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THE GENERALIZED RATE OF RETURN

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ERRATA AND CORRIGENDA

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An error in the statement of the theorem on page 9 of this paper was called to my attention by Professor Rubin Saposnik of the University of Buffalo. To make the statement correct requires only minor modification; to correct the proof requires a greater change in its second part. To make these changes, new pages 9 and 10 are enclosed herewith.

A typing error requires substitution of the word "inner" for the work "linear" on page 3 in the line following equation (1).

I'd appreciate your comments and having any other errors called to my attention.

H. Martin Weingartner

Theorem: The net present value of a vector of cash flows $\underline{Y}$, $V(\underline{Y})$, is nonnegative if and only if there exists an internal return vector $\underline{R}(\underline{Y})$ such that[14]

(10) $$\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m\underline{\underline{E}}(\underline{Y})$$

Proof: 1. (Sufficiency) If there exists an $\underline{R}(\underline{Y})$ such that $[\underline{R}(\underline{Y}), Y] = 0$ and $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m\underline{\underline{E}}(\underline{Y})$, then $V(\underline{Y}) = [\underline{R}^m, \underline{Y}] \geqq 0$.

From (10) we have

(11) $$R_t e_t \leqq R_t^m e_t$$

Next, substitute R_t^m for R_t, term by term, in the expression $[\underline{R}(\underline{Y}), \underline{Y}]$. By assumption, $R_0 = R_0^m = 1$. For $t \geqq 1$, if $y_t > 0$, $e_t = 1$ and the substitution will increase the scalar product above zero (or leave it at zero) since then $R_t^m \geqq R_t$. If $y_t < 0$, $e_t = -1$, it will increase the scalar product above zero (or leave it at zero) since then $R_t^m \leqq R_t$ and $R_t^m y_t \geqq R_t y_t$.

2. (Necessity) If $[\underline{R}^m, \underline{Y}] \geqq 0$, then there exists an $\underline{R}(\underline{Y})$ such that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m\underline{\underline{E}}(\underline{Y})$.

Let $[\underline{R}^m, \underline{Y}] = V(\underline{Y}) = a \geqq 0$. To prove the existence of an IRV with the required properties we construct one. Let $J' = \{t \mid y_t > 0\}$, i.e., J' is the set of t corresponding to positive elements of $\underline{Y}$; let $J = \{t \mid y_t = 0\}$; and let $J'' = \{t \mid y_t < 0, t \geqq 1\}$. The number of elements of J' is assumed to be n', the number of elements of J'' is assumed to be n'', with $n' + n'' \leqq n$. (As before, the number of elements of Y is n+1.) We define the elements of $\underline{R}(\underline{Y})$ as follows:

For $t = 0$, $\quad R_0 = R_0^m = 1$;

for $t \in J'$ (i.e., when $y_t > 0$), $\quad R_t = R_t^m - b/n'y_t$;

for $t \in J$ (i.e., when $y_t = 0$), $\quad R_t = R_t^m$;

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[14] The symbol for vector inequality, $\leqq$, follows the usual definition: each term on the left is less than or equal to the comparable term on the right. If at least one term is assumed to be strictly less than its counterpart on the right, the symbol $\leq$ is used.

for $t \varepsilon J''$ (i.e., when $y_t < 0$, $t \geq 1$), $\qquad R_t = R^m_t + c/n''y_t$;

where $b > 0$, $c < 0$, and $b - c = a$.[14a]

That $\underline{R}(\underline{Y})$ is an IRV may be seen from taking the inner product

$$[\underline{R}(\underline{Y}),\underline{Y}] = [\underline{R}^m,\underline{Y}] - \sum_{t \varepsilon J'} \frac{b}{n'y_t} y_t + \sum_{t \varepsilon J''} \frac{c}{n''y_t} y_t$$

$$= a - b + c$$
$$= 0.$$

To show that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leq \underline{R}^m\underline{\underline{E}}(\underline{Y})$, we prove this term by term. For $t = 0$, $R_0 = R^m_0 = 1$, by assumption. For $t \varepsilon J'$, $e_t = 1$ and $R_t = R^m_t - b/n'y_t < R^m_t$ since $b > 0$. For $t \varepsilon J$, $e_t = 1$ and $R_t = R^m_t$. For $t \varepsilon J''$, $e_t = -1$ and $R_t = R^m_t + c/n''y_t \geq R^m_t$ since both $c < 0$ and $y_t < 0$. However, $R_t \geq R^m_t$ implies $-R_t \leq - R^m_t$, or $R_t e_t \leq R^m_t e_t$.

QED.

Note that the theorem requires only that at least one IRV satisfying (1) be found and not that all IRV's have this property. An obvious corollary which we leave without proof is:

Corollary: If there exists an $\underline{R}(\underline{Y})$ such that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \geq \underline{R}^m\underline{\underline{E}}(\underline{Y})$, then $V(\underline{Y}) \leq 0$.

Comparison of Two Options

As is pointed out ubiquitously (though apparently not sufficiently for practitioners), the appropriate way of evaluating two rival options is either to compare their present values, computed by discounting at market rates, or to compute the vector of differences of cash flows and evaluate this. If $\underline{Y}^a$ and $\underline{Y}^b$ are the two options, the rate r^* such that

$$(12) \qquad \sum_{t=0}^{n} (y^a_t - y^b_t)(1 + r^*)^{-t} = 0,$$

[14a] The constant b must be chosen so that $R_t > 0$ for all $t \varepsilon J'$, in order that R lies in the first orthant as required of an IRV. Since b and c are only constrained in sign and by the condition $b-c=a$, this requirement can always be satisfied.

THE GENERALIZED RATE OF RETURN

H. Martin Weingartner

Introduction

Investment analysis, both for purposes of capital expenditures and for financial investments, is based on an evaluation of cash flows. This evaluation involves the application of interest rates in order to determine whether a given option--a series of cash flows--is profitable or not. For numerous reasons, primarily that of simplicity, it has been traditional to assume that the rates of interest used to measure the worth of an investment are constant. With this assumption it is possible to equate the two familiar investment criteria when investments are independent and outlays are not subject to expenditure constraints, i.e., when capital markets are taken to be perfect in the usual sense. An investment is profitable if its net present value is positive when discounting of cash flows uses the (constant) cost of capital, or if its (assumed unique) internal rate of return is greater than the cost of capital.[1] Equivalence of these two criteria is historically most frequently identified with Irving Fisher [3,4], and his two-period analysis, portrayed graphically, is generally utilized to establish the correctness of the equivalence of the criteria.

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[1] The internal rate of return, by now a familiar quantity, is that single rate of interest which makes the net present value of a series of cash flows equal to zero. Lack of uniqueness is a known difficulty with the concept, possibly arising when the sign of the cash flows reverses more than once. However, this issue is not one with which we shall be concerned here.

What if interest rates are not constant over time? Perfect capital markets do not imply a constant level of interest rates, and indeed, any yield curve[2] which is not absolutely constant in level, furnishes evidence that interest rates are not, in fact, expected to be constant. While the present value criterion applies to the same extent with nonconstant interest rates, the meaning of the rate of return criterion is completely lost. How can one compare a single rate of interest, the internal rate, with a set of rates? The answer to this question proves to be illuminating for a number of issues in investment analysis. For this purpose we shall generalize the notion of the rate of return to internal return vectors and show that these form a vector space. This description permits one to redefine the notion of preference under nonconstant rates using internal return vectors and to develop a meaningful criterion related to the standard utility analysis. This formulation leads to several theorems relevant to the evaluation of capital investment projects, and also provides some useful insights into the nature of the valuation process. More significantly, the generalized rate of return is employed to correct certain errors in the formal analysis of the term structure of interest rates, and indications are provided for using the refined formulation for the analysis of financial investments.

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[2]The yield curve is a graph which plots the yield on bonds, e.g., corporates, municipals, or U. S. Government obligations, against their term to maturity. The curve itself usually is drawn through the lowest point of each given maturity. See [1]; also below.

Formal Development

We begin by considering an option--a capital project or financial investment--as being characterized by a vector of cash flows,[3] one for each period, with the convention that an inflow has a positive sign, and outflow a negative sign. We assume further that each cash flow occurs at the end of the period,[4] and that the first cash flow, usually an outlay and hence negative, takes place at the end of period zero.

