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STATISTICAL THEORY APPLIED TO COMMUNICATION THROUGH MULTIPATH DISTURBANCES

ROBERT PRICE

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Research Laboratory of Electronics

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Robert Price

Group 34

Research Laboratory of Electronics
Technical Report No. 266

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ABSTRACT

This work is concerned with the synthesis and evaluation of integrated communication systems constructed specifically to perform in the presence of channel disturbances of the form encountered in multipath propagation. In the communication system considered, information is conveyed from source to user by encoding it into electrical symbols at the transmitter and decoding it at the receiver. Statistical theory, including information theory, is used throughout the paper. Gross aspects of multiple-mode propagation and ionospheric scattering are discussed, and previous efforts to improve systems affected by these disturbances are reviewed.

System capacities are computed for the transmission of band-limited white Gaussian signals through fixed multiple-mode disturbances, and the optimum spectral distribution of transmitter power is found. In general, presence of additional paths does not upset the message-handling capabilities of the channel if suitable wide-band signals are used. Quasi-stationary path fluctuations are considered, and a technique for measurement of the multipath characteristic by the receiver is suggested. Capacities are found for slow path delay fluctuations. Single-mode scatter propagation is considered as a complex multiplicative Gaussian process. The probability computing receiver is found analytically, and a physical realization is developed for small signal-to-noise ratios. Lower bounds to system capacity are found for a binary transmission through scatter. Incidental results are system capacities for a related multiplicative channel, and the capacity for a binary transmission through white Gaussian noise.

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To Dr. Wilbur B. Davenport, Jr., the supervisor of this thesis, the author is lastingly indebted. As a constant source of inspiration, a counselor of enduring patience, and a brilliant advocate of the statistical communication philosophy, Dr. Davenport has largely made this thesis possible. In its present content and form, this work represents many months of a close and enjoyable partnership.

The thesis problem was originally framed by Prof. Robert M. Fano, whose helpful advice and guidance have been sincerely appreciated. Dr. Bennett L. Basore gave the author much aid in the preliminary investigations and in the understanding of the general application of information theory. The author's close companionship with Paul E. Green, Jr., has given stimulation and careful consideration to the solution of many of the problems considered here.

The experimental work on ionospheric fluctuations would not have been possible without the generous help of Philip L. Fleck, Jr., and Jerome Elkind. Miss Roberta Gagnon has given her time to produce the original document.

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STATISTICAL THEORY APPLIED TO COMMUNICATION THROUGH MULTIPATH DISTURBANCES

CHAPTER I

I. INTRODUCTION

This work is concerned with the design and evaluation of integrated communication systems constructed specifically to perform in the presence of channel disturbances of the form encountered in multipath propagation. By a communication system is meant a combination of transmitter, channel, and receiver which has the general structure shown in Fig. 1. The transmitter contains an information source which generates a sequence of symbols drawn independently from a finite alphabet, and an encoder which translates these symbols into new symbols x_i, $i = 1, 2, \ldots, M$, whose character is appropriate to the channel being used. In this study, the x_i are voltage waveforms having duration T. The channel may, by performing general operations governed by random processes, perturb the transmitted symbols in such a way that observation of the received symbols is not sufficient to state with certainty which symbols were transmitted. Then, since the transmitted sequence cannot be known a priori by the receiver if any information is to be conveyed, the receiver is called upon to guess, in a detector, which symbols were transmitted. In making its decision, the receiver combines its observation of the perturbed y sequence with statistical knowledge of the transmitter and channel available to it a priori. It is important to note that the receiver is directly interested only in events at the transmitter, not in the channel. Ultimately, the information user is given a sequence of x_j related to the symbols generated by the source, but which may be in error to a certain degree. A slightly different model following the outline of Fig. 1 is one in which the receiver need not make decisions, but is required only to observe the y sequence and process it into a series of evaluations of the possibilities that the various x_i could have been transmitted. Our selection of the former model is conditioned by the practical aspects of familiar information users, such as teletype.

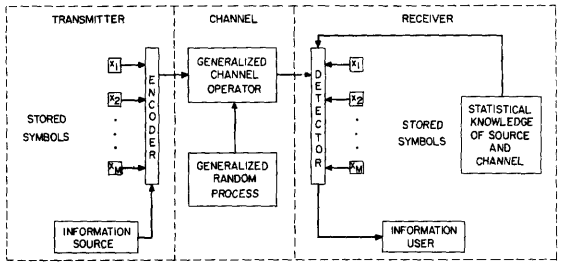

Fig. 1. The communication system model.

Since both the symbol sequence generation by the source and the channel disturbances are random processes, the theory appropriate to the communication problem is that of probability and statistics. In Sec. A of this chapter we shall discuss briefly a few aspects of the general approach, without specific reference to its application to multipath channels. In Sec. B a simplified description of ionospheric multipath is presented, together with a short history of the efforts which have been devoted to system improvement when multipath is present. Chapter II presents the results obtained in this paper for communication over fixed or quasi-stationary multiple ionospheric modes, and results for communication over a single scatter path are given in Chapter III. Appendices supplement these later two chapters.

A. Synthesis and Evaluation of Communication Systems

1. Criteria of System Performance

As stated in the Introduction, the operation of the communication system model of Fig. 1 is largely described in terms of statistics. It is natural, then, that acceptable measures of system performance should involve probabilities. For the symbols x_i fixed in number M and duration T, it appears that the best system, within additional constraints such as specified transmitter power, is one which minimizes the probability of error in the output sequence x_j.

If the number and duration of the transmitted symbols are allowed to vary, however, we have the problem of redefining system performance so that the rate at which symbols are being produced may enter. Since usually the probability of error increases with the rate, a criterion giving the best "trade" should be deduced. At this point it is pertinent to consider the communication system in the light of the mathematical theory of communication, or information theory, as developed by Shannon[1] and Fano.[2] Here entropy, a measure of the randomness of the output from a stochastic source, is a central concept used extensively in considering the flow of information within systems. Source rate, the rate of production of transmitted symbols, is defined as the entropy of the source H(x) per unit time. Equivocation $H_y(x)$ is a conditional entropy that is a measure of the average uncertainty in x, knowing y. Thus it is a measure of the information lost in transmission. Clearly, when the receiver is required to make a decision, the probability of error depends upon this uncertainty, and hence upon the equivocation. Per-unit-equivocation, defined as $H_y(x)/H(x)$, has been suggested as a useful performance criterion, since it involves both source rate and probability of error. Another parameter is the transmission rate, defined as $[H(x) - H_y(x)]/T$, a measure of the information which eventually reaches the receiver. For the particular system of Fig. 1, we have

$$H(x) = -\sum_{i=1}^{M} P(x_i) \log P(x_i) = \log M \quad , \tag{1-1}$$

$$H_y(x) = -\int_y \sum_{i=1}^{M} P(x_i/y)\, p(y) \log P(x_i/y)\, dy \tag{1-2}$$

$P(x_i)$ is the a priori probability of the i^{th} symbol, assumed equal to 1/M. $P(x_i/y)$ is the conditional, or a posteriori, probability of the i^{th} signal, knowing the received perturbed symbol y. In this paper, the logarithmic base is always taken as two, unless otherwise indicated. The unit measure of information is then the bit, a choice between two equally-likely possibilities.

Much of the work in this paper deals with systems of the form of Fig. 1, in which two assumptions are made. First, the symbols x_i to be used in the transmitter are allowed to approach infinity in length T. Secondly, the symbols are to be constructed by drawing samples of duration T at random from a given stochastic source S. This coding procedure does not seem unreasonable when we must deal with the problem of forming a very large alphabet. In general, as T increases indefinitely with the alphabet size M held fixed, probability of error converges to zero. At the same time, however, the source rate approaches zero. Shannon has shown, however, that with such random coding it is possible to increase M with T slowly enough that source rate remains constant, while probability of error and equivocation still converge to zero, on the average over all such random codes. Such a source rate is called an allowable source rate, and its upper bound, called the system capacity, may be found by computing entropies when the given code-generating source S drives the channel directly, as shown in Fig. 2. We then have

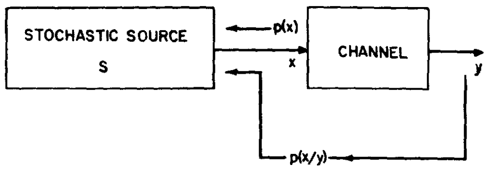

Fig. 2. Stochastic source driving a channel.

$$C = \lim_{T \to \infty} \frac{H(x) - H_y(x)}{T} \qquad (1\text{-}3)$$

with $H(x)$ and $H_y(x)$ found from expressions similar to Eqs.(1-1) and (1-2). Here we should emphasize that the model of Fig. 2 is not, in general, a communication system, but merely a device to enable the evaluation of C. A large part of the work in this paper deals with the computation of C for various sources and channels.

If now the source S, from which the symbols are derived, is changed, Eq.(1-3) is re-evaluated to find the new value of C. The maximum possible system capacity so obtainable for the given channel, with constraints included, is called the channel capacity C_c. This parameter is of prime concern to Shannon. C_c is the upper bound to the allowable source rate at which information can be transmitted over the channel, with constraints included, by any coding scheme. The proof consists simply in noting that since the transmitter of Fig. 1 is one possible S, any rate in excess of C_c would be in contradiction to the definitions of allowable source rate and C_c. Although in practice it is usually difficult to find C_c, one special case has been solved easily.[1] If the channel perturbation is additive white Gaussian noise of uniform spectral density N_o, and the transmitter is limited to bandwidth W and average power P,

$$C_c = W \log \left(1 + \frac{P}{N_o W}\right) \quad . \qquad (1\text{-}4)$$

The source giving C_c is a white Gaussian noise source of bandwidth W and average power P. Using transmitter symbols drawn from this source, the rate C_c can actually be achieved arbitrarily closely by letting the symbols increase indefinitely in length.

2. The Probability-Computing Receiver

Regardless of the system criterion accepted, an ideal receiver can be defined in an unambiguous way as one which extracts all the information present in the received perturbed symbol y. Within the framework of statistical analysis adopted for the study of systems of the form of Fig. 1, all the information which y bears about the x_i is contained in the set of a posteriori probabilities $P(x_i/y)$. It is then clear that the role of an ideal receiver is that of a probability computer, as recognized by Woodward and Davies.[4] When a decision must be made by the receiver, it naturally takes these probabilities into account. In the model considered, the symbol having the greatest a posteriori probability is selected as having been transmitted, in order to minimize the probability of error. This operation is equivalent to that of Siegert's[5] ideal observer, and will define the ideal receiver for this study. Other methods of making the decision, such as the Neyman-Pearson[6] test, have been examined, but they too use probability-computing techniques.

B. Ionospheric Propagation Channels

1. Fundamental Physical Aspects

Whenever communication between two points employs propagation of radio waves, there exists the possibility that more than one mode of transmission, or path, may be taken by the waves between transmitter and receiver. This phenomenon may arise from a variety of causes, such as reflections from tall buildings or atmospheric inversions of refractive index. This paper, however, is concerned mainly with ionospheric multipath, perhaps the most familiar and troublesome variety.

The mechanism by which multiple propagation modes may exist in the ionosphere has been understood for many years.[7,8] In summary, the physical model is as follows. Below a certain critical frequency which varies with time of day, season, and other factors, it is possible for an electromagnetic wave entering a densely ionized region of the ionosphere to be bent around and emerge on the same side from which it entered, producing reflection. Best reflection occurs near grazing incidence, and if the angle of arrival is successively increased toward the normal, a critical angle may be reached, beyond which there is substantially no reflection. This critical angle is closer to the normal, the lower the frequency. At a given frequency, therefore, there may be a number of possible sky-wave paths, consisting of alternate ionospheric and ground reflections, or "hops," connecting transmitter and receiver. When several ionospheric layers are present, there may, of course, be more than one family of "hops," as illustrated in Fig. 3. Here one- and two-hop transmissions are shown for the E and F ionospheric layers. In general, the number of hops will decrease as the frequency is increased, so that transmission can be restricted to a one-hop path by proper choice of operating frequency. Occasionally, the ground wave, also shown in Fig. 3, adds to the number of modes present.

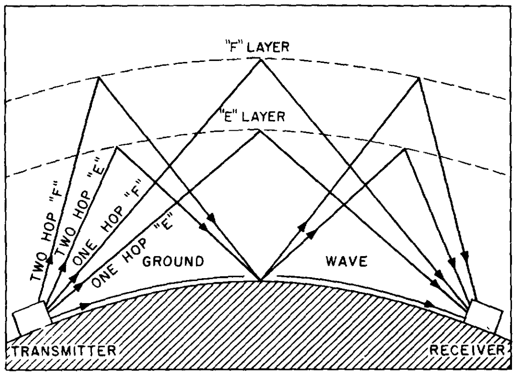

Fig. 3. Multi-hop propagation.

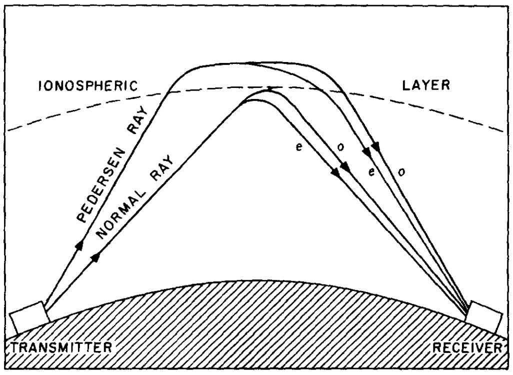

Fig. 4. Splitting of single-hop path.

It is found, however, that even when multi-hop and ground paths are suppressed, multiple-path effects are still present. This occurs because several distinct components are present in each "hop" mode (see Fig. 4). A study of ionospheric refraction admits two solutions, or possible paths by which a wave may be reflected in the absence of the earth's magnetic field. These are designated as the normal and Pedersen rays. Each ray, in turn, splits up into two modes, ordinary and extraordinary, when the earth's magnetic field is taken into account. Although elaborate techniques might be resorted to in order to suppress all modes but one, the philosophy of this paper is that the nuisance of multiple paths is accepted, and the studies deal with communication using models which approximate the gross features just outlined.

When it has been possible to isolate completely a single ionospheric mode, fluctuations have been observed which have led to a re-examination of the theory of ionospheric refraction. While previously the reflection was considered nearly specular, a scattering model has now been suggested by Booker, Gordon, and Ratcliffe.[9-11] The new theory postulates that the transmitted waves impinge upon a great number of randomly moving irregularities in the ionosphere, so that the received wave is the sum of many small scattered waves. Such a model has been found to fit some experimental observations, but more supporting data must be gathered before the theory is completely substantiated. Nevertheless, it is of interest to determine the effect of this type of propagation model on the communication system.

2. Previous Efforts to Combat Multipath

In systems which have been designed without regard to possible multipath disturbances, severe difficulties in operation have often arisen. For example, deep fading of the carrier relative to the sidebands in conventional AM produces serious nonlinear distortion at the output of the usual envelope detector. At other times, unpleasant distortions of the frequency response are apparent, as listening to transatlantic broadcasts readily shows. Both these effects are readily explained by the interference effects caused by multiple paths. Conventional FM systems also break down badly when more than one path is present.[12] In practice then, multipath can severely upset communication systems which have been designed without regard to its possible depredations. Naturally, considerable effort has been devoted to devising schemes which will improve present systems in this respect.

Design of transmission systems for operation under multipath conditions appears to fall into two main categories. First, systems originally designed without consideration of multipath may be modified so that improvement is obtained while simplicity is preserved. One such method in AM has been to transmit single-sideband-suppressed carrier, and to receive using a locally generated "exalted carrier" for detection.[13] This technique eliminates the nonlinear distortion previously mentioned, so that an order-of-magnitude improvement in intelligibility may be obtained. Nevertheless, frequency distortion is still present. Another technique useful for other systems as well as AM is to use diversity reception[14] to reduce deep fades in over-all strength. Diversity simply selects or gives greatest emphasis to the most powerful of a number of separate outputs from the various modes which may exist between transmitter and receiver. Representative forms of diversity systems are frequency, space, polarization, arrival, angle and combinations of these. Again, although improvement is surely obtained, it is by no means clear that such a technique is, in general, the best which could be used, and, in fact, the received signal still is generally at variance with what was transmitted. In FM, the work of Arguimbau and Granlund[15] has shown that this means of transmission may be made quite satisfactory when fewer than three paths are present, and in fact may be better than other existing systems in this special case. For three or more paths, however, their technique appears to break down.

The more logical design procedure is characterized by a system synthesis from the very foundations, rather than elaborations on a theme already established. Comparatively few contributions have been made in this field, the most conspicuous being the studies supervised by DiToro.[16] His work is concerned mainly with the question of best encoding when the message is subject to random, complete fades. The studies presented in this paper are in the spirit of this approach, and are guided by the statistical concepts of the communication process outlined in the preceding sections.

3. Applicability of Statistical Theory to Ionospheric Channels

The random nature of ionospheric fluctuations leads naturally to the consideration of Fig. 1 as the appropriate model to represent the communication system with a multipath channel. While, as stated earlier, it is then possible to specify the functional form of the receiver unambiguously, several possible performance criteria are available. The one which has been selected for study in this paper is the system capacity, a performance measure useful only when it is assumed that the symbols are drawn at random from a given stochastic source and allowed to approach infinity in length. In addition to studies of system capacity, a detailed analysis has been made of the ideal receiver for scatter propagation.

It is apparent that theoretical characterization of the channel is of major importance in a study of a communication system of the form shown in Fig. 1. The physical description of multipath presented in Sec. A of this chapter suggests that a hierarchy of channel models could be investigated, starting with statistically stationary "scatter" propagation over a single mode and proceeding all the way up through the combination of many such modes into a general nonstationary model of multipath in which the individual modes do not even fluctuate independently of each other. Various methods of analysis within the statistical domain may naturally be used more easily on some models than on others. The general difficulty encountered is, of course, that if the model is made realistic in every detail, mathematical intractability usually results. On the other hand, oversimplification yields a model that is meaningless or trivial.

In the work to follow, two main classes of multipath channel are considered. Chapter III considers these cases in which the reflection of an individual mode is considered specular, but where there may be multiple modes which vary slowly in amplitude and delay with respect to one another. Chapter IV treats just a single mode, but where the reflection is no longer purely specular, but rather a time-varying scattering process.

While nothing has been said of the presence of the most familiar channel disturbance, additive thermal or shot-effect noise, this must naturally be considered in the model in most cases.

CHAPTER II

I. MULTIPLE-MODE CHANNELS

In this chapter we are concerned with the application of statistical theory to communication through multiple-mode disturbances. Until the recent work on scatter propagation, a "classical" channel model consisting of multiple specularly-reflected modes was considered adequate. Referring to Figs. 3 and 4, we assume a number of specularly-reflected paths connecting the transmitter and receiver, which are fixed, or change in time slowly enough that a quasi-stationary analysis may be used. In Sec. A of this chapter, the time- and frequency-domain behavior of classical multipath is discussed, so that a suitable mathematical model may be constructed for the analysis in the later sections. Section B deals with the computation of system capacity for transmission of white Gaussian signals through fixed multiple-mode channels, and also indicates the optimum spectral distribution which the transmitter should have to achieve maximum capacity. Section C is concerned with communication when the multipath may be slowly varying, and contains both theoretical and experimental aspects. Section D treats system capacity for general random linear filters in the channel, and applies the results to multiple-mode disturbances in particular.

A. Time- and Frequency-Domain Representation of Classical Multipath

Since the aspects of the multipath channel considered are in nature linear, although time-varying, it is of interest to describe the channel in the two domains appropriate to linear systems, namely, the time and frequency domains. When the multipath is subject to slow variations, the frequency response is considered from a quasi-stationary point of view.

Considering first the case of fixed multipath, the time response, unlike the frequency response, can be defined rather arbitrarily. Since ionospheric transmission customarily uses bandpass signals, we will take a carrier pulse as the reference input. Assuming that the pulse is long compared to one cycle of the carrier, f_o, being transmitted, but short compared to the smallest time difference between paths, a response of the form of Fig. 5 might be obtained in a

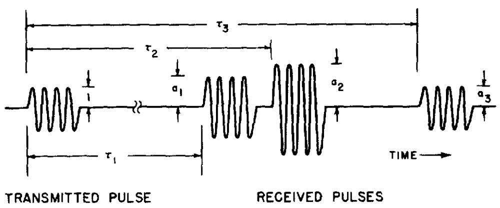

Fig. 5. Response of multipath to a carrier pulse.

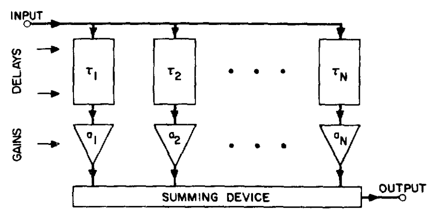

Fig. 6. A multipath model.

representative multipath situation. A nearly identical response could be obtained with the same input for the model shown in Fig. 6. Thus, for the frequency range of interest it is realistic to use such a model to represent multipath configurations, although the model departs radically from the actual system at other, and especially low, frequencies. The frequency response $H(\omega)$ for the model of Fig. 6 is easily obtained:

$$H(\omega) = \sum_i a_i \exp[-j\omega\tau_i] \tag{2-1}$$

where $\omega = 2\pi f$. Figure 7 shows a typical magnitude response $H(\omega)$ in the region of interest. The phase response is not shown, but has similar scallops. One can say that the scallops occur with a period roughly equal to the reciprocal of the length of the pulse train.

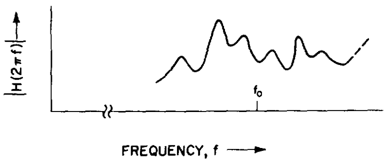

Fig. 7. Multipath frequency response.

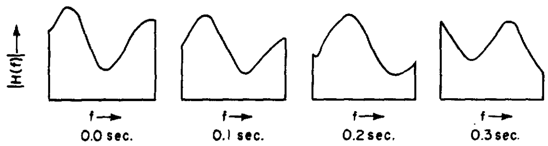

Fig. 8. Selective fading.

This section has so far considered only a static ionosphere, although extensive research has been conducted on dynamic ionospheric behavior.[17-19] Potter's measurements, in which he transmitted frequency groups across the Atlantic, show very clearly the phenomenon known as selective fading. Figure 8 presents a pattern of responses which might be observed in a particular frequency band at 0.1 second intervals. A steady drift of the pattern from left to right is observed in this case, but such continuity is not always apparent. In other cases absorption in all paths may increase, resulting in uniform fading across the band.

B. Fixed Multipath

1. System Capacity for White Gaussian Transmitted Signal and Channel Noise

The channel capacity given in Eq.(1-4) is obtained in the case of transmitter symbols drawn from a white, band-limited Gaussian source and additive white Gaussian channel noise. More generally, for arbitrary band-limited Gaussian symbols and Gaussian channel noise, we obtain for the system capacity[20]

$$C = \int_0^W \log\left(1 + \frac{S(f)}{N(f)}\right) df \quad , \tag{2-2}$$

where S(f) and N(f) are the signal and noise power spectra, respectively, and where the frequency band has arbitrarily been translated to low-pass. Fixed multipath, like any linear system, leaves a Gaussian time series still Gaussian, merely altering its spectral density. Equation (2-2) gives, therefore, an appropriate formula for the system capacity with Gaussian symbols transmitted through fixed multipath. If $S_o(f)$ is the original spectral distribution at the transmitter, then S(f) is given by

$$S(f) = S_o(f)\ |H(f)|^2 = S_o(f)\ \left|\sum_i a_i \exp[-j2\pi f\tau_i]\right|^2 \tag{2-3}$$

from Eq.(2-1).

