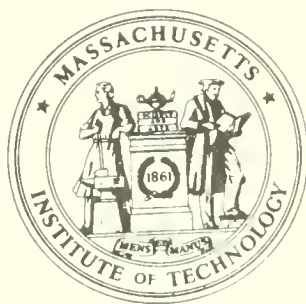


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GENERALIZED LAGRANGE MULTIPLIERS

IN INTEGER PROGRAMMING

by

Jeremy F. Shapiro

March 7, 1969

Working Paper 371-69

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## ABSTRACT

The integer programming problem is reformulated using group theory thereby allowing a new Lagrangian optimization problem for integer programming to be constructed. The properties of this problem and its relationship to group theoretic branch and bound and cutting plane algorithms are discussed. Necessary and sufficient conditions for the existence of optimal multipliers are also given.



GENERALIZED LAGRANGE MULTIPLIERS  
IN INTEGER PROGRAMMING

1. Introduction

Several authors ([3], [4], [8], [9],[10], [14]) have proposed generalized Lagrangian methods for finding good or optimal solutions to integer programming problems. The capital budgeting problem of Lorie and Savage [9], essentially the 0-1 multi-dimensional Knapsack problem, has received particular attention in this regard. In [9], Nemhauser and Ullman prove the somewhat negative result that the approach of Everett [4] applied to the capital budgeting problem by Kaplan in [8] can yield an optimal solution only if there is an optimal linear programming solution that is integer. In this paper, we use group theory ([5], [6], [7], [11], [12], [13]) to reformulate the integer programming problem, thereby obtaining a Lagrangian problem which appears to offer greater combinatorial resolution than previous methods. Conversely, the usefulness of the group theoretic approach is enhanced by the Lagrangian problem.

In the next section we construct the Lagrangian problem and discuss its properties. The following section is concerned with the relationship of cuts generated by the Lagrangian problem to the strong cuts developed in [6]. Necessary and sufficient conditions for the existence of optimal multipliers are also given. There follows a numerical example and a few concluding remarks. There is one appendix in which some of the previous results are specialized to the zero-one integer programming problem.





## 2. Construction of the Lagrangian Problem and Its Properties

Consider the integer programming problem in the form

$$\begin{array}{ll}
 \min & cw \\
 \text{s.t.} & Aw = b \\
 & w \text{ non-negative integer}
 \end{array} \tag{1}$$

where  $c$  is a  $1 \times (m+n)$  vector of integers,  $A$  is an  $m \times (m+n)$  matrix of integers, and  $b$  is an  $m \times 1$  vector of integers. The columns of  $A$  are denoted by  $a_j$ ,  $j = 1, \dots, m+n$ . Let  $B$  be an optimal LP basis for (1) and rearrange the columns of  $A$  so that it can be partitioned as  $(R, B)$  and  $c$  as  $(c_R, c_B)$ .<sup>1</sup> The generic  $n$ -vector of non-basic variables is denoted by  $x$  and is called a correction.

The columns of  $B^{-1}$  with addition modulo 1 generate a finite abelian group consisting of  $D$  elements (see [ 5 ]). Let  $G$  be the group and let  $\lambda_k$ ,  $k = 0, 1, \dots, D-1$ , be its elements where  $\lambda_0$  is the identity element. Finally, let  $g$  be the function which maps integer  $m$ -vectors into group elements. It can easily be shown that problem (1) is equivalent to

$$\min \sum_{j=1}^n \bar{c}_j x_j \tag{2a}$$

---

<sup>1</sup> If  $B$  is an arbitrary basis, the analysis below is valid with minor modifications.



