ON BASIC EXISTENCE THEOREMS IN NETWORK SYNTHESIS
IV. TRANSMISSION OF PULSES

M. V. CERRILLO
E. F. BOLINDER

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RESEARCH LABORATORY OF ELECTRONICS
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Abstract

This report is the fourth in a series of reports on basic existence theorems in network synthesis. The parameter representation of the time and frequency domain functions derived in earlier reports in the series has been used for an investigation of the delicate problem of transmission of narrow pulses through finite passive networks.

The report starts by stating and proving the necessary and sufficient condition for a transfer function to be rational. A simple and computationally appropriate relationship between the parameter function $U(0, \lambda)$ and the time function, which eliminates the integration, is then established by means of the so-called corner theorems. Due to discontinuous time derivatives, the Stieltjes integral representation has to be used. The $U(0, \lambda)$ function is then approximated in such a way that the famous phenomenon of "ringing" is completely controlled. A rational transfer function is obtained from the approximated U function by means of an interpolation procedure. Some illustrative examples are given. The important question of minimum number of elements is treated in connection with the correlation of the number of extremal points of the U function and the number of poles of a rational transfer function.

Finally, rational approximations of transfer functions are obtained by means of a powerful nonlinear process of summability which contains the Padé summability as a special case. The subject will be continued in Part V of this series (Technical Report No. 270).

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ON BASIC EXISTENCE THEOREMS IN NETWORK SYNTHESIS

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Necessary and Sufficient Condition for a Transfer Function to be Rational. On the Transmission of Impulses. On "Corner" Theorems. On the Question of Minimum Number of Elements. Ringing. Rational Approximations of Transfer Functions by Means of a Nonlinear Process of Summability.

This report is Part IV of a series of reports which partially describe the progress in this field. Part I appeared in the Quarterly Progress Report, January 15, 1952; Part II appeared in the Quarterly Progress Report, April 15, 1952; Part III has been published as Technical Report No. 233.

Section IV-1

A Basic Theorem on Rational Transfer Functions

The necessary and sufficient condition for a transfer function to be rational. Construction of rational transfer functions. Simple and multiple poles. Discrete networks.

IV-1.0 Introduction. In previous issues of the Quarterly Progress Report, references 1 and 2, we produced a series of theorems concerning transfer functions. These theorems have a constructive character; they lead to the simple "lattice" structure and demonstrate its generality for both concentrated-element and distributed-element networks. All these theorems are derived from a parametric integral representation of Laplace's direct and inverse transforms. See references 1 and 2.

Part III of these notes deals with the network constructions derived from "monogenic functions" having singular lines in the s plane. The singular lines are generated by an everywhere dense distribution of poles or sets of poles along such lines. The pole distributions have bounded density distribution functions. From this density distribution function we are able, by means of a series of basic theorems of existence, to construct the corresponding transfer function associated with such a singular line of pole distribution. Conversely, from a given transfer function we can find the corresponding density distribution function associated with prescribed lines which will constitute the natural boundaries of the monogenic representation of the given transfer function. In Part III, reference 3, we have shown, with ample constructive detail, the existence of singular lines which are formed by pole-zero chains. Discrete-finite networks can be derived from such pole-zero configuration by constructing sequences of pole-zero chains along those lines whose poles become denser and denser as the sequence index tends towards infinity. The limiting pole-zero chains have a density distribution function which is equal to the one corresponding to the given transfer function. The construction of such pole-zero chain sequences is shown in Part III.

From a formal point of view, the "monogenic" approach to transfer functions is a quite correct and illuminating process. As a matter of fact, Part III provides the foundation of the so-called potential-analog methods. From a practical point of view, however, the monogenic approach is inconvenient for producing a good approximation of one actual case, due to the fact that it necessitates, generally speaking, a very large number of pole-zero chains, and consequently a large number of network elements.

In Part IV, a set of existence theorems is produced in such a way that we do not need such pole-zero chains everywhere dense along the line. Here, we shall produce isolated singularities which do not crowd themselves in the limiting processes. We shall illustrate these ideas by considering the problem of pulse propagation. In this problem we find very important clues concerning the fundamental question of "the minimum number of elements" which are required in a synthesis process.

IV-1.1 The basic mathematical tool needed throughout the discussion conducted in Part IV is the integral representation of transfer functions and its inverse transformations. Let f(t) and F(s) be functions satisfying the requisites indicated in section 1, reference 1. The notation and the meaning of the literals intervening in the following integrals are the same as those used in reference 1.