Thus an option may be denoted by[5] $\underline{Y} = (y_0, y_1, y_2, \ldots, y_n)$ where n is the last period with a non-zero cash flow. The conventional internal rate of return would then be that rate r^* such that

$$(1) \qquad \sum_{t=0}^{n} y_t (1 + r^*)^{-t} = 0.$$

This equation can be written in terms of the linear product between the two vectors, $\underline{Y}$ and $\underline{R}^*$, viz., $[\underline{Y}, \underline{R}^*] = 0$ where $\underline{R}^* = (R_0^*, R_1^*, R_2^*, \ldots, R_n^*)$ and $R_t^* = (1 + r^*)^{-t}$. However, in general there are other vectors, $\underline{R}$, which have the property $[\underline{Y}, \underline{R}] = 0$. In fact, there is an infinite number of Internal Return Vectors[6] which make the present value of the stream of cash flows equal to zero.[7]

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[3] These would usually be after-tax cash flows, although questions of tax consequences are outside the present sphere of interest.

[4] The length of the period is arbitrary and needs only to be consistent with the dimensions of the interest rate.

[5] Vectors will be indicated by a single underscore; matrices by a double underscore. A scalar product of two vectors, $\underline{X}$ and $\underline{Y}$ will be denoted by $[\underline{X}, \underline{Y}]$.

[6] We shall henceforth abbreviate Internal Return Vector by IRV.

[7] Since these depend only on the option Y, they may also be called "internal."

To aid in understanding the nature of the generalized return vector, we may describe it concisely in terms of vector spaces. The vectors $\underline{R}$ which form a scalar product of zero with the given vector $\underline{Y}$ are called orthogonal to $\underline{Y}$, and they form a vector space.[8] That is, given any two such vectors, say $\underline{R}_a$ and $\underline{R}_b$, $\underline{R}_a \neq 0$, $\underline{R}_b \neq 0$, with $[\underline{Y},\underline{R}_a] = 0$, and $[\underline{Y},\underline{R}_b] = 0$, then any linear combination of $\underline{R}_a$ and $\underline{R}_b$ is also an IRV. That is, for arbitrary scalars[9] α and β

$$[\underline{Y},\alpha\underline{R}_a + \beta\underline{R}_b] = 0. \tag{2}$$

This follows since equation (2) may be rearranged,

$$\alpha[\underline{Y},\underline{R}_a] + \beta[\underline{Y},\underline{R}_b] = 0. \tag{3}$$

We wish to interpret the internal return vectors in terms of underlying interest rates, and so we must restrict the above formulation somewhat. It may readily be seen that the components of the $\underline{R}$-vectors, the R_t, may be decomposed into products of one-period discount rates, viz.,

$$\begin{aligned}
R_1 &= \frac{1}{1 + r_1} R_0 \\
R_2 &= \frac{1}{1 + r_2} \cdot \frac{1}{1 + r_1} R_0 = \frac{1}{1 + r_2} R_1 \\
&\ldots\ldots\ldots\ldots\ldots \\
R_n &= \frac{1}{1 + r_n} \cdot \frac{1}{1 + r_{n-1}} \cdots \frac{1}{1 + r_1} R_0 \\
&= \frac{1}{1 + r_n} R_{n-1}.
\end{aligned} \tag{4}$$

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[8] This description will be qualified, below.

[9] I.e., ordinary numbers.

Assuming interest to be finite, the range for r_t is $-1 < r_t < \infty$ which implies that $R_t > 0$, $t = 1,\ldots,n$. That is, we restrict our attention to interest rates for which the principal does not totally disappear, nor does it grow without bound. (There is no reason to limit attention only to positive rates of interest here.) The additional condition, $R_t > 0$, which is derived from the interpretation, means that we are concerned only with the positive orthant of the vector space of vectors orthogonal to the given Y-vector.

The term R_0 appears in all the expressions for R_t. It may be seen that R_0 is arbitrary, i.e., any vector which is an IRV will also be one if a different value of R_0 is substituted in its components. The interpretation of R_0 is straightforward. It is the dateline for the computation of the discounted value, utilizing the discount rates R_t. Thus, if R_0 is assumed to be unity, then the computation is as of the end of period zero, which is usually called a present value. However, it should be noted that any other dateline can be used, even involving a difference in time of fractional periods from the present. Also, an IRV (or, for that matter, an ordinary rate of return) can be computed on the basis of a terminal value of zero. Only a change in the value of R_0 is implied. For our present purposes, it is sufficient if $R_0 = 1$, which assumption will be assumed in what follows.

A formal development of the notion of the IRV's would also include the following. First, the given option $\underline{Y} = (y_0, y_1, \ldots, y_n)$ may be regarded as a point in the n+1 dimensional real vector space, V^{n+1}. That is, it lies in a 1-dimensional subspace of V^{n+1}. The vectors orthogonal to $\underline{Y}$

lie in the orthogonal complement of the subspace of $\underline{Y}$, which implies that the IRV's lie in an n-dimensional subspace of V^{n+1}. More specifically, given the nonnegativity condition on the IRV's, we see that they must lie in the positive orthant of this subspace. To help visualize this, it is only necessary to construct a basis for the subspace.

First, it is clear that with regard to the IRV's, multiplication of the option $\underline{Y}$ by a scalar will not have any effect. Hence it is possible to normalize $\underline{Y}$ by premultiplying it by $1/y_0$.[10] A further simplification may be achieved by assuming the first component of $\underline{Y}$ to be negative, i.e., an outflow. This will also mean no loss of generality since it is possible to multiply $\underline{Y}$ by -1. Thus we write $\underline{Y}' = (-1, y_1', y_2', \ldots, y_n')$, with $y_t' = -y_t/y_0$. A basis for the space of the IRV's is then

$$
(5) \quad
\begin{aligned}
\underline{R}^1 &= (1, \tfrac{1}{y_1'}, 0, 0, \ldots, 0) = (1, -\tfrac{y_0}{y_1}, 0, \ldots, 0) \\
\underline{R}^2 &= (1, 0, \tfrac{1}{y_2'}, 0, \ldots, 0) \quad = (1, 0, -\tfrac{y_0}{y_2}, 0, \ldots, 0) \\
&\cdots\cdots\cdots\cdots \\
\underline{R}^n &= (1, 0, \ldots, 0, \tfrac{1}{y_n'}) \qquad = (1, 0, \ldots, 0, -\tfrac{y_0}{y_n})
\end{aligned}
$$

It is clear that the $\underline{R}^j$ are linearly independent, that they are orthogonal to $\underline{Y}$, and that together they span the hyperplane of the IRV's. It should be noted, however, that the $\underline{R}^j$ do not necessarily lie in the positive orthant; hence they need not, themselves, be internal return vectors.

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[10] It is assumed that y_0 is nonzero, and that at least one y_t is strictly negative and at least one is strictly positive.

Whether they are or are not depends on the sign of y_0/y_t. Similarly there may be other orthogonal vectors which do not lie entirely in the positive orthant.

The Generalized Rate of Return in Investment Analysis

The present value criterion, allowing for different rates of discount for different periods, may be stated in terms of the vector notation developed in the following way. Let $\underline{R}^m$ be a vector of market discount rates where by assumption,

$$R_0^m = 1, \tag{6}$$

$$R_t^m = \frac{1}{1 + r_t^m} R_{t-1}^m, \qquad t=1,\ldots,n$$

and the r_t^m are 1-period market interest rates which apply to period t.[11] Thus the market vector may also be written in terms of the t-period discount rates, R_t^m, which discount a cash flow in period t to the present,

$$\underline{R}^m = (1, R_1^m, R_2^m, \ldots, R_n^m) \tag{7}$$

where n, the number of future periods, is arbitrary. The present value of an option $\underline{Y}$, $V(\underline{Y})$, is then simply the inner product of the market vector and the vector $\underline{Y}$, viz.,

$$V(\underline{Y}) = [\underline{R}^m, \underline{Y}]. \tag{8}$$

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[11] Within the present analysis it is assumed that the forward rates today are also the actual rates.