$$\text{s.t.} \quad \sum_{j=1}^n \bar{r}_{ij} x_j \leq \bar{b}_i, \quad i = 1, \dots, m, \quad (2b)$$

$$\sum_{j=1}^n \alpha_j x_j = \beta \quad (2c)$$

$$x_j = 0, 1, 2, \dots; \quad j = 1, \dots, n \quad (2d)$$

where  $\bar{c}_j = c_j - c_B B^{-1} a_j$ ,  $\bar{R} = (\bar{r}_{ij}) = B^{-1} R$ ,  $\bar{b} = (\bar{b}_i) = B^{-1} b$ ,  $\alpha_j = g(a_j)$  and  $\beta = g(b)$ . Condition (2b) is equivalent to the requirement that the basic variables (relative to B) be non-negative, and condition (2c) is equivalent to the requirement that they be integer.<sup>1</sup>

If we let

$$\bar{X} = \{x \mid x \text{ satisfies (2c) and (2d)}\},$$

Then problem (2) can be rewritten as

$$\begin{aligned} & \min \sum_{j=1}^n \bar{c}_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n \bar{r}_{ij} x_j \leq \bar{b}_i, \quad i = 1, \dots, m, \end{aligned} \quad (2')$$

$$x \in \bar{X}.$$

Problem (2') has the same form as Everett's problem [ 4 ] and problem (2) of [ 2 ]. The approach suggested by Everett is to weight the constraints (2b) by multipliers and place them in the objective function.

---

<sup>1</sup> We assume the existence of at least one  $x$  such that  $\sum_{j=1}^n \alpha_j x_j = \beta$ . Otherwise, problem (1) is infeasible.



In particular, for given  $u_i^0 \geq 0$ ,  $i = 1, \dots, m$ , satisfying  $\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij} \geq 0$ ,  $j = 1, \dots, n$ , we construct the new Lagrangian optimization problem

$$\begin{aligned} \min \quad & \sum_{j=1}^n (\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij}) x_j \\ \text{s.t.} \quad & x \in \bar{X} \end{aligned} \tag{4}$$

Problem (4) is a group optimization problem identical in form to the group optimization problem originally derived in [ 5 ]. See [ 11 ] and [ 12 ] for a further discussion of this problem. An optimal solution to (4) is denoted by  $x^0$ , or more generally by  $x(u)$  when  $u$  is the vector of multipliers in (4). The requirement that  $\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij} \geq 0$ ,  $j = 1, \dots, n$ , is necessary for (4) to have a bounded solution.

Notice that we could have constrained explicitly some variables in (1) to be zero or one. The modifications to the construction above in this case are given in Appendix A.





We begin our analysis with a restatement in the context of this paper of Everett's theorem 1 [ 4; p. 401]. First, for given  $u_i^0 \geq 0$ , let

$$P = \{i \mid u_i^0 > 0\} \quad (5)$$

LEMMA 1: An optimal solution  $x^0$  to (4) is optimal in (2) with  $\bar{b}_i$ ,  $i = 1, \dots, m$ , replaced by

$$(i) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0 \quad \text{if } i \in P$$

$$(ii) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0 + q_i \quad \text{if } i \notin P,$$

where  $q_i$  is any non-negative integer,  $i \notin P$ .

Proof: Suppose the contrary; that is, suppose there exists a non-negative integer vector  $y \in \bar{X}$  with the properties that

$$\sum_{j=1}^n \bar{c}_j y_j < \sum_{j=1}^n \bar{c}_j x_j^0,$$

and

$$\sum_{j=1}^n \bar{r}_{ij} y_j < \sum_{j=1}^n \bar{r}_{ij} x_j^0 \quad \text{for } i \in P.$$

Since  $u_i^0 \geq 0$ ,  $i = 1, \dots, m$ , there obtains



$$\sum_{j=1}^n (\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij}) y_j = \sum_{j=1}^n (\bar{c}_j + \sum_{i \in P} u_i^0 \bar{r}_{ij}) y_j$$

$$< \sum_{j=1}^n (\bar{c}_j + \sum_{i \in P} u_i^0 \bar{r}_{ij}) x_j^0 = \sum_{j=1}^n (\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij}) x_j^0$$

The last inequality is a contradiction to the optimality of  $x^0$  in (4).

