ***$(\rho+1)$st Order Constraints of Type C***

If $[\partial F_{m[\rho]+i}/\partial \underline{y} \mid \partial F_{m[\rho]+i}/\partial \underline{z}]$ is in the row space of $J^3\rho$, $F_{m[\rho]+i}$ is a $(\rho+1)$st order constraint of type C.□

Let there be

$\eta_{\rho+1}^{[A]} := \eta_{\rho+1}^{[A]}(x,\underline{y},\underline{z})$

$(\rho+1)$st order constraints of type A,

$\eta_{\rho+1}^{[B]} := \eta_{\rho+1}^{[B]}(x,\underline{y},\underline{z})$

of type B, and

$\eta_{\rho+1}^{[C]} := \eta_{\rho+1}^{[C]}(x,\underline{y},\underline{z})$

of type C, for all $(x,\underline{y},\underline{z}) \in \mathfrak{M}$. We will assume that the function defined by (3.1) is such that $\eta_{\rho+1}^{[A]}$, $\eta_{\rho+1}^{[B]}$, and $\eta_{\rho+1}^{[C]}$ are constants on $\mathfrak{M}$, with

$$\eta_{\rho+1}^{[A]} + \eta_{\rho+1}^{[B]} + \eta_{\rho+1}^{[C]} = \mathfrak{N}_{2\rho}. \qquad (3.21a)$$

There may exist linear dependence amongst the $(\wp+1)$st order constraints of type $\mathbf{L}$ on $\mathfrak{M}$, for $\mathbf{L}$ = A and B. Also, some of the $(\wp+1)$st order constraints of type $\mathbf{L}$ may be linearly dependent on some of the zeroth through $\wp$th order constraints. Therefore, we have $n_{\wp+1}^{[\mathbf{L}]} \le \mathscr{n}_{\wp+1}^{[\mathbf{L}]}$ linearly independent $(\wp+1)$st order constraints of type $\mathbf{L}$ on $\mathfrak{M}$, which are also linearly independent of the zeroth through $\wp$th order constraints, such that $\mathbf{L}$ = A, B. Also, let $n_{\wp+1}^{[C]}$ denote the number of linearly independent $(\wp + 1)$st order constraints of type C on $\mathfrak{M}$.

These linearly independent $(\wp + 1)$st order constraints are such that

$$n_{\wp+1}^{[A]} + n_{\wp+1}^{[B]} + n_{\wp+1}^{[C]} \le \mathfrak{N}_{2,\wp}, \qquad (3.21b)$$

and $n_{\wp+1}^{[\mathbf{L}]}$ ($\mathbf{L}$ = A, B, C) is constant on $\mathfrak{M}$. By construction, for $\mathbf{L}$ = A and B, if $\mathscr{n}_{\wp}^{[\mathbf{L}]} > 0$, we have

$$1 \le n_{\wp}^{[\mathbf{L}]} \le \mathscr{n}_{\wp}^{[\mathbf{L}]},$$

and if $\mathscr{n}_{\wp}^{[\mathbf{L}]} = 0$, then $n_{\wp}^{[\mathbf{L}]} = 0$ on $\mathfrak{M}$.

We will not discard any of our redundant or linearly dependent $(\wp + 1)$st order constraints because our classification is pointwise, because it may be difficult to numerically detect linear dependence, and because there is a notational advantage in preserving the size of $\underline{F}^{[\wp+1]}$.

4. Termination of Recursive Process

Our recursive process - primarily involving differentiations and inspections of row spaces - terminates if all the $(\wp+1)$st order constraints are of type C for $(x,\underline{y},\underline{z}) \in \mathbf{M}$. The proof of the following lemma verifies that our recursive process terminates at some finite stage.

Lemma 4.1:

There exists a nonnegative integer k, such that all the (k + 1)st order constraints are of type C on $\mathbf{W} := W_x \times R^n \times R^n$, or equivalently, there exists a nonnegative integer k, such that $R_{3,k} = R_{3,k+1}$ on $\mathbf{W}$. Also, if the nonnegative integer $\mathbf{k}$ is such that $R_{3,\mathbf{k}} = R_{3,\mathbf{k}+1}$ then $\mathbf{k} \geq k$.□

Proof of Lemma 4.1:

Because $J^3{}_\wp$ has only 2n columns, $R_{3,\wp}$ is bounded above by 2n: that is, $R_{3,\wp} \leq 2n$.

Information about the number of type A and B constraints on $\mathbf{W}$ can be utilized to express as follows:

$$R_{3,\wp} = R_{3,0} + \sum_{i=1}^{\wp} \left[n_i^{[A]} + n_i^{[B]} \right]. \qquad (4.1a)$$

Because $n_i^{[A]}$ and $n_i^{[B]}$ are nonnegative sequences, we can conclude from (4.1a) that

$$R_{3,\wp} < R_{3,\wp+1}, \qquad (4.1b)$$

unless we have reached a stage in our recursive process where all the constraints are of type C: that is,

$R_{3,\wp} = R_{3,\wp+1}$

Since $R_{3,\wp+1}$ is bounded from above by 2n, we know that there exists a nonnegative integer k, such that $\wp = k$, and

$$R_{3,k-1} < R_{3,k} = R_{3,k+1} = R_{3,k+2} = \ldots = R_{3,k+1} \square$$

By pointwise invoking Lemma 4.1, we know that our recursive process terminates at some stage $\wp = k$. When our process terminates we must determine if the initial conditions are consistent, and also must check for one of the following four situations on $\mathcal{M}$:

(i) $R_{2,k} = n$, and $R_{3,k} - R_{2,k} < R_{2,k}$.

(ii) $R_{2,k} < n$, and $R_{3,k} - R_{2,k} < R_{2,k}$.

(iii) $R_{2,k} = n$, and $R_{3,k} - R_{2,k} = R_{2,k}$.

(iv) $R_{2,k} < n$, and $R_{3,k} - R_{2,k} = R_{2,k}$.