We now assume that the transmitted spectral density and channel noise spectrum are uniform; $S_o(f) = S_o$ and $N(f) = N_o$. The evaluation of Eq.(2-2) for this case is considered in Appendix I, Sec. A. Under the assumption that the τ_i are commensurable, it is possible to divide the argument of the logarithm in Eq.(2-2) into simple factors, so that the integral becomes the sum of a number of simpler integrals. Evaluation of the separate integrals is possible if either $W = m/(2\tau)$ or $W >> 1/(2\tau)$, where τ is the greatest common divisor of the τ_i. For two paths of relative strength unity and a, we obtain from Eq.(A1-17), under these assumptions, the system capacity C_2:

$$C_2 = W \log \frac{1}{2} \left\{ 1 + R(1 + a^2) + \sqrt{[1 + R(1 + a^2)]^2 - 4a^2R^2} \right\} . \qquad (2\text{-}4)$$

R is the signal-to-noise ratio S_o/N_o when $a = 0$. For $R >> 1$, we have

$$C \approx \begin{cases} W \log R; \ 0 \leq |a| \leq 1 \\ W \log a^2R; \ 1 \leq |a| \end{cases} . \qquad (2\text{-}5)$$

Thus for large signal-to-noise ratios, the channel capacity is substantially that due to the strongest path alone. Actually, the presence of a second path never decreases the channel capacity, and usually results in a slight increase, although not so much as would be obtained if both paths combined to form one of strength $(1 + a)$.

Under similar bandwidth assumptions, some three-path capacities have been calculated, for a special case only. The results are presented in Fig. 9. Here it is assumed that there are two paths of strength unity and a third of strength a, with delays 0, τ, and 2τ, respectively. The signal-to-noise ratio R is 10, taken for one of the unity-gain paths alone. For reference, the capacity for just the first path present is also included. It is seen that although the channel capacity is reduced for some values of a by introduction of the third path, it does not fall below that given by the strongest path alone. When further paths are introduced, the analysis becomes considerably more complex. Two particular cases, for four and five paths present, have been evaluated. It is assumed that each path has strength unity, and that the paths are delayed 0, τ, 2τ, 3τ, and 4τ, respectively. $R = S_o/N_o$ is taken as 10. The results appear in Fig. 10,

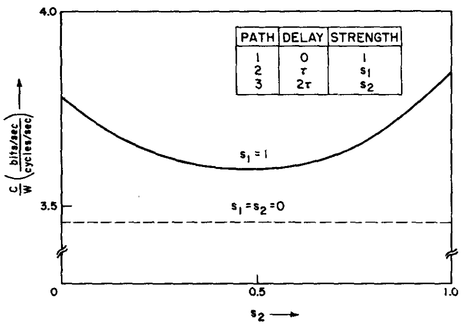

Fig. 9. System capacity for three paths.

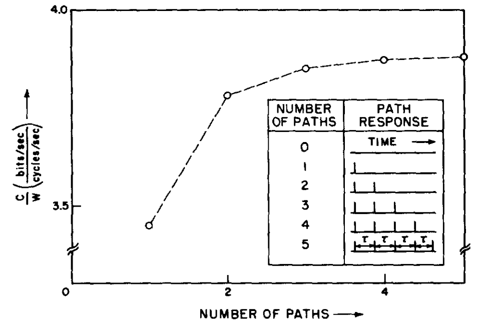

Fig. 10. System capacity for multiple paths.

where they are compared with similar one-, two-, and three-path cases. Again, the system capacity does not decrease below the single-path value as more paths are added. While no analytic proof has been found, it appears likely that this condition holds true in general. Intuitive support for this statement may be found in the plots of $|H(f)|$ for various path structures presented in Fig. 11. The valleys produced by destructive interference appear to be well offset by the peaks of constructive interference, providing the transmission takes place over a wide enough band.

From these sample computations we may conclude that, providing a wide transmission band is permitted, one need not suffer the decrease in message-handling capability so often observed in conventional systems when additional paths appear. By choosing the right code, such destructive interference may apparently be avoided.

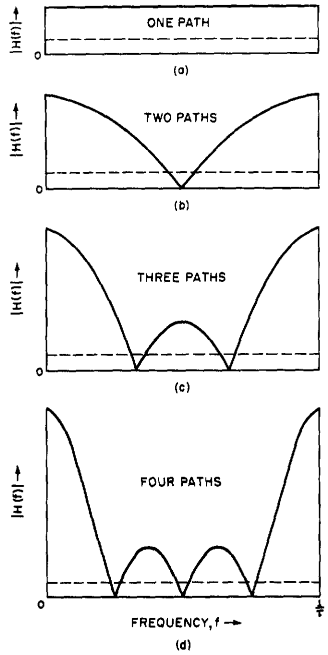

Fig. 11. Frequency response for multiple delays; path structure is that shown in Fig. 10.

During the consideration of this topic, it has been implicitly assumed that the receiver has knowledge of the fixed multipath characteristic so that it can apply correction to its stored x_i, thus in effect reducing the channel to simply an additive noise perturbation. This assumption will be re-examined in a later section.

2. The Optimum Transmitted Spectrum

If it is assumed that the transmitter as well as the receiver has knowledge of the multipath characteristic, further improvement in the capacity, for a fixed average transmitter power, can be obtained by allowing the transmission to depart from uniform spectral density. The maximum, or channel, capacity for a fixed average power is achieved by a variational technique similar to that used by Shannon to find the best distribution of power for a non-white channel noise. Details are given in Appendix II, Sec. B. Given the noise density N(f), the multipath response H(f), and the average transmitter power P, restricted to band W, we arrange $S_o(f)$ so that either:

$$\frac{N(f)}{|H(f)|^2} + S_o(f) = K, \text{ a constant} ,$$

or

$$S_o(f) = 0, \text{ for } \frac{N(f)}{|H(f)|^2} > K \quad . \tag{2-6}$$

A graphical interpretation of Eq.(2-6), following Shannon,[20] appears in Fig. 12. The power is "poured" into the profile until the entire P is exhausted, keeping the level of the "liquid" constant.

3. Probability of Error for Gaussian Signals and Gaussian Noise of Arbitrary Spectra

The preceding results have dealt solely with channel capacity, and are useful in establishing upper bounds to the rate at which information may be transmitted with vanishing probability of error. It is of interest to examine the case of Gaussian symbols in Gaussian noise to see how this upper bound can actually be approached, as indicated by Shannon's fundamental theorem.

For the case in which the signal spectrum S(f) and the noise spectrum N(f) are uniform, Eq.(2-2) reduces to Eq.(1-4), and proofs that the channel capacity can be achieved (arbitrarily closely) have been supplied rigorously by Rice[21] and more intuitively by Shannon.[20] The work given here constitutes an extension of their results.

In the more general case, let the band-limited stochastic source, from which the transmitter symbols of length T are drawn, and the channel noise source both be Gaussian with step-function power spectra, as shown in Fig. 13. The spectra can then be broken up into a finite number of sections n, in each of which the signal and noise spectra remain constant, as indicated.

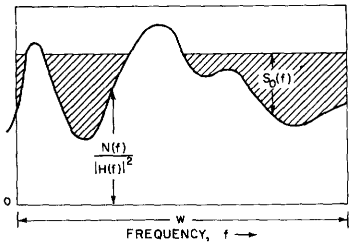

Fig. 12. Optimization of $S_0(f)$ to achieve channel capacity.

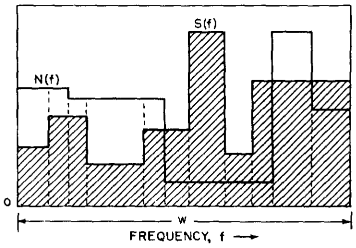

Fig. 13. Generalized Spectral distributions.

Let S_i, N_i, and W_i be the signal and noise spectral densities, and bandwidth, respectively, of the i^{th} section. The capacity is then, from Eq.(2-2),

$$C = \sum_{i=1}^{n} C_i$$

where

$$C_i = W_i \log \left(1 + \frac{S_i}{N_i}\right) \tag{2-7}$$

Suppose it is desired to transmit at a rate $C - \delta$, where $0 < \delta < C$. We shall do this by transmitting independent information in each spectral section. K_1 possible symbols will be transmitted in the first section, K_2 in the second, and so on. Let the K_i be such that

$$K_i = \eta_i(T) \exp \left[C_i - \frac{\delta}{n}\right]T \tag{2-8}$$

and $\eta_i(T)$ is chosen so that K_i is the closest integer to $\exp\left[C_i - \frac{\delta}{n}\right]T$:

$$\frac{\exp\left[C_i - \frac{\delta}{n}\right]T - 1}{\exp\left[C_i - \frac{\delta}{n}\right]T} < \eta_i(T) < \frac{\exp\left[C_i - \frac{\delta}{n}\right]T + 1}{\exp\left[C_i - \frac{\delta}{n}\right]T} \quad . \tag{2-9}$$

Then if $K = \prod_{i=1}^{n} K_i$ possible symbols are transmitted in the whole band, the desired rate is achieved in the limit $T \to \infty$.

$$\lim_{T\to\infty} \frac{\log K}{T} = \Sigma \left(C_i - \frac{\delta}{n}\right) + \lim_{T\to\infty} \frac{\sum \log \eta_i(T)}{T} = C - \delta \quad .$$

Assuming that the information to be transmitted is in the form of one of K equally-likely events, so that the coding is one-to-one, the following scheme may be used: Divide K into K_1 equally-likely groups labeled $\alpha_1, \alpha_2, \ldots \alpha_{K_1}$, divide each of the K_1 groups into K_2 equally-likely subgroups labeled $\beta_1, \beta_2, \ldots, \beta_{K_2}$, and so on until the K_n division is achieved.

Then each member will have been resolved and denoted by a sequence $\alpha_i\beta_j \ldots \omega_v$. Draw K_1 symbols at random from a white Gaussian source representing the first spectral section, K_2 from one representing the second, and so on. These symbols are then assigned to the α_i, $\beta_j, \ldots \omega_v$, respectively, so that to each of the K events there corresponds a set of n symbols occupying adjacent parts of the spectrum. To generate the symbol to be transmitted for any event, its component set is merely added together, so that a symbol is obtained which is drawn from a Gaussian source with the power spectrum given originally.

At the receiver, each spectral zone is observed separately with an ideal receiver to recover the α_i, β_j, $\ldots \omega_v$ which specify the transmitted event. What is the same, since the α_i, β_j, $\ldots \omega_v$ are statistically independent by the completeness of the coding scheme, the entire spectrum may as well be observed directly with an appropriate receiver. In any event, the probability of error p(e) is given by

$$p(e) = 1 - [1 - p_1(e)]\,[1 - p_2(e)] \ldots [1 - p_n(e)] \leq p_1(e) + p_2(e) + \ldots + p_n(e) \qquad (2\text{-}10)$$

where $p_1(e)$ is the probability that α_i will be in error, $p_2(e)$ is that for β_j, etc. But since the transmission in each spectral section considered separately is at a rate less than the channel capacity for that section, and uses white Gaussian noise symbols in white Gaussian channel noise, the $p_i(e)$ can be made to vanish, on the average over the random coding used, for sufficiently great T. Thus for non-white Gaussian signal and noise of the spectral forms considered, the fundamental theorem holds true. It should extend also to smooth spectra, but the second limiting operation required has not been studied.

The coding devised to obtain the entire transmitted symbol, comprising the sum of the transmitted symbols in each spectral section, differs somewhat from the completely random coding studied by Shannon. Although the codes for the separate sections are drawn at random, a large family of complete transmitted symbols may share an identical symbol in any one spectral zone. Nevertheless, the theory of random coding developed by Shannon has made the preceding analysis possible.

C. Quasi-Stationary Multipath

There is a serious question whether the assumption that the receiver has knowledge of the ionospheric characteristic, as implicitly assumed in the last section, is realistic. Certainly if the multipath is fixed, it takes a vanishingly small amount of channel capacity to convey this information to the receiver. When the paths vary, however, as is usually the case, it seems appropriate to build into the transmitted symbols a portion which is known completely to the receiver ahead of time. From observation of this portion at the receiver, correction can then be made on the stored symbols x_i in accordance with the state of the ionosphere. While there is by no means any indication that this would optimize the transmitted symbols, such a technique enables the previous analysis to carry over into a quasi-stationary channel, in which path fluctuations are slow compared to the bandwidths of the transmitted symbols. Of course, the state of the ionosphere cannot be conveyed exactly, since there are always perturbations caused by channel noise. The effects of random errors of this sort have not yet been analyzed.

1. The Transmission of Frequency Groups as an Ionospheric Tracker

One possible method of constructing this redundant portion is suggested here. If a group of cosine waves, equally spaced in frequency starting from zero, and of equal amplitudes is summed, the resulting waveform has certain interesting properties. Let the sum be denoted:

$$S_n(t) = \frac{1}{2} \sum_{k=-n}^{k=n} \exp[-jkt] = \frac{1}{2} + \cos t + \cos 2t + \ldots + \cos nt \qquad (2\text{-}11)$$

where the angular frequency spacing has arbitrarily been normalized to unity. It is found that a trigonometric identity is[22]

$$S_n(t) = \frac{\sin (2n+1)\frac{t}{2}}{2 \sin \frac{t}{2}} \quad . \qquad (2\text{-}12)$$

Figure 14 presents the waveform and frequency analysis of $S_n(t)$ for n = 5. It is seen that $S_n(t)$ is similar to the familiar sin x/x function, in that its zeros are evenly spaced, and has certain orthogonal properties in which we are interested.

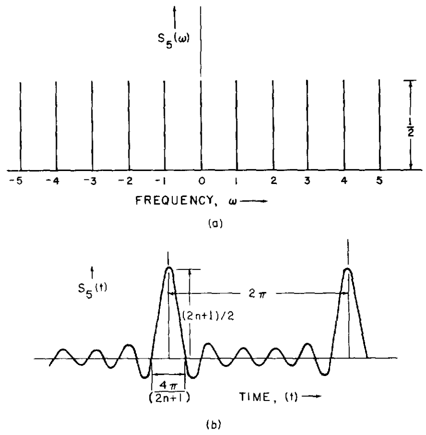

Fig. 14. Time and frequency representations of $S_5(t)$ (from Guillemin).

Suppose this frequency group is sent as a test signal through an arbitrary linear physical filter. We may synthesize a nearly-identical filter from observation of the output, which filter is then the one used at the receiver to correct the stored x_i. Synthesis is accomplished using a delay line with taps brought out in intervals of $(2\pi/2n + 1)$, coinciding with the sampling times. The output from each tap is made proportional to the value of the sample observed at the corresponding time, and all outputs are added together. Then, if the frequency group is sent through each filter, the output waveforms will be identical, except perhaps in scale (see Fig. 15). A simple proof may be stated as follows. When the $S_n(t)$ waveform is applied to the input of the delay-line filter, the outputs at the sample times have the same relative amplitude as the corresponding outputs from the original filter. This results from the coincidence of the delay-line taps with the evenly-spaced zeros of $S_n(t)$. But the coincidence of the output waveforms at these selected points guarantees the identity everywhere. For the output of the delay-line filter is given completely in terms of the amplitudes and phases of the (n + 1) frequencies present in the given input, resulting therefore in (2n + 1) degrees of freedom for both the time and frequency representations. Since the output waveforms coincide at the (2n + 1) distinct sampling points, they must be identical everywhere.

For input waveforms other than that of the given frequency group, the outputs of the given and synthesized filters may not coincide, since the matching was performed only at certain discrete frequencies. By spacing the components closer together and increasing n, the frequency response for all frequencies as well as just those of $S_n(t)$ will generally come into coincidence.

Figure 16 shows the agreement with a given filter of pure delay, for n = 1 and n = 2. In the limit, the technique carries over directly into the generation of an arbitrary band-limited waveform by using 2TW samples in conjunction with sin x/x pulses.

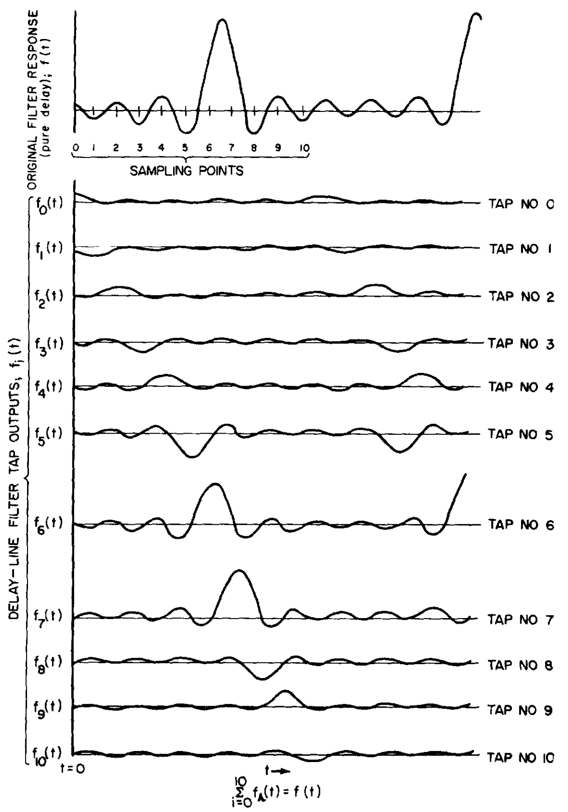

Fig. 15. Delay-line synthesis for pure-delay filter.

Of course, delay-line synthesis of an arbitrary physical filter is not new, but has usually been accomplished by synthesizing to the impulse response of the given filter. Here, however, is a simple method of correcting just within the frequency band of interest, using a test signal which is confined to the band. If the band is band-pass, it must be heterodyned to low-pass in order for sampling to be used.

2. Experimental Work on Ionospheric Fluctuations

In order to implement the preceding suggestion of transmitting a frequency group and isolating it for sampling at the receiver, many practical questions arise. There are, for example, the problems of specifying the optimum number of frequencies to use and their power relative to that of the message, and the bandwidth of the "comb" filter used at the receiver to isolate them. It is necessary in this connection to know the degree of spreading of the transmitted spectral lines produced by ionospheric fluctuation. More generally, measurements of the dynamic performance of the ionosphere are of interest in many other applications, and also provide data for ionospheric theory. To this end an experiment has been devised to extend our knowledge of ionospheric fluctuations. At present, the only autocorrelation studies which have been made on ionospheric fluctuation have been performed on the envelopes of received signals.[19,23] Even some of these experiments cannot resolve fluctuations faster than about 1 cps. The present experiment has attempted to measure the carrier directly, with a bandwidth of 4 cps either side of the carrier.

The equipment which has been constructed is shown in Fig. 17. A more detailed description will be found in Appendix IV. The transmitted sine wave is the 5 mcps transmission of WWV in Beltsville, Md. Locally, it is heterodyned with a 5.000020 mcps wave generated from a General Radio primary frequency standard. The fluctuating 20 cycle wave thus obtained contains all the phase and amplitude information of the received carrier from WWV, and analysis of its spectrum is equivalent to performing an analysis at 5 mcps directly. Provision is made for fluctuations up to 4 cps by extending the bandwidth of the filter following the multiplier from 16 to 24 cps. The Air Force Cambridge Research Center low frequency correlator (LOCO) is

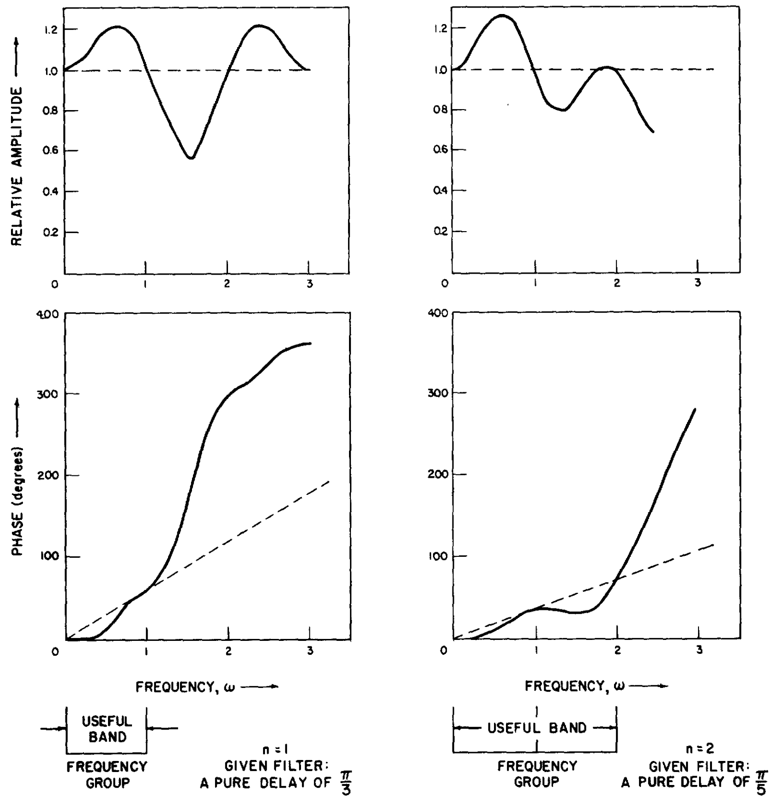

Fig. 16. Comparison of given delay filter with synthesized delay-line filter. (Broken lines indicate the response of the given delay filter.)

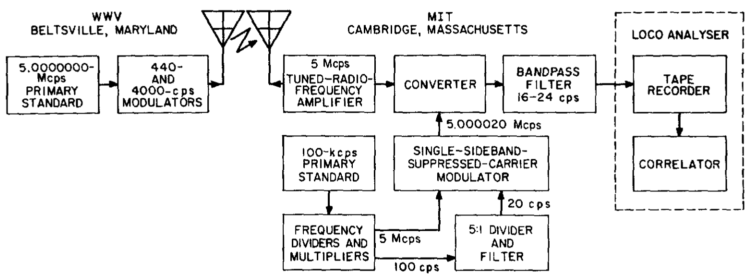

Fig. 17. Measurement of ionospheric fluctuations.

used to find the autocorrelation function of the 20 cycle note, and thus its power spectrum. Two pen recordings of the 20 cycle note are shown in Fig. 18. A portion of the autocorrelation function is presented in Fig. 19, and a plot of the envelope of the autocorrelation function is given in Fig. 20. Assuming that the spectrum is symmetric, as indicated by the uniform zero crossings of the autocorrelation function, the transform of the envelope gives the power spectrum, presented in Fig. 21. Experimental inaccuracies probably account for the slightly negative regions. It is seen also that there is a strong specular component present. It is felt that this is more likely due to pickup of a harmonic of the primary standard at the antenna directly, than to pure specular reflection in the ionosphere.

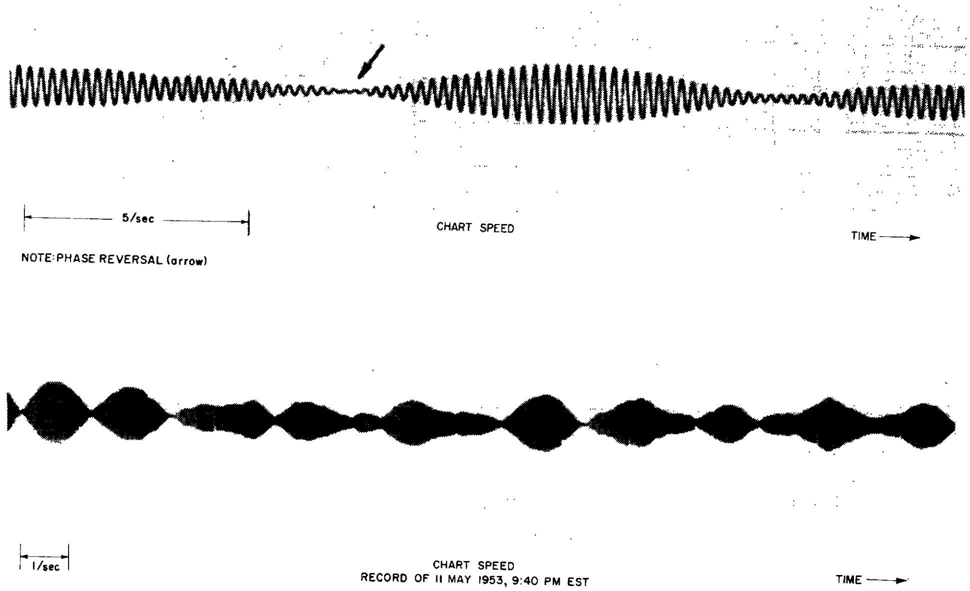

Fig. 18. Pen recordings of WWV fluctuations.