The usual present value criterion for acceptance or rejection of an option is simply: reject option $\underline{Y}$ if $V(\underline{Y}) < 0$.[12] It is possible to relate the present value criterion to a proposition about IRV's of the option $\underline{Y}$ in a basic theorem. Before stating it, we require an additional definition.

The option $\underline{Y}$ has not been constrained in any way. Specifically, we have not assumed that $\underline{Y}$ is of the "standard" variety--with a negative term at the start followed by nonnegative terms exclusively. In order to allow for generality we define the elementary matrix[13] $\underline{\underline{E}}(\underline{Y})$ in which the off-diagonal terms are all zero, and the diagonal contains terms e_t are defined by

(9)
$$e_0 = 1$$
$$e_t = \begin{cases} 1 \text{ if } y_t \geq 0 \\ -1 \text{ if } y_t < 0 \end{cases} \quad , \ t=1,\ldots,n.$$

The y_t are, as before, the components of $\underline{Y}$. It may help to visualize $\underline{\underline{E}}(\underline{Y})$ as a transformation which is equivalent to turning an arbitrary option into a standard one. It is required to prove the following:

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[12] The foundation for this criterion in terms of utility have been developed axiomatically by Williams and Nassar in [8]. However, by selecting axioms relevant primarily for consumption preferences, they leave open questions which may be related to exchange possibilities, such as in inflation in which a deferred inflow may be preferred to an earlier one. This possibility would depend, of course, on market rates, in which a real interest rate may be negative.

[13] A matrix is denoted by a double underlining.

Theorem: The net present value of a vector of cash flows $\underline{Y}$, $V(\underline{Y})$, is nonnegative if and only if there exists an internal return vector $\underline{R}(\underline{Y})$ such that[14]

$$\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m. \tag{10}$$

Proof:

1. (Sufficiency) If there exists an $\underline{R}(\underline{Y})$ such that $[\underline{R}(\underline{Y}),Y] = 0$ and $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m$, then $V(\underline{Y}) = [\underline{R}^m,\underline{Y}] \geqq 0$.

From (10) we have

$$R_t e_t \leqq R_t^m \tag{11}$$

Next, substitute R_t^m for R_t, term by term, in the expression $[\underline{R}(\underline{Y}),\underline{Y}]$. If $y_t > 0$, this will increase the scalar product above zero (or leave it at zero) since $R_t^m \geqq R_t$. If $y_t < 0$, it will increase the scalar product above zero (or leave it at zero) since $R_t^m < R_t$ and $R_t^m y_t \geqq R_t y_t$.

2. (Necessity) If $[\underline{R}^m,\underline{Y}] \geqq 0$, then there exists an $\underline{R}(\underline{Y})$ such that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m$.

Let $[\underline{R}^m,\underline{Y}] = V(\underline{Y}) = a \geqq 0$. Also, let y_k be the first value of y_t different from zero, $t \geqq 1$. (The term y_0 is necessarily different from zero.) An IRV is then

$$\underline{R}(\underline{Y}) = (1, R_1^m, R_2^m, \ldots, R_{k-1}^m, R_k^m - a/y_k, R_{k+1}^m, \ldots, R_n^m).$$

That this is an IRV can be seen from the factorization

[14] The symbol for vector inequality, $\leqq$, follows the usual definition: each term on the left is less than or equal to the comparable term on the right. If at least one term is assumed to be strictly less than its counterpart on the right, the symbol $\leq$ is used.

$$[\underline{R}(\underline{Y}),\underline{Y}] = [\underline{R}^m,\underline{Y}] - (a/y_k)y_k = a - a = 0.$$

To show that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) \leqq \underline{R}^m$, we prove this term by term.

For $t = 0$, $R_0 = R_0^m = 1$ (by assumption).

For $t=1,\ldots,k-1$, $y_t = 0$, and hence $e_t = 1$. Therefore $R_t e_t = R_t^m$, $t=1,\ldots,k-1$. $R_k = R_k^m - a/y_k$. If $y_k > 0$, $e_k = 1$, and $R_k e_k = R_k < R_k^m$. If $y_k < 0$, $e_k = -1$. Then $R_k > R_k^m$, but $R_k e_k = -R_k < R_k^m$. For $t=k+1,\ldots,n$, $R_t e_t = R_t^m$ if $y_t \geqq 0$; $R_t e_t = - R_t^m < R_t^m$ if $y_t < 0$ since $R_t^m \geqq 0$. QED.

Note that the theorem requires only that at least one IRV satisfying (1) be found and not that all IRV's have this property. An obvious corollary which we leave without proof is:

Corollary: If there exists an $\underline{R}(\underline{Y})$ such that $\underline{R}(\underline{Y})\underline{\underline{E}}(\underline{Y}) > \underline{R}^m$, then $V(\underline{Y}) \leqq 0$.

Comparison of Two Options

As is pointed out ubiquitously (though apparently not sufficiently for practitioners), the appropriate way of evaluating two rival options is either to compare their present values, computed by discounting at market rates, or to compute the vector of differences of cash flows and evaluate this. If $\underline{Y}^a$ and $\underline{Y}^b$ are the two options, the rate r^* such that

$$\sum_{t=0}^{n} (y_t^a - y_t^b)(1 + r^*)^{-t} = 0, \tag{12}$$

may be compared with the market rate of interest, if the latter is constant over the n periods. If it is not constant, applying the IRV criterion to the vector $(\underline{Y}^a - \underline{Y}^b)$ will lead to correct solutions. However, the virtue of the original internal rate of return criterion, namely that the appropriate cut-off rate r^m did not have to be determined with precision, is no longer as strong a virtue.

In the instance of strong dominance, when $\underline{Y}^a \geq \underline{Y}^b$, every vector $\underline{R}$ orthogonal to $(\underline{Y}^a - \underline{Y}^b)$ satisfies $\underline{R} \leq \underline{R}^m$. In particular, except for $t = 0$, $R_t < R_t^m$, assuming only that $R_t^m > 0$. That this must be the case may be seen from the fact that (a) the components of $(\underline{Y}^a - \underline{Y}^b)$ are all positive, (b) that a basis for the space of the IRV's constructed as in equations (5) consists of vectors none of which lies entirely in the first orthant (since $-y_0/y_t < 0$) and (c) any linear combination of these basis vectors has components with both positive and negative signs, and hence does not lie within the first orthant. This establishes the conclusion. Under our earlier definition of the IRV's, such an orthogonal vector $\underline{R}$ is not strictly an IRV. We may conclude that in the absence of strong dominance, reference to the market vector $\underline{R}^m$ is required.[15]

The Uniform Perpetual Rate of Return

On occasions when it is desirable to make comparisons between standard options in the form of rates of return, the uniform perpetual rate

[15] On a diagram of present value against discount rate, the condition of strong dominance for a constant interest rate analysis would show the curve for option $\underline{Y}^a$ to lie entirely above the curve for $\underline{Y}^b$. The usual internal rate of return for each option separately is found at the intersection of its curve with the interest-axis. The internal rate of return for the difference between the options, usually located at their crossing point, would not exist in this example.

will give valid answers. In the standard option, the first term is negative while the following ones are positive. The first term, y_0, may therefore be called its cost. Given the interest rate on perpetuities, r_∞, and $\underline{R}^m$ (with as many terms as the longest of the options to be compared) one obtains first the present value of the returns of an option,[16] $[\underline{R}^m, \underline{Y}] + y_0$, and converts it into a perpetual return of $r_\infty([\underline{R}^m, \underline{Y}] + y_0)$ per period, and into a rate of return

$$r = \frac{r_\infty([\underline{R}^m, \underline{Y}] + y_0)}{y_0} . \tag{13}$$

Obviously, if $r > r_\infty$, the option should be accepted since this is equivalent to the statement that $[\underline{R}^m, \underline{Y}] = V(\underline{Y}) > 0$, as some transposing of terms will show. Ranking of independent options according to their respective uniform perpetual rates of return is equivalent to present value ranking, and also to ranking by the "Excess Present Value Index" or benefit/cost ratio. In fact, r is only a scalar multiple of the benefit/cost ratio, the multiplier being r_∞.[17]

Analysis of the Term Structure of Interest Rates

In the preceding sections we have considered the evaluation of streams of cash flows in terms of vectors of discount rates and have applied the framework to capital projects. In the remaining parts we shall utilize it for an analysis of the term structure of interest rates. First

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[16] The positive sign before y_0 results from the assumption that y_0 is negative.