Situations (i) and (ii), which will be referred to as our primary categorizations, are the most likely ones to occur. If our primary categorizations occur, we will have utilized our constraint theory to transform (3.2) to an explicit ordinary differential equations initial value problem. Situations (iii) and (iv), which will be referred to as our secondary categorizations, are desirable, but special cases. If these secondary categorizations occur, we will have utilized our constraint theory to transform (3.2) to a nonlinear algebraic system.

We will utilize the necessary conditions for a solution to systems of the form (3.2), which were obtained via our nonlinear constraint analysis, to derive sufficient conditions for a solution to (3.2). We were led by our constraint analysis as to how we should define $\underline{F}^{[k]}$ on a trajectory in D; that is, $\underline{F}^{[k]}$ as defined in (3.12). We have assumed that the function defined by (3.12) is well-defined on $\mathcal{M}$. This assumption is explicitly stated in the following definition:

Definition 4.1. The function $\underline{F}^{[k]}$, defined in (3.12), is ***well-defined*** on $\mathcal{M}$ if the following is true:

(i) For $\wp = k$, the vector $[\partial \underline{F}^{[k]}/\partial x]$ is a smooth function of x.

(ii) For $\wp = k$, the Jacobian matrices defined in (3.13) are smooth, and of constant rank and nullity on $\mathcal{M}$.

(iii) For $\mathbf{L}$ = A, B, C and $0 \leq \wp \leq k+1$, $n_\wp^{[\mathbf{L}]}$ is a constant on $\mathcal{M}$.□

Let us extend the definition of $\underline{F}^{[k]}$ for arbitrary x, $\underline{y}$, and $\underline{z}$ on the open domain $D_0 \subset \mathbf{R}^{2n+1}$; that is, let us consider a function of the form

$$\underline{F}^{[k]}(x,\underline{y},\underline{z}), \tag{4.2}$$

such that

$(x,\underline{y},\underline{z}) \in D_0$,

$\underline{F}^{[k]}: D_0 \rightarrow \mathbf{R}^{m[k]}$,

and $m[k] \geq n$. The corresponding Jacobian matrices are as previously defined in (3.13).

4.1 Termination of Process: Maximal Rank Differential Equations Case

First we will discuss nonlinear systems of the form (3.2) and (3.12), for which our recursive process terminates with $R_{2,k} = n$ and $R_{3,k} - R_{2,k} < R_{2,k}$ on $\mathcal{M}$. We will see that if in addition to having consistent initial conditions, we also have $R_{2,k} = n$, $R_{3,k} - R_{2,k} < R_{2,k}$, and certain smoothness properties hold, then we will be able to obtain a unique local solution to nonlinear systems of differential/algebraic equations of the form (3.2). The criteria, which give us conditions for the existence and uniqueness of a local solution to rectangular nonlinear systems of differential/algebraic equations of the form (3.2), are stated in the next two theorems.

Theorem 4.1:

Let $W_x \subset [x_0, x_{fin}]$ and $W_{\underline{y}} \subset \mathbf{R}^n$ be neighborhoods of x_0 and $\underline{y}_0$, respectively. Suppose there exists a $\underline{z}_0 \in W_{\underline{z}} \subset \mathbf{R}^n$, such that

$$\underline{F}^{[k]}(x_0, \underline{y}_0, \underline{z}_0) = \underline{0}, \tag{4.3}$$

$R_{2,k} := \mathcal{R}[J^{2,k}(x_0, \underline{y}_0, \underline{z}_0)] = n$. Let $\mathcal{M} := W_x \times W_{\underline{y}} \times W_{\underline{z}} \subset D$ be a neighborhood of $(x_0, \underline{y}_0, \underline{z}_0) \in D$. Suppose the function $\underline{F}$, defined by (3.1), and the corresponding well-defined function $\underline{F}^{[k]}$, defined by (4.2), exist on $\mathcal{M}$ and are of class $C^2(\mathcal{M})$.

Then $\underline{z}_0$ can be uniquely expressed in terms of x_0 and $\underline{y}_0$. □

Proof of Theorem 4.1

Consider the following ball about $\underline{z}_0$ of radius ρ, such that ρ > 0, that is,

$B(\underline{z}_0,\rho) := \{\underline{z} \in \mathbf{R}^n : \|\underline{z}-\underline{z}_0\| < \rho\}$

We claim that $\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}) \neq \underline{0}$, for $\underline{z} \in B(\underline{z}_0,\rho) - \{\underline{z}_0\}$, or equivalently, $\|\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z})\| > 0$, for $\underline{z} \in B(\underline{z}_0,\rho) - \{\underline{z}_0\}$.

Utilizing the fact that the function $\underline{F}^{[k]}$ is of class $C^2(\mathcal{M})$, and taking the Taylor's expansion of $\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z})$ about the point $(x_0,\underline{y}_0,\underline{z}_0)$ yields

$\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}) = \underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0) + [\partial\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0)/\partial\underline{z}](\underline{z}-\underline{z}_0) + O(\|\underline{z}-\underline{z}_0\|^2)$.

Taking norms of the previous expansion, utilizing the reverse triangle inequality, and utilizing the definitions of the order symbol and singular values, yields

$$\begin{aligned}
\|\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z})\| &\geq \|[\partial\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0)/\partial\underline{z}](\underline{z}-\underline{z}_0)\| - O(\|\underline{z}-\underline{z}_0\|^2)\\
&\geq \|[\partial\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0)/\partial\underline{z}](\underline{z}-\underline{z}_0)\| - c\|\underline{z}-\underline{z}_0\|^2\\
&\geq s_n\|\underline{z}-\underline{z}_0\| - c\|\underline{z}-\underline{z}_0\|^2\\
&= \|\underline{z}-\underline{z}_0\|(s_n + c(-\|\underline{z}-\underline{z}_0\|))
\end{aligned}$$

We know that $\|\underline{z}-\underline{z}_0\| > 0$ for $\underline{z} \in B(\underline{z}_0,\rho) - \{\underline{z}_0\}$. We also know that $(s_n + c(-\|\underline{z}-\underline{z}_0\|)) > 0$ if $\|\underline{z}-\underline{z}_0\| < s_n/c$: that is, if $\rho < s_n/c$.