We will use the fundamental set of integrals

$$\left.\begin{aligned} f(t) &= \frac{2}{\pi}\int_{\Gamma} e^{\gamma t}\, U(\gamma,\lambda)\left[\cos \lambda t\, d\lambda + \sin \lambda t\, d\gamma\right] \\ F(s) &= \frac{2}{\pi}\int_{\Gamma} \left[\lambda\, d\gamma + (s-\gamma)\, d\lambda\right] \frac{U(\gamma,\lambda)}{(s-\gamma)^2+\lambda^2} \\ U(\gamma,\lambda) &= \int_0^{\infty} e^{-\gamma t} f(t) \cos \lambda t\, dt \end{aligned}\right\} \qquad 1,(\text{IV-1.1})$$

We shall make continuous use of the set of integrals 1,(IV-1.1) for the particular contour Γ_o. This contour is a semi-infinite line parallel to the imaginary axis of the s plane, a line which is located at a distance $\gamma = \gamma_o$ = constant, from this axis. Let c_o

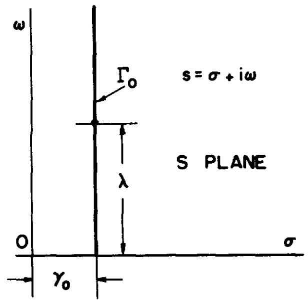

Fig. 1, (IV-1.1)
The contour Γ_o.

be the abscissa of convergence of F(s), as used in connection with Laplace transforms. Then, the contour Γ_o must satisfy the condition $c_o \leq \gamma_o$. See Fig. 1, (IV-1.1). The integrals 1, (IV-1.1) can then be written as Stieltjes integrals as

$$\left.\begin{aligned} f(t) &= \frac{2}{\pi} e^{\gamma_o t} \int_{\Gamma_o} \cos \lambda t \, d\phi(\gamma_o, \lambda) \\ F(s) &= \frac{2(s - \gamma_o)}{\pi} \int_{\Gamma_o} \frac{d\phi(\gamma_o, \lambda)}{(s - \gamma_o)^2 + \lambda^2} \\ U(\gamma_o, \lambda) &= \int_0^\infty \cos \lambda t \, d\tau(\gamma_o, t) \end{aligned}\right\} \qquad 2, (IV-1.1)$$

alternate

$$U(\gamma_o, \lambda) = \int_0^\infty e^{-\gamma_o t} \cos \lambda t \, dT(t) \qquad 3, (IV-1.1)$$

where the distribution functions are given by

$$\left.\begin{aligned} \phi(\gamma_o, \lambda) &= \int_0^\lambda U(\gamma_o, \eta) \, d\eta \\ \tau(\gamma_o, t) &= \int_0^t f(\mu) \, e^{-\gamma_o \mu} \, d\mu \end{aligned}\right\} \qquad 4, (IV-1.1)$$

alternate

$$T(t) = \int_0^t f(\mu) \, d\mu \qquad 5, (IV-1.1)$$

Sometimes we shall make use of this set of integrals when Γ_o coincides with the positive part of the imaginary axis of the s plane. Then, $\gamma_o = 0$

$$\left.\begin{aligned} f(t) &= \frac{2}{\pi}\int_0^\infty \cos \lambda t \, d\phi(0, \lambda) \\ F(s) &= \frac{2s}{\pi}\int_0^\infty \frac{d\phi(0, \lambda)}{s^2 + \lambda^2} \\ U(0, \lambda) &= \int_0^\infty \cos \lambda t \, dT(t) \end{aligned}\right\} \qquad 6, (IV\text{-}1.1)$$

with the corresponding expression for the distribution functions.

IV-1.2 We must recall a property of the distribution function associated with transfer functions.

1) Let F(s) be a transfer function. Then, as was shown in Part I, reference 1

$$U(\gamma, \lambda) = \text{Real } F(s)\Big)_{s\epsilon\Gamma} \qquad 1, (IV\text{-}1.2)$$

2) The integrals 1,(IV-1.1), or their equivalent forms, 2, 4, and 6,(IV-1.1), express, among other things, the necessary and sufficient condition for a function to be transfer. The center integral in each set tells us the method for constructing a transfer function from a given $U(\gamma, \lambda)$ function. The corresponding discussion is conducted in references 1 and 3. It is convenient, however, to illustrate with a simple example the nature of these integrals. Consider the function

$$\frac{1}{s - a}; \quad \infty > a > 0 \text{ Real} \qquad 2, (IV\text{-}1.2)$$

This function is not a transfer function because it has a pole to the right of the imaginary axis. Let us compute the real part of 2,(IV-1.2), say along the imaginary axis $\gamma_o = 0$.