[17] For a discussion of the use and limitations of this ratio for analysis of capital rationing problems, see [7].

we shall define some additional notation and utilize it to prove an elementary, if not universally known, proposition about bond yields. Then we shall indicate the source and extent of certain errors in empirically estimated forward interest rates.

Since we shall confine our discussion to the structure of interest rates prevailing at one time, and not to its changes over time, we shall again denote future one-period (spot) rates prevailing in the market in period t by r_t^m, and the discount rates applying to future cash flows by R_t^m. We shall also require rates which express, on an annual basis, the discount rates R_t^m, by use of the following definition, to be enlarged upon below,

$$(14) \qquad R_t^m = \left(\frac{1}{1+\hat{r}_t}\right)^t \qquad \text{or} \qquad \hat{r}_t = (R_t^m)^{-1/t} - 1$$

The yield-to-maturity on a bond, i.e., the internal rate of return, will again be denoted by r^* and the yield vector by $\underline{R}^*$. To simplify the arguments, we shall scale a bond so that it has a maturity value of 1 and denote its coupon rate by r^c. By assuming that the frequency of coupons is one per interest period, we have, in effect, an equivalence between the coupon rate and the amount of the coupon. Unless explicitly stated to the contrary, we shall discuss bonds only ex-coupon on a coupon date, and hence we are able to describe bonds as options with the following vector of cash flows:

$$(15) \qquad \underline{Y} = (-p,\ r^c,\ r^c,\ \ldots,\ r^c,\ 1 + r^c)$$

The price of the bond is p, the periodic coupon is r^c, and the final payment is its unity par value plus one coupon, i.e., $1 + r^c$.

The first proposition we wish to establish is that two bonds having the same term-to-maturity and the same frequency of coupons but having different coupon rates cannot have the same yield-to-maturity unless the market yield curve is absolutely flat.[18] Let $\underline{Y}^a$ and $\underline{Y}^b$ be two otherwise identical bonds; their yields, expressed as vectors, are $\underline{R}^*(\underline{Y}^a)$ and $R^*(\underline{Y}^b)$, and $\underline{R}^m$ is the market vector.[19] If the market vector determines their prices, then it must be that a) $[\underline{R}^m, \underline{Y}^a] = [\underline{R}^m, \underline{Y}^b] = 0$. By assumption b) $[\underline{R}^*(\underline{Y}^a), \underline{Y}^a] = 0$ and c) $[\underline{R}^*(\underline{Y}^b), \underline{Y}^b] = 0$; and also d) $\underline{Y}^a \neq \underline{Y}^b$. Further, by saying that the yield curve is not flat we mean that $r^m_t = r^m_q$ is not true for all t and q less than the maturity of the bonds. Since by definition of the yield vector $\underline{R}^*$,

$$R^*_t = \left(\frac{1}{1+r^*}\right)^t, \quad t=1,\ldots,n \tag{16}$$

i.e., all components of $\underline{R}^*$ are functions of the single short rate r^*, this also implies that $\underline{R}^m \neq \underline{R}^*(\underline{Y}^a)$ and $\underline{R}^m \neq \underline{R}^*(\underline{Y}^b)$. From their definitions and also, from the previously adopted convention (implied by referring to <u>present</u> values) that $R_0 = 1$ for all IRV's, we know also that $\underline{R}^m \neq \alpha\underline{R}^*(\underline{Y}^a)$ and $\underline{R}^m \neq \beta\underline{R}^*(\underline{Y}^b)$, for arbitary scalars α and β. That is, the market vector and the yield vectors are not linearly dependent. By (a) and (b), $\underline{R}^m$ and $\underline{R}^*(\underline{Y}^a)$ are elements of the orthogonal complement of $\underline{Y}^a$, and as just established, they are linearly independent. If the two bonds have the same yield to maturity, then $\underline{R}^*(\underline{Y}^a) = \underline{R}^*(\underline{Y}^b) = \underline{R}^*$, which means that $\underline{R}^*$ and $\underline{R}^m$ are also independent elements of the orthogonal complement of $\underline{Y}^b$.

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[18] In this discussion and also that below, we shall ignore the effects of taxation on the valuation of bonds. The results we obtain are independent, or rather, in addition to such effects.

[19] It clearly does not matter for our present analysis whether the market vector does or does not include a liquidity premium.

Put another way, the vectors $\underline{Y}^a$ and $\underline{Y}^b$ are both elements of the orthogonal complement of the subspace of $\underline{R}^m$ and $\underline{R}^*$. Since the latter space is n-dimensional, the space of $\underline{Y}^a$ and $\underline{Y}^b$ is one-dimensional, which implies that $\underline{Y}^a = \gamma\underline{Y}^b$ for some scalar γ, contrary to the hypothesis.

This theorem is pertinent when the yield curve is utlized to discuss the term structure of interest rates, and when evidence is marshalled for or against the expectations hypothesis. Specifically, it is useful to make clear the distinction between a schedule of yield-to-maturity of bonds versus maturities, and a schedule of n-period interest rates, i.e., the term structure. In the former, bond coupons play a specific role. The n-periods rates, however, do not correspond to rates on contracts in the usual financial markets. Instead, they express trade-offs across the time axis in most simple terms, viz., as the periodic interest rate applicable to a loan made at the end of period 0 which is to be repaid, <u>together with all interest</u>, compounded once per period, at the end of period t. Thus a loan of one dollar made for k periods requires repayment of $(1+\hat{r}_k)^k$ at the end, and the yield on a k-period bond is neither identical with $\hat{r}_k$ nor is it a function of it alone.[20]

Bond yields and n-period rates are related, but the relationship involves the coupon rate and the term structure of interest rates up to the maturity date of the bond. We shall demonstrate this in two parts. First we derive the relationship between coupon rates and bond yields, given a term structure, and illustrate it graphically for some hypothetical

[20] This incorrect identity is maintained by Meiselman [6], and Kessel [5], among others.

term-structures. Then we shall derive the difference between the yield on an n-period bond and the n-period forward rate, and also provide some examples.

Bond Yields and Coupon Rates

Given the fact that both the market vector and the yield vector of a bond are IRV's, it is a simple matter to relate the yield to the coupon rate, given the market vector.[21] We do so by writing the bond price, p, as the discounted value of coupon payments and payment of par value at the end, when discounting is done first by use of market discount rates, R_t^m, and second, using its yield to maturity,[22] r^*. In the latter case, the price is the sum of an annuity of r^c per period for n periods plus the present value of $1, n periods hence. The two expressions are

$$p = r^c \sum_{t=1}^{n} R_t^m + R_n^m \tag{17}$$

$$p = \frac{r^c}{r^*} [1 - (1 + r^*)^{-n}] + (1 + r^*)^{-n}. \tag{18}$$

After transposing and rearranging, the desired expression is obtained:

$$r^c = \frac{(1 + r^*)^{-n} - R_n^m}{\sum_{t=1}^{n} R_t^m - \frac{1}{r^*} [1 - (1 + r^*)^{-n}]} . \tag{19}$$

[21] There is a one-to-one correspondence between the components of the market vector and the n-period rates which comprise the term structure.

[22] All interest rates, here and below, are assumed to be expressed as decimals and not as percentages.

For obvious reasons of mathematical simplicity, the expression shows the coupon rate as a function of yield, rather than the reverse. The main implications of the expression need not, however, be lost. First, the relationship between yield and coupon rate involves also the entire schedule of n-period interest rates, expressed here in terms of R_t^m. Justification for this observation is intuitively easy to grasp. Since coupons are received periodically between the purchase date and maturity, these coupons are discounted to the present at rates expressing time trade-offs between the coupon dates and the present, and not between the maturity date and the present.[23]

Apart from understanding the general role of the n-period rates in expression (19), it is less easy to visualize the shape of the relationship. For this reason it has been graphed for three separate term-structures, two of them rising, one falling.[24] The same term structures are also utilized in later discussions, and are presented in Table I and Figure I.