Thus, we have proven that $\|\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z})\| > 0$, for all $\underline{z} \in B(\underline{z}_0,\rho) - \{\underline{z}_0\}$, such that,

$B(\underline{z}_0,\rho) = \{\underline{z} \in R^n : \|\underline{z}-\underline{z}_0\| \le \rho < s_n/c\}$,

where $s_n > 0$ is the smallest singular value of $[\partial \underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0)/\partial \underline{z}]$, and $c > 0$ is defined via the Taylor's expansion of $\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z})$ about $(x_0,\underline{y}_0,\underline{z}_0)$.□

Thus, if the hypothesis of Theorem 4.1 is satisfied, we can prove that a locally unique set of consistent initial conditions for the initial value problem (3.2) exists. We will now obtain conditions for the existence of a unique local solution to system (3.2) by integrating the system (3.15), subject to the initial conditions

$(x_0, \underline{y}_0, \underline{z}_0) := (x_0, \underline{y}(x_0), \underline{z}(x_0))$.

Let us define $\underline{\varepsilon}: W_x \longrightarrow W_{\underline{y}} \times W_{\underline{z}}$ as follows:

$$\underline{\varepsilon} := [\underline{y}^T, \underline{z}^T]^T. \tag{4.4}$$

By definition of our (k+1)st order constraints (that is, (3.20) with $\mu = k$), and the Fredholm Alternative Theorem, we can express $\underline{\varepsilon}'$ in the system (3.15) as a function of x and $\underline{\varepsilon}$. More explicitly, because certain consistency conditions are satisfied at each $(x,\underline{y},\underline{z}) \in \mathcal{M}$ - that is, the (k+1)st order constraints (3.20) hold at each $(x,\underline{y},\underline{z}) \in \mathcal{M}$, the Fredholm Alternative Theorem can be invoked to prove the existence of a solution to the implicit system (3.15) at each $(x,\underline{y},\underline{z}) \in \mathcal{M}$.

Thus, we can rewrite the implicit system (3.15) as the following explicit system

$$\underline{\varepsilon}' = \underline{h}(x,\underline{\varepsilon}), \tag{4.5}$$

subject to the initial conditions

$(x_0, \underline{z}_0, \underline{z}_0) = (x_0, \underline{y}(x_0), \underline{z}(x_0)),$

such that

$x \in W_x,$

$\underline{z}: W_x \rightarrow W_{\underline{y}} \times W_{\underline{z}},$

and

$\underline{h}: \mathfrak{M} \rightarrow W_{\underline{y}} \times W_{\underline{z}}$

is defined as

$$\underline{h}(x, \underline{z}) = \begin{bmatrix} \underline{z} \\ \left[J^{2,k}(x, \underline{y}, \underline{z})\right]^{+} \underline{\alpha}^{[k]} \end{bmatrix}.$$

The results of the following existence and uniqueness theorem for the initial value problem (4.5) will be deduced for all $(x, \underline{z})$ in a region $\mathcal{D} \subset \mathfrak{M}$, which is defined as follows:

$$\mathcal{D} := W_{x'} \times W_{\underline{z}'} \tag{4.6}$$

such that

$W_{x'} := [x_0, x_0 + a],$

$W_{\underline{z}'} := \{\underline{z} : \|\underline{z} - \underline{z}_0\| \leq b\},$

where $a > 0$, $b > 0$, $W_{x'} \subset W_x$, and $W_{\underline{z}'} \subset W_{\underline{y}} \times W_{\underline{z}}$. If $\underline{h}$ is of class $\mathcal{C}^0(\mathcal{D})$, then $\underline{h}$ is bounded there. Let

$$\mathcal{M} := \sup_{(x, \underline{z}) \in \mathcal{D}} \|\underline{h}\|, \tag{4.7}$$

and

$$\epsilon := \min(a, (b/\mathcal{M})). \tag{4.8}$$

Theorem 4.2:

Suppose the hypothesis of Theorem 4.1 is satisfied. Suppose the initial value problem (4.5) is such that $\underline{h} \in (\mathcal{C}^0, \text{Lip})$ in $\mathcal{D}$, with the Lipschitz constant

$$L := \sup_{(x,\underline{z}) \in \mathcal{D}} \| \mathbf{\mathfrak{L}}(x,\underline{y},\underline{z}) \|, \tag{4.9}$$

where

$$\mathbf{\mathfrak{L}}(x,\underline{y},\underline{z}) := \begin{bmatrix} 0 & I \\ 0 & -\left[J^{2,k}(x,\underline{y},\underline{z})\right]^{+} J^{1,k}(x,\underline{y},\underline{z}) \end{bmatrix}$$

for all $(x,\underline{y},\underline{z}) \in \mathbf{M}$.

Then, there exists a unique solution

$$\underline{\Phi} := \underline{\Phi}(x), \quad x \in [x_0, x_0+\epsilon] \tag{4.10}$$

of (4.5), such that $\underline{\Phi}(x_0) = \underline{z}_0$. Also, the unique solution to the nonlinear system (3.2) is the first n components of $\underline{\Phi}$; that is, the solution to (3.2) is

$$\underline{\Phi}_* := \underline{\Phi}_*(x) := [\Phi_1(x), \ldots, \Phi_n(x)]^T. \square \tag{4.11}$$

Proof of Theorem 4.2:

Invoke the Picard-Lindelöf Theorem.□

4.2 Termination of Process: Rank-Deficient Differential Equations Case

Next we will discuss nonlinear systems of the form (3.2) and (3.12) for which our recursive process terminates with $R_{2,k} < n$, and $R_{3,k} - R_{2,k} < R_{2,k}$ on $\mathcal{M}$. We will see that if, in addition to having consistent initial conditions for the problem (3.2), we also have $R_{2,k} < n$, and certain smoothness properties hold on $\mathcal{M}$, then we will only be able to prove existance of a solution to (3.2). This particular situation is summarized in the following two theorems.