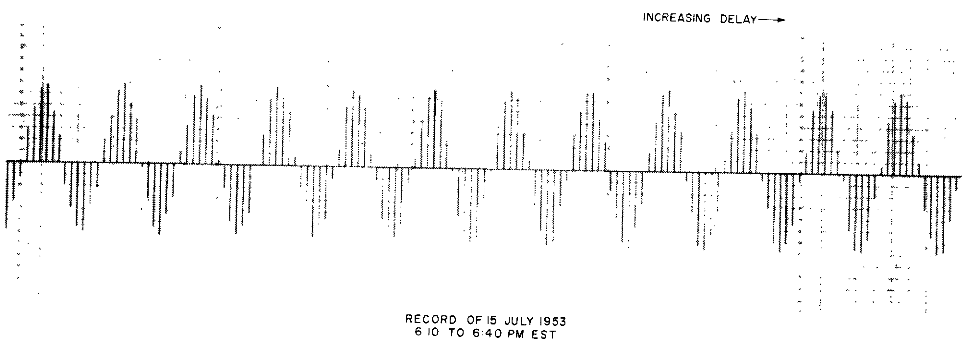

Fig. 19. Portion of measured autocorrelation function. This section includes delays between approximately 0.06 and 0.75 seconds; the incremental delay is 0.004 sec. Approximately seven hours were taken by LOCO to obtain this portion.

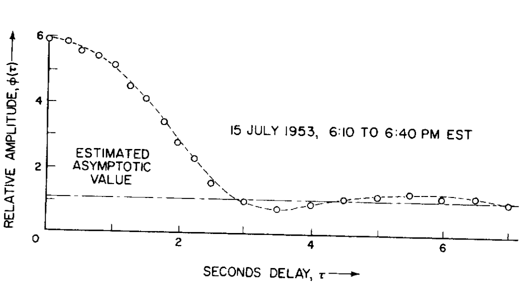

Fig. 20. Envelope of the autocorrelation function.

D. System Capacity for Channels Having Random Linear Filters

We now consider a channel which consists of a linear filter which is varying in a stationary-random manner, followed by additive white Gaussian channel noise, as shown in Fig. 22. Zadeh's work on time varying linear systems has led to the specification of such filters by their impulse response $h(t;\ \tau)$, which is the response, at time t, to a unit impulse applied at time τ. The frequency response $H(\omega;\ t)$ is defined as:

$$H(\omega;\ t) = \int_{-\infty}^{+\infty} h(t;\ \tau) \exp\left[-j\omega(t-\tau)\right] d\tau \quad . \tag{2-13}$$

It has been possible, with the detailed analysis presented in Appendix II, to obtain system capacities for arbitrary spectra $S_x(\omega)$ transmitted through channels of the form of Fig. 22. Using Zadeh's analysis, we find the portion of the input signal which appears unperturbed in the filter output. The remainder of the output signal, also found by Zadeh's method, appears as noise to an observer who has a priori knowledge of the input, or transmitted, signal. Introduction of additive white Gaussian channel noise, of band $2\pi W$ and spectral density N_o, permits us to invoke Shannon's entropy-absorption theorem, providing $S_x(\omega) << N_o$. A general expression for the system capacity C is then obtained, which is applied, with certain restrictions, to multiple-mode models of the ionospheric channel.

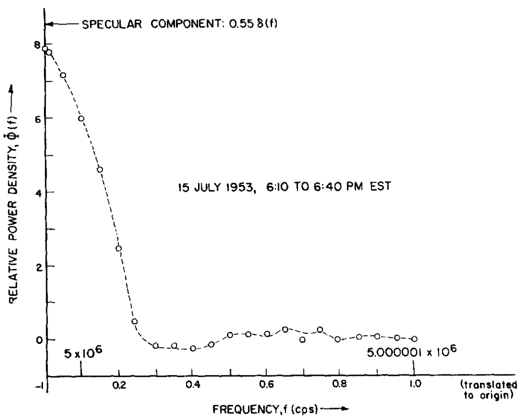

Fig. 21. Spectrum of WWV carrier, showing upper side of the symmetric spectrum.

From Eq.(A2-24), we find that the general expression for the capacity of the system shown in Fig. 22 is:

$$C \approx \frac{W \log_2 e \int_{-\infty}^{+\infty} S_x(\omega)\, \left|\overline{H(\omega)}\right|^2 d\omega}{2\pi W N_o + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Delta'(\omega';\ \omega)\, S_x(\omega')\, d\omega'\, d\omega} \tag{2-14}$$

where

$$\Delta'(\omega';\ \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\{\overline{h(t;\ \tau)h(t+\sigma;\ \tau')} \exp\left[j\omega\sigma\right] - \overline{h(t;\ \tau)h(t;\ \tau')}\right\} \times \exp\left[j\omega(\tau-\tau') + j\sigma(\omega'-\omega)\right] d\tau\, d\tau'\, d\sigma \tag{2-15}$$

and

$$\overline{H(\omega)} = \int_{-\infty}^{+\infty} \overline{h(t;\ \tau)} \exp\left[-j\omega(t-\tau)\right] d\tau \tag{2-16}$$

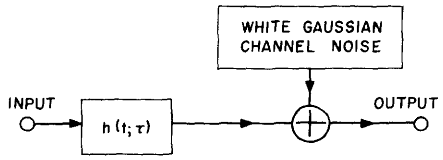

Fig. 22. Channel with randomly varying filter.

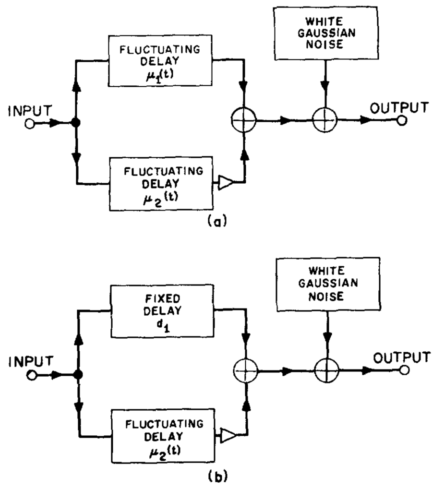

Fig. 23. (a) Two fluctuating paths; (b) one fluctuating path, one fixed path.

All spectral densities are in units of (power/radian per second). The bandwidth of the noise is assumed wider than that of $S_x(\omega)$. The bar denotes averaging over the ensemble of filters.

In Appendix II, Eq.(2-14) is applied to channel filters of the form shown in Fig. 23. In the first model, Fig. 23(a), there are two paths, of strength unity and a, which have mean delays d_1 and d_2, respectively. Each path fluctuates about its mean independently of the other and of the transmitted signal. The probability distributions of the fluctuations are, at least up to the second order, stationary Gaussian and identical. Following the multipath, there is additive white Gaussian channel noise, of spectral density N_o, which fluctuates independently of the transmitted signal and the random filter. The transmission is assumed to occupy a band from $\omega_o - \pi W_s$ to $\omega_o + \pi W_s$, wide compared to the delay fluctuations but narrow compared to its center frequency ω_o. Its spectral density $S_x(\omega)$ is assumed to be small compared to N_o. An approximate expression for system capacity C is obtained in Eq.(A2-36) of Appendix II, and is:

$$C \approx \frac{\left\{\exp\left[-\theta_d^2\right]\right\}\left[(1+a^2)P + 2a\int_{\omega_o-\pi W_s}^{\omega_o+\pi W_s} \cos\omega(d_1-d_2)S_x(\omega)d\omega\right]}{2\pi N_o}\log_2 e \quad . \tag{2-17}$$

Here

$$P = \int_{\omega_o-\pi W_s}^{\omega_o+\pi W_s} S_x(\omega)\, d\omega$$

is the transmitted power, and $\theta_d^2 = \omega_o^2 \phi_\mu(0)$ is the variance in phase of the carrier, due to delay fluctuations. For uniform $S_x(\omega) = S_o$, and $W_s|d_1 - d_2| >> 1$, Eq.(2-17) reduces to

$$C \approx \frac{W_s S_o}{N_o} \exp\left[-\theta_d^2\right](1+a^2)\log_2 e \quad . \tag{2-18}$$

The limits are:

$$C \rightarrow \begin{cases} 0 & \text{as } \theta_d^2 \rightarrow \infty \\ \dfrac{W_s S_o}{N_o}(1+a^2)\log_2 e & \text{as } \theta_d^2 \rightarrow 0 \end{cases} .$$

The limit for $\theta_d^2 \rightarrow 0$ agrees with Eq.(2-4) for small $R = S_o/N_o$.

Under the same assumptions, except that the path of strength unity is now fixed [Fig. 23(b)], we obtain, from Eq.(A2-37) of Appendix II,

$$C \approx \frac{W_s S_o}{N_o} \left(1 + a^2 \exp\left[-\theta_d^2\right]\right) \log_2 e \quad . \tag{2-19}$$

Thus

$$C \rightarrow \begin{cases} \frac{W_s S_o}{N_o} \log_2 e & \text{as } \theta_d^2 \rightarrow \infty \\ \frac{W_s S_o}{N_o} (1 + a^2) \log_2 e & \text{as } \theta_d^2 \rightarrow 0 \end{cases} \quad .$$

Here the fixed path still provides facility for communication even when the other path has become completely incoherent, and contributes only noise. Since low signal-to-noise ratio was assumed originally, this added noise is vanishingly small compared to the channel noise, and does not appear in Eq.(2-19).

CHAPTER III

I. SINGLE-MODE CHANNELS

A. Scatter as a Complex Gaussian Multiplicative Process

In analyzing the effect of scatter on a communication system, we use the physical model considered by Rice.[24] Here there are a large number of irregularities, all of equal size, moving at random. Thus, if a sine wave is transmitted, the receiver observes the sum of a great number of small contributions, represented by the vector addition of many small vectors which are equal in length but random in direction. This is the so-called "random-walk" problem in two dimensions which has been studied by Rayleigh.[25] He showed that in the limit of a great number of vectors, the projection of the resultant on any given axis has a Gaussian distribution of probability across the ensemble of possible random walks, or, in this case, scattering configurations. More generally, this changing projection, representing the received signal, is Gaussian in all distributions as the scattering centers move in time, through the central limit theorem. The received signal, then, is statistically identical to filtered thermal noise with the same spectrum.

Let us assume that the signal $z_o(t)$ received when a sine wave $x_o(t) = \sin \omega_o t$ is transmitted is narrow-banded and has a symmetric spectrum $Y(\omega)$ as present theory indicates. Then, according to Rice,[26] $z_o(t)$ can be written

$$z_o(t) = y_s(t) \sin \omega_o t + y_c(t) \cos \omega_o t \quad , \tag{3-1}$$

where $y_s(t)$ and $y_c(t)$ are independent Gaussian waveforms, both with autocorrelation $\phi(\tau)$.

$$\phi(\tau) = \int_o^{\infty} Y(\omega) \cos (\omega - \omega_o)\tau \, d\omega \quad . \tag{3-2}$$

Suppose, now, that instead of a sine wave, an actual information-bearing waveform is transmitted. Assuming that the signal $x(t)$ is narrow-band with respect to its center frequency ω_o, it can be written

$$x(t) = x_s(t) \sin \omega_o t + x_c(t) \cos \omega_o t \quad . \tag{3-3}$$

Then the signal $z(t)$ observed following scatter would be

$$z(t) = z_s(t) \sin \omega_o t + z_c(t) \cos \omega_o t \quad , \tag{3-4}$$

where

$$z_s(t) = x_s(t)\, y_s(t) - x_c(t)\, y_c(t) \quad , \tag{3-5}$$

and

$$z_c(t) = x_s(t)\, y_c(t) + x_c(t)\, y_s(t) \quad . \tag{3-6}$$

Formally, some of the arguments in Eq.(3-5) and Eq.(3-6) should read $(t - \pi/2\omega_o)$, but the indicated approximation is good so long as $x(t)$ remains narrow-band. Thus, by Eq.(3-5) and Eq.(3-6), scatter may be pictured as a complex multiplicative process.

Following scatter, there is additive channel noise n(t) to be considered. Assuming that it is Gaussian and limited to a symmetric band B which is wide compared to that of z(t) but still narrow compared to its center ω_o, we may write:

$$n(t) = n_s(t) \sin \omega_o t + n_c(t) \cos \omega_o t \quad , \tag{3-7}$$

where again $n_s(t)$ and $n_c(t)$ are independent Gaussian waveforms, both with autocorrelation

$$\phi_n(\tau) = \int_0^{\infty} N(\omega) \cos (\omega - \omega_o)\tau \, d\omega \quad , \tag{3-8}$$

where $N(\omega)$ is the power spectrum of the noise. It is assumed that the additive channel noise is independent of the scatter. Combining scatter and noise, the signal w(t) finally observed at the receiver is

$$w(t) = w_s(t) \sin \omega_o t + w_c(t) \cos \omega_o t \quad , \tag{3-9}$$

where

$$w_s(t) = x_s(t)\, y_s(t) - x_c(t)\, y_c(t) + n_s(t) \quad , \tag{3-10}$$

and

$$w_c(t) = x_s(t)\, y_c(t) + x_c(t)\, y_s(t) + n_c(t) \quad . \tag{3-11}$$

Since $y_s(t)$, $y_c(t)$, $n_s(t)$, and $n_c(t)$ are independent Gaussian waveforms, $w_s(t)$ and $w_c(t)$ share a joint Gaussian distribution, assuming that x(t) is known to the observer of w(t).

1. The Inverse-Probability-Computing Receiver

We now have the channel model illustrated in Fig. 24. Assuming that the transmitter is specified as well, it is possible to determine the operation of the ideal, probability-computing receiver to use with this system. We shall show that the appropriate receiver performs matrix operations on the received signal w(t). The parameters of the matrices are determined by the various possible transmitted symbols available in storage at the receiver. Let there be a finite number of possible transmitted symbols $x^{(k)}(t)$, k = 1, 2, ..., M, selected independently, as indicated in Fig. 1. In actually computing the probabilities $P[x^{(k)}(t)/w(t)]$, it is usually convenient to use Baye's Theorem on Inverse Probability, which is really an identity.

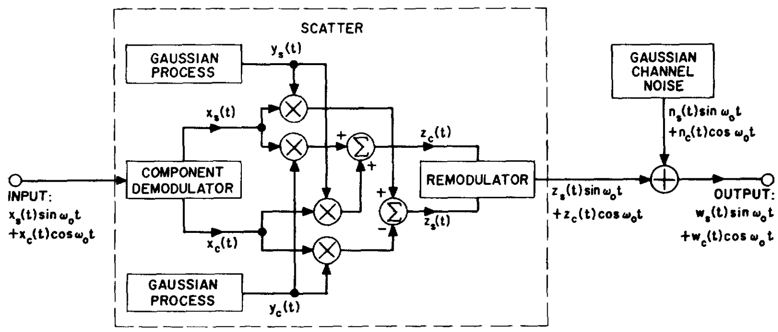

Fig. 24. Channel model for scatter propagation.

$$P[x^{(k)}(t)/w(t)] = \frac{p[w(t)/x^{(k)}(t)]\, P[x^{(k)}(t)]}{p[w(t)]} \quad . \tag{3-12}$$

There has occasionally been criticism of the use of this expression, but the theorem is fully justified when the a priori $P[x^{(k)}(t)]$ are known, as postulated in the model. We shall assume in the following analysis that the $P[x^{(k)}(t)]$ are all equal, although this work can be generalized without difficulty. Then, we have, from Eq.(3-12),

$$P[x^{(k)}(t)/w(t)] = K(w)\; p[w(t)/x^{(k)}(t)] \quad , \tag{3-13}$$

where $K(w)$ is constant for all K, with any given $w(t)$.

The computation of

$$p[w(t)/x^{(k)}(t)] = p[w_s(t), w_c(t)/x^{(k)}(t)]$$

is generally simplified if the $w_s(t)$ and $w_c(t)$ are conditionally independent, so that

$$p[w(t)/x^{(k)}(t)] = p[w_s(t)/x^{(k)}(t)]\; p[w_c(t)/x^{(k)}(t)] \quad . \tag{3-14}$$

We find that in general

$$\overline{w_s(t_1)w_c(t_2)}\Big|_{x^{(k)}(t)} = [x_s^{(k)}(t_1)\, x_c^{(k)}(t_2) - x_s^{(k)}(t_2)\, x_c^{(k)}(t_1)]\, \phi(t_1 - t_2) \neq 0 \quad , \tag{3-15}$$

where ϕ is given by Eq.(3-2). In the special case in which

$$\frac{x_c^{(k)}(t)}{x_s^{(k)}(t)} = c, \text{ a constant} \quad , \tag{3-16}$$

it happens that

$$\overline{w_s(t_1)w_c(t_2)}\Big|_{x^{(k)}(t)} = 0 \quad ,$$

so that then $w_s(t)$ and $w_c(t)$ are conditionally independent, since they have a joint Gaussian conditional distribution, as stated earlier. The particular transmission represented by Eq.(3-16) is amplitude-modulation, for then

$$x(t) = x_s(t) \sin \omega_o t + c x_s(t) \cos \omega_o t = x_s(t) \sqrt{1 + c^2} \cos (\omega_o t + \theta) \quad . \tag{3-17}$$

When more general transmissions are used, however, $w_s(t)$ and $w_c(t)$ may be conditionally dependent. We then seek two new variables $f(t)$ and $g(t)$ through a linear transformation of $w_s(t)$ and $w_c(t)$ so that

$$\overline{f(t_1)g(t_2)}\Big|_{x^{(k)}(t)} = 0 \quad .$$

It has been found that such a transformation must depend on $x^{(k)}(t)$ itself, so that a new transformation must be made for each new k, in finding the $p[w(t)/x^{(k)}(t)]$. One such transformation is

$$f(t) = x_s^{(k)}(t)\, w_s(t) + x_c^{(k)}(t)\, w_c(t)$$

$$g(t) = x_s^{(k)}(t)\, w_c(t) - x_c^{(k)}(t)\, w_s(t) \quad , \qquad (3\text{-}18)$$

which gives

$$\overline{f(t_1)g(t_2)}\Big|_{x^{(k)}(t)} = [x_c^{(k)}(t_1)\, x_s^{(k)}(t_2) - x_s^{(k)}(t_1)\, x_c^{(k)}(t_2)]\, \phi_n(t_1 - t_2) \quad . \qquad (3\text{-}19)$$

We now assume that the noise has uniform spectral density in its band B, and that the waveforms f(t) and g(t) are sampled at intervals of 1/B. Then Eq.(3-19) vanishes for t_1 and t_2 at any i^{th} or j^{th} sampling point; $\overline{f_i g_j} = 0$. Thus, under these assumptions, conditional independence is achieved.

More generally, any "rotation" of f(t) and g(t) will still give conditional independence. Let

$$f'(t) = af(t) + bg(t)$$

$$g'(t) = ag(t) - bf(t) \quad , \qquad (3\text{-}20)$$

where $a^2 + b^2 = 1$. Then

$$\overline{f_i' g_j'}\Big|_{x^{(k)}(t)} = 0 \quad .$$

In the preceding, sampling points have been introduced only in the case where a general transmission x(t) is considered. At this point, however, it is convenient to introduce sampling-point analysis for all transmissions, so that $p[w(t)/x^{(k)}(t)]$ may be computed explicitly. Accordingly, a common grid of sampling points will be established at transmitter and receiver. The various time series which have been considered will now be denoted with the subscript i, to indicate sample value at the i^{th} point. In the AM transmission case, the sampling interval Δ may be taken arbitrarily, but in order to preserve conditional independence of the f_i and g_j for the general transmission, we must assume that $\Delta = 1/B$, as shown in the last paragraph. Then, letting u_i and v_j represent the pairs of variables w_{si}, w_{cj}; f_i, g_j; or f_i', g_j', according to the case considered, we have, for all i and j,

$$\overline{u_i u_j}\Big|_{x^{(k)}(t)} = \overline{v_i v_j}\Big|_{x^{(k)}(t)} = m_{ij}$$

$$\overline{u_i v_j}\Big|_{x^{(k)}(t)} = 0 \quad .$$

This definition of m_{ij} is now used in computing probabilities.

We now give detailed expressions which approximate $p[w(t)/x^{(k)}(t)]$ from observation of the waveforms $x^{(k)}(t)$ and w(t) at their sampling points only. Since the u and v are independent,

$$p[u, v/x^{(k)}(t)] = p[u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n/x^{(k)}(t)]$$

$$= p[u/x^{(k)}(t)]\; p[v/x^{(k)}(t)] \qquad (3\text{-}21)$$

$$= (2\pi)^{-n} |M_n|^{-1} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{M_n^{ij}}{|M_n|} (u_i u_j + v_i v_j)\right] \quad ,$$

where M_n^{ij} is the cofactor of m_{ij} in M_n, and $|M_n|$ is the determinant of the matrix M_n:

$$M_n = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{12} & m_{22} & \cdots & m_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ m_{1n} & m_{2n} & \cdots & m_{nn} \end{bmatrix} . \tag{3-22}$$

For symbols T long, n = BT. The values for m_{ij} are as follows.

For amplitude-modulated x(t):

$$m_{ij} = \overline{w_{si}w_{sj}}\Big|_{x^{(k)}(t)} = r_i^{(k)} r_j(k)\, \phi_{ij} + \phi_{nij} \quad , \tag{3-23}$$

where we have defined, from Eq.(3-17), $r_i^{(k)} = \sqrt{1 + c^2}\, x_{si}^{(k)}$. Also

$$\phi_{ij} = \overline{y(t_i)y(t_j)} = \phi[(i - j)\,\Delta]$$

$$\phi_{nij} = \overline{n(t_i)n(t_j)} = \phi_n[(i - j)\,\Delta] \quad .$$

For the transformed variables f_i, g_j and f_i', g_j' in the case of more general transmission,

$$m_{ij} = \overline{f_i f_j}\Big|_{x^{(k)}(t)} = \overline{f_i' f_j'}\Big|_{x^{(k)}(t)} = r_i^{(k)} r_j^{(k)} (r_i^{(k)} r_j^{(k)} \phi_{ij} + N\delta_{ij}) \quad , \tag{3-24}$$

where $[r_i(k)]^2 = [x_{ci}^{(k)}]^2 + [x_{si}^{(k)}]^2$. In this case, it is necessary to assume that $\Delta = 1/B$, B being the bandwidth of the noise, as mentioned earlier. Here $N = BN_o$, where N_o is the noise spectral density. δ_{ij} is the Kronecker δ – function: $\delta_{ii} = 1$; $\delta_{ij} = 0$, $i \neq j$.

Once the appropriate $p[u, v/x^{(k)}(t)]$ have been computed, as indicated, all the $P[x^{(k)}(t)/w(t)]$ are immediately available from Eq.(3-13)

$$P[x^{(k)}(t)/w(t)] = K(w)\; p[u, v/x^{(k)}(t)] \quad . \tag{3-25}$$

Thus we see that the probability-computing receiver performs a series of operations, as illustrated in Fig. 25. The received signal w(t) is first separated into its two components $w_s(t)$ and $w_c(t)$. If a general transmission is used, rather than AM, $w_s(t)$ and $w_c(t)$ must undergo next a preliminary linear transformation which gives f(t) and g(t). Finally, sample values are taken of the $w_s(t)$ and $w_c(t)$, or f(t) and g(t), and matrix operations performed on these samples to obtain the $p[x^{(k)}(t)/w(t)]$. One matrix operation is performed for each stored $x^{(k)}(t)$, the parameters of the matrix being sample values of the $x^{(k)}(t)$.