We have

$$\text{Real } \frac{1}{s - a} = \frac{\sigma - a}{(\sigma - a)^2 + \omega^2}$$

and form the distribution function along Γ_o; $\sigma = \gamma_o = 0$; $\omega = \lambda$

$$U(0, \lambda) = \frac{-a}{a^2 + \lambda^2} \qquad 3, (IV\text{-}1.2)$$

Our aim is to construct a transfer function from 3,(IV-1.2) which is to be taken as a density distribution function.

Since 3,(IV-1.2) remains regular and bounded for all values of λ, then the integral for F(s) in 6, (IV-1. 1) exists as a Riemann integral.

Hence, by direct computation

$$F(s) = \frac{-2}{\pi} \int_0^\infty \frac{a}{a^2 + \lambda^2} \times \frac{s}{s^2 + \lambda^2} \, d\lambda \tag{4, (IV-1. 2)}$$

By partial expansion of the integrand and by using the well-known integral

$$\int_0^\infty \frac{\alpha dx}{\alpha^2 + x^2} = \frac{\pi}{2} \tag{5, (IV-1. 2)}$$

one gets

$$F(s) = -\frac{1}{s + a} \tag{6, (IV-1. 2)}$$

which is evidently a transfer function.

One can at once verify the property 1, (IV-1. 2) for the transfer function F(s) in 6, (IV-1. 2).

We have

$$\text{Real } F(s) = -\frac{\sigma + a}{(\sigma+a)^2 + \omega^2} \tag{7, (IV-1. 2)}$$

Hence, along the contour Γ_o on the imaginary axis

$$\text{Real } F(s) = -\frac{a}{a^2 + \lambda^2}$$

as it should be.

Of course

$$\text{Real}\left(\frac{1}{s - a}\right) \quad \text{and} \quad \text{Real}\left(\frac{-1}{s + a}\right)$$

coincide only along the imaginary axis of the s plane. Therefore, the property 1, (IV-1. 2) is not valid for the function 1/(s-a).

The reader must note the transforming effect of the integral 6,(IV-1.1) in the generation of transfer functions. In the illustration above we start from a function definitely not transfer and the result of the integration is a transfer function. A generalization of this simple illustration will be used in a subsequent set of theorems.

3) Finally, we shall recall to the reader that when the density distribution function is regular and bounded in the immediate neighborhood of every point of the imaginary axis, the corresponding transfer function F(s) has no singularities on the ω axis. The converse property is true. If, for example, F(s) has poles (which must be simple) on the imaginary axis, then $U(0, \lambda)$ has impulses with finite area at such points, and conversely.

We will make continuous use of the first two integrals 2, (IV-1. 1) taken along a contour Γ_o. If the distribution function remains regular and bounded in the immediate vicinity of every point on the Γ_o contour, then these first two integrals exist in the Riemann sense. Otherwise, they must be taken in the Stieltjes sense. For transfer functions, we have shown in reference 1 that the density distribution function is necessarily regular and bounded in the immediate vicinity of every point of the contour Γ_o when $\gamma_o > 0$.

IV-1.3 We proceed to the formulation of a set of theorems concerning the necessary and sufficient condition for a transfer function to be rational. For a reader who is accustomed to dealing always with rational transfer functions, the main theorem will appear trivial, but when we start with a synthesis problem in the time domain, this main theorem is far from trivial. It plays a primary role in the theory of networks.

For a better presentation we shall first consider two separate cases:

(a) Transfer functions having simple poles only

(b) Transfer functions that may have poles of multiplicity other than 1. (General case.)

IV-1.3, 1 The direct theorem is almost obvious; it reads: "Let F(s) be a rational transfer function whose numerator is a polynomial of degree n, $\infty > n \geqslant 0$, and whose denominator is a polynomial in s of degree m, $\infty > m \geqslant 0$. Then, the distribution function is necessarily a rational function."

To prove the theorem we set

$$\text{Real } F(s) = \text{Real}\left[\frac{P_n(s)}{Q_m(s)}\right] = \frac{P_n(s)\, Q_m(\bar{s}) + P_n(\bar{s})\, Q_m(s)}{2\,|Q_m(\bar{s})|^2}$$

which is evidently a real rational function.