When (4) is solved by the algorithm of [11], the algorithm also finds optimal solutions to (4) when the group constant  $\beta$  in (2c) is replaced by  $\lambda_k$ ,  $k = 0, 1, \dots, D-1$ ,  $\lambda_k \neq \beta$ . Let  $x^0(\lambda_k)$  be the optimal solution when  $\beta$  is replaced by  $\lambda_k$ .

COROLLARY 1: An optimal solution  $x^0(\lambda_k)$  to (4) with group constant  $\lambda_k$  is optimal in (2) with  $\bar{b}_i$ ,  $i = 1, \dots, m$ , replaced by

$$(i) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0(\lambda_k) \quad \text{if } i \in P$$

$$(ii) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0(\lambda_k) + q_i \quad \text{if } i \notin P$$

where  $q_i$  is any non-negative integer,  $i \notin P$ .

Thus, problem (4) is algorithmically useful in several ways. Suppose (4) with  $u^0 = 0$  (i.e., the group optimization problem of [5]) fails to solve (2) because for at least one  $i$ ,  $\sum_{j=1}^n \bar{r}_{ij} x_j^0 > \bar{b}_i$ . Then (4) can be resolved with new  $u^1$  for which some  $u_i^1 > 0$ ; e.g.,  $u_i^1 > 0$  only if the  $i^{\text{th}}$  constraint in (2b) with  $x = x^0$  is violated. Let  $x^1$  be the new optimal solution to (4).



There are three cases to consider. First, suppose  $\sum_{j=1}^n \bar{r}_{ij} x_j^1 \geq \bar{b}_i$ ,  $i \in P$ . In this case,  $\bar{c}x^1$  is a lower bound on  $z^*$ , the cost of an optimal correction to (2). This lower bound is greater than  $\bar{c}x^0$  because of the constraints  $\sum_{j=1}^n \bar{r}_{ij} x_j^1 \leq \sum_{j=1}^n \bar{r}_{ij} x_j^0$ ,  $i \in P$ , implied by (4) which were not in effect when (4) was previously solved with  $u^0 = 0$ . The improvements of bounds could be very important to the algorithm of [12] in which group theoretic bounds are used to limit the search of the non-basics for an optimal correction.

The second possibility is that  $\sum_{j=1}^n \bar{r}_{ij} x_j^1 \leq \bar{b}_i$ ,  $i = 1, \dots, m$ , or  $x^1$  is a feasible correction. Since  $\bar{c}x^0 \leq z^*$ , the non-negative quantity  $\bar{c}x^1 - \bar{c}x^0$  is an upper bound on the loss from terminating with  $x^1$ .

Finally, if neither of the two cases above obtain, the cut

$$(\bar{c} + u^1 \bar{R})x \geq (\bar{c} + u^1 \bar{R})x^1 \quad (6)$$

is a valid cut which could be added to (1). The relationship of the Lagrangian methods to the cutting plane method are discussed in the next section.

If the requirements vector  $b$  can be relaxed, then a suitable solution to (1) may be derived by replacing  $b$  with  $Rx^0(\lambda_k) + Bq$  for some  $\lambda_k$  where  $q$  can be any integer vector satisfying  $q \geq 0$  and  $q_i = 0$  if  $i \in P$ .<sup>1</sup> The optimal solution to the new (approximate) problem is  $(x^0(\lambda_k), q)$ .

---

<sup>1</sup>

We are now assuming  $u^0$  is an arbitrary non-negative vector in (4).