2. Approximations to the Probability-Computing Receiver at Small Signal-to-Noise Ratios

In general, the inversion of a high-order matrix, such as M_n of Eq.(3-22), is a tedious process. This is essentially the operation required, however, to compute the M_n^{ij} of

Eq.(3-21), and so obtain the parameters necessary to specify the functional operation of the probability-computing receiver. In the limit of $n \to \infty$ for a fixed length of observation T, the sampling points become very dense, and all the statistical information is extracted. Inversion of the matrix M_n then goes over into the solution of an integral equation. While such infinite-order matrices have been successfully inverted, such as those considered by Reich and Swerling,[27] the form of the M_{ij} considered here apparently does not yield a tractable solution. It has been possible, however, to obtain an approximation to the operation of the ideal receiver in the limiting case of small signal-to-noise ratios. The method consists of finding an expansion for M_n^{ij} in which all but the first term become negligible in this case.

Let us write M_n^{ij} as

$$M_n^{ij} = -\prod_{m \neq i,j} m_{mm} \left(m_{ij} - \sum_{h \neq i,j} \frac{m_{ih} m_{jh}}{m_{hh}} + \ldots \right) \quad , \tag{3-26}$$

in which the neglected terms contain successively more of the diagonal terms m_{hh} in the denominator. Examining Eq.(3-23) and Eq.(3-24), we see that if

$$N >> n\, r^2_{max} \left[\frac{\phi_{ih}\phi_{jh}}{\phi_{ij}} \right]_{max} \quad , \tag{3-27}$$

then the summation terms in Eq.(3-26) are negligible in comparison with m_{ij}. Here r^2_{max} is the greatest value of $[r_i^{(k)}]^2$. Now since the observation interval is taken as B, where B is the bandwidth of the noise, $N = BN_o = (n/T)\,N_o$. r^2_{max} is of the order of WS_o, where S_o is the average signal power density in the transmission bandwidth W. Thus we require:

$$\frac{S_o}{N_o} << \frac{1}{TW} \left[\frac{\phi_{ij}}{\phi_{ih}\phi_{jh}} \right]_{min} \quad . \tag{3-28}$$

If this is satisfied, then we may write

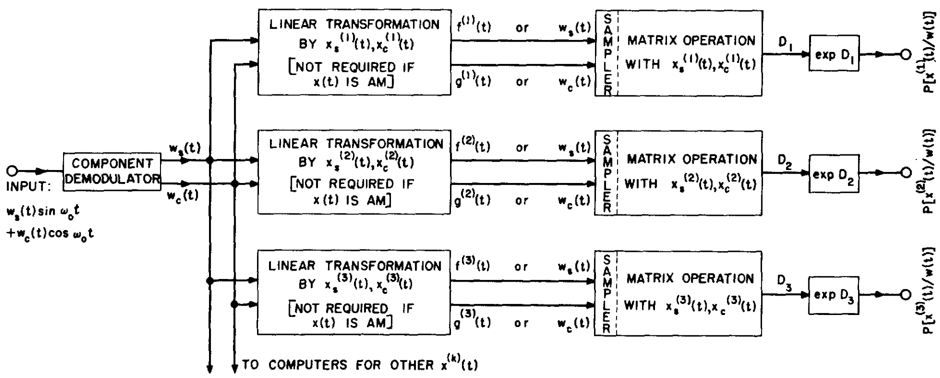

Fig. 25. Probability-computing receiver for scatter channel.

$$p(u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n/x_1, x_2, \ldots x_n) \approx (2\pi)^{-n} |M_n|^{-1}$$

$$\times \exp\left[-\frac{\pi m_{kk}}{2|M_n|} \Sigma\Sigma \frac{m_{ij}(u_iu_j + v_iv_j)}{m_{ii}m_{jj}}\right] , \qquad (3\text{-}29)$$

since $N >> r^2_{max}$, $m_{ii} \approx N$, and thus the receiver is reduced to computing

$$q(k) = \sum_1^n \sum_1^n m_{ij}^{(k)}(u_iu_j + v_iv_j) \quad , \qquad (3\text{-}30)$$

for each k, corresponding to the various possible transmitted symbols. $P[x^{(k)}(t)/u(t), v(t)]$ is a monotonic function of q(k). In the limit of infinite n for finite T, Eq.(3-30) goes over into an integral form Q(k). When amplitude modulation is transmitted,

$$Q_a(k) = \int_o^T\int_o^T \phi(\tau_1 - \tau_2)\, r(\tau_1)\, r(\tau_2)\, [w_s(\tau_1)w_s(\tau_2) + w_c(\tau_1)w_c(\tau_2)]\, d\tau_1\, d\tau_2 \quad , \qquad (3\text{-}31)$$

and for the more general transmission,

$$Q(k) = \int_o^T\int_o^T \phi(\tau_1 - \tau_2)\, r^2(\tau_1)\, r^2(\tau_2)\, [f(\tau_1)f(\tau_2) + g(\tau_1)g(\tau_2)]\, d\tau_1\, d\tau_2 \quad , \qquad (3\text{-}32)$$

where f and g may be primed or unprimed. If $\phi = 1$, that is, vanishingly small change in the scatter during the symbol length, we obtain:

$$Q_a(k) = \left[\int_o^T r^{(k)}(\tau_1)w_s(\tau_1)d\tau_1\right]^2 + \left[\int_o^T r^{(k)}(\tau_1)w_c(\tau_1)d\tau_1\right]^2 \quad , \qquad (3\text{-}33)$$

$$Q(k) = \left[\int_o^T r^{2(k)}(\tau_1)f(\tau_1)d\tau_1\right]^2 + \left[\int_o^T r^{2(k)}(\tau_1)g(\tau_1)d\tau_1\right]^2 \quad . \qquad (3\text{-}34)$$

In terms of physical equipment, the integrating operations indicated in Eq.(3-33) and Eq.(3-34) may be carried out by linear filters with impulse responses $h(t) = r^{(k)}(T - t)$ or $r^{2(k)}(T - t)$, into which $w_s(t)$, $w_c(t)$, or f(t), g(t) are fed, respectively. More generally, it is possible to reduce the double integrations of Eq.(3-31) and Eq.(3-32) to operations involving linear filters. We invoke the fact that if F(s, t) is a symmetric function, that is, F(s, t) = F(t, s), in the range $0 \leq t \leq T$ and $0 \leq s \leq T$, then

$$\int_o^T\int_o^T F(s,t)\, ds\, dt = 2\int_o^T\int_o^s F(s,t)\, dt\, ds \quad . \qquad (3\text{-}35)$$

The integrands of $Q_a(k)$ and Q(k) satisfy the conditions, since $\phi(t - s) = \phi(s - t)$.
Thus

$$\frac{Q_a(k)}{2} = \int_o^T r^{(k)}(\tau_1)\, w_s(\tau_1) \int_o^{\tau_1} \phi(\tau_1 - \tau_2)\, r^{(k)}(\tau_2)\, w_s(\tau_2)\, d\tau_2\, d\tau_1$$

$$+ \int_o^T r^{(k)}(\tau_1)\, w_c(\tau_1) \int_o^{\tau_1} \phi(\tau_1 - \tau_2)\, r^{(k)}(\tau_2)\, w_c(\tau_2)\, d\tau_2\, d\tau_1 \quad , \qquad (3\text{-}36)$$

$$\frac{Q(k)}{2} = \int_0^T r^{2(k)}(\tau_1)\, f(\tau_1) \int_0^{\tau_1} \phi(\tau_1 - \tau_2)\, r^{2(k)}(\tau_2)\, f(\tau_2)\, d\tau_2\, d\tau_1$$
$$+ \int_0^T r^{2(k)}(\tau_1)\, g(\tau_1) \int_0^{\tau_1} \phi(\tau_1 - \tau_2)\, r^{2(k)}(\tau_2)\, g(\tau_2)\, d\tau_2\, d\tau_1 \quad . \tag{3-37}$$

To evaluate one of the integrals of Eq.(3-36) or Eq.(3-37) the following technique might be used. Let us assume that it is the first integral of Eq.(3-36) which is to be found. We multiply the received $w_s(\tau_1)$ by the stored $r^{(k)}(\tau_1)$ and pass the product to one of the inputs of a second multiplier. The other input is this same product after it has been passed through a filter with response $h(t) = \phi(t),\ t \geq 0$. The output from the second multiplier is fed into an integrating filter with a step-function response. Sampling the output of this second filter at time T then gives the required integral. Both filters must be discharged before the integration for the next symbol is performed. A block diagram of the operation is given in Fig. 26.

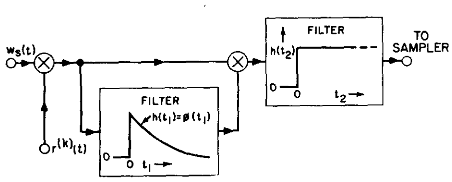

Fig. 26. Element of approximate probability-computing receiver.

It is also of interest to see how f(t) and g(t) might be generated from $W_s(t)$ and $W_c(t)$ by physical means. Examination of Eq.(3-18) reveals that if the stored $x^{(k)}(t)$, in bandpass form, is multiplied by the received signal w(t) directly, and the double carrier-frequency terms are filtered out, $f^{(k)}(t)$ is obtained. Similarly, multiplying w(t) by the Hilbert transform $\tilde{x}^{(k)}(t)$ of x(t), and filtering gives $g^{(k)}(t)$. $\tilde{x}^{(k)}(t)$ is obtained by passing x(t) through a ninety-degree phase shift network at the carrier frequency.

$$\tilde{x}^{(k)}(t) = x_s(t) \cos \omega_o t - x_c(t) \sin \omega_o t \quad . \tag{3-38}$$

The transformation Eq.(3-20) is obtained if the signal x(t) is stored with a carrier phase shift.

3. Lower Bounds to System Capacity for Scatter Transmission of a Binary-Modulated Carrier

Before dealing with a particular transmission and channel, it is useful to consider a general approach to establishing a lower bound for system capacity. Let us assume that the transmitter generates stochastically a time series x(t) which is completely determined by the specification of its parameters at points equally spaced Δ apart in time. Likewise, let the receiver observe the time series w(t), resulting from the transmission of x(t) through the channel, but at another series of equally-spaced sampling points, not necessarily with the same interval as for x(t) Also, since this will be the case later, assume that the parameters of x(t) can take on only discrete values, while the sampling-point measurements of w(t) may arise from a continuum. Then, providing that all random processes involved in the transmitter and channel are stationary, the system capacity C^S is given by:

$$C^S = \lim_{n\to\infty, L\to\infty} {}_{n,L}C^S \tag{3-39}$$

where the information $\Delta_{n,L} C^S$ in the interval Δ is given by:

$$\Delta_{n,L}C^{s} = {}_{n}H(x) - {}_{n,L}H_{w}(x) \quad , \tag{3-40}$$

and

$$ {}_{n}H(x) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1x_2\ldots x_n)\log P(x_1/x_2\ldots x_n) \quad , \tag{3-41}$$

$$ {}_{n,L}H_{w}(x) = -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n}\int_{w_I}\int_{w_{II}}\int_{w_L}\begin{bmatrix}P(x_1x_2\ldots x_n/w_Iw_{II}\ldots w_L)\\ \times\ p(w_Iw_{II}\ldots w_L)\end{bmatrix} \times \log P(x_1/x_2\ldots x_n, w_Iw_{II}\ldots w_L)\,dw_I\,dw_{II}\ldots dw_L \quad . \tag{3-42}$$

The superscript s denotes that a sample-point analysis is being performed. The points 1 and I represent the same instant of time, which may be chosen arbitrarily because of the assumption of stationarity. Increasing subscripts denote sample points occurring at successively earlier times with respect to 1 and I. If there is more than one parameter at a sampling point, $x_1 \ldots x_n$ and $w_I \ldots w_L$ may be taken to be vectors, the coordinates being the parameters.

Now it has been shown that as n and L are allowed to increase, that is, the duration of observation grows, ${}_{n,L}H_w(x)$ either remains constant or decreases.[28] Thus if the x_i are generated independently, so that ${}_{n}H(x) = H(x)$, a constant independent of n, any ${}_{n,L}C^s$ is a lower bound to C^s. If, furthermore, a new system is studied in which the sampling points at the receiver are made more dense, all other features remaining the same, C^s is a lower bound to the new capacity. Carrying the receiver to the limit of vanishingly small sampling intervals, we obtain the ultimate capacity C for the given transmitter and channel. Thus ${}_{n,L}C^s$ is a lower bound of C, providing the x_i are independent. In many cases, such as the one to be considered, computing such a lower bound is the only practical means of studying the system capacity.

Returning now to our particular problem, we are transmitting a narrow-band signal x(t) through scatter and additive channel noise. Two components, $w_s(t)$ and $w_c(t)$, are present in the received signal w(t), as discussed previously, and may be observed separately. In order to make the problem tractable, it has been necessary to assume that x(t) is of the form of a carrier amplitude-modulated by a binary source. That is, at the i^{th} sampling interval of x(t),

$$x(t_i) = x_i \cos(\omega_o t_i + \phi) \quad . \tag{3-43}$$

The x_i are drawn from a stochastic source which generates a sequence of independent amplitudes $\pm a$, with probability one-half for either sign. ϕ is an arbitrary phase angle.

Since the two components of w(t) may be isolated and observed separately at the receiver, it is of interest to find how each separately contributes to the total information about the transmitted signal. If, for example, almost the entire information is contained in one component alone, the other adding very little extra, some simplification in receiving equipment might be justified. In particular, we shall find the lower bounds ${}_{2,2}C_1^s$ and ${}_{2,2}C_2^s$ for one and both components observed at the receiver, respectively. The two sampling points at the transmitter and the two at the receiver will be made the same. Using common subscripts, we have if just $w_s(t)$ or $w_c(t)$ is observed,

$$\Delta_{2,2}C_1^{\,s} = \sum_{x_1}\sum_{x_2}\int_{w_1}\int_{w_2} P(x_1x_2/w_1w_2)\,P(w_1w_2)\log\frac{P(x_1/x_2,w_1,w_2)}{P(x_1)}\,dw_1\,dw_2 \quad , \qquad (3\text{-}44)$$

where w_i denotes w_{si} or w_{ci}, whichever is observed. ${}_{2,2}C_2^{\,s}$ for both components observed is

$$\Delta_{2,2}C_2^{\,s} = \sum_{x_1}\sum_{x_2}\int_{w_{s1}}\int_{w_{s2}}\int_{w_{c1}}\int_{w_{c2}} P(x_1x_2/w_{s1}w_{s2}w_{c1}w_{c2})\,P(w_{s1}w_{s2}w_{c1}w_{c2}) \times \log\frac{P(x_1/x_2w_{s1}w_{s2}w_{c1}w_{c2})}{P(x_1)}\,dw_{s1}\,dw_{s2}\,dw_{c1}\,dw_{c2} \quad . \qquad (3\text{-}45)$$

Naturally, we would prefer if possible to find ${}_{3,3}C^s$ or even higher-order entropy differences, but these do not appear to be tractable even for the simple form of transmission considered. On the other hand, ${}_{1,1}C_o = 0$, so that ${}_{2,2}C^s$ represents a significant improvement over any simpler calculation.

The application of Eq.(3-44) and Eq.(3-45) to the binary transmission in scatter has been studied in Appendix III, and we find from Eq.(A3-13) and Eq.(A3-31), respectively,

$$\Delta_{2,2}C_1^{\,s} = -\frac{1}{\sqrt{2\pi F}}\int_{-\infty}^{\infty}\exp\left[-a^2/2F\right]da\int_{-\infty}^{+\infty}p(s,a)\log p(s,a)\,ds - \log\sqrt{2\pi e} \quad , \qquad (3\text{-}46)$$

$$\Delta_{2,2}C_2^{\,s} = -\frac{1}{F}\int_{0}^{\infty}\exp\left[-a^2/2F\right]da\int_{-\infty}^{+\infty}p(s,a)\log p(s,a)\,ds - \log\sqrt{2\pi e} \quad , \qquad (3\text{-}47)$$

where

$$p(s,a) = \frac{1}{2\sqrt{2\pi}}\left\{\exp\left[-(s-a)^2/2\right] + \exp\left[-(s+a)^2/2\right]\right\} \quad . \qquad (3\text{-}48)$$

We shall call F the <u>figure of merit</u> of the channel, for ${}_{2,2}C_1^{\,s}$ and ${}_{2,2}C_2^{\,s}$ will be shown to be monotonically increasing functions of F. F is given by

$$F = \frac{\beta^2}{1-\beta^2} \quad , \quad \beta = \frac{a^2\phi_1 + \phi_{n1}}{a^2\phi_o + \phi_{no}} \qquad (3\text{-}49)$$

where

$$\phi_{ni} = \overline{n_s(t_j)n_s(t_{j+i})} = \phi_{nj,j+i}$$

$$\phi_i = \phi_{j,j+i} = \overline{y_s(t_j)y_s(t_{j+i})} \quad .$$

Assuming that $\phi_o = 1$, and that the signal x(t) and white noise n(t) are limited to the same band $W = 1/\Delta$, so that $\phi_{n1} = 0$, we have

$$\beta = \frac{a^2\phi_1}{a^2 + N} = \frac{R\phi_1}{1+R} \quad , \qquad (3\text{-}50)$$

where $N = \phi_{no}$, and $R = a^2/N$, the power signal-to-noise ratio.

It has been possible to evaluate the integrals of Eq.(3-46) and Eq.(3-47) through the expression of $\int_{-\infty}^{+\infty} p(s,a)\log p(s,a)\,ds$ as an infinite series (see Appendix IV). The result is, from Eq.(A4-10),

$$G(\alpha) = -\int_{-\infty}^{+\infty} p(s, \alpha) \log p(s, \alpha)\, ds = \log 2\sqrt{2\pi e} + \left\{ -\sqrt{\frac{2}{\pi}}\, \alpha^2 \exp\left[-\alpha^2/2\right] \right.$$

$$\left. + (2\alpha^2 - 1)\, \mathrm{erf}\,(|\alpha|) + \sum_{k=1}^{\infty} \frac{(-1)^k \exp\left[2\alpha^2 k(k+1)\right] \mathrm{erf}\left[|\alpha|\,(2k+1)\right]}{k(k+1)} \right\} \log_2 e \quad . \tag{3-51}$$

From Eq.(3-48) and Eq.(3-51) it is seen that $G(\alpha)$ is an entropy. This is discussed in more detail in Appendix IV. $G(\alpha)$ is plotted in Fig. 27. Using Eq.(3-51) the integration on α indicated in Eq.(3-46) can be performed, term by term, as shown in Appendix III. From Eq.(A3-21) we have

$$\frac{{}_{2,2}C_1^{\,s}}{W} = 1 + \frac{\log_2 e}{\pi}\left[-2\sqrt{F} + (2F - 1)\tan^{-1}(1/\sqrt{F}) \right.$$

$$\left. \begin{cases} + \sum \dfrac{(-1)^n \tan^{-1}\sqrt{[1 - 4n(n+1)F]/(2n+1)^2 F}}{n(n+1)\sqrt{1 - 4n(n+1)F}} & \text{when } 1 \leq n \leq n_o \\[2ex] + \sum \dfrac{(-1)^n \tanh^{-1}\sqrt{[4n(n+1)F - 1]/(2n+1)^2 F}}{n(n+1)\sqrt{4n(n+1)F - 1}} & \text{when } n_o < n < \infty \end{cases} \right] \tag{3-52}$$

where $4n_o(n_o + 1)F = 1$. A plot of ${}_{2,2}C_1^{\,s}/W$ versus F appears in Fig. 28. As expected, for large F the ratio approaches the asymptotic value of one bit. When F is small,

$$\frac{{}_{2,2}C_1^{\,s}}{W} \approx \frac{F}{2} \log_2 e \qquad ; \ F \ll 1 \quad , \tag{3-53}$$

and if $R \ll 1$,

$$\frac{{}_{2,2}C_1^{\,s}}{W} \approx \frac{R^2 \phi_1^{\,2}}{2} \log_2 e \quad ; \ R \ll 1 \quad . \tag{3-54}$$

Note that the power signal-to-noise ratio appears as the square in Eq.(3-54). This is in marked contrast to the linear dependence of capacity observed at low signal-to-noise ratios for Gaussian channel noise without scatter. Of course, ${}_{2,2}C_2^{\,s}$ is only a lower bound to C, the actual system capacity, but its behavior is nevertheless interesting.

When both components are observed at the receiver, Eq.(3-47) is the appropriate expression. Integration term-by-term, in a similar manner to that used in Eq.(3-46), yields, from Eq.(A3-37) of Appendix III,

$$\frac{{}_{2,2}C_2^{\,s}}{W} = 1 + \log_2 e \left\{ (2F - \tfrac{1}{2}) - \frac{(2F + \frac{1}{2})}{\sqrt{(1/F) + 1}} \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{(-1)^k [1 - (2k+1)]/\sqrt{(1/F) + 1}}{2k(k+1)[(1/F) - 4k(k+1)]} \right\} \quad . \tag{3-55}$$

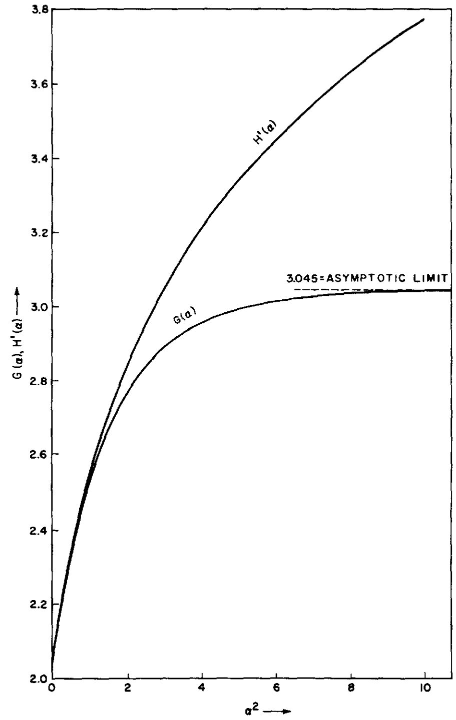

Fig. 27. Entropy of a split Gaussian distribution.

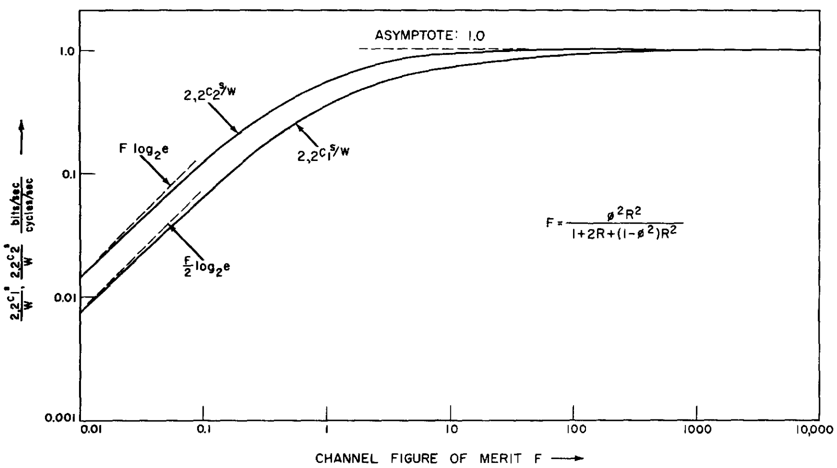

Fig. 28. Performance of binary transmission through scatter.

The result for $({}_{2,2}C_2^{S}/W)$ versus F appears in Fig. 28. Again, the asymptotic value of this ratio for large F is one bit, but now for small F

$$\frac{{}_{2,2}C_2^{S}}{W} \approx F \log_2 e \qquad ; \; F \ll 1 \quad , \tag{3-56}$$

and for $R \ll 1$,

$$\frac{{}_{2,2}C_2^{S}}{W} \approx R^2 \phi_1^2 \log_2 e \; ; \; R \ll 1 \quad . \tag{3-57}$$

Roughly speaking, at small F each component carries information which is not contained in the other, with the result that the lower bound doubles when both components are observed. At higher F, the information loses this disjointness, until finally at $F = \infty$ the entire information can be carried in just one component.