Theorem 4.3:

Let $W_x \subset [x_0, x_{fin}]$ and $W_{\underline{y}} \subset \mathbf{R}^n$ be neighborhoods of x_0 and $\underline{y}_0$, respectively. Suppose there exists a $\underline{z}_0 \in \mathbf{R}^n$, such that

$$\underline{F}^{[k]}(x_0, \underline{y}_0, \underline{z}_0) = \underline{0} \tag{4.12}$$

$r := R_{2,k} := \mathcal{R}[J^{2,k}(x_0, \underline{y}_0, \underline{z}_0)] < n$. Let $\mathcal{M} := W_x \times W_{\underline{y}} \times W_{\underline{z}} \subset D$ be a neighborhood of $(x_0, \underline{y}_0, \underline{z}_0)$. Suppose the function $\underline{F}$, defined by (3.1), and the corresponding well-defined function $\underline{F}^{[k]}$, defined by (4.2), exist on $\mathcal{M}$ and are of class $\mathcal{C}^1(\mathcal{M})$.

Without loss of generality, we will assume locally that r components of $\underline{z}_0$ can be uniquely expressed in terms of x_0, $\underline{y}_0$, and the remaining n-r components of $\underline{z}_0$. More explicitly, let us relabel the components of $\underline{z}$. Then $\underline{z}_{r,0} \in \mathbf{R}^r$ (r components of $\underline{z}_0$) can be uniquely expressed in terms of x_0, $\underline{y}_0$, and $\underline{z}_{n-r,0} \in \mathbf{R}^{n-r}$ (remaining n-r components of $\underline{z}_0$).□

Proof of Theorem 4.3

Let us relabel the system (3.12) with ρ = k as follows:

$$\underline{\mathbf{f}}^{[k]}(x, \underline{\mathbf{y}}, \underline{\mathbf{z}}_{n-r}, \underline{\mathbf{z}}_r) = \underline{0}, \qquad (4.13)$$

such that

$\underline{\mathbf{y}} = \underline{\mathbf{z}} = [\underline{\mathbf{z}}_{n-r}\ \underline{\mathbf{z}}_r]^T$,

$\underline{\mathbf{f}}^{[k]}(x_0, \underline{\mathbf{y}}_0, \underline{\mathbf{z}}_{n-r,0}, \underline{\mathbf{z}}_{r,0}) = \underline{0}$, and $\partial\underline{\mathbf{f}}^{[k]}/\partial\underline{\mathbf{z}}$ - evaluated at $(x_0, \underline{\mathbf{y}}_0, \underline{\mathbf{z}}_{n-r,0}, \underline{\mathbf{z}}_{r,0})$ - is of rank r.

Then by the Implicit Function Theorem, we know that there exists a neighborhood U $\subset$ $\mathbf{R}^{2n-r+1}$ of $(x_0, \underline{\mathbf{y}}_0, \underline{\mathbf{z}}_{n-r,0})$, and a neighborhood V $\subset$ $\mathbf{R}^r$ of $\underline{\mathbf{z}}_{r,0}$, and a unique function $\underline{\gamma}$:U $\rightarrow$ V, such that

$$\underline{\mathbf{f}}^{[k]}(x, \underline{\mathbf{y}}, \underline{\mathbf{z}}_{n-r}, \underline{\gamma}(x, \underline{\mathbf{y}}, \underline{\mathbf{z}}_{n-r})) = \underline{0} \qquad (4.14)$$

for all $(x, \underline{\mathbf{y}}, \underline{\mathbf{z}}_{n-r}) \in$ U. Furthermore, $\underline{\gamma}$ is of class C^1.□

Thus if the hypothesis of Theorem 4.3 is satisfied we can obtain an entire surface $\underline{\Gamma} \subset \mathbf{R}^{2n+1}$ of consistent initial conditions for the initial value problem (3.2). Now we will obtain conditions for the existence of a local solution to the initial value problem (3.2) by integrating the system (3.15), subject to any point on the surface $\underline{\Gamma}$.

Because certain consistency conditions are satisfied - that is, the (k + 1)st order constraints ((3.20) with ρ = k) hold, the Fredholm Alternative Theorem can be invoked to prove the existence of a solution to the implicit system (3.15). Suppose we can choose a smooth function $\underline{g}$: $\mathcal{M} \rightarrow \mathbf{R}^n$, such that $\underline{g}$ is in the null space of $J^{2,k}$ on $\mathcal{M}$, and the initial conditions on $\underline{\Gamma}$ are consistent for the following explicit system:

$$\underline{\zeta}' = \underline{h}(x, \underline{\zeta})$$

$$:= \begin{bmatrix} \underline{z} \\ \left[J^{2,k}(x, \underline{y}, \underline{z})\right]^{+} \underline{\alpha}^{[k]}(x, \underline{y}, \underline{z}) + \underline{g} \end{bmatrix} \tag{4.15}$$

where

$x \in W_x$,

$\underline{g}: \mathfrak{M} \rightarrow R^n$,

$\underline{\zeta}: W_x \rightarrow R^{2n}$,

and

$\underline{h}: \mathfrak{M} \rightarrow R^{2n}$.

The local results of the following existence theorem for the initial value problem (4.15) will be deduced for all $(x, \underline{\zeta})$ in a region $\mathcal{D} \subset \mathfrak{M}$, which is defined by (4.7), (4.8), and (4.9).

Theorem 4.4:

Suppose the hypothesis of theorem 4.3 is satisfied. Suppose the initial value problem (4.15) is such that $\underline{h} \in (C^0, \text{Lip})$ in $\mathcal{D}$, with the Lipschitz constant (4.9).