In order to put this relaxation scheme in better perspective, consider the isomorphic representation of  $G$  as  $M(I)/M(B)$  where  $M(I)$  is the group consisting of all integer points in  $m$ -space with ordinary addition, and  $M(B)$  is the subgroup consisting of all integer points which can be spanned by integer combinations of the basic activities. The element  $\beta$  in (2c) is then an equivalence class of  $m$ -vectors. Because of the combinatorial structure of the given problem, it may be that vectors from  $\beta$  are awkward as  $b$  vectors in (1); e.g., low cost corrections for group right hand side  $\beta$  may tend to be long and infeasible in (2).

It is tempting to try to construct an algorithm which, starting with  $u^0 = 0$ , solves (4) with a monotonic increasing sequence of  $u^k$ ,  $k = 0, 1, 2, \dots$ , so that  $\bar{R}x^k$  converges to  $b^k$  from above. Unfortunately, the  $\bar{R}x^k$  vectors are not sufficiently well behaved to make such an algorithm feasible. There is, however, a certain degree of regular behavior exhibited by the  $\bar{R}x^k$ .

LEMMA 2: Any optimal solutions  $x^0$  and  $x^1$  to (4) with non-negative multipliers  $u^0$  and  $u^1$  respectively must satisfy

$$(u^1 - u^0) \bar{R} (x^0 - x^1) \geq 0$$



Proof. The proof is immediate from the inequalities

$$(\bar{c} + u^0 \bar{R})x^0 \leq (\bar{c} + u^0 \bar{R})x^1,$$

and

$$(\bar{c} + u^1 \bar{R})x^1 \leq (\bar{c} + u^1 \bar{R})x^0.$$

Thus, if  $u_k^1 > u_k^0$  and  $u_i^1 = u_i^0$ ,  $i = 1, \dots, m$ ,  $i \neq k$ , then  $\sum_{j=1}^n \bar{r}_{ij} x_j^1 \leq \sum_{j=1}^n \bar{r}_{ij} x_j^0$ . Unfortunately, with  $u^1$  so defined there is no way to explicitly control the magnitude of the sums  $\sum_{j=1}^n \bar{r}_{ij} x_j^1$ ,  $i \neq k$ . Although future research may lead to algorithms with well behaved properties for using (4) to solve (2), it appears at this time that the primary usefulness of (4) in solving (2) is as a heuristic. In particular, we are currently implementing an adaptive integer programming system based on the ideas in [7], and we plan to experiment with the setting of multipliers in (4) in the near future.

Finally, we remark that the multipliers in (4) have an unusual interpretation for the integer programming problem with the formulation (2). In particular,

$$\bar{c}_j + u \bar{a}_j = c_j - \Pi a_j + u \bar{a}_j = c_j - (\Pi - \psi) a_j$$

where  $\psi = uB^{-1}$ . The usual LP interpretation of  $\Pi$  is that  $\Pi_i$  reflects the cost savings to be gained by reducing  $b_i$  by one unit. In the above expression,  $\Pi_i$  is adjusted by  $-\psi_i$  which can be interpreted as an added cost because the distance with respect to the  $i^{\text{th}}$  basis activity  $a_{n+i}$  from  $b$  to the boundary



of the cone  $\{\ell: B^{-1}\ell \geq 0\}$  is insufficient. This last statement can be made more precise by the following lemma from [13]. The lemma states: Given any set  $I \subset \{1, \dots, m\}$ , there exist non-negative integers  $t^*$  such that if  $\bar{b}_i \geq t_i^*$ ,  $i \in I$ , then (2) with rows  $i \in I$  omitted in (2b) solves (1) in the sense that there is an optimal solution to the reduced problem which is an optimal correction in (1). Suppose (2) satisfies the condition of this theorem for some set  $I$ . Then problem (4) formed from the reduced problem (2) is the same as problem (4) formed from the original problem (2) with  $u_i^0 = 0$ ,  $i \in I$ .





### 3. Cutting Plane and Lagrangian Methods

In [ 6 ], Gomory uses the group theoretic approach to develop theorems for characterizing and constructing strong cuts for the cutting plane method. If we let  $[\bar{X}]$  denote the convex hull of  $\bar{X}$ , then the cuts of interest are the faces of  $[\bar{X}]$ . Our purpose in this section is to relate problem (4) and the Lagrangian cuts to these faces.