In Fig. 29, both ${}_{2,2}C_1^{S}/W$ and ${}_{2,2}C_2^{S}/W$ are shown as functions of R and ϕ_1. For comparison, the channel capacity which would be obtainable in the absence of scatter is shown. For slow scatter fluctuations relative to the transmission bandwidth, a condition often met in practice, $\phi_1 \approx 1$. For $\phi_1 = 1$, ${}_{2,2}C_2^{S}$ is twice as great as ${}_{2,2}C_1^{S}$ for small R, but the two converge to 1 as $R \rightarrow \infty$. At smaller ϕ_1, however, ${}_{2,2}C_2^{S}$ remains nearly double ${}_{2,2}C_1^{S}$ over the entire range of R, and for $R \rightarrow \infty$ asymptotic values of less than unity are reached. For comparison, there are shown, in Fig. 28, capacities which can be obtained when the scatter multiplier is no longer random, but a constant equal to its previous root-mean-square strength. The channel is then essentially the classical one in which the only perturbation is additive white Gaussian noise. Three types of transmission through this channel are considered. The first, using white Gaussian signals, achieves the maximum system capacity, or channel capacity, $C_g = W \log(1 + R)$.

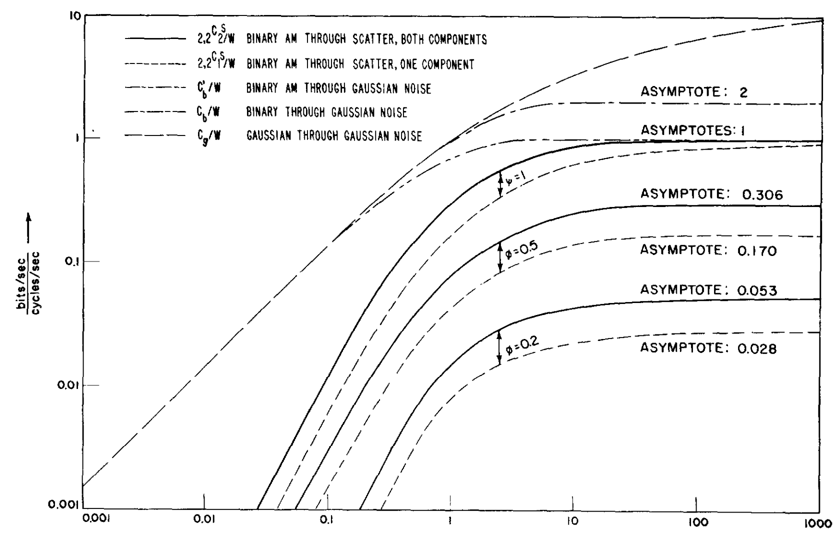

Fig. 29. Comparison of transmissions and channels.

The other two transmissions are of the binary type discussed in Appendix IV. C_b' corresponds to an AM transmission identical to that considered in the scatter channel, while C_b has additional freedom between the two transmitted components. Note that for $\phi_1 \gtrsim 1$ and $R \approx 1$, the binary transmission in scatter can achieve system capacities quite comparable to the best that can be obtained, C_g, without scatter.

B. Related Results for Another Multiplicative Model

In the course of studying the scatter channel discussed in Sec. A of this chapter, the following channel model was also investigated, using a sampling-point analysis. The channel consists of a random multiplier, followed by additive channel noise. The values of the multiplier and noise are assumed statistically independent of each other, and both are independent from sampling-point to sampling-point. Assuming that the transmitted signal also has this independence, the entropy calculations used to obtain system capacity may be computed on the basis of just first-order probability distributions.

In particular, we shall assume that the multiplier has a Rayleigh distribution of unit variance, and that the noise is white Gaussian of variance σ_n^2 and bandwidth W. The transmission also occupies bandwidth W and its power is normalized to unity. A block diagram of the channel appears in Fig. 30. Although this particular representation has not been found to be a realistic model in any actual channel environment which has been encountered, the following results are of general interest.

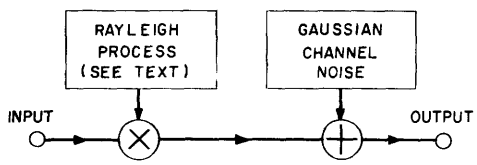

Fig. 30. Channel with Rayleigh multiplier.

We have considered the system capacity which is obtained for two types of transmission through the given channel. We first consider a binary transmission, and then one having "sine-statistics" at the sampling points. Bounds to the system capacity are established by invoking entropy inequalities deduced by Shannon.

For the case of binary transmission, we assume that the transmitted sampling-point value x may, with equal probability, have the value +1 or −1. Now the probability density p(y) for the multiplier y has been assumed to be

$$p(y) = \begin{cases} 0 & ; \ y < 0 \\ 2y \exp[-y^2] & : \ y \geq 0 \end{cases} \quad . \tag{3-58}$$

Thus the probability density p(z) for z = xy, the output of the multiplier, is

$$p(z) = \sum_x p[y = z(x)] \frac{P(x)}{|x|} = |z| \exp[-z^2] \quad . \tag{3-59}$$

The conditional probability density p(z/x) for z, given x, is

$$p(z/x) = \frac{p(y = \frac{z}{x})}{|x|} = \begin{cases} \frac{2|z|}{x^2} \exp[-z^2/x^2] \ ; & zx \geq 0 \\ 0 & ; \ zx < 0 \end{cases} \quad . \tag{3-60}$$

Following multiplication, we have additive Gaussian noise n, so that the final received signal is w = xy + n. We wish to find the entropy difference H per degree of freedom, since this yields the system capacity C_B.

$$C_B = 2WH \quad , \tag{3-61}$$

$$H = H(w) - H_x(w) \quad , \tag{3-62}$$

where

$$H(w) = -\int_w p(w) \log p(w)\, dw \quad , \tag{3-63}$$

$$H_x(w) = \begin{cases} -\sum_x \int_w p(w/x)\, P(x) \log p(w/x)\, dw & ; \text{ if } x \text{ is discrete} \\ -\int_x \int_w p(w/x)\, p(x) \log p(w/x)\, dw\, dx & ; \text{ if } x \text{ is continuous} \end{cases} \tag{3-64}$$

At this point it is convenient to introduce the concept of entropy power, in order to apply some of Shannon's relationships to the case considered. It has been shown that the distribution having the maximum entropy for a given variance, or power, is the Gaussian distribution.[29] The corresponding entropy[29] is

$$H_g = \log \sqrt{2\pi e N} \quad ; \tag{3-65}$$

where N is the power. We then define the entropy power $\overline{N}$ of a random variable r as the power of a Gaussian distribution having the same entropy H_r as that of the given random variable. Thus, from Eq.(3-65)

$$\overline{N} = \frac{1}{2\pi e} 2^{2H_r} \quad , \tag{3-66}$$

where

$$H_r = -\int_{-\infty}^{+\infty} p(r) \log p(r)\, dr \quad . \tag{3-67}$$

Shannon has shown[30] that the entropy power $\overline{N_{u+v}}$ of the sum of two independent random variables u and v satisfies the inequalities

$$\overline{N}_u + \overline{N}_v \leq \overline{N_{u+v}} \leq N_u + N_v \tag{3-68}$$

where $\overline{N}_u$ and $\overline{N}_v$ are the entropy powers of u and v, respectively, and N_u and N_v are their respective powers. If u or v have a non-zero mean, the powers are taken as the variances.

In the particular application of Eq.(3-68) being considered, u is z and v is n. Since n is Gaussian, $\overline{N}_v = N_v = \sigma_n^2$. The entropy H(z) of z found from Eq.(3-59):

$$\begin{aligned} H(z) &= -\int_{-\infty}^{+\infty} \left\{|z| \exp\left[-z^2\right]\right\} \log \left\{|z| \exp\left[-z^2\right]\right\} dz \\ &= (1 + \gamma/2) \log_2 e \\ &= \log \sqrt{2\pi e \overline{N}_u} \quad , \end{aligned} \tag{3-69}$$

where γ is Euler's constant, γ = .5772157.... Since z has zero mean, its variance N_u is unity.

Thus, from Eq.(3-68),

$$H(w) = \log \sqrt{2\pi e \, \overline{N_{u+v}}}$$

is bounded by

$$\log \sqrt{2\pi e \left\{ \frac{\exp[1+\gamma]}{2\pi} + \sigma_n^2 \right\}} \leq H(w) \leq \log \sqrt{2\pi e (1 + \sigma_n^2)} \quad . \tag{3-70}$$

In finding the conditional entropy $H_x(w)$, we first compute $H_x(z)$.

$$H_x(z) = - \sum_x P(x) \int_z p(z/x) \, dz \quad . \tag{3-71}$$

Using Eq.(3-60), we find:

$$H_x(z) = (1 + \frac{\gamma}{2}) \log_2 e - 1 = \log \sqrt{2\pi e \, \overline{N_u}\big|_x} \quad . \tag{3-72}$$

The conditional variance of z, knowing x, $N_u\big|_x$, is

$$N_u\big|_x = \overline{z^2}\big|_x - \overline{z}^2\big|_x = 1 - \frac{\pi}{4} \quad . \tag{3-73}$$

The bounds on $H_x(w)$ are established in a similar manner to those for H(w).

$$\log \sqrt{2\pi e \left\{ \frac{\exp[1+\gamma]}{8\pi} + \sigma_n^2 \right\}} \leq H_x(w) \leq \log \sqrt{2\pi e (1 - \frac{\pi}{4} + \sigma_n^2)} \quad . \tag{3-74}$$

Using Eqs.(3-61), (3-62), (3-70), and (3-74), we find the bounds for C_B:

$$\log \left\{ \frac{\sigma_n^2 + \frac{\exp[1+\gamma]}{2\pi}}{\sigma_n^2 + 1 - \frac{\pi}{4}} \right\} \leq \frac{C_B}{W} \leq \log \left\{ \frac{\sigma_n^2 + 1}{\sigma_n^2 + \frac{\exp[1+\gamma]}{8\pi}} \right\} \quad . \tag{3-75}$$

The bounds in Eq.(3-75) are plotted in Fig. 31 as a function of $R = 1/\sigma_n^2$, the signal-to-noise ratio. For $\sigma_n^2 = 0$, H(w) = H(z) and $H_x(w) = H_x(z)$, so C_B/W = 2 bits.

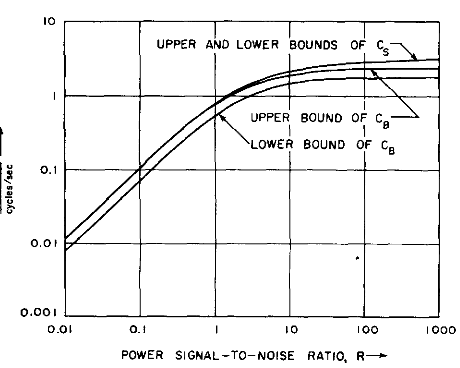

Fig. 31. Performance of channel with Rayleigh multiplier.

One other case which has been studied is the transmission of a signal having "sine-statistics." That is, x is drawn from a distribution which would be observed for amplitudes drawn at random times from a sine wave. This distribution p(x) is

$$p(x) = \begin{cases} \frac{1}{\pi\sqrt{2 - x^2}} & : |x| \leq \sqrt{2} \\ 0 & ; |x| > \sqrt{2} \end{cases} \quad . \tag{3-76}$$

It is easily shown that the product z = xy of two independent variables is Gaussian, when x has the distribution of Eq.(3-76) and y is Rayleigh distributed. In turn, the sum w = xy + n of two independent Gaussian variables has a Gaussian distribution. Thus for this particular transmission, it is simple to find H(w) exactly. From Eq.(A5-8),

$$H(w) = \log \sqrt{2\pi e(1 + \sigma_n^2)} \quad . \tag{3-77}$$

In order to determine the bounds on $H_x(w)$, we first find the entropy bounds assuming that x is fixed, and then average over all x, as in Eq.(3-64). We have, by analogy to Eq.(3-74), in which x is fixed at unity

$$\begin{aligned} \log \sqrt{2\pi e \left(\frac{\exp[1+\gamma]}{8\pi} x^2 + \sigma_n^2 \right)} &\leq - \int_W p(w/x) \log p(w/x)\, dw \\ &\leq \log \sqrt{2\pi e[(1 - \tfrac{\pi}{4})x^2 + \sigma_n^2]} \quad . \end{aligned} \tag{3-78}$$

We now integrate over x, as indicated in Eq.(3-64). We have need of:

$$\begin{aligned} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\log(ax^2 + b)}{\pi\sqrt{2 - x^2}}\, dx &= \int_{-1}^{1} \frac{\log[ay + (b + a)]}{\pi\sqrt{1 - y^2}}\, dy \\ &= \frac{1}{\pi} \int_0^{\pi} \log[a \cos z + (b + a)]\, dz \\ &= 2 \log \left[\frac{\sqrt{b} + \sqrt{2a + b}}{2} \right] \quad . \end{aligned} \tag{3-79}$$

In integrating Eq.(3-79), we have made the substitutions: $(x^2 - 1) = y = \cos z$. The last integral of Eq.(3-79) is evaluated by recourse to Eq.(A1-15). Applying Eq.(3-79) to Eq.(3-64) and Eq.(3-78), we obtain

$$\begin{aligned} \log \sqrt{2\pi e} \left[\frac{\sigma_n + \sqrt{\sigma_n^2 + \{\exp[1 + \gamma]/4\pi\}}}{2} \right] &\leq H_x(w) \\ &\leq \log \sqrt{2\pi e} \left[\frac{\sigma_n + \sqrt{\sigma_n^2 + 2 - \pi/2}}{2} \right] \quad . \end{aligned} \tag{3-80}$$

From Eqs.(3-61), (3-62), 3-77), and (3-80),

$$\log \frac{4(1 + \sigma_n^2)}{\left[\sigma_n + \sqrt{\sigma_n^2 + 2 - \pi/2}\right]^2} \leq \frac{C_S}{W} \leq \log \frac{4(1 + \sigma_n^2)}{\left[\sigma_n + \sqrt{\sigma_n^2 + \{\exp[1 + \gamma]/4\pi\}}\right]^2} \tag{3-81}$$

where S has replaced B, in order to indicate that "sine-statistics" apply to the transmission. The bounds in Eq.(3-81) are plotted in Fig. 31 as a function of $R = 1/\sigma_n^2$. When $\sigma_n^2 = 0$, C_S is exactly equal to its upper limit, and $C_S/W = 3.38$ bits. This represents a slight improvement over the binary transmission at infinite signal-to-noise ratio.

CHAPTER IV

I. CONCLUSIONS

We have endeavored, in this paper, to present a systematic approach to the problem of communicating efficiently through multipath disturbances. Statistical analysis has been found to be a useful tool for conducting studies of the multipath channel, just as it has been in the case of the elementary channel affected by additive noise alone. Two major aspects of the synthesis of optimum communication systems have been investigated through a number of special cases. First, there is the problem of specifying the optimum receiver, given the transmitter and channel. For the communication system model which has been adopted in this paper, the optimum receiver can be stated unambiguously as one which computes certain probabilities and makes decisions based on these computations. The second major aspect of our work has been the evaluation of over-all system performance through the criterion of system capacity. This measure has been selected in preference to others equally plausible mainly for reasons of mathematical tractability. The criterion of system capacity is appropriate only to the transmission of very long symbols, and thus there are very real limitations on the practical aspects of our treatment. Probably the most serious gap in system optimization theory is the lack of knowledge about optimum codes for finite duration symbols. We naturally cannot expect to obtain coding theorems of this sort for the multipath channel until they are at least available for much simpler channel perturbations, such as additive white Gaussian noise.

Studies were first conducted on a multipath channel model consisting of a number of fixed paths, of various delays and strengths, connected in parallel and followed by additive white Gaussian noise. System capacities for white Gaussian transmitted symbols were found as a function of the various channel parameters, with the assumption that the receiver knew the multipath structure. Providing that the transmission occupies a wide enough band, it appears from these results that system performance for fixed multipath can be made at least as good as that which would be obtained with the strongest path alone. This is quite in contradiction to the destructive interference produced by the extra paths which is so often encountered in conventional, present-day systems. If the spectral shape of the transmission is allowed to vary, we have shown how the spectrum may be optimized to achieve the maximum system capacity, subject to power and bandwidth constraints, for any fixed multipath condition. Future work with the fixed multipath model might investigate more complex path configurations than have been considered here, as well as models in which the path responses are not pure delayed impulses but have some dispersion in time. An incidental result in connection with the work on fixed multipath has been a direct proof of Shannon's fundamental theorem for the particular case of Gaussian signals and noise of arbitrary spectra.

The next channel model to be considered was one of similar structure to that for fixed multipath, but in which the parameters were allowed to vary slowly in time. Results for fixed multipath may carry over, approximately, to this quasi-stationary model if provision is made that the receiver be kept informed of the multipath characteristic. We are thus led to postulate that a good transmitted signal for slowly-varying multipath might well contain a portion which is known a priori to the receiver. This portion would be used to "sense" the ionosphere, enabling the receiver to correct its stored symbols in step with the multipath fluctuations. The use of such a technique is merely a suggestion, but one possible method of constructing and utilizing the redundant part of the transmission has been described. Further work should be done on the effect of imperfect ionospheric measurement by this and other such schemes, on communication capabilities of the channel.

In connection with the suggested method of ionospheric measurement, experimental work was initiated to measure the rates of fluctuation encountered in the ionosphere. Such experiments seem to support the contention that these fluctuations are slow compared to the usual transmission bandwidths. One result indicated that fluctuations contained no frequencies higher, at the time of the experiment, than one-quarter cycle per second. Of course, a fairly wide range of fluctuation rates would probably be measured under a variety of conditons. Further experimental studies of this sort should be conducted at various geographical locations, times of day and seasons, and during ionospheric storms.

We now return to the quasi-stationary model described earlier. It has been possible to apply some results on randomly varying filters to the model in which slow variations are occurring in the multipath, but these changes are not known directly to the receiver. It appears, at least for low signal-to-noise ratio, that reasonable system capacity may be obtained. The particular models considered, however, contained only fluctuation in path delay. Further investigation in which both strengths and delays vary, perhaps not even independently, would no doubt yield greater insight for more general multipath problems.

The last multipath model which has been investigated is single-mode "scatter" propagation. Here it has been possible to specify theoretically the ideal probability-computing receiver, together with simpler approximations to it. It is hoped that further work will be done in this direction, with the aim of improving the transmission of information through scatter. At the same time, lower bounds to system capacity have been established for transmission through scatter. It appears that the scatter channel may give fairly efficient performance, relative to a channel perturbed by additive Gaussian noise alone. A system capacity may be obtained which is at least 30 per cent of that for the additive noise channel, providing that scatter is changing slowly and the signal-to-noise ratio is about unity. There is much room for further work in the capacity computations, since only rather rough lower bounds to system capacity have been obtained. Furthermore, generalization of these results to multiple scatter-modes would be of use in dealing realistically with a greater variety of channels.

Incidental theoretical conclusions have been obtained during the multipath study which are of interest in their own right. In particular, the study of a channel containing a random positive multiplier was investigated, partly because of its connection with multipath channels. The results have most meaning, however, when they are viewed as arising from an application of information theory to a new, and as yet rather academic, channel model. Another incidental result has been the derivation of system capacities for binary transmissions through additive Gaussian noise. These latter results obey Shannon's conclusion that all comparable transmissions have equal merit at low signal-to-noise ratios.

In general, further work on the application of statistical theory to communication through multipath disturbances would be most worth while. There seems to be quite a dearth of results in this field appearing in the literature. Despite the recognized random nature of multipath disturbances, apparently little has been done on this problem using the appropriate mathematical tool, statistics. From its relative success in dealing with the simpler additive noise channel, one would suppose that the application of statistical theory as a logical attack on the multipath channel might have been more widely discussed. It is hoped that future work may generalize many of the results obtained in this paper and lead to a new understanding of this perennial communication problem.

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APPENDIX I

I. ANALYSIS OF CAPACITY FOR FIXED MULTIPATH AND ADDITIVE GAUSSIAN NOISE

A. Capacity Calculations for Specified Path Strengths and White Gaussian Noise

Referring to Eqs. (2-2) and (2-3), we seek to find the integral

$$C = \int_o^W \log \left(1 + \frac{S_o \left| \sum_i \alpha_i \exp \left[- j2\pi f \tau_i \right] \right|^2}{N_o} \right) df \quad , \tag{A1-1}$$

where a spectrum of uniform power density S_o and bandwidth W has been transmitted over a finite number of paths of strength α_i and delay τ_i, with additive noise of uniform power density N_o in the channel. The most apparent way to evaluate Eq. (A1-1) is to divide the expression in brackets into simple factors. Under certain assumptions, we find this can be done. The factors thus obtained are of the form $(c_k + d_k \cos cf)$, where k = 0, 1, 2, ... n, and c is a constant. If $W = m\pi/c$, where m is an integer, Eq. (A1-1) can then be obtained in closed form by evaluation of the separate integrals. A good approximation is obtained if n is nonintegral, but very large, $n >> 1$.

In order to reduce Eq. (A1-1) to a sum of simple integrals, let us assume that the τ_i are commensurable, τ being the largest common divisor. Then in general the bracket B in Eq. (A1-1) may be written

$$B = \frac{1}{2}(1 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx) \quad , \tag{A1-2}$$

where the first term has been set arbitrarily to 1/2 by adding an appropriate constant to Eq. (A1-1). Here $x = 2\pi f \tau = cf$. We can then factor B in the form

$$B = (\frac{1}{2} + b \cos x) [\frac{b_o}{2} + b_1 \cos x + b_2 \cos 2x + \ldots + b_{n-1} \cos (n-1)x] \ . \tag{A1-3}$$

The process may be repeated with the second bracket of Eq. (A1-3), and so on until we obtain

$$B = (\frac{1}{2} + b \cos x)(c_1 + d_1 \cos x)(c_2 + d_2 \cos x) \ldots (c_n + d_n \cos x) \quad , \tag{A1-4}$$

and the factorization is complete. The factorization of Eq. (A1-3) is carried out as follows. We equate the coefficients of the cosine terms of equal arguments in Eqs. (A1-2) and (A1-3), using the identity: $\cos mx \cos nx = 1/2 \cos (m+n)x + 1/2 \cos (m-n)x$. The system of equations is obtained:

$$\begin{aligned}
\frac{1}{2} b_o + bb_1 &= 1 \\
bb_o + b_1 + bb_2 &= a_1 \\
bb_1 + b_2 + bb_3 &= a_2 \\
bb_2 + b_3 + bb_4 &= a_3 \\
bb_{n-3} + b_{n-2} + bb_{n-1} &= a_{n-2} \\
bb_{n-2} + b_{n-1} &= a_{n-1} \\
bb_{n-1} &= a_n \ .
\end{aligned} \tag{A1-5}$$

We have (n + 1) equations in (n + 1) unknowns. We find b by solving for b_o and b_1 in terms of b and then substituting in the first equation of Eq. (A1-5). Using the last (n − 1) equations of Eq. (A1-5), we find b_1.

$$b_1 \begin{vmatrix} b & 1 & b & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & b & 1 & b & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & b & 1 & b & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \dots & b & 1 & b \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & b & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & b \end{vmatrix} = \begin{vmatrix} a_2 & 1 & b & 0 & 0 & \dots & 0 & 0 & 0 \\ a_3 & b & 1 & b & 0 & \dots & 0 & 0 & 0 \\ a_4 & 0 & b & 1 & b & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ a_{n-2} & 0 & 0 & 0 & 0 & \dots & b & 1 & b \\ a_{n-1} & 0 & 0 & 0 & 0 & \dots & 0 & b & 1 \\ a_n & 0 & 0 & 0 & 0 & & 0 & 0 & b \end{vmatrix} \quad . \qquad \text{(A1-6)}$$

We expand b_1 along the column of a's in the numerator determinant, using Laplace's development.[1]

$$b_1 b^{n-1} = a_2 b^{n-2} - a_3 b^{n-3} + a_4 \begin{vmatrix} 1 & b \\ b & 1 \end{vmatrix} b^{n-4} - a_5 \begin{vmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{vmatrix} b^{n-5} + \dots$$

$$+ (-1)^k a_k \Delta_{k-2}^{(b)} b^{n-k} + \dots + (-1)^n a_n \Delta_{n-2}^{(b)} = \sum_{i=2}^{n} a_i b^{n-i} (-1)^i \Delta_{i-2}^{(b)} \quad . \qquad \text{(A1-7)}$$

where $\Delta_k^{(b)}$ has k rows and k columns in the form:

$$\Delta_k^{(b)} = \begin{vmatrix} 1 & b & 0 & 0 & \dots & 0 & 0 & 0 \\ b & 1 & b & 0 & \dots & 0 & 0 & 0 \\ 0 & b & 1 & b & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & b & 1 & b \\ 0 & 0 & 0 & 0 & \dots & 0 & b & 1 \end{vmatrix} \quad , \quad \Delta_o^{(b)} = 1 \quad . \qquad \text{(A1-8)}$$

$k \neq 0$

Similarly, solving for b_o using the last n equations of Eq. (A1-5),

$$b_o b^n = \sum_{i=1}^{n} a_i b^{n-i} (-1)^{i+1} \Delta_{i-1}^{(b)} \quad . \qquad \text{(A1-9)}$$

Substituting Eqs. (A1-7) and (A1-9) in the first equation of Eq. (A1-5), we find the polynomial equation which determines b.

$$\sum_{i=2}^{n} a_i b^{n-i} (-1)^{i+1} [\Delta_{i-1}^{(b)} - 2b^2 \Delta_{i-2}^{(b)}] + a_1 b^{n-1} - 2b^n = 0 \quad . \qquad \text{(A1-10)}$$

There is a useful recursion formula[2] for $\Delta_i^{(b)}$

$$\Delta_i^{(b)} = \Delta_{i-1}^{(b)} - b^2 \Delta_{i-2}^{(b)} \quad ; \quad i \geq 2 \quad . \qquad \text{(A1-11)}$$

The specific polynomial equation in b, given by Eq. (A1-9), good for n from 1 to 4 inclusive, is

$$(2b)^4 (a_2 - a_4 - 1) + (2b)^3 (a_1 - 3a_3) + (2b)^2 (8a_4 - 2a_2) + (2_b) 4a_3 + 8a_4 = 0 \quad . \tag{A1-12}$$

After b is found from Eq. (A1-10), b_o may be obtained by (A1-9) and then the equations of Eq. (A1-5) are used in sequence to obtain all the b_i.