Then, there exists a unique solution

$$\underline{\Phi g} := \underline{\Phi g}(x), \quad x \in [x_0, x_0 + \epsilon] \tag{4.16}$$

of (4.15), such that $\underline{\Phi g}(x_0) = \underline{\zeta}_0$. The notation $\underline{\Phi g}$ denotes that the solution is unique once a smooth $\underline{g}$ has been chosen for which the initial conditions are consistent. Also a solution to the nonlinear system (3.2) gives the first n components of $\Phi\mathfrak{q}$; that is, a nonunique solution to (3.2) is

$$\underline{\Phi}_* := \underline{\Phi}_*(x) := [\Phi\mathfrak{q}_1(x), \ldots, \Phi\mathfrak{q}_n(x)]^T \square \qquad (4.17)$$

Proof of Theorem 4.4:

Invoke the Picard-Lindelöf Theorem.□

4.3 Termination of Process: Algebraic Systems Case

Now that we have discussed our primary categorizations, we wish to move on to our secondary categorizations. It can be pointwise proven, that the Jacobian matrix $J^{3,k}(x,\underline{y},\underline{z})$ can be transformed, via Gaussian elimination with pivoting, to a block matrix of the form

$$\begin{bmatrix} \dfrac{\mathfrak{J}^{1,k}(x,\underline{y})}{\mathcal{J}^{1,k}(x,\underline{y},\underline{z})} & \Bigg| & \dfrac{0}{\mathcal{J}^{2,k}(x,\underline{y},\underline{z})} \end{bmatrix}, \qquad (4.18)$$

such that

$\mathcal{J}^{1,k}, \mathcal{J}^{2,k} : \mathcal{M} \to \mathbf{R}^{m[k]-r \times n}$,

$r := R_{3,k} - R_{2,k}$,

$\mathfrak{J}^{1,k} : W_x \times W_{\underline{y}} \to \mathbf{R}^{r \times n}$,

and $\mathfrak{J}^{1,k}$ has rank r on $W_x \times W_{\underline{y}}$. The criteria, which gives us conditions for the exisitence, and in some instances, the uniqueness of a local solution to rectangular nonlinear systems of differential/algebraic equations of the form (3.2), for which (4.18) holds, are stated in the following theorem.

Theorem 4.5:

Let $W_x \subset [x_0,x_{fin}]$ and $W_{\underline{y}} \subset \mathbf{R}^n$ be neighborhoods of x_0 and $\underline{y}_0$, respectively. Suppose there exists a $\underline{z}_0 \in W_{\underline{z}} \subset \mathbf{R}^n$, such that

$(x_0,\underline{y}_0,\underline{z}_0) \in \mathcal{M} := W_x \times W_{\underline{y}} \times W_{\underline{z}} \subset D$,

$$\underline{F}^{[k]}(x_0,\underline{y}_0,\underline{z}_0) = \underline{0}, \qquad (4.19)$$

and $r = R_{3,k} - R_{2,k}$. Suppose the function $\underline{F}$, defined by (3.1), and the corresponding well-defined function $\underline{F}^{[k]}$, defined by (4.3), exist on $\mathfrak{M}$ and are of class $C^1(\mathfrak{M})$.

The system (3.12), with $\rho = k$, can be transformed to the following decoupled system:

$$\underline{F}^{[k]}(x,\underline{y},\underline{z}) := \begin{bmatrix} \underline{\mathbf{F}}^{r}(x,\underline{y}) \\ F^{m[k]-r}(x,\underline{y},\underline{z}) \end{bmatrix}, \tag{4.20}$$

such that

$J^{1,k} = \partial\underline{\mathbf{F}}^r/\partial\underline{\mathbf{y}}$,

$(x,\underline{y},\underline{z}) \in \mathfrak{M}$,

$\underline{\mathbf{F}}^r : W_x \times W_{\underline{y}} \to \mathbf{R}^r$,

$\underline{F}^{m[k]-r} : \mathfrak{M} \to \mathbf{R}^{m[k]-r}$

Then at each $(x,\underline{y},\underline{z}) \in \mathfrak{M}$, we can solve the first r equations of the nonlinear system (4.20) for r components of $\underline{y}$ in terms of x.□

Proof of Theorem 4.5:

Suppose we relabel the components of $\underline{y}$. Let $\mathfrak{U} := W_x \times W_{\mathbf{y}_{n-r}} \times W_{\mathbf{y}_r}$ be an open set in $\mathbf{R}^{n+1}$. Suppose we extend the definition of $\underline{\mathbf{F}}^r$ for arbitrary $(x,\mathbf{y}_{n-r},\mathbf{y}_r) \in \mathfrak{U}$: that is, let us consider a function of the form

$$\underline{\mathbf{F}}^r(x,\mathbf{y}_{n-r},\mathbf{y}_r), \tag{4.21}$$

and

$J^{1,k} = \partial \underline{F}^r / \partial \underline{y}$

is of rank r.

Then by the Implicit Function Theorem we know that there exists a neighborhood $[W_{x'} \times W'_{\mathbf{y}_{n-r}}] \subset \{W_x \times W_{\mathbf{y}_{n-r}}\}$ of $(x_0, \mathbf{y}_{n-r,0})$, and a neighborhood $W'_{\mathbf{y}_r}$ of $\mathbf{y}_{n-r,0}$ in $W_{\mathbf{y}_r}$, and a unique function $\underline{\Phi}_r: [W_{x'} \times W'_{\mathbf{y}_{n-r}}] \rightarrow W'_{\mathbf{y}_r}$, such that

$$\underline{F}^r(x, \mathbf{y}_{n-r}, \underline{\Phi}_r(x, \mathbf{y}_{n-r})) = \underline{0}, \qquad (4.22a)$$

for all $(x, \mathbf{y}_{n-r}) \in \{W_{x'} \times W'_{\mathbf{y}_{n-r}}\}$.