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[23]The one circumstance in which the coupon rate has no effect on yield is $R_n^m = (1 + r^*)^{-n}$, i.e., when the term-structure is flat. From equations (17) and (18) we see that this must be the case since then also

$$\sum_{t=1}^{n} R_t^m = \frac{1}{r^*} [1 - (1 + r^*)^{-n}]$$

which is equivalent to

$$\sum_{t=1}^{n} [\prod_{\tau=1}^{t} (1 + r_\tau)]^{-t} = \sum_{t=1}^{n} (1 + r^*)^{-t}.$$

[24]Instead of the R_t^m, the graphs portray the $\hat{r}_t$, defined by expression (14), which are more readily understood.

TABLE I

Three Assumed Term Structures of Interest Rates for Illustration of the Relationship Between Bond Yields and Bond Coupons

t	Rising $\hat{r}_t$	Rapidly Rising $\hat{r}_t$	Falling $\hat{r}_t$
1	0.02230	0.02320	0.05300
2	.02400	.02600	.05140
3	.02550	.02800	.05020
4	.02660	.02960	.04930
5	.02770	.03120	.04840
6	.02875	.03230	.04760
7	.02965	.03320	.04680
8	.03045	.03415	.04620
9	.03120	.03485	.04560
10	.03195	.03575	.04500
11	.03270	.03645	.04455
12	.03330	.03700	.04395
13	.03390	.03760	.04350
14	.03450	.03825	.04305
15	.03500	.03870	.04270
16	.03555	.03905	.04240
17	.03600	.03945	.04210
18	.03650	.03965	.04180
19	.03700	.03980	.04155
20	.03740	.03990	.04135
21	.03775	.03995	.04120
22	.03810	.04000	.04102
23	.03840	.04000	.04085
24	.03868	.04000	.04069
25	.03895	.04000	.04054
26	.03918	.04000	.04040
27	.03938	.04000	.04028
28	.03955	.04000	.04018
29	.03968	.04000	.04010
30	.03979	.04000	.04004

FIGURE I

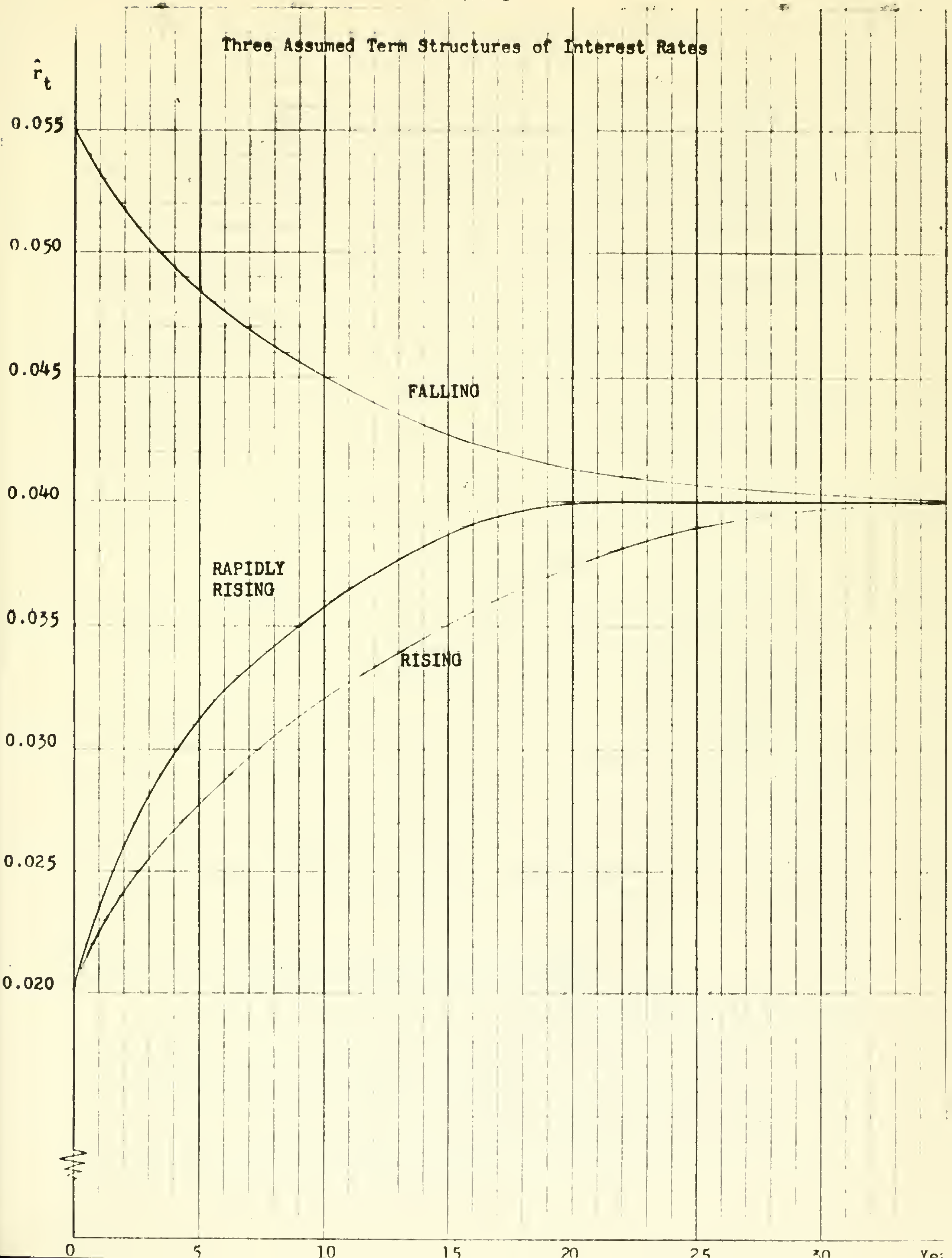

Three Assumed Term Structures of Interest Rates
r̂t
0.055
0.050
0.045
0.040
0.035
0.030
0.025
0.020
FALLING
RAPIDLY
RISING
RISING
0
5
10
15
20
25
30

Figure II presents the relationship between r^* and r^c for these three term-structures and for several different maturities. Two observations are most striking. First, contrary to the empirical relationship found by Durand [2], in which many other considerations, especially taxes, play a role, the curves corresponding to the rising term structures fall, though at decreasing rates, i.e., the higher the coupon, the lower the corresponding yield for a given rising term structure. The decrease is more pronounced for the more gradually rising term structure. By contrast, the falling term structure produces a positive relationship between coupon rate and yield, though also one whose rate of charge decreases. That is, when the t-period interest rate decreases for increasing t, the effect of a higher coupon rate is to increase the yield.[25]

One way in which these relationships are sometimes visualized is in terms of "average" or "effective" maturity. A higher coupon rate implies receipt of a larger proportion of the total payments of a bond by a given date than one with a lower coupon rate. By shortening the "effective" maturity, the yield on a bond (which is also an average--a weighted average) changes in the same direction as a change in the t-period interest rate from a <u>later</u> to an <u>earlier</u> date. This corresponds to a decrease for a rising term structure, an increase for a falling one.[26]

