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CRITERIA FOR DOCILE BEHAVIOR OF FEEDBACK AMPLIFIERS

Samuel J. Mason

Abstract

A docile amplifier is one that remains stable when connected to an arbitrary passive network of a specified type. Docility criteria are developed for end-loading, for ideal-transformer feedback, and for an arbitrary passive feedback network.

INTRODUCTION

In the design of amplifier circuits to be used in conjunction with passive loading or coupling networks, the question of docility arises. A docile amplifier is defined here as one that remains stable when connected to any passive network of a specified type. Figure 1 shows the amplifier Z and the attached passive two-port (two-terminal-pair) network N. The amplifier is represented as a two-port linear active device with open-circuit impedances Z_{11}, Z_{12}, Z_{21}, Z_{22}. Network N loads each end of the amplifier and may also provide external feedback.

The amplifying system is presumably driven by an independent signal source but the drive may be ignored in a stability analysis. We shall need only the equations of the amplifier

$$E_1 = Z_{11}I_1 + Z_{12}I_2 \tag{1}$$

$$E_2 = Z_{21}I_1 + Z_{22}I_2 \tag{2}$$

These may be recast as the matrix equation

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} Z_{11} Z_{12} \\ Z_{21} Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \tag{3}$$

or, in more abbreviated form,

$$E = ZI \tag{4}$$

The net complex power delivered to the amplifier is very conveniently expressed as the matrix product

$$E_t\bar{I} = (E_1 E_2)\begin{pmatrix} \bar{I}_1 \\ \bar{I}_2 \end{pmatrix} = E_1\bar{I}_1 + E_2\bar{I}_2 \tag{5}$$

where the subscript t denotes transposition. Docility can be assured by requiring that the real power shall be positive, and this requirement leads to conditions upon Z. We shall first state the conditions and then present proofs.

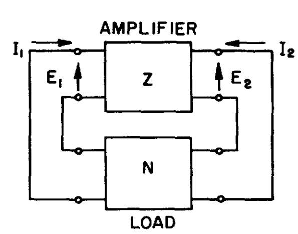


Fig. 1
The loaded amplifier.

DOCILITY CRITERIA

Table I shows the three types of passive loading considered here: arbitrary feedback, ideal-transformer feedback, and end loading. For a given load type, the amplifier is docile if and only if:

(a) The amplifier is open-circuit stable, that is, Z_{11}, Z_{12}, Z_{21}, and Z_{22} have no poles in (or touching) the right half of the complex-frequency plane.

(b) The open-circuit input and output resistances, R_{11} and R_{22}, are positive at all real frequencies.

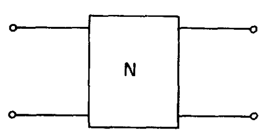
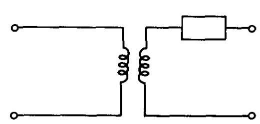
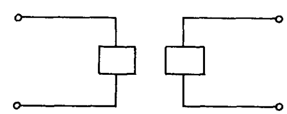
(c) The inequality shown in Table I is satisfied at all real frequencies.

Moreover, if the amplifier is not docile for a given load type, instability can always be produced by a lossless network of that type.

Conditions (a) and (b) are obvious. Taken together, (a) and (b) insure the positiveness of R_{11} and R_{22} in the right half plane, also necessary for stability under passive loading. The same is true of the inequality (c). Conditions (a) and (c), taken together, imply the satisfaction of the inequality in the right half plane. It remains to justify the table.

Table I

Docility Criteria for Three Load Types.