Also, because we can only uniquely obtain r components of $\underline{y}$ in terms of x and the remaining n-r components of $\underline{y}$, the remaining m[k]-r equations of (4.20) are functionally dependent on the first r equations of (4.20). Thus, the function $\underline{\Phi}_r$ also satisfies these equations: that is, we also have

$$\underline{F}^{m[k]-r}(x, \mathbf{y}_{n-r}, \underline{\Phi}_r(x, \mathbf{y}_{n-r}), \underline{z}) = \underline{0}. \square \qquad (4.22b)$$

5. Illustration of Nonlinear Constraint Theory (Rigid Pendulum Problem)

Suppose a pendulum swings in a plane with rectangular coordinates $(y_1(t), y_2(t))$, and corresponding velocities $(y_3(t), y_4(t))$. The Lagrangian is $L := (y_3^2 + y_4^2)/2$ for a unit mass, while the external force is $\underline{f} = [0, -1]^T$ for a unit gravitational force in the negative y_2 direction. The constraint or algebraic equation is $c := y_1^2 + y_2^2 - 1 = 0$ if the mass is a unit distance from the pivot. Also, the authors in [2] interpret the term $-2y_5(t)$ to be the tension in the pendulum rod. Therefore, the rigid pendulum problem can be represented by a decoupled system of differential/algebraic equations of the form (3.2). That is,

$$F(x,\underline{y}(x),\underline{z}(x)) := \begin{bmatrix} F_1(x,\underline{y}(x),\underline{y}'(x)) \\ F_2(x,\underline{y}(x),\underline{y}'(x)) \\ F_3(x,\underline{y}(x),\underline{y}'(x)) \\ F_4(x,\underline{y}(x),\underline{y}'(x)) \\ F_5(x,\underline{y}(x),\underline{y}'(x)) \end{bmatrix} = \begin{bmatrix} y_1' - y_3 \\ y_2' - y_4 \\ y_3' + 2y_1y_5 \\ y_4' + 2y_2y_5 - 1 \\ y_1^2 + y_2^2 - 1 \end{bmatrix} = \underline{0}. \quad (5.1)$$

The first order constraint for system (5.1) is:

$$F_6 := -2v_5^{(1)}(y_1y_1' + y_2y_2') = 0.$$

For sake of simplicity we will let $2v_5^{(1)} = -1$.

Because

$$\partial F_6/\partial \underline{z} = [y_1 \;\; y_2 \;\; 0 \;\; 0 \;\; 0]$$

is in the row space of

$$J^{2,0} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix},$$

and

$$[\, \partial F_6/\partial \underline{y} \mid \partial F_6/\partial \underline{z} \,] = [y'_1 \;\; y'_2 \;\; 0 \;\; 0 \;\; 0 \mid y_1 \;\; y_2 \;\; 0 \;\; 0 \;\; 0]$$

is not in the row space of

$$J^{3,0} = \left[\begin{array}{ccccc|ccccc} 0 & 0 & -1 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & -1 & 0 & & 1 & & & \\ 2y_5 & 0 & 0 & 0 & 2y_1 & & & 1 & & \\ 0 & 2y_5 & 0 & 0 & 2y_2 & & & & 1 & \\ 2y_1 & 2y_2 & 0 & 0 & 0 & & & & & 0 \end{array}\right]$$

F_6 is a first order constraint of type B, for all $(x, \underline{y}, \underline{z}) \in \mathcal{M}$.

Now we can form $\underline{F}^{[1]}$.

$$\underline{F}^{[1]} = \begin{bmatrix} \underline{F}^{[0]} \\ F_6 \end{bmatrix} = \begin{bmatrix} y_1' - y_3 \\ y_2' - y_4 \\ y_3' + 2y_1y_5 \\ y_4' + 2y_2y_5 - 1 \\ y_1^2 + y_2^2 - 1 \\ y_1y_1' + y_2y_2' \end{bmatrix} = \underline{0} \qquad (5.2)$$

Differentiating the nonlinear system (5.2) with respect to x, and utilizing the substitution $\underline{z} := \underline{y}'$, yields a system of the form (3.15), for $\rho = 1$,

$$J^{2,1} := \begin{bmatrix} J^{2,0} \\ \partial F_6/\partial \underline{z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\underline{\alpha}^{[1]} := -J^{1,1}\underline{z},$$

such that

$$J^{1,1} = \begin{bmatrix} J^{1,0} \\ \partial F_6/\partial \underline{y} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 2y_5 & 0 & 0 & 0 & 2y_1 \\ 0 & 2y_5 & 0 & 0 & 2y_2 \\ 2y_1 & 2y_2 & 0 & 0 & 0 \\ y'_1 & y'_2 & 0 & 0 & 0 \end{bmatrix}.$$

The rank-deficient matrix $J^{2,1}$ is such that $R_{2,1} = 4$, and $\mathcal{N}_{2,1} := m[1] - R_{2,1} = 2$. Thus $[J^{2,1}]^T$ has two null vectors of the form

$[\underline{v}_1^{(1)}]^T = [0, 0, 0, 0, 1, 0]$,

$[\underline{v}_1^{(2)}]^T = [y_1, y_2, 0, 0, 0, -1]$,

which are such that (3.19) holds, for $\rho = 1$ and $i = 1,2$. Thus we obtain the following second order constraints:

$F_7 := [\underline{v}_1^{(1)}]^T \underline{\alpha}^{[1]} = -2(y_1 y'_1 + y_2 y'_2)$,

and

$F_8 := [\underline{v}_1^{(2)}]^T \underline{\alpha}^{[1]} = y_1 y'_3 + y_2 y'_4 + y'_1 y'_1 + y'_2 y'_2$.

For the complete demonstration of the application of our constraint theory in order to prove the existence and uniqueness of the solution to this problem, see [4]. Successive differentiations with respect to x, etc. yield a system of the form (3.15), for

$\rho = 3$,

$$J^{2,3} = \begin{bmatrix} J^{2,2} \\ \partial F_9/\partial \underline{z} \\ \partial F_{10}/\partial \underline{z} \\ \partial F_{11}/\partial \underline{z} \\ \partial F_{12}/\partial \underline{z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & 0 & 0 \\ -2y_1 & -2y_2 & 0 & 0 & 0 \\ 2y'_1 & 2y'_2 & y_1 & y_2 & 0 \\ -2y_1 & -2y_2 & 0 & 0 & 0 \\ 2y'_1 & 2y'_2 & y_1 & y_2 & 0 \\ 4y'_1 & 4y'_2 & 2y_1 & 2y_2 & 0 \\ 3y'_3-2y_1y_5 & 3y'_4-2y_2y_5 & 3y'_1 & 3y'_2 & -2(y_1^2+y_2^2) \end{bmatrix}$$