Suppose  $\lambda x \geq \lambda_0$  is a face of  $[\bar{X}]$ . Gomory shows [ 6 ; p. 16] that  $\lambda \geq 0$  and  $\lambda_0 \geq 0$ . It is natural to ask if it is possible to obtain a Lagrangian cut (6) equivalent to this face. Since for any  $k > 0$ ,  $k\lambda x \geq k\lambda_0$  is an equivalent cut, the correct mathematical statement of this problem is: Does there exist a  $u \geq 0$ ,  $k > 0$ , such that

$$u\bar{R} - k\lambda = -\bar{c} \quad (7)$$

The system (7) consists of  $n$  equations in  $m+1$  variables which does not always possess a solution of the form we seek.

Although this inability to produce selected strong cuts appears to be a drawback of the Lagrangian method, we remark that the faces of  $[\bar{X}]$  are generated with respect only to the group identities of the non-basic activities  $a_j$ ,  $j = 1, \dots, n$ , and the requirements vector  $b$ . The Lagrangian approach, on the other hand, incorporates information about the real space identities of the  $a_j$  and  $b$ . A synthesis of the two approaches would be to (i) select a boundary point  $x^0$  of  $[\bar{X}]$  by solving problem (4) for a specified  $u^0 \neq 0$  chosen according to the magnitude of  $\bar{b}_1$ , and (ii) use the methods of [ 6 ] to generate cuts beginning with those which contain  $x^0$ .



Of course, a more fundamental existential question is: Does there exist a  $u \geq 0$  such that (4) solves (2)? To answer this, a definition is needed. A solution  $t \in \bar{X}$  is said to be irreducible if  $s \leq t$  and  $s \in \bar{X}$  implies  $s = t$ . Let  $T$  be the set of all irreducible solutions. It is easy to demonstrate that  $T$  is finite; say  $T = \{t^k\}_{k=1}^K$ .

Suppose  $x^*$  is an optimal correction. Then it is clear that  $x^*$  is an optimal solution in (4) if and only if there exists a  $u \geq 0$  such that

$$\bar{c} + u\bar{R} \geq 0, \quad (8)$$

and

$$(\bar{c} + u\bar{R})t^k \geq (\bar{c} + u\bar{R})x^*, \quad k = 1, \dots, K. \quad (9)$$

Note that each  $x \in \bar{X}$  can be decomposed in at least one way as  $x = t + s$  where  $t \in T$  and  $s$  is non-negative integer. Thus,  $(\bar{c} + u\bar{R})x \geq (\bar{c} + u\bar{R})t \geq (\bar{c} + u\bar{R})x^*$  and condition (9) is sufficient as well as necessary. The following lemma is a direct consequence of Farkas' lemma applied to (8) and (9) for  $u \geq 0$ .

LEMMA 3: Suppose  $x^*$  is an optimal correction. Then there exists a  $u \geq 0$  such that  $x^*$  is optimal in (4) if and only if for every  $K+n$  vector  $v \geq 0$  and  $v \neq 0$  such that

$$\sum_{k=1}^K \bar{R}t^k \frac{v_k}{\sum_{k=1}^K v_k} + \frac{\sum_{j=1}^n \bar{a}_j v_{K+j}}{\sum_{k=1}^K v_k} \leq \bar{R}x^*,$$



there obtains

$$\sum_{k=1}^K \bar{c}^k \frac{v_k}{\sum_{k=1}^K v_k} + \frac{\sum_{j=1}^n \bar{c}_j v_{k+j}}{\sum_{k=1}^K v_k} \geq \bar{c}x^*.$$