Having thus factored Eq. (A1-2) into the form Eq. (A1-4), the expression Eq. (A1-1) becomes, for commensurable τ_i,

$$C = K_o + \frac{1}{2\pi\tau} \sum_{j=1}^{n} \int_o^{2\pi W\tau} \log (c_j + d_j \cos x) \, dx \quad , \tag{A1-13}$$

where the K_o, c_j, and d_j are related to S_o, N_o, and the a_i, and τ is the smallest common divisor of the τ_i. We seek integrals of the form

$$I = \int_o^A \log (1 + \beta \cos x) \, dx \quad . \tag{A1-14}$$

Since the argument of the logarithm in Eq. (A1-1) is always positive, none of the arguments of the logarithm in Eq. (A1-13) may vanish. Thus β is either complex or $|\beta|<1$. If complex, the β appear in complex conjugate pairs, for the product of the factors in Eq. (A1-4) is always real. For real β, $|\beta|<1$ and $A = m\pi$, m an integer, Eq. (A1-14) can be found in closed form.[3]

$$\int_o^{m\pi} \log (1 + \beta \cos x) \, dx = m\pi \left(\log \frac{1 + \sqrt{1 - \beta^2}}{2} \right) \quad . \tag{A1-15}$$

For complex β, Eq. (A1-15) still applies, providing that $\sqrt{1 - \beta^2}$ is taken to lie in the right half of the complex plane. If $W >> 1/\tau$ so that $A >> 1$, Eq. (A1-15) holds approximately for any large m, since the integrand is repetitive with period 2π. In the remaining work, we shall assume for simplicity that $W = m/2\tau$, where m is an integer, although the same results apply to a good approximation for $W >> 1/\tau$.

If two paths of strength unity and a are present, Eq. (A1-1) takes the form of Eq. (A1-15) immediately.

$$C_2 = \frac{1}{2\pi\tau} \int_o^{m\pi} \log \left[\frac{N_o + S_o(1 + a^2)}{N_o} + \frac{2aS_o}{N_o} \cos x \right] dx \quad . \tag{A1-16}$$

From Eq. (A1-15), this gives

$$C_2 = W \log \frac{1}{2} \left\{ 1 + R(1 + a^2) + \sqrt{[1 + R(1 + a^2)]^2 - 4a^2R^2} \right\} \quad , \tag{A1-17}$$

where $R = S_o/N_o$. This is the expression quoted in Eq. (2-4). If only the path of strength unity were present, we would have

$$C_1 = W \log (1 + R) \quad .$$

If $\tau = 0$ so that both paths are combined into one, of strength $(1 + a)$,

$$C_2' = W \log [1 + R(1 + a^2)] \quad .$$

It is straightforward to show that $C_1 \leq C_2 \leq C_2$.

The three-path case is more difficult algebraically. We shall merely quote the result. For paths of strength 1, s_1 and s_2, with delays 0, τ, and 2τ, respectively,

$$C_3 = W \log\left(\frac{1}{4} \left\{1 + R[s_1^2 + (1 - s_2)^2]\right\} \times \left\{1 + \frac{\sqrt{[1 + R(1 + s_1^2 + s_2^2)]^2 - R^2[2s_1^2(1 + s_2)^2 + 4s_2^2] - 2Rs_1(1 + s_2)\sqrt{R^2(1 - s_2)^2(s_1^2 - 4s_2) - 4Rs_2}}}{1 + R[s_1^2 + (1 - s_2)^2]}\right\} \times \left\{1 + \frac{\sqrt{[1 + R(1 + s_1^2 + s_2^2)]^2 - R^2[2s_1^2(1 + s_2)^2 + 4s_2^2] + 2Rs_1(1 + s_2)\sqrt{R^2(1 - s_2)^2(s_1^2 - 4s_2) - 4Rs_2}}}{1 + R[s_1^2 + (1 - s_2)^2]}\right\}\right) \quad \text{(A1-18)}$$

In Fig. 9 of Chap. II, C_3 has been plotted as a function of the strength s_2 of the third path, with unity strength in the first and second, $s_1 = 1$, and with $R = 10$.

For still more paths, the calculation becomes succeedingly complex. Only two particular cases have been studied for more than three paths present. These are for four and five paths present, where each path has strength unity and the paths are delayed 0, τ, 2τ, 3τ, and 4τ, respectively. Again $R = 10$. Results are presented in Fig. 10 of Chap. II.

B. Optimization of Transmitted Spectrum to Achieve Maximum Capacity

If we are permitted to transmit band-limited Gaussian noise with an arbitrary spectrum $S_o(f)$, it is possible to maximize the system capacity for fixed multipath and Gaussian channel noise of spectrum $N(f)$, that is, find the code-generating source statistics which yield channel capacity. The constraint of fixed average power P is put on the transmitter. Thus, from Eq. (2-2), we seek to maximize the system capacity C.

$$C = \int_o^W \log\left[\frac{S_o(f)\,|H(f)|^2 + N(f)}{N(f)}\right] df \quad , \quad \text{(A1-19)}$$

where $H(f)$ is the multipath frequency response. We have the constraints

$$P = \int_o^W S_o(f)\, df \quad , \quad S_o(f) \geq 0 \quad . \quad \text{(A1-20)}$$

By the calculus of variations, we must maximize, with respect to $P(f)$,

$$I = \int_o^W \left\{ \log\left[\frac{S_o(f)\,|H(f)|^2 + N(f)}{N(f)}\right] + \lambda\, S_o(f) \right\} df \quad , \quad \text{(A1-21)}$$

subject to $S_o(f) \geq 0$. λ is an arbitrary constant. Following Shannon's[20] similar solution for the best transmitter spectrum for arbitrary Gaussian channel noise, we obtain

$$S_o(f) + \frac{N(f)}{|H(f)|^2} = K, \text{ a constant} \quad , \quad \text{(A1-21a)}$$

or

$$S_o(f) = 0, \text{ for } \frac{N(f)}{|H(f)|^2} > K \quad . \tag{A1-21b}$$

K is found from the power constraint P.

$$K = \frac{P}{W} + \frac{1}{W} \int_{W_i} \frac{N(f)}{|H(f)|^2} df \quad , \tag{A1-22}$$

where the W_i are the frequency bands in which Eq. (A1-21a) holds. This is the result quoted in Eq. (2-6) of Chap. II. A graphical interpretation appears in Fig. 12.

Using the $S_o(f)$ given by Eq. (A1-21), the channel capacity C_c is obtained.

$$C_c = \int_{W_i} \log \frac{K|H(f)|^2}{N(f)} df = \int_{W_i} \log \left\{ \frac{|H(f)|^2}{WN(f)} \left[P + \int_{W_i} \frac{N(f')df'}{|H(f')|^2} \right] \right\} df \quad . \tag{A1-23}$$

So long as the channel noise is Gaussian, C_c is the upper bound to system capacity for any code-generating source. This is so because Eq. (2-2) then is an upper bound for all systems having the given transmitted spectrum, $S_o(f)$.

REFERENCES

1. E. A. Guillemin, *The Mathematics of Circuit Analysis* (John Wiley and Sons, Inc., New York, 1949), p. 7.
2. *Ibid.*, p. 21.
3. P. Franklin, *Methods of Advanced Calculus* (McGraw-Hill Book Company, Inc., New York, 1944), Problem 97, p. 195.

APPENDIX II

I. SYSTEM CAPACITY ANALYSIS FOR CHANNELS WITH RANDOM LINEAR FILTERS

We shall first consider the formulation of system capacity for a general time-varying linear network in the channel, and then apply the results to a few special cases appropriate to multipath propagation. Some results of Zadeh's work on time-varying linear systems, together with an important entropy-absorption theorem of Shannon's, provide a method of computing system capacity for such channels when the signal-to-noise ratio is low. For the present, let us neglect the additive channel noise. Suppose that we transmit a known signal x(t) through the network and observe its output z(t). Then, following Zadeh,[1]

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega;t)\, X(\omega)\, \exp[j\omega t]\, d\omega \quad , \qquad \text{(A2-1)}$$

where $X(\omega)$ is the transform of x(t), and $H(\omega;t)$ is the system function of the variable linear network.

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, \exp[-j\omega t]\, dt \quad , \qquad \text{(A2-2)}$$

$$H(\omega;t) = \int_{-\infty}^{+\infty} h(t;\tau)\, \exp[-j\omega(t-\tau)]\, d\tau \quad . \qquad \text{(A2-3)}$$

$h(t;\tau)$ is the response of the network at time t to a unit impulse applied at time τ. If now $h(t;\tau)$ is varying randomly, so that it is known only in a statistical sense to the observer, we have for the expected output $\overline{z(t)}|_x$,

$$\overline{z(t)}|_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{H(\omega;t)}\, X(\omega)\, \exp[j\omega t]\, d\omega \quad , \qquad \text{(A2-4)}$$

where

$$\overline{H(\omega;t)} = \int_{-\infty}^{+\infty} \overline{h(t;\tau)}\, \exp[-j\omega(t-\tau)]\, d\tau \quad . \qquad \text{(A2-5)}$$

$\overline{h(t;\tau)}$ is the expected response of the network, at time t, to a unit impulse applied at time τ. The subscript x implies that $\overline{z(t)}|_x$ is the average of z(t), conditional upon knowing x(t).

Now the z(t) that is actually observed differs from the expected $\overline{z(t)}|_x$ by a random variable z'(t) having zero mean. In order to describe z'(t) statistically, we now suppose that x(t) is a stationary random process, and that $h(t;\tau)$ has stationarity in the sense that

$$\begin{aligned} \overline{h(t_1;\tau)} &= \overline{h[t_2;\tau + (t_2 - t_1)]} \quad , \\ \overline{h(t_1;\tau)\, h(t_1+\tau;\tau')} &= \overline{h[t_2;\tau+(t_2-t_1)]\, h[t_2+\sigma;\tau'+(t_2-t_1)]} \quad , \end{aligned} \qquad \text{(A2-6)}$$

for all t_1, t_2, σ, τ, and τ'. Then the output z(t) has the autocorrelation function[2] $\phi_z(\sigma)$,

$$\phi_z(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi_H(\sigma;\omega)\, S_x(\omega)\, \exp[j\omega\sigma]\, d\omega \quad , \tag{A2-7}$$

where

$$\psi_H(\sigma;\omega) = \overline{H(\omega;t)\, H(-\omega;t+\sigma)} \quad , \tag{A2-8}$$

and

$$S_x(\omega) = \int_{-\infty}^{+\infty} \overline{x(t')\, x(t'+\lambda)}\, \exp[-j\omega\lambda]\, d\lambda \quad . \tag{A2-9}$$

It is assumed that the fluctuations in $h(t;\tau)$ are independent of those of $x(t)$. From Eq. (A2-7), the power spectrum $S_z(\omega)$ of $z(t)$ is obtained.

$$S_z(\omega) = \int_{-\infty}^{+\infty} \Delta(\omega';\omega)\, S_x(\omega')\, d\omega' \quad , \tag{A2-10}$$

where

$$\Delta(\omega';\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi_H(\sigma;\omega)\, \exp[j\sigma(\omega'-\omega)]\, d\sigma \quad . \tag{A2-11}$$

$\Delta(\omega';\omega)$ is what Zadeh calls a bifrequency function. It is, in effect, the Green's function in the frequency domain. We may divide it into two parts. Let

$$\psi_H(\omega) = \lim_{\sigma\to\infty} \psi_H(\sigma;\omega) = \overline{H(\omega;t)}\;\overline{H(-\omega;t)} \quad , \tag{A2-12}$$

$$\psi_H'(\sigma;\omega) = \psi_H(\sigma;\omega) - \psi_H(\omega) \quad . \tag{A2-13}$$

Then

$$\Delta(\omega';\omega) = \psi_H(\omega)\,\delta(\omega-\omega') + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi_H'(\sigma;\omega)\, \exp[j\sigma(\omega'-\omega)]\, d\sigma \quad , \tag{A2-14}$$

where $\delta(\omega - \omega')$ is a unit impulse in the frequency domain.

Now we have for the average frequency response $\overline{H(\omega)}$,

$$\overline{H(\omega)} = \overline{H(\omega;t)} \quad , \tag{A2-15}$$

since from Eqs. (A2-5) and (A2-6), $\overline{H(\omega;t)}$ does not depend on t. Thus, from Eq. (A2-12),

$$\psi_H(\omega) = |\overline{H(\omega)}|^2 \quad . \tag{A2-16}$$

The first member on the right side of Eq. (A2-14) may therefore be interpreted as a contribution of the average frequency response. Rewriting Eq. (A2-10) with the help of Eqs. (A2-14) and (A2-16), we have

$$S_z(\omega) = S_x(\omega)|\overline{H(\omega)}|^2 + \int_{-\infty}^{+\infty} \Delta'(\omega';\omega)\, S_x(\omega')\, d\omega' \quad , \tag{A2-17}$$

where

$$\Delta'(\omega';\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi_H'(\sigma;\omega)\, \exp\,[j\sigma(\omega'-\omega)]\, d\sigma \quad . \tag{A2-18}$$

The first member of the right side of Eq. (A2-17) represents the power in the fluctuations of $\overline{z(t)}|_x$, due to the fluctuations in x(t). These, however, are known a priori to an observer who has complete knowledge of x(t). Thus the only "noise" which such an observer sees in the output z(t) is that due to the second member of the right side of Eq. (A2-17). We shall denote this by $S_z'(\omega)$.

$$S_z'(\omega) = \int_{-\infty}^{+\infty} \Delta'(\omega';\omega)\, S_x(\omega')\, d\omega' \quad . \tag{A2-19}$$

$S_z'(\omega)$ is the spectral density of $z'(t) = z(t) - \overline{z(t)}|_x$.

In order to determine the system capacity, we now introduce additive white Gaussian noise of spectral density N_o and band $2\pi W$ (radians/sec) in the channel, following the random filter. Then, to find the system capacity, we invoke Shannon's entropy-absorption theorem.[3] Paraphrasing, this states that the entropy rate H/τ of the sum of a Gaussian noise of spectral density N_o and band $2\pi W$, and a time series drawn from any other ensemble of power P is given by

$$\frac{H}{T} = W \log 2\pi e(2\pi W N_o + P) \quad . \tag{A2-20}$$

The theorem assumes that the spectral density $S(\omega)$ of the time series satisfies $S(\omega) << N_o$, and that $S(\omega) = 0$ outside the band W. Thus, assuming that $S_z(\omega)$ and $S_z'(\omega)$ are both $<< N_o$, we have

$$\frac{H(x)}{T} = W \log 2\pi e\left[2\pi W N_o + \int_{-\infty}^{+\infty} S_z(\omega)\, d\omega\right] \quad , \tag{A2-21}$$

and

$$\frac{H_y(x)}{T} = W \log 2\pi e\left[2\pi W N_o + \int_{-\infty}^{+\infty} S_z'(\omega)\, d\omega\right] \quad . \tag{A2-22}$$

The system capacity C, from Eqs. (A2-17), (A2-21) and (A2-22) is therefore

$$C = \frac{H(x) - H_y(x)}{T} = W \log \left[1 + \frac{\int_{-\infty}^{+\infty} S_x(\omega)\, |\overline{H(\omega)}|^2\, d\omega}{2\pi W N_o + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Delta'(\omega';\omega)\, S_x(\omega')\, d\omega'\, d\omega}\right], \tag{A2-23}$$

where the code-generating source at the transmitter has spectral density $S_x(\omega)$. Taking into account the small signal-to-noise ratio, we expand the logarithm, using $\log_2 (1 + x) \approx x \log_2 e$.

$$C \approx \frac{[\log_2 e]\, W \int_{-\infty}^{+\infty} S_x(\omega)\, |\overline{H(\omega)}|^2\, d\omega}{2\pi W N_o + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Delta'(\omega';\omega)\, S_x(\omega')\, d\omega'\, d\omega} \quad . \tag{A2-24}$$

Equation (A2-24) is the general formula for system capacity, at low signal-to-noise ratios, for a channel consisting of a "stationary-random" linear filter followed by additive white Gaussian noise. From Eqs. (A2-8), (A2-12), (A2-13) and (A2-18),

$$\Delta'(\omega';\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\overline{h(t;\tau)\, h(t+\sigma;\tau')}\, \exp[j\omega\sigma] - \overline{h(t;\tau)}\;\overline{h(t;\tau')}] \times \exp[j\omega(\tau-\tau') + j\sigma(\omega'-\omega)]\, d\tau\, d\tau'\, d\sigma \quad , \tag{A2-25}$$

and from Eq. (A2-5)

$$|\overline{H(\omega)}|^2 = \left|\int_{-\infty}^{+\infty} \overline{h(t;\tau)}\, \exp[-j\omega(t-\tau)]\, d\tau\right|^2 \quad . \tag{A2-26}$$

Now applying Eq. (A2-24) to particular channels, it has been possible to obtain Eq. (A2-25) in closed form for channels of the kind shown in Fig. 23. Here the filter consists of a number of pure delays which fluctuate independently of each other and of the transmitted signal and channel noise. The delay distributions are, up to the second order, assumed to be stationary Gaussian and identical. Thus,

$$h(t;\tau) = \sum_i a_i \delta[t - \tau - \mu_i(t)] \quad , \tag{A2-27}$$

where $\mu_i(t)$ has mean d_i, and $\mu_i'(t) = \mu(t) - d_i(t)$ has the second distribution

$$P[\mu_1 < \mu_i'(t_1) < \mu_1 + d\mu_i; \mu_2 < \mu_i'(t_2) < \mu_2 + d\mu_2] = \frac{\exp\left\{-\frac{1}{2}[\phi_\mu(0)(\mu_1^2 + \mu_2^2) - 2\phi_\mu(t_1 - t_2)\mu_1\mu_2]\right\}}{2\pi[\phi_\mu^2(0) - \phi_\mu^2(t_1 - t_2)]}\, d\mu_1\, d\mu_2 \quad . \tag{A2-28}$$

Here, $\phi_\mu(\tau) = \overline{\mu_i'(t)\,\mu_i'(t+\tau)}$. Substituting Eq. (A2-27) in Eq. (A2-25),

$$\Delta'(\omega';\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\sum_i\sum_j a_i a_j \left(\overline{\exp\left\{j\omega[\mu_i(t+\sigma) - \mu_j(t)]\right\}} - \overline{\exp[j\omega\mu_i(t)]} \times \overline{\exp[-j\omega\mu_j(t)]}\right)\right] \exp[j\sigma(\omega'-\omega)]\, d\sigma \quad . \tag{A2-29}$$

From Eq. (A2-28), we obtain

$$\Delta'(\omega';\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_i a_i^2 \left(\exp\left\{-\omega^2[\phi_\mu(0) - \phi_\mu(\sigma)]\right\} - \exp[-\omega^2\phi_\mu(0)]\right) \exp[j\sigma(\omega'-\omega)]\, d\sigma \quad . \tag{A2-30}$$

These results are a generalization of Zadeh's.[4] Similarly,

$$|\overline{H(\omega)}|^2 = \left|\sum_i a_i \exp[-j\omega d_i]\right|^2 \exp[-\omega^2\phi_\mu(0)] \quad . \tag{A2-31}$$

Using Eq. (A2-30), we may write

$$P_I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Delta'(\omega';\omega)\, S_x(\omega')\, d\omega'\, d\omega = \sum_i a_i^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\exp\left\{-\omega^2[\phi_\mu(0)-\phi_\mu(\sigma)]\right\} - \exp[-\omega^2\phi_\mu(0)]\right)\phi_x(-\sigma)\exp[-j\omega\sigma]\, d\sigma\, d\omega \quad , \tag{A2-32}$$

where $\phi_x(\tau)$ is the inverse transform of $S_x(\omega)$. Let us now assume that the delay fluctuations are slow compared to the bandwidth of $S_x(\omega)$. Then the term $[\phi_\mu(0) - \phi_\mu(\sigma)]$ in Eq. (A2-32) remains substantially constant while the integration on σ is carried out over the range in which $\phi_x(-\sigma)$ is substantially different from zero. We obtain

$$P_I \approx \sum_i a_i^2 \int_{-\infty}^{+\infty}\left\{1 - \exp[-\omega^2\phi_\mu(0)]\right\} S_x(\omega)\, d\omega \quad . \tag{A2-33}$$

If now $S_x(\omega)$ is assumed narrow-band compared to its center frequency ω_o,

$$P_I \approx \sum_i a_i^2\, P \left\{1 - \exp[-\omega_o^2\phi_\mu(0)]\right\} << N_o W \quad , \tag{A2-34}$$

where $P = \int_{-\infty}^{+\infty} S_x(\omega)\, d\omega$ is the transmitted power. We obtain for the system capacity, using Eqs. (A2-24), (A2-31) and (A2-34),

$$C \approx \frac{W\,[\log_2 e]\exp[-\theta_d^2]\int_{\omega_o-\pi W_s}^{\omega_o+\pi W_s}\left|\sum_i a_i \exp[-j\omega d_i]\right|^2 S_x(\omega)\, d\omega}{2\pi W N_o} \quad , \tag{A2-35}$$

where $\theta_d^2 = \omega_o^2\phi_\mu(0)$ is the mean-square phase deviation of the carrier ω_o, and the signal bandwidth is assumed to be $2\pi W_s$.

Applying Eq. (A2-35) to the two-path case shown in Fig. 23a,

$$C \approx \frac{\exp[-\theta_d^2]\,[(1+a^2)\,P + 2a\int_{\omega_o-\pi W_s}^{\omega_o+\pi W_s}\cos\omega(d_1-d_2)\, S_x(\omega)\, d\omega]\log_2 e}{2\pi N_o} \quad . \tag{A2-36}$$

This is the result quoted in Eq. (2-17).