and

$$\underline{\alpha}^{[3]} := -J^{1,3}\underline{z},$$

such that

$$J^{1,3} := \begin{bmatrix} J^{1,2} \\ \partial F_9/\partial \underline{y} \\ \partial F_{10}/\partial \underline{y} \\ \partial F_{11}/\partial \underline{y} \\ \partial F_{12}/\partial \underline{y} \end{bmatrix}$$

$$
= \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
2y_5 & 0 & 0 & 0 & 2y_1 \\
0 & 2y_5 & 0 & 0 & 2y_2 \\
2y_1 & 2y_2 & 0 & 0 & 0 \\
y'_1 & y'_2 & 0 & 0 & 0 \\
-2y'_1 & -2y'_2 & 0 & 0 & 0 \\
y'_3 & y'_4 & 0 & 0 & 0 \\
-2y'_1 & -2y'_2 & 0 & 0 & 0 \\
y'_3 & y'_4 & 0 & 0 & 0 \\
2y'_3 & 2y_4 & 0 & 0 & 0 \\
-2y_5y'_1-4y_1y'_5 & -2y_5y'_2-4y_2y'_5 & 0 & 0 & -2(y_1y'_1+y_2y'_2)
\end{bmatrix}
$$

We observe that the matrix $J^{2,3}$ is of full rank, that is, $R_{2,3} = n = 5$. Also, the matrix $[J^{2,3}]^T$ has $\mathbf{N}_{2,3} := m[3] - R_{2,3} = 7$ null vectors of the form

$[\underline{v}_3^{(1)}]^T = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$,

$[\underline{v}_3^{(2)}]^T = [y_1\ y_2\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 0]$,

$[\underline{v}_3^{(3)}]^T = [2y_1\ 2y_2\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0]$,

$[\underline{v}_3^{(4)}]^T = [2y'_1\ 2y'_2\ y_1\ y_2\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0]$,

$[\underline{v}_3^{(5)}]^T = [2y_1\ 2y_2\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$,

$[\underline{v}_3^{(6)}]^T = [2y'_1\ 2y'_2\ y_1\ y_2\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0]$,

and

$[\underline{v}_3^{(7)}]^T = [4y_1\ 4y_2\ 2y_1\ 2y_2\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0]$,

which are such that (3.19) holds, for $\rho = 3$ and $1 \le i \le 7$. Thus we obtain the following fourth order constraints:

$F_{13} := [\underline{v}_3^{(1)}]^T \underline{\alpha}^{[3]} = F_7$,

$F_{14} := [\underline{v}_3^{(2)}]^T \underline{\alpha}^{[3]} = F_8$,

$F_{15} := [\underline{v}_3^{(3)}]^T \underline{\alpha}^{[3]} = 2F_8$,

$F_{16} := [\underline{v}_3^{(4)}]^T \underline{\alpha}^{[3]} = F_{12}$,

$F_{17} := [\underline{v}_3^{(5)}]^T \underline{\alpha}^{[3]} = 2F_8$,

$F_{18} := [\underline{v}_3^{(6)}]^T \underline{\alpha}^{[3]} = F_{12}$,

and

$F_{19} := [\underline{v}_3^{(7)}]^T \underline{\alpha}^{[3]} = 2F_{12}$.

Thus we see that we have seven nontrivial fourth order constraints, which are all of type C.

Therefore, our recursive process, applied to the system of differential/algebraic equations (5.1), representing the rigid pendulum problem, has terminated with all type C fourth order constraints, $R_{2,3} = n = 5$, and $R_{3,3} - R_{2,3} = 2$, for all $(x,\underline{y},\underline{z})$ in a neighborhood $\mathcal{N}$ of a point $(x_0,\underline{y}_0,\underline{z}_0)$. Thus we can invoke Theorems 4.1 and 4.2 to prove the existence of a unique local solution to (5.1).

6. Extension of Local Results to Semi-Local Results

The Picard-Lindelöf Theorem (or the method of successive approximations) is used to investigate conditions under which an initial valued ordinary differential equation has a solution and that that solution is unique. In the previous sections of this article, we extended that idea and obtained necessary and also sufficient conditions for the existence, and in some instances, the uniqueness of a local solution to nonlinear systems of differential/algebraic equations. Now we will discuss the extension of our local results to semi-local results. We will eschew the extension of our local results to semi-local results for systems of differential/algebraic equations with discontinuities and/or turning points, because these systems require that certain problem-dependent matching conditions be supplied.

We utilized our constraint theory to prove the existence, and in some cases, the uniqueness of linear and nonlinear systems of differential/algebraic equations via transforming them to explicit systems of differential equations of the form

$$\underline{\mathcal{Z}}' = \underline{h}(x,\underline{\mathcal{Z}}), \qquad (6.1)$$

subject to the initial conditions

$$\underline{\mathcal{Z}}_0 := \underline{\mathcal{Z}}(x_0) = [\underline{y}^T(x_0), \underline{z}^T(x_0)]^T,$$

such that

$$x \in W_x,$$

$$\underline{\mathcal{Z}}: W_x \to \mathbf{R}^{2n},$$

$\underline{h}: \mathcal{M} \to R^{2n}$,

$\underline{h} \in \mathcal{C}^0(\mathcal{M})$,

$\mathcal{M} := W_x \times W_{\underline{y}} \times W_{\underline{z}}$

where $W_x \subset [x_0, x_{fin}]$, $W_{\underline{y}} \subset R^n$, and $W_{\underline{z}} \subset R^n$ are neighborhoods of x_0, $\underline{y}_0$, and $\underline{z}_0$, respectively.