Now, the theorem follows from the property 1, (IV-1.2) which states that

$$U(\gamma, \lambda) = \text{Real } F(s)\Big)_{s \in \Gamma}$$

IV-1.3, 2 Let us study the converse proposition. We will show that if $U(\gamma, \lambda)$ is a real rational function, which satisfies the conditions

(a) to be bounded for all values of $\gamma > 0$

(b) to be symmetric with respect to the real axis

then the transfer function generated by $U(\gamma, \lambda)$ is a rational function. The reader may find the reasons for these stipulations in reference 1. The condition of symmetry in (b) is immediately satisfied by assuming that U is a function of λ^2, λ real. Therefore we can write

$$U(\gamma, \lambda) = \frac{A_n(\gamma, \lambda^2)}{B_m(\gamma, \lambda^2)} \qquad 1, (\text{IV-1.3, 2})$$

where A is a real polynomial of degree n in λ^2 and B is a real polynomial of degree m in λ^2. The condition (a) of boundedness is satisfied for $m \geqslant n$.

For mere convenience we first prove the theorem for the case in which all the roots of $B_m(\gamma, \lambda^2)$ are of multiplicity one. The general case is considered in subsection IV-1.4. In proving the theorem it is enough to take a contour Γ_o, $\gamma = \gamma_o$ = constant, as can be seen from reference 1. The following notation will be used.

Let us denote the roots of

$$\left.\begin{array}{c} B_m(\gamma_o, \lambda^2) = 0 \\ \\ \text{by} \qquad \mu_1^2, \mu_2^2, \ldots, \mu_m^2 \end{array}\right\} \tag{2, (IV-1.3, 2)}$$

and the corresponding Encke roots (the roots with opposite sign) by

$$\lambda_k^2 = -\mu_k^2 \tag{3, (IV-1.3, 2)}$$

Finally, we will assume, without loss of generality, that the contour Γ_o is placed to the right of the abscissa c_o of convergence. That is $\gamma_o > c_o$.

By partial expansion of 1, (IV-1.3, 2), γ_o = constant, we can express

$$U(\gamma_o, \lambda) = \frac{A_n(\gamma_o, \lambda^2)}{B_m(\gamma_o, \lambda^2)} = K_o + \sum_{k=1}^{k=m} \frac{K_k}{\lambda^2 + \lambda_k^2} \tag{4, (IV-1.3, 2)}$$

The constant K_o is different from zero only when m = n; otherwise it is zero. $K_1, \ldots,$ K_m are, respectively, the residues at the poles.

The transfer function associated with the rational distribution 4, (IV-1.3, 2) can be found by using the second integral of 2, (IV-1.1). Since $\gamma_o > c_o$, then the integral given above exists in the Riemann sense. Hence we get

$$F(s) = \frac{2}{\pi} \mathscr{s} K_o \int_0^\infty \frac{d\lambda}{\mathscr{s}^2 + \lambda^2} + \frac{2}{\pi} \mathscr{s} \sum_{k=1}^{k=m} K_k \int_0^\infty \frac{d\lambda}{(\lambda^2 + \lambda_k^2)(\mathscr{s}^2 + \lambda^2)} \tag{5, (IV-1.3, 2)}$$

where momentarily we set $\mathscr{s} = s - \gamma_o$.

By new partial fraction expansion one gets

$$\frac{1}{(\lambda^2 + \lambda_k^2)(\mathscr{s}^2 + \lambda^2)} = \frac{1}{\mathscr{s}^2 - \lambda_k^2}\left(\frac{1}{\lambda^2 + \lambda_k^2} - \frac{1}{\lambda^2 + \mathscr{s}^2}\right) \tag{6, (IV-1.3, 2)}$$

By using the integral 5, (IV-1.2) we get, after simple operations,

$$F(s) = K_o + \sum_{k=1}^{k=m} \frac{\left(\frac{K_k}{\lambda_k}\right)}{s + (-\gamma_o + \lambda_k)} \tag{7, (IV-1.3, 2)}$$

which is a rational function. Then, the theorem is proved for simple roots in λ^2 of B_m.