Type of Passive Load N / Class of Amplifier	Arbitrary Feedback 	Ideal-Transformer Feedback 	End Loading 
general: $Z_{12} \neq Z_{21}$	$\left\lvert \frac{Z_{21} + \overline{Z}_{12}}{2} \right\rvert^2 < R_{11}R_{22}$	$\left(\frac{R_{21} + R_{12}}{2}\right)^2 < R_{11}R_{22}$	$\left(\mathrm{Re}\sqrt{Z_{21}Z_{12}}\right)^2 < R_{11}R_{22}$
reciprocal: $Z_{12} = Z_{21}$	$R_{21}^2 < R_{11}R_{22}$	$R_{21}^2 < R_{11}R_{22}$	$R_{21}^2 < R_{11}R_{22}$
antireciprocal: $Z_{12} = -Z_{21}$	$X_{21}^2 < R_{11}R_{22}$	no restriction	$X_{21}^2 < R_{11}R_{22}$
unilateral: $Z_{12} = 0$	$\left\lvert \frac{Z_{21}}{2} \right\rvert^2 < R_{11}R_{22}$	$\left(\frac{R_{21}}{2}\right)^2 < R_{11}R_{22}$	no restriction
resistive: $X = 0$	$\left(\frac{R_{21} + R_{12}}{2}\right)^2 < R_{11}R_{22}$	$\left(\frac{R_{21} + R_{12}}{2}\right)^2 < R_{11}R_{22}$	$R_{21}R_{12} < R_{11}R_{22}$

ARBITRARY FEEDBACK

The real power delivered to the amplifier is

$$P = \mathrm{Re}\left[E_t\bar{I}\right] = \frac{1}{2}\left(E_t\bar{I} + I_t\bar{E}\right) \tag{6}$$

From Eq. 4 we find

$$E_t = I_t Z_t \tag{7}$$

so that

$$P = \frac{1}{2}\left(I_t Z_t \bar{I} + I_t \bar{Z}\bar{I}\right) = I_t \left(\frac{Z_t + \bar{Z}}{2}\right)\bar{I} \tag{8}$$

In order for currents I_1 and I_2 to exist in the undriven system, the power delivered to the amplifier must equal the power supplied by the feedback network. Since a passive network cannot supply power for complex frequencies in or touching the right half plane, the system can be made unstable if and only if P is negative somewhere in that frequency domain. Moreover, the feedback network can certainly be chosen to give any desired magnitude ratio and phase difference between I_1 and I_2. Hence matrix I is completely arbitrary. The docility condition, therefore, is that P must be positive at all real frequencies. (Open-circuit stability of Z then assures positiveness in the right half plane.) The square matrix appearing in Eq. 8 is

$$\begin{pmatrix} R_{11} & \dfrac{Z_{21} + \bar{Z}_{12}}{2} \\ \dfrac{Z_{12} + \bar{Z}_{21}}{2} & R_{22} \end{pmatrix} \tag{9}$$

The determinant and principal minors are real. In order for P to be positive for all I, this matrix must be positive definite, that is

$$R_{11} > 0, \quad R_{22} > 0 \tag{10}$$

$$R_{11}R_{22} - \left|\frac{Z_{21} + \bar{Z}_{12}}{2}\right|^2 > 0 \tag{11}$$

This completes the proof for arbitrary feedback. The result, though different in form, is identical with Tellegen's general passivity condition. (See Bibliography.)

IDEAL-TRANSFORMER FEEDBACK

The transformer feedback network shown in Table I differs from arbitrary feedback in that the quotient of I_1 and I_2 must be real. Hence the current matrix may be written as a real matrix I multiplied by an exponential phase shift $\exp(j\theta)$, and Eq. 8 becomes

$$P = I_t \left(\frac{R_t + R}{2} \right) I \qquad (12)$$

where I is real. For transformer feedback, therefore, Eq. 11 is replaced by

$$R_{11} R_{22} - \left(\frac{R_{21} + R_{12}}{2} \right)^2 > 0 \qquad (13)$$

END LOADING

To investigate end loading, we shall attach a passive impedance Z_2 to the amplifier output and then examine the resulting input impedance Z_1' appearing at the opposite port. From Eqs. 1 and 2

$$Z_1' = \frac{E_1}{I_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_2} \qquad (14)$$

where

$$Z_2 = - \frac{E_2}{I_2} \qquad (15)$$

It follows that

$$\frac{Z_{22} + Z_2}{(Z_{12} Z_{21})^{1/2}} = \frac{(Z_{12} Z_{21})^{1/2}}{Z_{11} - Z_1'} \qquad (16)$$

or, in normalized form,

$$\frac{1 + \left(\frac{Z_2 + jX_{22}}{R_{22}} \right)}{\left(\frac{Z_{12} Z_{21}}{R_{11} R_{22}} \right)^{1/2}} = \frac{\left(\frac{Z_{12} Z_{21}}{R_{11} R_{22}} \right)^{1/2}}{1 - \left(\frac{Z_1' - jX_{11}}{R_{11}} \right)} \qquad (17)$$