Recall that our existence and uniqueness results for the initial value problem (6.1) were deduced for all $(x, \underline{z})$ in a region $\mathcal{D} \subset \mathcal{M}$, where $\mathcal{D}$ is defined in (4.6), (4.7), and (4.8). Via invoking the Picard-Lindelöf Theorem, we proved that the initial value problem (6.1) had a solution $\underline{\Phi}$, which exists on a finite interval $[x_0, \mathbf{x}) \subset [x_0, x_{fin}]$, and passes through the point $(x_0, \underline{\mathcal{E}}_0) \in \mathcal{M}$. The authors in [1] prove that if $\|\underline{h}\|$ is bounded by $M < \infty$ on $\mathcal{D}$, then the following limit exists:

$$\underline{\Phi}(\mathbf{x} - 0) = \lim_{x \to \mathbf{x}-0} \underline{\Phi}(x). \tag{6.2}$$

Suppose that the point $(\mathbf{x}, \underline{\Phi}(\mathbf{x} - 0))$ is in $\mathcal{D}$. If $\underline{\Psi}$ is the function defined by

$$\underline{\Psi}(x) = \underline{\Phi}(x), \qquad x \in [x_0, \mathbf{x}) \tag{6.3a}$$

$$\underline{\Psi}(x) = \underline{\Phi}(\mathbf{x} - 0), \quad x = \mathbf{x}, \tag{6.3b}$$

then $\underline{\Psi}$ is a solution to the initial value problem (6.1) of class $\mathcal{C}^1$ on $[x_0, x_1]$. The function $\underline{\Psi}$ is called a ***continuation*** or ***extension*** of the solution $\underline{\Phi}$ to $[x_0, \mathbf{x}]$. The authors in [1] discuss other extensions of the solution $\underline{\Phi}$. If it is known that there exists at most one solution through

$(\mathbf{x}, \underline{\Phi}(\mathbf{x} - 0))$, then one can speak of *the* continuation of $\underline{\Phi}$ to $[x_0, \mathbf{x}+\delta] \subset [x_0, x_{fin}]$, for $\delta > 0$. In general, if a continuation of a solution $\underline{\Phi}$ on $[x_0, \mathbf{x})$ exists on some interval containing $[x_0, \mathbf{x}) \subset [x_0, x_{fin}]$, then we can say $\underline{\Phi}$ can be continued (extended), or has a continuation (extension).

The previous remarks are summarized in the following theorem from [1].

Theorem 6.1:

Let the system in (6.1) be such that $\underline{h} \in C^0$ in a domain $D \subset \mathcal{M}$, and suppose $\underline{h}$ is bounded on D. If $\underline{\Phi}$ is a solution to the initial valued problem (6.1) on an interval $[x_0, \mathbf{x}) \subset [x_0, x_{fin}]$, then the limit

$$\underline{\Phi}(\mathbf{x} - 0) = \lim_{x \to \mathbf{x}-0} \underline{\Phi}(x)$$

exists. If $(\mathbf{x}, \underline{\Phi}(\mathbf{x} - 0))$ is in D then the solution $\underline{\Phi}$ may be extended to the right of $\mathbf{x}$. □

We can repeatedly apply Theorem 6.1 to continue a local solution to a system of differential/algebraic equations to a semi-local solution until the criteria that allowed us to invoke the Picard-Lindelöf Theorem no longer hold: that is, until one or more of the following occur.

(i) A discontinuity of $\underline{F}$ is encountered on $\mathcal{M}$.

(ii) A turning point of $\underline{F}$ is encountered on $\mathcal{M}$: that is, a point is encountered such that the rank of $[\partial \underline{F}/\partial \underline{z}]$ varies on $\mathcal{M}$.

(iii) A discontinuity of $\underline{F}^{[k]}$ is encountered on $\mathfrak{M}$.

(iv) Discontinuities of $[\partial\underline{F}^{[k]}/\partial x]$, and of the Jacobian matrices $[\partial\underline{F}^{[k]}/\partial\underline{y}]$, and/or $[\partial\underline{F}^{[k]}/\partial\underline{z}]$ are encountered on $\mathfrak{M}$.

(v) A point is encountered such that the ranks of the Jacobian matrices $[\partial\underline{F}^{[k]}/\partial\underline{y}]$, and/or $[\partial\underline{F}^{[k]}/\partial\underline{z}]$ vary on $\mathfrak{M}$.

(vi) The number of constraints are variable on $\mathfrak{M}$: for example, a type A constraint at a point $(x_1,\underline{y}_1,\underline{z}_1) \in \mathfrak{M}$ becomes a type B or type C constraint at another point $(x_2,\underline{y}_2,\underline{z}_2) \in \mathfrak{M}$.

(vii) The function $\underline{n}$ in (6.1) does not satisfy a Lipschitz condition on $\mathfrak{M}$

The situations in (i), (ii) and (vii) are the most common and natural ones for which the hypothesis of the Picard-Lindelöf Theorem breaks down. The remaining situations are consequences of our constraint analysis. In particular, the situation described in (vi), which may be caused by one or all of the previous situations, motivates us to define the following:

Definition 6.1. A nonlinear function of the form (6.1) has ***higher order turning points*** if the number of constraints are variable on $\mathfrak{M}$ □

Thus we can conclude that if the situations (i) through (vii) do not occur, we can utilize our constraint analysis, the Picard-Lindelöf theorem, etc., to continue a local solution to a system of differential/algebraic equations to a semi-local (and perhaps a global) solution.

References

[1]

Coddington, E.A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).

[2]

Gear, C.W. and Petzold, L.R., *Singular Implicit Ordinary Differential Equations and Constraints*, in Sparse Matrix Techniques, edited by Dold, A. and Eckmann, B., Lecture Notes in Mathematics, Springer Verlag, Copenhagen, (1976), pp. 120-127.

[3]

Hanson, A., Regge, T. and Teitelboim, C., *Constrained Hamiltonian Systems*, Academia Nazionale Dei Lincei, Rome, (1976).

[4]

Mack, I. M., *Block Implicit Methods for Solving Smooth and Discontinuous Systems of Differential/Algebraic Equations*, Doctoral Thesis, (1986).

[5]

Marsden, J.E., *Elementary Classical Analysis*, W.H. Freeman and Company, San Francisco, (1974).

[6]

Ortega, J.M. and Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[7]

Strang, G., *Linear Algebra and its Applications, Third Edition,* Harcourt, Brace, and Jovanoich, (1988).

[8]

Sudarsahn, E.C.S. and Mukunda, N., *Classical Dynamics: A Modern Perspective,* John Wiley and Sons, Inc., New York, (1974).

12-31-91