This lemma is difficult to interpret as it stands. Suppose, however, that there exists a vector  $v \geq 0$  and  $v \neq 0$  with  $v_{k+j} = 0$ ,  $j = 1, \dots, n$ , and also such that

$$\sum_{k=1}^K \bar{R}t^k \frac{v_k}{\sum_{k=1}^K v_k} \leq \bar{R}x^*, \quad (10)$$

and

$$\sum_{k=1}^K \bar{c}t^k \frac{v_k}{\sum_{k=1}^K v_k} < cx^*. \quad (11)$$

In this case, there does not exist a  $u$  such that  $x^*$  is an optimal solution in (4). Conditions (10) and (11) state, of course, that  $x^*$  is not optimal in (4) if there is a convex combination of the  $t^k$  which is less costly and uses no more resource than  $x^*$ . The interested reader should compare this with Everett's discussion on the source of gaps which can be found on page 408 of reference 4.



Lemma 3 was derived without regard to whether or not the given optimal correction  $x^*$  is irreducible. If  $x^*$  is reducible, then necessary and sufficient conditions that  $x^*$  is optimal in (4) is that there exist a  $u \geq 0$  such that for at least one  $t \in T$  for which  $x^* = t + s$  ( $s \geq 0$ ,  $s \neq 0$ )

$$\bar{c} + u\bar{R} \geq 0$$

$$(\bar{c} + u\bar{R})t^k \geq (\bar{c} + u\bar{R})t \quad k = 1, \dots, k$$

$$(\bar{c} + u\bar{R})s = 0$$

Since  $g(t) = g(x^*) = \beta$ , we have  $g(s) = 0$ . Thus, the condition that  $(\bar{c} + u\bar{R})s = 0$  can be interpreted as the requirement that there be at least one costless circuit in the shortest route network representation of (4). Any such circuit can be built up from costless elementary circuits which can be found by the algorithm of [11].





#### 4. Numerical Example

Consider the integer programming problem

$$\begin{aligned}
 \min \quad & 21x_2 + 6x_4 + 4x_5 \\
 \text{s.t.} \quad & -1x_1 + 13x_2 + 5x_4 + 2x_5 = 16 \\
 & 10x_2 - 1x_3 + 1x_4 + 3x_5 = 8
 \end{aligned} \tag{12}$$

$$x_j \text{ non-negative integer, } j = 1, 2, 3, 4, 5.$$

Activities  $a_4$  and  $a_5$  constitute an optimal LP basis. Problem (4) derived with respect to this basis is (all fractions were eliminated by multiplying by 13)

$$\begin{aligned}
 \min \quad & (14 - 3u_1 + 1u_2)x_1 + (11 + 19u_1 + 37u_2)x_2 + (8 + 2u_1 - 5u_2)x_3 \\
 \text{s.t.} \quad & -3x_1 + 19x_2 + 2x_3 \leq 32 \\
 & 1x_1 + 37x_2 - 5x_3 \leq 24 \\
 & 1x_1 + 11x_2 + 8x_3 \equiv 11 \pmod{13}
 \end{aligned} \tag{13}$$

$$x_j \text{ non-negative integer, } j = 1, 2, 3$$

The optimal solution to (13) with  $u_1^0 = 0$ ,  $u_2^0 = 0$ , is  $x_1^0 = 0$ ,  $x_2^0 = 1$ ,  $x_3^0 = 0$ . This correction is not feasible in (12) as the second inequality constraint in (13) is violated. Setting  $u_1^1 = 0$ ,  $u_2^1 = 1$ , yields the solution  $x_1^1 = 0$ ,  $x_2^1 = 0$ ,  $x_3^1 = 3$ , which is the optimal correction we seek. In fact, this correction is optimal for any  $u \geq 0$  with  $u_1 = 0$ ,  $9/13 \leq u_2 \leq 8/5$ .