When the path of strength unity does not fluctuate (Fig. 23b), we have, by a similar analysis,

$$C \approx \frac{\left(\left\{1 + a^2\exp[-\theta_d^2]\right\} P + 2a\exp[-\theta_d^2/2]\int_{\omega_o-\pi W_s}^{\omega_o+\pi W_s}\cos\omega(d_1-d_2)\, S_x(\omega)\, d\omega\right)\log_2 e}{2\pi N_o} \quad , \tag{A2-37}$$

which yields Eq. (2-19).

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APPENDIX III

I. CAPACITY ANALYSIS FOR THE SCATTER TRANSMISSION OF A BINARY-MODULATED CARRIER

In this appendix, we evaluate the capacities ${}_{2,2}C_1^s$ and ${}_{2,2}C_2^s$, as given by Eqs. (3-44) and (3-45). As explained in Chap. III, Sec. A, 4, in the system being considered a binary-modulated carrier is transmitted through a channel consisting of scatter and additive white Gaussian noise. ${}_{2,2}C_1^s$ and ${}_{2,2}C_2^s$ are lower bounds to the respective system capacities attained when either one or both components of the received signal are observed. We shall first consider the evaluation of ${}_{2,2}C_1^s$, and then perform the more complicated analysis for ${}_{2,2}C_2^s$.

A. Evaluation of ${}_{2,2}C_1^s$

In performing this analysis, we shall first rewrite Eq. (3-44) in more convenient form as a sum of two integrals and then evaluate these separately. One integrand involves an expression for the entropy of a split Gaussian distribution, found in Appendix IV in the form of an infinite series. Integration is performed term-by-term to obtain the final expression for ${}_{2,2}C_1^s$, quoted in Eq. (3-52).

From Eq. (3-44), we have

$$\Delta_{2,2}C_1^s = -A_1 + A_2 \quad , \tag{A3-1a}$$

where

$$A_1 = -\sum_{x_1}\sum_{x_2} P(x_1)\, P(x_2) \int_{w_1}\int_{w_2} p(w_1, w_2/x_1, x_2) \log p(w_1, w_2/x_1, x_2)\, dw_1\, dw_2 \quad , \tag{A3-1b}$$

$$A_2 = -\sum_{x_2} P(x_2) \int_{w_1}\int_{w_2} p(w_1, w_2/x_2) \log p(w_1, w_2/x_2)\, dw_1\, dw_2 \quad . \tag{A3-1c}$$

From Eqs. (3-21), (3-22) and (3-23),

$$p(w_1, w_2/x_1, x_2) = (2\pi)^{-1} |M_2|^{-1/2} \exp\left[-\frac{1}{2} \sum_{i=1} \sum_{j=1} \frac{M_2^{ij}}{|M_2|} w_i w_j\right] \quad , \tag{A3-2}$$

where

$$M_2 = \begin{bmatrix} (x_1^2 \phi_o + \phi_{no}) & (x_1 x_2 \phi_1 + \phi_{n1}) \\ (x_1 x_2 \phi_1 + \phi_{n1}) & (x_2^2 \phi_o + \phi_{no}) \end{bmatrix} \quad . \tag{A3-3}$$

Here we have set c = 0 in Eq. (3-23), with no actual loss in generality. Also, $\phi_o = \phi_{11} = \phi_{22}$; $\phi_1 = \phi_{12}$, and similarly for ϕ_n. Expanding Eq. (A3-2) for the particular case considered, in which $x_1 = \pm a$, $x_2 \pm a$,

$$p(w_1, w_2/x_1, x_2) =$$

$$\frac{\exp\left\{-[w_1^2 (a^2 \phi_o + \phi_{no}) - 2w_1 w_2 (x_1 x_2 \phi_1 + \phi_{n1}) + w_2^2 (a^2 \phi_o + \phi_{no})]/2[(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2]\right\}}{2\pi[(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2]^{1/2}} , \tag{A3-4}$$

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and the conditional entropy is[1]

$$H_{x_1x_2} = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(w_1, w_2/x_1, x_2) \log p(w_1, w_2/x_1, x_2)\, dw_1\, dw_2$$

$$= \log 2\pi e\, |M_2|^{1/2} \qquad \text{(A3-5)}$$

$$= \log 2\pi e[(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2]^{1/2} \quad .$$

Substituting Eq. (A3-5) in Eq. (A3-1a), we see that $A_1 = H_{x_1x_2}$ since Eq. (A3-5) does not involve x_1 or x_2 explicitly.

From Eq. (A3-4), we obtain $p(w_1, w_2/x_2)$.

$$p(w_1, w_2/x_2) = \sum_{x_1} p(w_1, w_2/x_1, x_2)\, P(x_1)$$

$$= \frac{\exp[-w_1^2/2\sigma_1^2]\left\{\exp[-(w_2 - \beta w_1)^2/2\sigma_2^2] + \exp[-(w_2 + \beta w_1)^2/2\sigma_2^2]\right\}}{4\pi\sigma_1\sigma_2} \quad , \qquad \text{(A3-6)}$$

where

$$\sigma_1^2 = a^2 \phi_o + \phi_{no} \quad , \qquad \text{(A3-7)}$$

$$\sigma_2^2 = \frac{(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2}{a^2 \phi_o + \phi_{no}} \quad , \qquad \text{(A3-8)}$$

$$\beta = \frac{a^2 \phi_1 + \phi_{n1}}{a^2 \phi_o + \phi_{no}} \quad . \qquad \text{(A3-9)}$$

We have used the assumption that $P(x_1 = a) = 1/2 = P(x_1 = -a)$. Substituting Eq. (A3-6) in Eq. (A3-1c), we find

$$A_2 = -\frac{1}{\sqrt{2\pi F}}\int_{-\infty}^{+\infty} \exp[-\alpha^2/2F]\, d\alpha \int_{-\infty}^{+\infty} p(s,\alpha) \log p(s,\alpha)\, ds$$

$$-\int_{-\infty}^{+\infty} \frac{\exp[-w_1^2/2\sigma_1^2]}{\sqrt{2\pi}\,\sigma_1} \log \frac{\exp[-w_1^2/2\sigma_1^2]}{\sqrt{2\pi}\,\sigma_1\sigma_2}\, dw_1 \quad , \qquad \text{(A3-10)}$$

where

$$p(s,\alpha) = \frac{1}{2\sqrt{2\pi}}\left\{\exp[-(s-\alpha)^2/2] + \exp[-(s+\alpha)^2/2]\right\} \quad . \qquad \text{(A3-11)}$$

Here $s = w_2/\sigma_2$, $\alpha = \beta w_1/\sigma_2$, and

$$F = \frac{\gamma^2}{1-\gamma^2} \quad ; \quad \gamma^2 = \frac{a^2 \phi_1 + \phi_{n1}}{a^2 \phi_o + \phi_{no}} \quad . \qquad \text{(A3-12)}$$

Combining Eqs. (A3-1a), (A3-5) and (A3-10), we have

$$\triangle_{2,2}C_1^S = -\frac{1}{\sqrt{2\pi F}}\int_{-\infty}^{+\infty} \exp[-a^2/2F]\, da \int_{-\infty}^{+\infty} p(s,a) \log p(s,a)\, ds - \log\sqrt{2\pi e} \quad , \text{(A3-13)}$$

which is the result quoted in Eq. (3-46).

From Appendix IV, Eq. (A4-10), we have

$$G(a) = -\int_{-\infty}^{+\infty} p(s,a) \log p(s,a)\, ds = \log 2\sqrt{2\pi e} + \log_2 e \left\{-\sqrt{\frac{2}{\pi}}\,|a| \exp[-a^2/2] \right. \quad \text{(A3-14)}$$

$$\left. + (2a^2 - 1)\operatorname{erf}(|a|) + \sum_{k=1}^{\infty} \frac{(-1)^k \exp[2a^2 k(k+1)] \operatorname{erf}[|a| (2k+1)]}{k(k+1)} \right\} \quad .$$

We now have the problem of finding the integral, from Eq. (A3-13).

$$I = \frac{1}{\sqrt{2\pi F}}\int_{-\infty}^{+\infty} \exp[-a^2/2F]\, G(a)\, da \quad . \quad \text{(A3-15)}$$

We may do this by performing the integration on Eq. (A3-14), term by term. The only integrals which are not straightforward have the form

$$J_k = \int_0^{\infty} \exp[-\frac{a^2}{2F} + 2a^2 k(k+1)] \operatorname{erf}[a(2k+1)]\, da \quad . \quad \text{(A3-16)}$$

Referring to Eq. (A4-11) of Appendix IV, we may write

$$J_k = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{a(2k+1)}^{\infty} \exp[-\frac{a^2}{2F} + 2a^2 k(k+1) - \frac{y^2}{2}]\, dy\, da \quad . \quad \text{(A3-17)}$$

We make the substitution $y = a(2k+1)\,z$, obtaining

$$J_k = \frac{(2k+1)}{\sqrt{2\pi}}\int_1^{\infty} dz \int_0^{\infty} a \exp\left\{-\frac{a^2}{2}[\frac{1}{F} - 4k(k+1) + z^2(2k+1)^2]\right\} da \quad . \quad \text{(A3-18)}$$

The inner integral is easily evaluated, giving

$$J_k = \frac{(2k+1)}{\sqrt{2\pi}}\int_0^{\infty} \frac{dz}{\frac{1}{F} - 4k(k+1) + z^2(2k+1)^2} \quad . \quad \text{(A3-19)}$$

According to whether $[(1/F) - 4k(k+1)]$ is positive or negative, Eq. (A3-19) has two forms.

$$J_k = \begin{cases} \dfrac{\tan^{-1}[\sqrt{(1/F) - 4k(k+1)}/(2k+1)]}{\sqrt{2\pi[(1/F) - 4k(k+1)]}} \quad ; & \dfrac{1}{F} \geq 4k(k+1) \\[2ex] \dfrac{\tan h^{-1}[\sqrt{4k(k+1) - (1/F)}/(2k+1)]}{\sqrt{2\pi[4k(k+1) - (1/F)]}} \quad ; & \dfrac{1}{F} < 4k(k+1) \end{cases} \quad . \quad \text{(A3-20)}$$

Thus from Eqs. (A3-13), (A3-16) and (A3-20),

$$\Delta_{2,2}C_1^s = 1 + \frac{\log_2 e}{\pi}\left[-2F + (2F-1)\tan^{-1}(1/\sqrt{F})\right.$$

$$+\left[\begin{array}{ll} \sum_k \dfrac{(-1)^k \tan^{-1}\sqrt{1-4k(k+1)F/(2k+1)^2 F}}{k(k+1)\sqrt{1-4k(k+1)F}} & \text{when } 1 \leqslant k \leqslant k_o \\ \sum_k \dfrac{(-1)^k \tanh^{-1}\sqrt{4k(k+1)F-1/(2k+1)^2 F}}{k(k+1)\sqrt{4k(k+1)F-1}} & \text{when } k_o < k < \infty \end{array}\right] \qquad \text{(A3-21)}$$

where $4k_o(k_o+1)F = 1$. This is the result quoted in Eq. (3-52) with $\Delta = 1/W$.

B. Evaluation of ${}_{2,2}C_2^s$

The analysis for the two-component case follows practically the same method as for observation of one component. From Eq. (3-45), we have

$$\Delta_{2,2}C_2^s = -B_1 + B_2 \quad , \qquad \text{(A3-22a)}$$

where

$$B_1 = -\sum_{x_1}\sum_{x_2} P(x_1)\,P(x_2) \int_{w_{s_1}}\int_{w_{s_2}}\int_{w_{c_1}}\int_{w_{c_2}} p(w_{s1},w_{s2},w_{c1},w_{c2}/x_1,x_2) \qquad \text{(A3-22b)}$$

$$\times \log p(w_{s1},w_{s2},w_{c1},w_{c2}/x_1,x_2)\, dw_{s1}\, dw_{s2}\, dw_{c1}\, dw_{c2} \quad ,$$

$$B_2 = -\sum_{x_2} P(x_2) \int_{w_{s_1}}\int_{w_{s_2}}\int_{w_{c_1}}\int_{w_{c_2}} p(w_{s1},w_{s2},w_{c1},w_{c2}/x_2) \log p(w_{s1},w_{s2},w_{c1},w_{c2}/x_2) \qquad \text{(A3-22c)}$$

$$\times\, dw_{s1}\, dw_{s2}\, dw_{c1}\, dw_{c2} \quad .$$

From Eqs. (3-21), (3-22) and (3-23),

$$p(w_{s1},w_{s2},w_{c1},w_{c2}/x_1,x_2) = (2\pi)^{-2}\,|M_2|\,\exp\left[-\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{M_2^{ij}}{|M_2|}\right](w_{si}\,w_{sj} + w_{ci}\,w_{cj}) \quad , \qquad \text{(A3-23)}$$

where M_2 is given in Eq. (A3-3). The conditional entropy corresponding to Eq. (A3-23) is just twice that of Eq. (A3-5), and, again, is equal to B_1 itself.

$$B_1 = -\int_{w_{s_1}}\int_{w_{s_2}}\int_{w_{c_1}}\int_{w_{c_2}} p(w_{s1},w_{s2},w_{c1},w_{c2}/x_1,x_2) \log p(w_{s1},w_{s2},w_{c1},w_{c2}/x_1,x_2) \qquad \text{(A3-24)}$$

$$\times\, dw_{s1}\, dw_{s2}\, dw_{c1}\, dw_{c2} = 2\log 2\pi e\left[(a^2\phi_o + \phi_{no})^2 - (a^2\phi_1 + \phi_{n1})^2\right]^{1/2} \quad .$$

From Eq. (A3-23), we obtain $p(w_{s1},w_{s2},w_{c1},w_{c2}/x_2)$.

$$p(w_{s1}, w_{s2}, w_{c1}, w_{c2}/x_2) = \sum_{x_1} p(w_{s1}, w_{s2}, w_{c1}, w_{c2}/x_1, x_2)\, P(x_1)$$

$$= \sum_{x_1} \frac{\exp - \dfrac{(w_{s1}^2 + w_{c1}^2)(a^2 \phi_o + \phi_{no}) - 2(w_{s1} w_{s2} + w_{c1} w_{c2})(x_1 x_2 \phi_1 + \phi_{n1}) + (w_{s2}^2 + w_{c2}^2)(a^2 \phi_o + \phi_{no})}{2[(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2]}}{(2\pi)^2 [(a^2 \phi_o + \phi_{no})^2 - (a^2 \phi_1 + \phi_{n1})^2]} P(x_1)$$

$$= \frac{\exp[-(w_{s1}^2 + w_{c1}^2/2\sigma_1^2]\left(\exp\left\{[-(w_{s2} - \beta w_{s1})^2 - (w_{c2} - \beta w_{c1})^2]/2\sigma_2^2\right\} + \exp\left\{[-(w_{s2} + \beta w_{s1})^2 - w_{c2} + \beta w_{c1})^2]/2\sigma_2^2\right\}\right)}{2(2\pi)^2 \sigma_1^2 \sigma_2^2},$$

(A3-25)

where σ_1^2, σ_2^2, and β are given in Eqs. (A3-7), (A3-8) and (A3-9), respectively. To simplify Eq. (A3-25), we perform a rotation transformation. Let

$$w_1 = w_{s2} \frac{w_{s1}}{\sqrt{w_{s1}^2 + w_{c1}^2}} + w_{c2} \frac{w_{c1}}{\sqrt{w_{s1}^2 + w_{c1}^2}} ,$$

(A3-26)

$$w_2 = -w_{s2} \frac{w_{c1}}{\sqrt{w_{s1}^2 + w_{c1}^2}} + w_{c2} \frac{w_{s1}}{\sqrt{w_{s1}^2 + w_{c1}^2}} .$$

Then

$$p(w_1, w_2, w_{s1}, w_{s2}/x_2) = \frac{\exp\left[-\dfrac{w^2}{2\sigma_1^2} - \dfrac{w_2^2}{2\sigma_2^2}\right]\left\{\exp[-(w_1 - \beta w)^2/2\sigma_2^2] + \exp[-(w_1 + \beta w)^2/2\sigma_2^2]\right\}}{2(2\pi)^2 \sigma_1^2 \sigma_2^2}$$

$$= p(w_1, w_2, w) , \qquad \text{(A3-27)}$$

where

$$w = \sqrt{w_{s1}^2 + w_{s2}^2} . \qquad \text{(A3-28)}$$

We may now write the integration of Eq. (A3-22c) in the form

$$B_2 = 2\pi \int_0^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w\, p(w_1, w_2, w) \log p(w_1, w_2, w)\, dw_1\, dw_2\, dw . \qquad \text{(A3-29)}$$

Let $s = w_1/\sigma_2$, $\alpha = \beta w/\sigma_2$. Then

$$B_2 = \frac{-1}{F} \int_0^{\infty} \alpha \exp[-\alpha^2/2F]\, d\alpha \int_{-\infty}^{+\infty} p(s,\alpha) \log p(s,\alpha)\, ds - \int_{-\infty}^{+\infty} \frac{\exp[-w_2^2/2\sigma_2^2]}{\sqrt{2\pi}\,\sigma_2}$$

$$\times \log \frac{\exp[-w_2^2/2\sigma_2^2]}{\sqrt{2\pi}\,\sigma_2} dw_2 - \int_0^{+\infty} \frac{w \exp[-w^2/2\sigma_1^2]}{\sigma_1^2} \log \exp[-w^2/2\sigma_1^2]\, dw$$

$$+ \log 2\pi\sigma_1^2 \sigma_2 , \qquad \text{(A3-30)}$$

where $p(s,\alpha)$ is given by Eq. (A3-11) and F by Eq. (A3-12). Combining Eqs. (A3-22a), (A3-24) and (A3-30),

$$\Delta_{2,2}C_2^S = -\frac{1}{F}\int_0^\infty \alpha \exp\left[-\alpha^2/2F\right] d\alpha \int_{-\infty}^{+\infty} p(s,\alpha) \log p(s,\alpha)\, ds - \log\sqrt{2\pi e} \quad , \qquad \text{(A3-31)}$$

which is quoted in Eq. (3-47).

As in Sec. A of this Appendix, the inner integral of Eq. (A3-31) may be given as the infinite series Eq. (A3-14), expressing the entropy of a split Gaussian distribution. The second integration may then be performed term-by-term. We have the problem of integrating expressions like the following.

$$N_k = \int_0^\infty \alpha \exp\left[-\frac{\alpha^2}{2F} + 2\alpha^2 k(k+1)\right] \operatorname{erf}\left[\alpha(2k+1)\right] d\alpha \qquad \text{(A3-32)}$$

As in Sec. A, we write the error-function in its integral representation.

$$N_k = \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_{\alpha(2k+1)}^\infty \alpha \exp\left[-\frac{\alpha^2}{2F} + 2\alpha^2 k(k+1) - \frac{y^2}{2}\right] dy\, d\alpha \quad . \qquad \text{(A3-33)}$$

Letting $y = \alpha(2k+1)\, z$, we have

$$N_k = \frac{(2k+1)}{\sqrt{2\pi}}\int_1^\infty dz \int_0^\infty \alpha^2 \exp\left\{-\frac{\alpha^2}{2}\left[\frac{1}{F} - 4k(k+1) + z^2(2k+1)^2\right]\right\} d\alpha \quad . \qquad \text{(A3-34)}$$

Evaluating the inner integral,

$$N_k = (k + \tfrac{1}{2})\int_1^\infty \frac{dz}{\left[\frac{1}{F} - 4k(k+1) + z^2(2k+1)^2\right]^{3/2}} \quad . \qquad \text{(A3-35)}$$

Finally,

$$N_k = \frac{1 - (2k+1)/\sqrt{(1/F) + 1}}{2[(1/F) - 4k(k+1)]} \quad . \qquad \text{(A3-36)}$$

From Eqs. (A3-31), (A3-32) and (A3-36), we obtain

$$\Delta_{2,2}C_2^S = 1 + \log_2 e \left\{(2F - \tfrac{1}{2}) - \frac{(2F + 1/2)}{\sqrt{(1/F) + 1}} + \sum_{k=1}^{\infty} \frac{(-1)^k\left[1 - (2k+1)/\sqrt{(1/F) + 1}\right]}{2k(k+1)\left[(1/F) - 4k(k+1)\right]}\right\} . \qquad \text{(A3-37)}$$

This expression is quoted in Eq. (3-55), with $\Delta = 1/W$.

APPENDIX IV

I. THE ENTROPY OF SPLIT GAUSSIAN DISTRIBUTION

A. Analysis

In the course of establishing the lower bounds to channel capacity for scatter transmission, we encounter the integral, in Eqs. (3-46), (3-47), (A3-13) and (A3-31),

$$G(\alpha) = -\int_{-\infty}^{+\infty} p(s,\alpha) \log p(s,\alpha)\, ds \quad , \qquad \text{(A4-1)}$$

where

$$p(s,\alpha) = \frac{1}{2\sqrt{2\pi}} \left\{ \exp\left[-(s-\alpha)^2/2\right] + \exp\left[-(s+\alpha)^2/2\right] \right\} \quad . \qquad \text{(A4-2)}$$

$G(\alpha)$ is recognized as Shannon's definition of entropy for the continuous probability distribution $p(s,\alpha)$. $p(s,\alpha)$, which we shall call a split Gaussian distribution, may be interpreted as the distribution of the sum of a binary variable x and an independent Gaussian variable n. The variable x assumes the values $\pm\alpha$ with equal probability, while n has zero mean and unit variance.