Errors in the Empirically Measured Term Structures

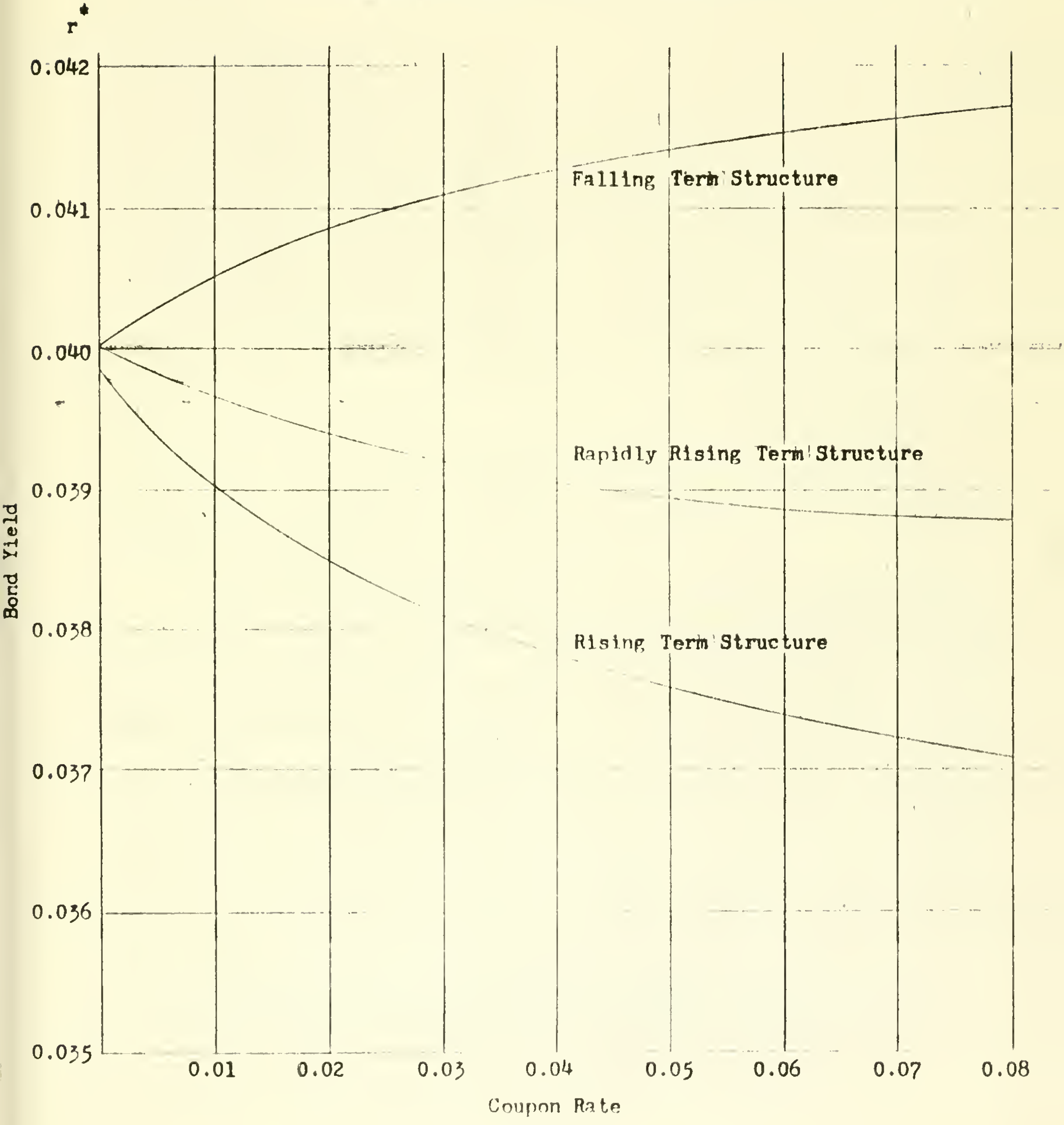
As indicated, many writers fail to make the distinction between the term structure and the yield curve. As a result, tests of theories about

[25] A flat term-structure does not give rise to a curve since, in that case, $\hat{r}_t = r^*$ for all t, and both numerator and denominator of (19) vanish.

[26] The leveling-off of all curves in Figure II is due to the fact that the assumed term structures all approach a constant level.

FIGURE II

RELATIONSHIP BETWEEN YIELD AND COUPON RATES FOR THREE ASSUMED TERM STRUCTURES OF INTEREST RATES AND A MATURITY OF THIRTY PERIODS



the nature of the term structure as, for example, for or against the expectations hypothesis, cannot be carried out accurately. This concluding section is devoted to the derivation of the error resulting from use of the yield curve for obtaining the term structure, and to indicate its size by use of some examples.

The yield curve is generally obtained by drawing a free-hand line on a graph of the yield to maturity against term to maturity. For a variety of reasons, most of which are outside our present interest, such a graph generally shows several yields for a given maturity. For a class of bonds of generally homogeneous risk of default, as, for example, government bonds or certain municipal bonds, all noncallable and having no special features, an important source of difference between yields is the difference between coupons. The common practice is to draw the yield curve through the points of bonds with lowest coupons, which are also generally the lowest points for given maturity.[27]

There is nothing objectionable in this procedure for graphing the yield curve. However, it must be borne in mind that the lowest coupon rate may differ from one maturity to another, and that the lowest rate is not zero. Because of this, the whole set of preceding t-period rates, $t=1,\ldots,n-1$, enters into the estimation of the n-period rate. This rate may again be derived from the two expressions for the price of a bond, (17) as a function of the term structure, and (18) in terms of its yield to maturity.

[27] This rule is not rigidly adhered to by Durand [1,2], whose curves are most frequently referenced. Durand is extremely careful, and he explains his deviations from this rule in specific instances.

Once more combining these relationships and solving for R_n^m, after rearranging terms we obtain[28]

$$R_n^m = \left(\frac{r^c}{1+r^c}\right)\left\{\frac{1+\left(\frac{r^*}{r^c}-1\right)(1+r^*)^{-n}}{r^*} - \sum_{t=1}^{n-1} R_t^m\right\}. \tag{20}$$

Making use of the relation between the discount rates and the n-period rates, viz., $\hat{r}_n = (1/R_n^m)^{1/n} - 1$, we obtain

$$\hat{r}_n = \left\{\left(\frac{r^c}{1+r^c}\right)\left[\frac{1+\left(\frac{r^*}{r^c}-1\right)(1+r^*)^{-n}}{r^*} - \sum_{t=1}^{n-1} R_t^m\right]\right\}^{-1/n} - 1 \tag{21}$$

As this expression shows, the rate to year n implied by a given yield depends also on the market discount rates, R_t^m, t=1,...,n-1, as well as on the coupon rate, r^c.

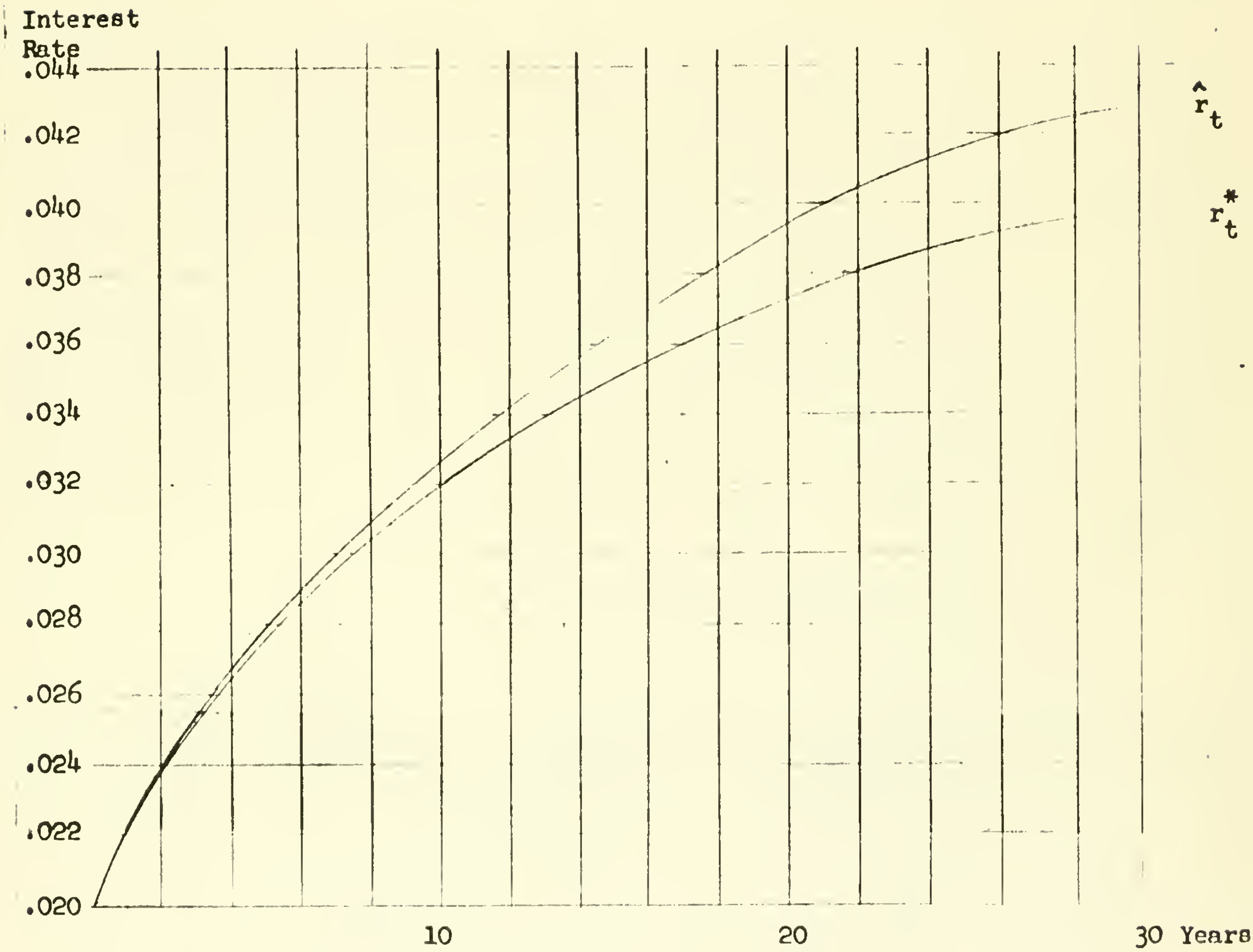
The relationship between bond yields and the implied t-period rate has been graphed in two ways. In Figure III the t-period rates are plotted against the term together with the yield curves from which they were derived. A uniform 5% coupon was assumed. The t-period rate in each instance was computed from the yield of the same value of t, and from the values of $\hat{r}_t$, t=1,...,t-1 obtained before. I.e., expression (21) was applied recursively. To keep the examples simple, the yield curves used here are the