Equation 17 may be interpreted geometrically as the inversion shown in Fig. 2. Point p is inverted about point r to obtain point q. As p moves up the $R_2 = 0$ line, q describes a clockwise circle, as shown. The conformality of the inversion makes the circle tangent to a vertical line when angle α reaches zero. At this point Eq. 17 is real, and the dimensions indicated in the figure follow directly. It is tacitly assumed that R_{11} and R_{22} are positive at real frequencies.

For stability under arbitrary passive end-loading we must have R_1' positive. Hence the necessary and sufficient condition for docile behavior is

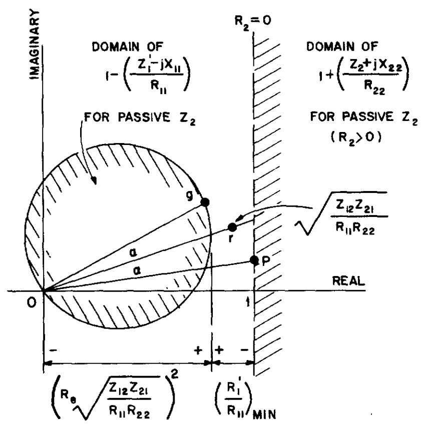


Fig. 2

The input-impedance diagram.

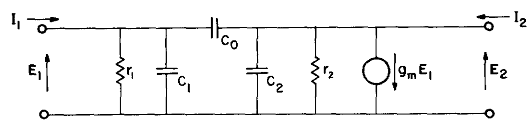


Fig. 3

A two-port linear vacuum-tube circuit.

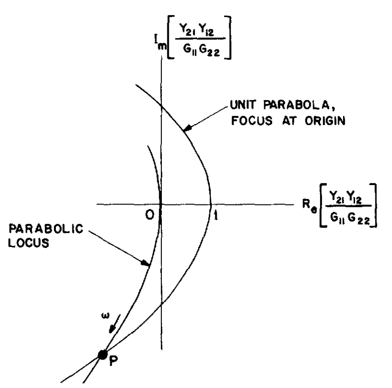


Fig. 4

Complex locus of $\frac{Y_{21}Y_{12}}{G_{11}G_{22}}$ for real frequencies.

$$R_{11}R_{22} - \left[\mathrm{Re}(Z_{12}Z_{21})^{1/2}\right]^2 > 0 \qquad (18)$$

at all real frequencies.

AN ILLUSTRATIVE EXAMPLE

Since the linear vacuum-tube model shown in Fig. 3 is short-circuit stable, it will be convenient to analyze it on the admittance basis. Suppose that we want to find the lowest possible frequency of oscillation under passive end-loading. The admittance matrix is

$$Y = \begin{pmatrix} \frac{1}{r_1} + j\omega(C_1 + C_o) & -j\omega C_o \\ g_m - j\omega C_o & \frac{1}{r_2} + j\omega(C_2 + C_o) \end{pmatrix} \qquad (19)$$

The dual form of the criterion given in Table I is, after a slight rearrangement,

$$\left| \mathrm{Re} \left(\frac{Y_{21}Y_{12}}{G_{11}G_{22}} \right)^{1/2} \right| < 1 \qquad (20)$$