## 5. Conclusion

In this paper we have tried to demonstrate how the Lagrangian problem (4) is useful when trying to solve integer programming problems. We saw that an optimal solution to (4) could be used in a variety of ways to find good or optimal solutions to (1). Unfortunately, the theoretical results obtained thus far are not sufficiently strong to enable us to algorithmically control the properties of optimal solutions to (4). Future research may lead to an improvement in these procedures. We emphasize once again, however, that we expect to incorporate problem (4) into the adaptive group theoretic algorithm of [7] and experiment with heuristics for setting the multipliers  $u_i$ . It seems clear that the power of the group problem from [5] can be enhanced by the approach here.

Certain results from the literature on generalized Lagrange multipliers were not specialized for our problem although it is possible to do so. In particular, Everett's lambda and epsilon theorems in reference 4 were ignored because they appear to be not directly relevant to our study. Similarly, the linear programming problem in [2] and Brooks theorems in [3] have been ignored. In addition, there may be some connection between the results here and those of Balas in [1] on duality theory in integer programming.



## Appendix A

## The Zero-one Integer Programming Problem

Suppose there is a subset  $S \subset \{1, \dots, m+n\}$  such that  $x_j = 0$  or  $1$ ,  $j \in S$ , and  $x_j = 0, 1, 2, \dots, j \in S^c$ . Suppose further that the first  $m_1$ -basic variables are 0-1 variables. Then (2) becomes

$$\min \sum_{j=1}^n \bar{c}_j x_j \quad (14a)$$

$$\text{s.t.} \quad \sum_{j=1}^n \bar{r}_{ij} x_j \leq \bar{b}_i, \quad i = 1, \dots, m \quad (14b)$$

$$\sum_{j=1}^n (-\bar{r}_{ij}) x_j \leq 1 - \bar{b}_i, \quad i = 1, \dots, m_1 \quad (14c)$$

$$\sum_{j=1}^n \alpha_i x_j = \beta \quad (14d)$$

$$x_j = 0, 1, 2, \dots, \quad j \in S^c \quad (14e)$$

$$x_j = 0 \text{ or } 1, \quad j \in S \quad (14f)$$

As before, define

$$Z = \{x \mid x \text{ satisfies (14d), (14e), (14f)}\}.$$

For a given  $u^0$  where we now require  $u_i^0 \geq 0$  only if  $i \geq m_1 + 1$ , we define the Lagrangian problem (4)



$$\min \sum_{j=1}^n (\bar{c}_j + \sum_{i=1}^m u_i^0 \bar{r}_{ij}) x_j \quad (15)$$

$$x \in Z$$

Problem (15) is a zero-one group optimization problem which can be solved by the zero-one group algorithm of [7]. We have the following corollary to lemma 1.

COROLLARY 2: Let  $x^0$  be an optimal solution to (15) with  $u^0$  such that  $u_i^0 \geq 0$ ,  $i = m_1 + 1, \dots, m$ . Then  $x^0$  is optimal in (14) with (a)  $\bar{b}_i$ ,  $i = 1, \dots, m$ , replaced by

$$(i) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0 \quad \text{if } u_i^0 > 0$$

$$(ii) \quad \sum_{j=1}^n \bar{r}_{ij} x_j^0 + q_i \quad \text{if } u_i^0 \leq 0;$$

and (b)  $1 - \bar{b}_i$ ,  $i = 1, \dots, m_1$ , replaced by

$$(iii) \quad \sum_{j=1}^n (-\bar{r}_{ij}) x_j^0 \quad \text{if } u_i^0 < 0$$

$$(iv) \quad \sum_{j=1}^n (-\bar{r}_{ij}) x_j^0 + q_i \quad \text{if } u_i^0 \geq 0,$$

where the  $q_i$  can be any non-negative integers.

Problems (14) and (15) are the ones applicable to the Lorie-Savage capital budgeting model. We mention in passing that the zero-one group algorithm solved the example in [8] with  $u^0 = 0$ ; that is, without resort to explicit use of the multipliers of this paper.





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