[28] To clarify, it may help to write (17) as

$$p = r^c \sum_{t=1}^{n-1} R_t^m + (1+r^c)R_n^m .$$

FIGURE III

Relationship Between Yields (r_t^*) and Implied Interest Rates ($\hat{r}_t$) For Three Hypothetical Yield Curves of 5% Coupon Bonds with Terms-to-Maturity from One to Thirty Years



a. Rising Yield Curve*

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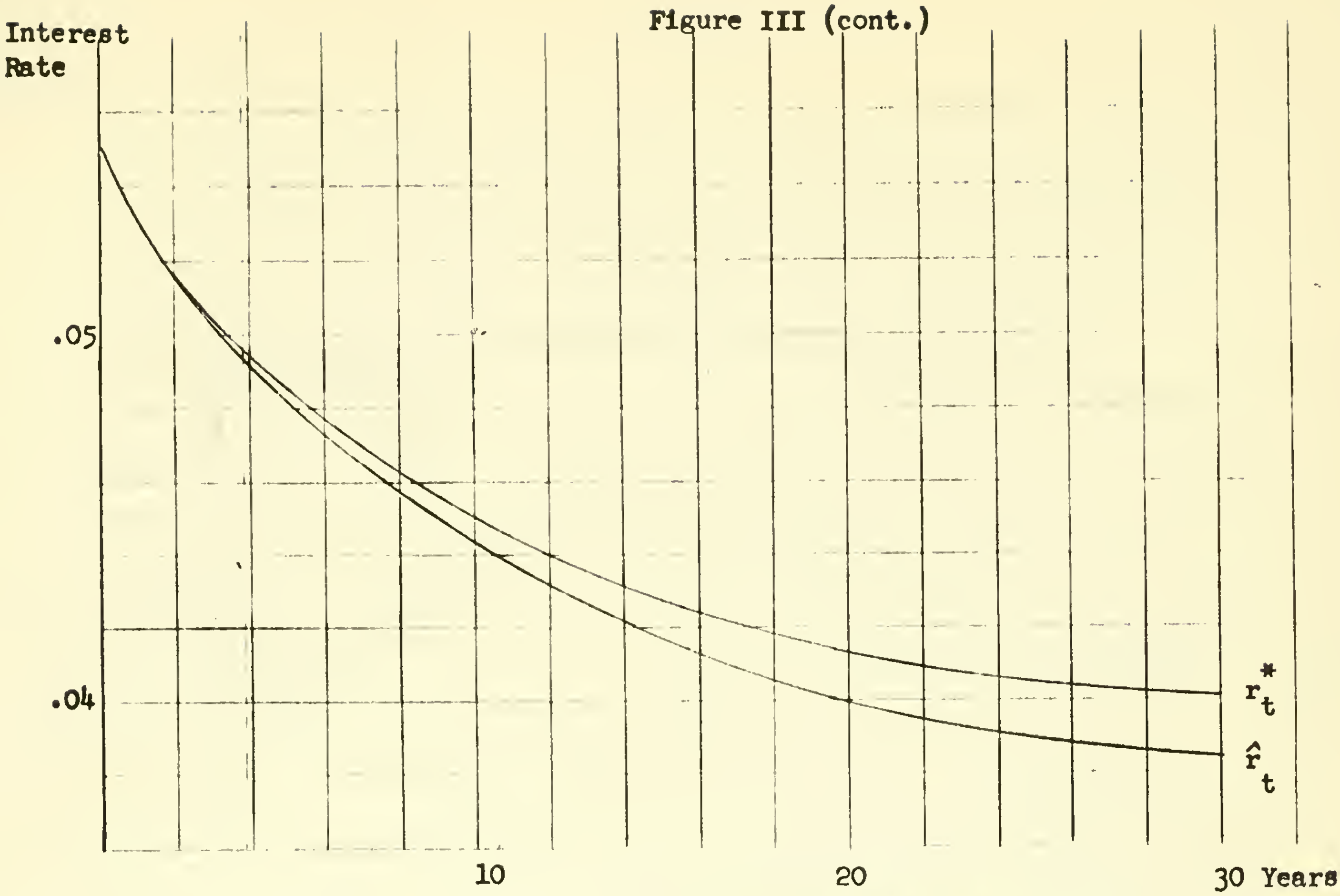
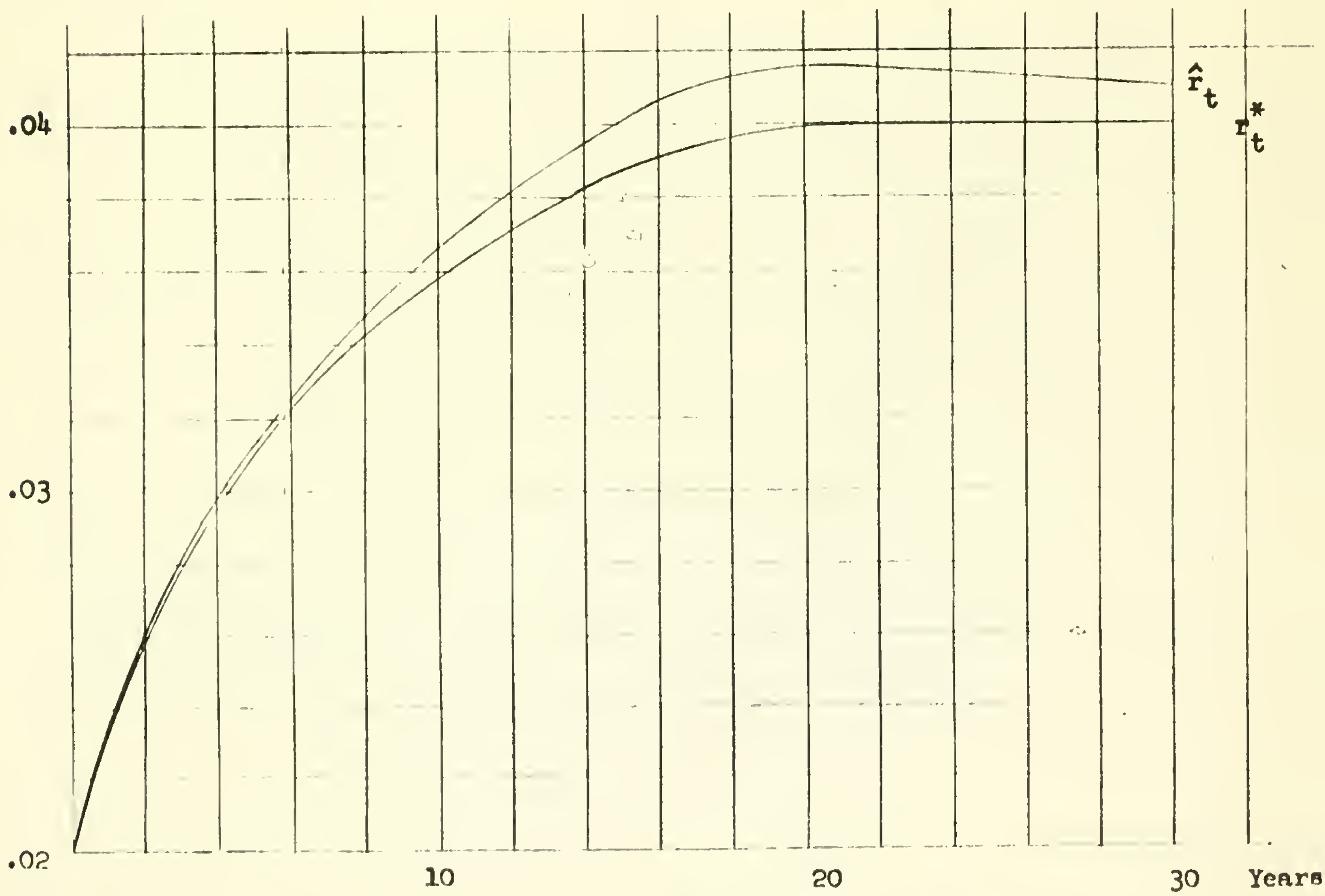
* $\hat{r}_t$ is computed by recursive application of equation (21) in which the yield (r_t^*) is taken from Table I. Since it is much simpler to compute the n-period interest rate than the yield, the numbers in Table I were used as yields, and not as n-period rates. Thus the graphs of r_t^* in Figure III are identical with those of the term structure ($\hat{r}_n$) in Figure I.