It is easy to demonstrate that the inequality is satisfied at those real frequencies for which the complex locus of the quantity $Y_{21}Y_{12}/G_{11}G_{22}$ remains on the concave side of the unit parabola shown in Fig. 4. To plot the locus we evaluate

$$\frac{Y_{21}Y_{12}}{G_{11}G_{22}} = -j\omega C_o\,(g_m - j\omega C_o)\, r_1 r_2 \qquad (21)$$

and recognize the real and imaginary parts as the coordinates of a parabola, whose focal length

$$f = \frac{1}{4}\, g_m^2\, r_1 r_2 \qquad (22)$$

happens to be equal to the low-frequency matched power gain of the amplifier. The parabolas intersect at point p where the frequency ω reaches the value

$$\omega_o = \frac{g_m}{2C_o}\left[f(f-1)\right]^{-1/2} \qquad (23)$$

For a power gain f considerably larger than unity (the usual case), relation 23 may be rewritten as the power-gain docile-bandwidth product

$$f\,\omega_o \approx \frac{g_m}{2C_o} \qquad (24)$$

The critical value ω_o is the lowest real frequency at which the amplifier will oscillate under passive end-loading.

For a chosen load the oscillations may start at a complex frequency whose imaginary part ω is less than ω_o. As the amplitude grows, however, the nonlinear behavior of the actual vacuum tube will bring about a reduction in the effective value of g_m, and the oscillations will therefore settle at a real frequency greater than ω_o, since a smaller g_m means a smaller focal length f. Such a gross simplification of the nonlinear effects is legitimate when the equilibrium amplitude of oscillations is not too large and when the steady-state waveform is nearly sinusoidal – in short, when the system is quasi-linear.

For completely docile behavior, of course, the focal length of the locus must be less than unity.

$$\frac{1}{4} g_m^2 r_1 r_2 < 1 \tag{25}$$

Incidentally, the docility requirement is the same for arbitrary feedback loading in this particular problem.

GENERAL DISCUSSION

For reciprocal amplifiers the criteria of all three load types reduce to the classical Gewertz condition. Hence, a docile reciprocal amplifier has all the properties of a passive two-port network, insofar as measurements at the available ports can show. Moreover, if the criterion is violated, any one of the three load types can be adjusted to cause instability.

The antireciprocal (or gyratory) amplifier is always stable under transformer feedback, provided conditions (a) and (b) above are met. Its transfer reactance is restricted in the same manner by both end-loading and arbitrary feedback loading. Hence a gyrator having purely resistive transfer impedances is universally docile.

A unilateral amplifier meeting conditions (a) and (b) is always docile for passive end-loading. With arbitrary feedback applied, the docility condition is recognizable as a requirement that the available power gain must not exceed unity. Otherwise a suitable feedback matching network could be chosen to make the system a feedback oscillator.

For the general amplifier, the end-loading criterion imposes a restriction upon the product $Z_{12}Z_{21}$. With transformer feedback or arbitrary feedback, however, we see that the restriction is upon the sum of two transfer quantities. Under end-loading, docility is assured if either of the two transfer impedances can be made sufficiently small, since the internal feedback loop runs forward through Z_{21} and returns via Z_{12} to close upon itself. With external feedback, however, either Z_{12} or Z_{21} can produce appreciable loop transmission. Hence it is not surprising that each must be restricted even when the other vanishes.

It is apparent that the docility criterion for arbitrary feedback is a severe restriction and that an amplifier satisfying such a requirement is indeed a very docile animal and not really a power amplifier at all. For this load type the criterion becomes useful in the design of feedback oscillators, since we may wish to calculate the nondocile real-frequency range over which a tuned feedback network can produce oscillations.

The criterion for end-loading is of great interest in the analysis of cascaded amplifiers. An amplifier docile under end-loading is a respectable and dependable device. Passively loaded at one end, it presents a passive impedance at its opposite end. Hence a cascade of such devices is itself docile for loading at the two extreme terminations. The design may violate the criterion in a chosen band of real frequencies, provided the terminations are controlled in this region with a sufficient margin of safety. If the device satisfies the criterion outside this band, we can be assured of stability no matter how the passive terminating or interstage impedances behave at extra-band frequencies.

Most grounded-base and grounded-emitter junction transistors, incidentally, fail to meet the end-loading docility requirement in a midband of real frequencies, although they are both open-circuit stable and short-circuit stable and docile at both high and low frequencies.

Acknowledgment

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