We obtain $G(\alpha)$ by a perturbation method. Suppose, in general, that we have a given $p_o(s)$ and its corresponding entropy H_o.

$$H_o = -\int_{-\infty}^{+\infty} p_o(s) \log p_o(s)\, ds \quad . \qquad \text{(A4-3)}$$

We now cause a perturbation $\Delta p(s)$ in $p_o(s)$, and ask for the new entropy $H_o + \Delta H$.

$$H_o + \Delta H = -\int_{-\infty}^{+\infty} [p_o(s) + \Delta p(s)] \log [p_o(s) + \Delta p(s)]\, ds \quad . \qquad \text{(A4-4)}$$

Let us assume that $|\Delta p(s)| \leq p_o(s)$, a condition which does not always hold, but which can be applied in the particular case considered. Then we may expand the logarithm of Eq. (A4-4) in a convergent series, obtaining

$$H_o + \Delta H = H_o - \int_{-\infty}^{+\infty} \Delta p(s) \log p_o(s)\, ds + \log_2 e \int_{-\infty}^{+\infty} [p_o(s) + \Delta p(s)] \times \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\frac{\Delta p(s)}{p_o(s)}\right]^k ds \quad . \qquad \text{(A4-5)}$$

The last member of Eq. (A4-5) simplifies. Interchanging summation and integration,

$$\Delta H = -\int_{-\infty}^{+\infty} \Delta p(s) \log p_o(s)\, ds - \log_2 e \int_{-\infty}^{+\infty} \Delta p(s)\, ds + \log_2 e \times \sum_{k=1}^{\infty} \frac{\int_{-\infty}^{+\infty} \Delta p(s) \left[\frac{\Delta p(s)}{p_o(s)}\right]^k ds}{(-1)^k\, k(k+1)} \quad . \qquad \text{(A4-6)}$$

To apply the general formula Eq. (A4-6) to our particular case, we let

$$p_o(s) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}} \exp[-(s+\alpha)^2/2] & s \leq 0 \quad , \\ 0 & s > 0 \end{cases} \tag{A4-7}$$

$$\Delta p(s) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}} \exp[-(s-\alpha)^2/2] & ; \quad s \leq 0 \\ 0 & ; \quad s > 0 \end{cases} \quad . \tag{A4-8}$$

Because of the symmetry of $p(s,\alpha)$, it is only necessary to perform the integration over the half-line in order to obtain $G(\alpha)$. In this range, $\Delta p(s) \leq p_o(s)$, as required, assuming that α is positive. Substituting Eqs. (A4-7) and (A4-8) in Eq. (A4-6),

$$\frac{G(\alpha)}{2} = H_o + \Delta H = -\int_{-\infty}^{o} \left\{ \frac{\exp[-(s+\alpha)^2/2] + \exp[-(s-\alpha)^2/2]}{2\sqrt{2\pi}} \right\} \quad .$$

$$\times \log \frac{\exp[-(s+\alpha)^2/2]}{2\sqrt{2\pi}} ds - \frac{\log_2 e}{2\sqrt{2\pi}} \int_{-\infty}^{o} \exp[-(s-\alpha)^2/2] \, ds \tag{A4-9}$$

$$+ \sum_{k=1}^{\infty} \log_2 e \frac{(-1)^k \int_{-\infty}^{o} \exp[2\alpha sk - \frac{(s-\alpha)^2}{2}] \, ds}{2\sqrt{2\pi}\, k(k+1)} \quad .$$

Performing the indicated integrations gives

$$G(\alpha) = \log 2\sqrt{2\pi e} + \log_2 e - \sqrt{\frac{2}{\pi}} \alpha \exp[-\alpha^2/2] + (2\alpha^2 - 1) \operatorname{erf}(\alpha)$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k \exp[2\alpha^2 k(k+1)] \operatorname{erf}[\alpha(2k+1)]}{k(k+1)} \quad , \tag{A4-10}$$

where

$$\operatorname{erf} x = \frac{1}{2\pi} \int_{x}^{\infty} \exp[-y^2/2] \, dy \quad . \tag{A4-11}$$

The series (A4-10) is quoted in Eqs. (3-51) and (A3-14) where α appears as $|\alpha|$. For $|\alpha| \gtrsim 1$, a good approximation to Eq. (A4-10) is obtained by expanding error functions in asymptotic series

$$G(\alpha) \approx \log 2\sqrt{2\pi e} + \log_2 e \left\{ -0.7979 |\alpha| \exp[-\alpha^2/2] + (2\alpha^2 - 1) \operatorname{erf}(|\alpha|) \right.$$

$$- \frac{\exp[4\alpha^2] \operatorname{erf}(3|\alpha|)}{2} + \frac{\exp[-\alpha^2/2]}{\alpha} [0.010003 - \frac{0.0004564}{\alpha^2} \tag{A4-12}$$

$$\left. + \frac{0.0000622}{\alpha^4}] \right\} \quad ; \quad |\alpha| \gtrsim 1 \quad .$$

In Figure 27, $G(\alpha)$ is plotted as a function of $|\alpha|$. For comparison, the entropy $H'(\alpha)$, which would be obtained if s had a Gaussian distribution of variance $(1 + \alpha^2)$, is also plotted.

$$H'(\alpha) = -\int_{-\infty}^{+\infty} \frac{\exp[-s^2/2(1+\alpha^2)]}{\sqrt{2\pi(1+\alpha^2)}} \log \frac{\exp[-s^2/2(1+\alpha^2)]}{\sqrt{2\pi(1+\alpha^2)}} ds$$
$$= \log \sqrt{2\pi e(1+\alpha^2)} \quad . \tag{A4-13}$$

For $|\alpha|$ less than about 1, the agreement between $G(\alpha)$ and $H'(\alpha)$ is very good. This is a demonstration of Shannon's entropy-absorption theorem (Appendix II, Reference 3).

B. System Capacity for Binary Digits in Gaussian Noise

In this section, we consider the system capacity obtained when the code-generating source at the transmitter has a binary output, and the channel noise n(t) is white Gaussian. That is, we assume that both the noise and coding source are limited to the band from zero to W, and that the source generates a time series x(t) having the values $\pm\alpha$ at points 1/2W apart in time. These two values are equally-likely and statistically independent from sampling-point to sampling-point. The noise is normalized to have unit variance. Then the system capacity C_b is given by

$$C_b = 2W[H(s) - H_x(s)] \quad , \tag{A4-14}$$

where H(s) is the entropy of the received time series s(t) = x(t) + n(t), for just one sampling point. Similarly, $H_x(s)$ is the conditional entropy, again for just one sampling point, or degree of freedom. From the discussion in Section A of this appendix, we see that $H(s) = G(\alpha)$. Further, $H_x(s)$ is simply the noise entropy per degree of freedom.

$$H_x(s) = -\int_{-\infty}^{+\infty} \frac{\exp[-n^2/2]}{\sqrt{2\pi}} \log \frac{\exp[-n^2/2]}{\sqrt{2\pi}} dn = \log \sqrt{2\pi e} \quad . \tag{A4-15}$$

Combining Eqs. (A4-10), (A4-14), and (A4-15),

$$\frac{C_b}{2W} = 1 + \log_2 e \left\{ \sqrt{\frac{2}{\pi}} |\alpha| \exp[-\alpha^2/2] + (2\alpha^2 - 1) \operatorname{erf} |\alpha| + \sum_{k=1}^{\infty} \frac{(-1)^k \exp[2\alpha^2 k(k+1)] \operatorname{erf}(|\alpha| \, 2k+1)}{k(k+1)} \right\} \quad . \tag{A4-16}$$

This expression is plotted in Figure 29, where it may be compared with the optimum Gaussian transmission for the additive Gaussian noise channel, and also with binary transmission through scatter and noise. Here, $R = \alpha^2$. For small R, the two types of transmission, through the channel perturbed by additive noise only, are seen to agree closely in system capacity.

The preceding analysis has assumed that both the signal and noise are low-pass. If the transmission is of bandwidth W and is bandpass, instead, the sampling points are taken 1/W apart in time. Two components, the sine and cosine, are now observed at each sampling point, thus preserving the number of degrees of freedom. Furthermore, the noise components are independent, so that if the amplitudes of the signal components at the sampling points are selected independently to be $\pm\alpha$, the analysis which leads to Eq. (A4-16) is still applicable. If, however, the bandpass signal transmitted is required to be amplitude-modulated, so that the cosine component is proportional to the sine component, the number of degrees of freedom is halved. Since the vector representing the transmitted signal now remains fixed in direction, we may simplify the problem of finding the system capacity for an AM transmission by choosing new Euclidean reference axes so that effectively just one component is being transmitted. The receiver need only observe the component of the received signal in phase with the transmitted signal in order to extract all information about the transmitted signal. Observation of the quadrature component of the received signal yields no information about either the signal or the in-phase component of the noise. For the same signal-to-noise ratio, the amplitude of the single transmitted component in the binary AM case is now $\sqrt{2}$ that of the separate components in the general binary transmission first considered. Thus we have the same result for the system capacity C'_b of a binary AM transmission in Gaussian noise that we would obtain for a low-pass binary transmission with twice the power and half the bandwidth.

$$\frac{C'_b}{W} = 1 + \log_2 e \left\{ \sqrt{\frac{2}{\pi}} \, |2\alpha| \, \exp\left[- 2\alpha^2\right] + (8\alpha^2 - 1) \, \text{erf}\,(|2\alpha|) + \sum_{k=1}^{\infty} \frac{(-1)^k \exp\left[2\alpha^2 k(k+1)\right] \text{erf}\,(|2\alpha| \; 2k+1)}{k(k+1)} \right\} . \qquad \text{(A4-17)}$$

Equation (A4-17) is plotted in Figure 29, where it may be compared with other system capacities. Again, $R = \alpha^2$. For small R, the capacities for binary AM, general binary, and Gaussian transmissions through Gaussian noise are seen to agree closely.

APPENDIX V

I. EXPERIMENTAL STUDY OF IONOSPHERIC FLUCTUATIONS

In Sec. C, 2 of Chapter II, a technique for measuring ionospheric fluctuations is discussed, together with particular results which have been obtained using that technique. The measurement consists of transmitting a sinusoidal carrier over the path to be studied, and performing a statistical analysis of the incoming signal at the receiving point. In the previous section receiving equipment details were omitted for simplicity, the block diagram of Fig. 17 sufficing to describe the general operation. Since there may be interest in constructing similar equipment for extending these measurements to other geographical locations and frequencies, this appendix is concerned with details of some of the more complex receiving apparatus.

First, we shall mention that there is no control over the transmitted signal in our measurement, since WWV is selected as the stable transmission. Although the 440 and 4000 cps modulations present in this signal are filtered out, and hence are of no consequence in the measurement, WWV departs in another respect from pure carrier transmission. When one-minute voice announcements are made over the station, the carrier has been observed to change slightly from its normal amplitude, and small fluctuations coherent with the voice modulation have even been observed. The effect of this imperfection is probably negligible in the measurement; nevertheless it would be desirable, in future measurements, to have a transmitting station with the stability of WWV, but with no modulation. At the minimum, the stability should be better by a factor of 10 than that expected for the ionosphere.

At the receiver a conventional single-wire antenna is used, tapped near the center, unbalanced, and about 200 feet long. Before proceeding into the tuned-radio frequency amplifier indicated in Fig. 17, the signal is prefiltered by a crystal with approximately 100 cps bandwidth. This is necessary because of strong local interference which overloads amplifier stages preceding the final noise-suppressing filter. The amplifier itself is quite straightforward, having a bandwidth of about 40 kcps and an available gain of approximately 50,000. The stages are synchronously single-tuned, and a detector is provided so that WWV may be monitored aurally.

The local standard used for heterodyning is a General Radio Primary Standard, type 1101-A. The nominal stability accuracy is 1 part in 10^7, but for short periods, such as during the half-hour measurement made for the results presented in Chapter II, Sec. C, 2, the stability is more like 1 part in 10^8. Since it is necessary to generate locally a frequency of 5.000020 mcps derived from the 100.00000 kcps standard, and since pulling of the standard itself cannot be permitted, a rather complex frequency converter is employed. The procedure decided upon is to single-sideband modulate a 5.000000 mcps signal, obtained by frequency multiplication from the standard, with a 20 cps wave generated by frequency division. The frequency multiplier is conventional,[1] the required factor of 50 being obtained in stage multiplications of 2, 5, and 5. Divider circuits provided by General Radio (model 1102-A) with the primary standard yield a 100 cps output. The 100 cps is divided into another factor of 5 and filtered to obtain a pure 20 cycle signal, as required. The circuits to perform this last operation are shown in schematic form in Fig. 32. A twin-T feedback amplifier[2] is used for the narrow-band filtering necessary.

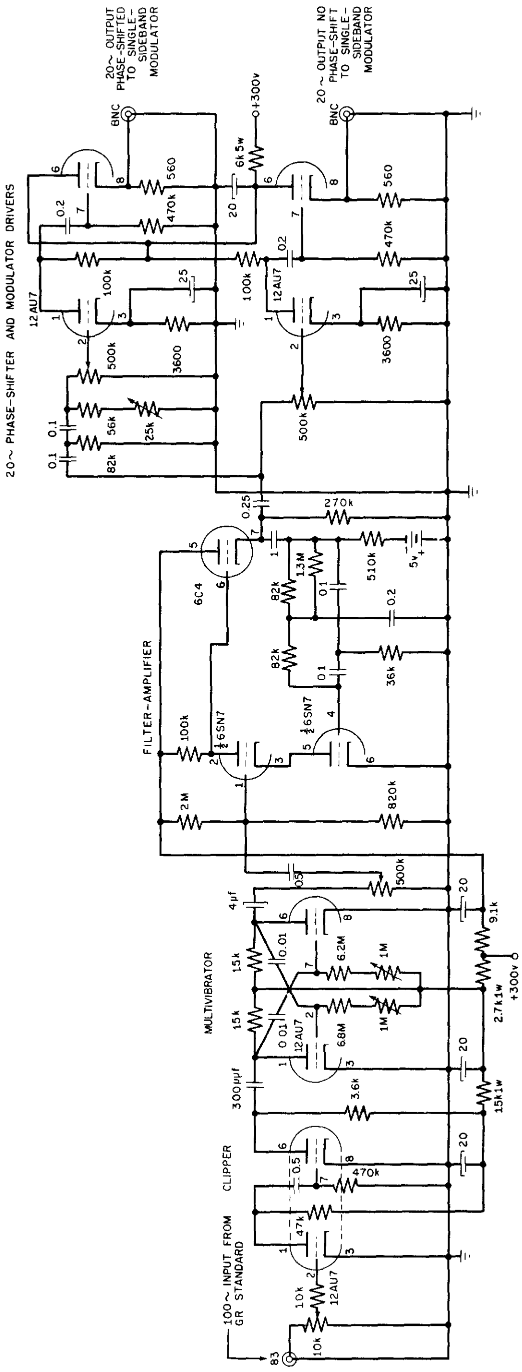

Fig. 32. The 100 cps to 20 cps divider.

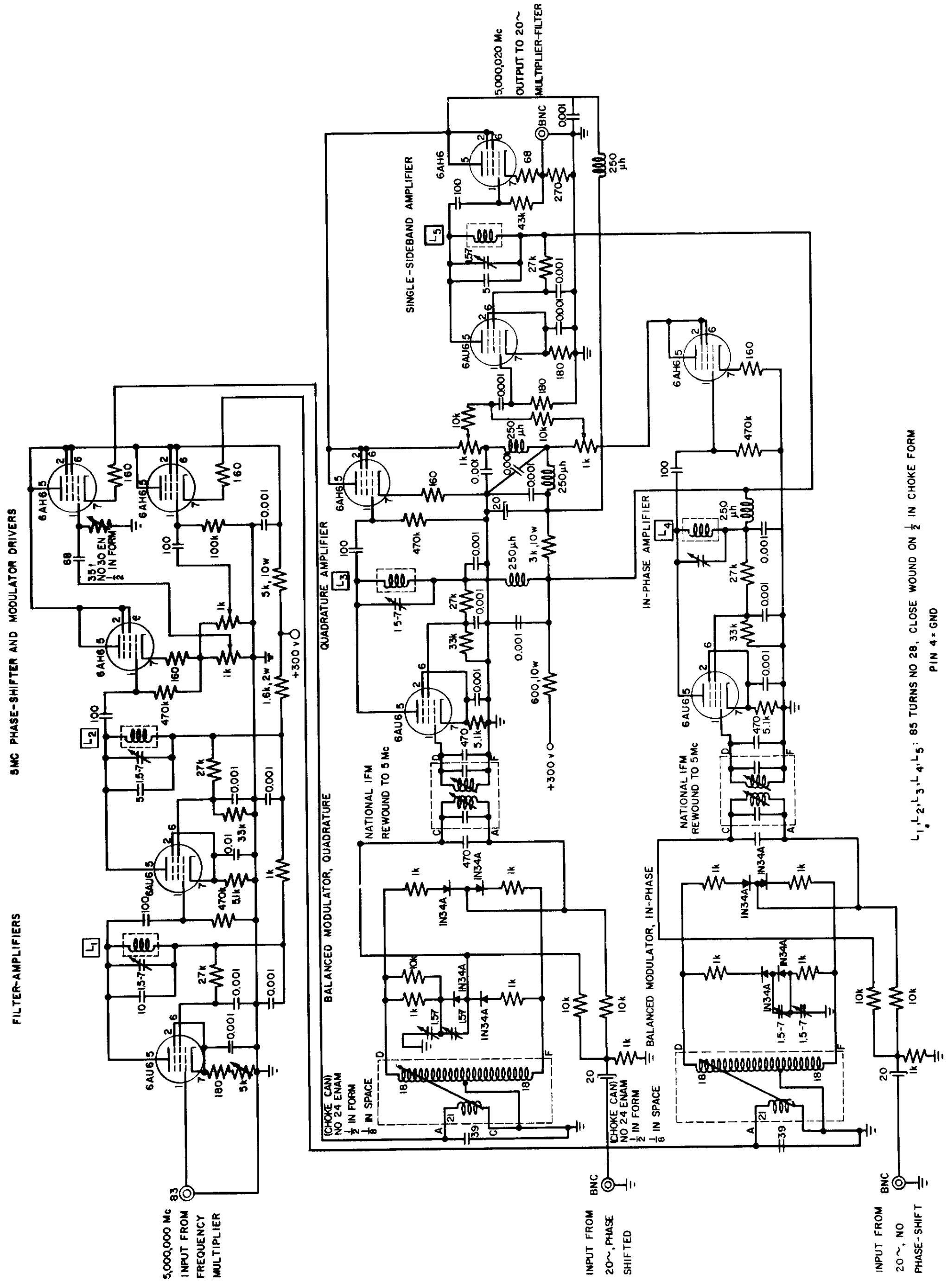

Fig. 33. The single-sideband modulator.

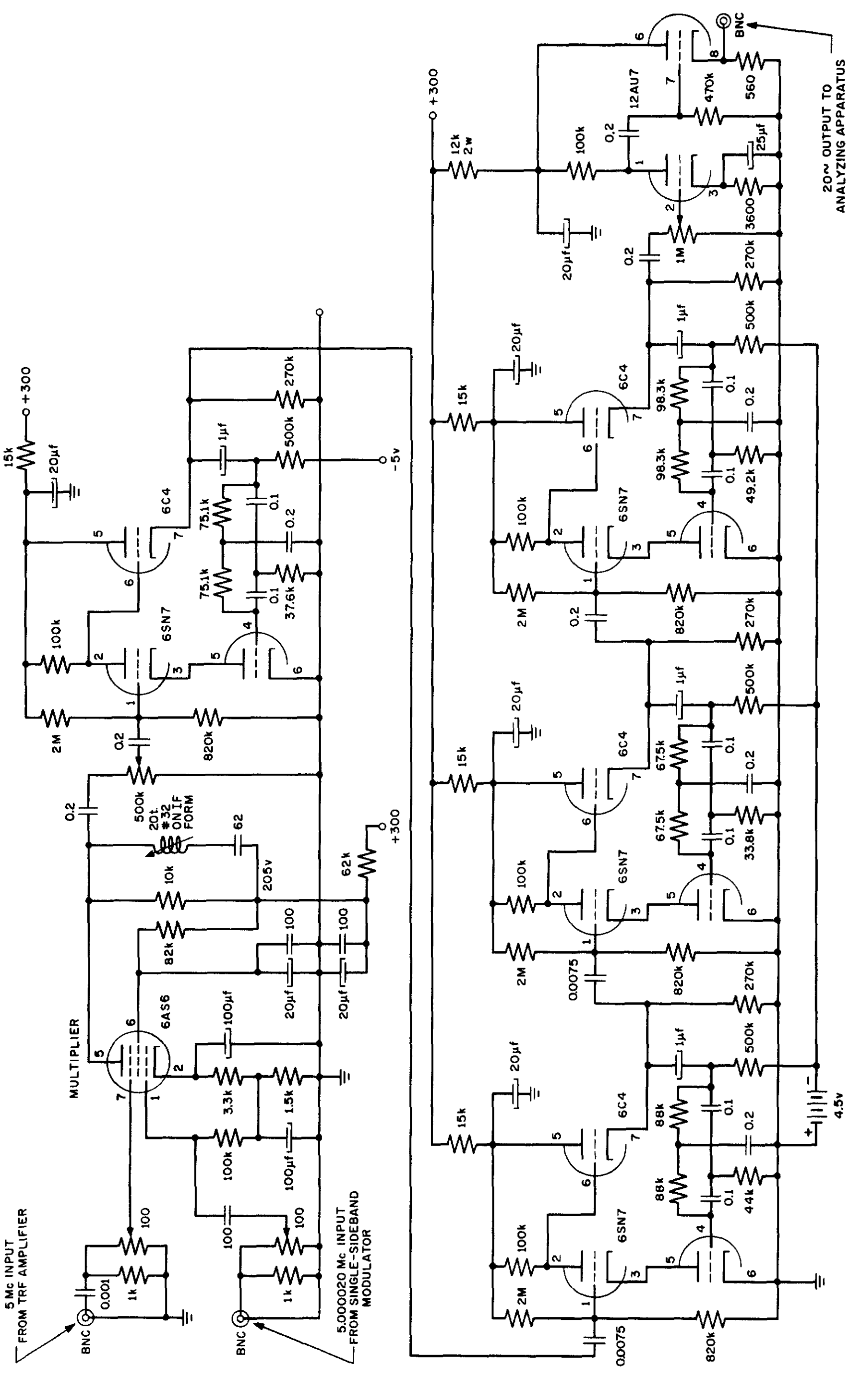

Fig. 34. The 20 cps multiplier-filter.

As stated earlier, the 5.000000 mcps and 20 cps signals are now mixed in a single-sideband modulator. The modulator, shown in Fig. 33, performs the following mathematical operation literally.

$$\cos 2\pi(x + y)t = \cos 2\pi xt \cos 2\pi yt - \sin 2\pi xt \sin 2\pi yt \quad . \qquad \text{(A5-1)}$$

That is, letting x = 20 cps and y = 5 mcps, we obtain a signal, $\cos 2\pi(x + y)t$, at 5.000020 mcps by subtracting two carriers which have been suppressed-carrier-AM modulated. The first term of Eq. (A5-1) is obtained by amplitude-modulating the 5.000000 mcps signal with the 20 cps, in a balanced modulator so that the carrier is suppressed. The second term is obtained by an identical operation, except that both signals are now in quadrature to the first pair being mixed. 4.999980 mcps is obtained by simply adding, rather than subtracting, the terms in Eq. (A5-1). Of course, either 4.999980 mcps or 5.000020 mcps is a satisfactory frequency with which to heterodyne the received 5 mcps wave down to a 20 cps wave.

Having now obtained a 5.000020 (or 4.999980) mcps signal, we convert the received signal from WWV, centered at 5.000000 mcps, to a signal centered at 20 cps, and then filter to suppress adjacent channel noise or other interference. The equipment performing these functions is shown in Fig. 34. A simple multi-grid converter is sufficiently linear for the heterodyning process. The following filter, however, is complex enough that a detailed description will be given here.

We wish to have a bandpass filter, centered at 20 cps, which will pass all frequencies between, say, 16 and 24 cps with little change in gain, while greatly suppressing all other frequencies. A faithful measurement of receiver power spectra up to 8 cycles wide, about the 5 mcps carrier, will then be possible. There are no requirements on phase linearity of the filter, since it is only the power spectrum which is desired. A satisfactory bandpass filter, for our purpose, is one which has a maximally-flat,[3] or Butterworth, characteristic. Filters of this class have responses H(f) which can be written

$$H(f) = \frac{K}{1 + \frac{1}{a^2}\left(\frac{f}{f_o} - \frac{f_o}{f}\right)^{2n}} \qquad \text{(A5-2)}$$

where K is a constant, f_o is the geometric mean of the half-power frequencies, and af_o is the half-power bandwidth. The order, n, of Eq. (A5-2), determines the complexity of the filter, and the steepness with which the gain drops off at the edges of the passband. The filter which has

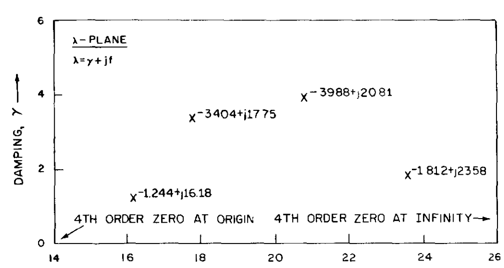

Fig. 35. Pole-and-zero diagram of bandpass filter (showing only the upper left quadrant of the λ-plane).

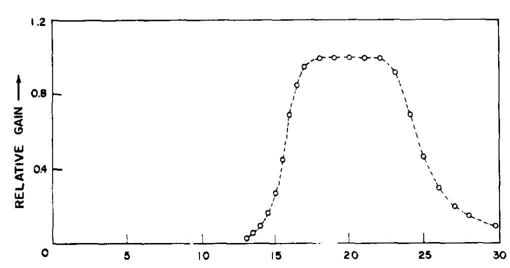

Fig. 36. Measured response of bandpass filter.

been constructed is designed for n = 4, with half-power cutoff frequencies of 16 and 24 cps. The corresponding pole-and-zero diagram is shown in Fig. 35. The poles nearly fall on a semicircle.

The physical realization of the given filter involves the construction of resonant circuits having moderate Q at very low frequencies. It was decided not to use inductances, both for reasons of bulk and Q requirements. Rather, the resonances, which give the poles of Fig. 35, are synthesized with twin-T feedback amplifiers.[2] A series of four such stages in cascade provide the desired characteristic of Fig. 35. A measurement of the frequency response which has been achieved experimentally is presented in Fig. 36.

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3. *Ibid.*, Sec. 4.7.