Figure III (cont.)



b. Falling Yield Curve



c. Rapidly Rising Yield Curve

three curves of Figure I, given in Table I as term-structures. In Figure IV the relationship between the coupon rate and the implied t-period rate is plotted for a fixed maturity of thirty years and for three given yields, those in Figure III, and for their respective term-structures.[29]

Figure III indicates that the direction of the error in estimating the t-period rate from bond yields depends on the slope of the yield curve itself: if it is rising, the yield underestimates the t-period rate; if it is falling, it overestimates it.[30] Thus an inversion in the yield curve, from rising to falling or vice versa, with the long-term yields unchanged, implies a change in the long-term interest rate.

A curious fact which emerges from Figure IIIc is that although the yield curve here is monotonically non-decreasing, the term structure falls slightly at the end. This result can be attributed to the leveling off of the yield curve.

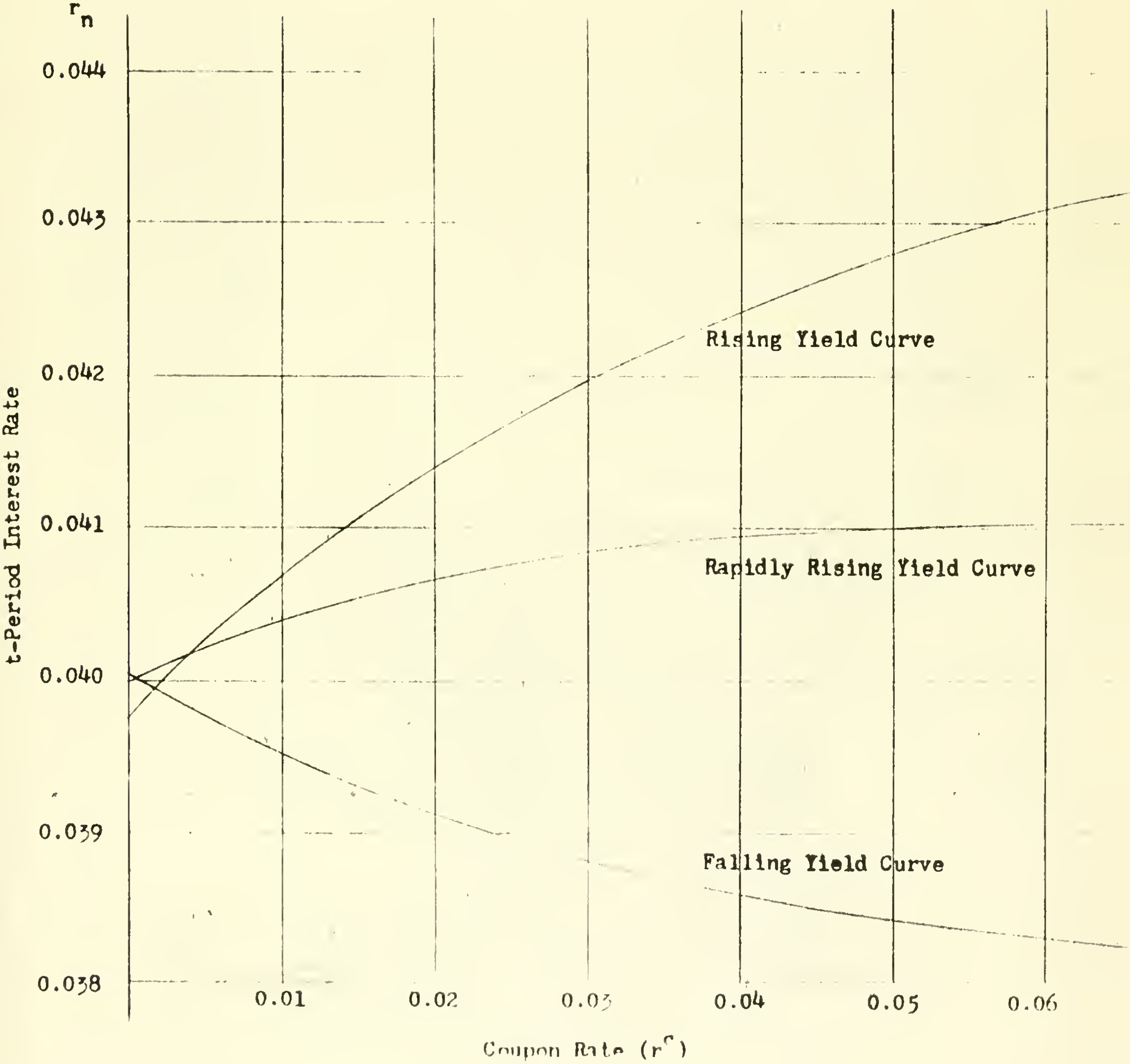
Figure IV gives an indication of the magnitude of the error which the coupon introduces in estimation of the term structure from bond yields. For the rising term structure (Figure IIIa) a 2% coupon implies a 16 basis point overestimate, a 5% coupon implies a 30 basis point overestimate. The falling term-structure shows a nine basis point underestimate with a 2% coupon and a 16 basis point underestimate with a 5% coupon. The chief implication of these graphs is that while it would be better to estimate the term structure from bond yields by drawing a curve below the points with

[29] Note that these are not identical with the term structures of Table I, as explained above.

[30] A "humped" yield curve implies a change from an underestimate of the term structure to an overestimate, the two curves crossing.

Figure IV

Relationship Between Coupon Rate and the Implied t-Period Interest Rate for Three Given Term-Structures and Yields and a Thirty-Period Term



lowest coupons when the term structure is rising, when it is falling the curve should not be below but somewhat above the lowest points. The exact deviations can be computed recursively from expression (21).

Conclusions

The objective of this paper has been to generalize the notion of the internal rate of return for situations of nonconstant market interest rates and to apply it to the evaluation of streams of cash flows for investment decisions by firms; and to utilize the framework for an analysis of the term structure of interest rates. We have established the relationship between net present value and internal return vectors and the requirements for ranking streams of cash flows, and also related it to a one-dimensional criterion, the uniform perpetual rate of return which may be used with standard streams of an outflow followed by inflows.

The notation has also proved useful for establishing the relationship between bond yields and coupon rates for given term structures of interest rates, and to indicate the need for a clear distinction between the schedule of bond yields to maturity and the term structure. The method was applied to measure the error introduced by using the yield curve in place of the term structure, and it was possible to show that the sign of the error depends on the term structure itself.

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