IV-1.3, 3 Expression 7, (IV-1.3, 2) provides a simple method of constructing the rational transfer function corresponding to the rational distribution case of simple poles, 4, (IV-1.3, 2). Both roots and residues depend on the selection of Γ_o, but the function F(s) remains invariant as a whole with the position of Γ_o, provided that $\Gamma_o \geqslant c_o$. The condition of boundedness of $U(\gamma_o, \lambda)$ implies that $c_o \leqslant 0$. Consequently γ_o can be made equal to zero or negative. Hence, the expression 7, (IV-1.3, 2) and the invariances of the poles of F(s) assure the zero or positive value of

$$\text{Real } (-\gamma_o + \lambda_k)$$

The reader must note that in Eq. 7, (IV-1.3, 2) we have λ_k and not λ_k^2. This means that in 7, (IV-1.3, 2) λ_k must be taken as the root of λ_k^2 which has zero or negative real part.

We close this subsection with the following numerical example.

Along $\gamma_o = 0$ (imaginary axis) the following density distribution function is given:

$$U(0, \lambda) = \frac{-(\lambda^2 - 1)(\lambda^2 - 9)(\lambda^2 - 25)}{1.498\,\lambda^6 - 25.98\,\lambda^4 + 147.44\,\lambda^2 + 45} \qquad 1, (\text{IV-1.3, 3})$$

Proceeding in order we find

$$\mu_1^2 = -\lambda_1^2 = -0.290$$

$$\mu_2^2 = -\lambda_2^2 = 8.816 + i5.08$$

$$\mu_3^2 = -\lambda_3^2 = 8.816 - i5.08$$

$$K_0 = -0.6675$$

$$K_1 = 1.8607$$

$$K_2 = 5.09\, e^{-i13.82°}$$

$$K_3 = 5.09\, e^{+i13.82°}$$

From which we get

$$F(s) = -0.6675 + \frac{3.46}{s + 0.538} + \frac{1.595\, e^{+i61.12°}}{s + 3.185\, e^{-i75°}} + \frac{1.595\, e^{-i61.12°}}{s + 3.185\, e^{+i75°}}$$

The rational function given above is a rational transfer function. The negative constant term is a positive resistance in the longitudinal branches of the corresponding lattice structure. See references 1 and 3.

IV-1. 3, 4 In accordance with the direct theorem given in subsection IV-1. 3, 1, the real part of the transfer function F(s) in 7, (IV-1. 3, 2) must coincide along Γ_o with 4, (IV-1. 3, 2). As an exercise, the reader may verify that this is the case, observing that

1) Because of the real character of $U(\gamma_o, \lambda)$, the residues K_k and the roots λ_k^2 must necessarily appear in conjugate pairs.

2) The real part of the sum of a pair of terms in F(s) containing K_k, $\overline{K}_k$, λ_k, $\overline{\lambda}_k$ is given by

$$\text{Real}\left[\frac{K_k/\lambda_k}{(s-\gamma_o)+\lambda_k}+\frac{\overline{K}_k/\overline{\lambda}_k}{(s-\gamma_o)+\overline{\lambda}_k}\right]_{s=\gamma_o+i\lambda} = \frac{\lambda^2(K_k+\overline{K}_k)+\overline{K}_k\lambda_k^2+K_k\overline{\lambda}_k^2}{\lambda^4+|\lambda_k|^4+\lambda^2\left[\lambda_k^2+\overline{\lambda}_k^2\right]}$$

3) The term above is equal to the sum of the terms

$$\frac{K_k}{\lambda^2+\lambda_k^2}+\frac{\overline{K}_k}{\lambda^2+\overline{\lambda}_k^2}$$

in 4, (IV-1. 3, 2).

IV-1. 4 We now proceed to prove the general theorem concerning the existence of rational transfer functions.

Theorem 1, (IV-1. 4) "The necessary and sufficient condition for a function F(s) to be a rational transfer function is that F(s) shall satisfy an integral representation of the type 1, (IV-1. 1) or the equivalent form 2, (IV-1. 1), together with a rational density distribution function in λ^2 whose distribution function along the contour Γ is bounded."

For the allocations of the contour Γ and the definition of the distribution function, the reader may consult the general theorem on transfer functions, as given in reference 1.

Proof. It is sufficient to prove the theorem for the contour Γ_o and for $\gamma_o > c_o$. Under these circumstances 2, (IV-1. 1) exists in the Riemann sense.

I. The condition is necessary. The proof of this part was given in subsection IV-1. 3, 1.

II. The condition is sufficient. Let $U(\gamma_o, \lambda)$ be represented by the rational expression

$$U(\gamma_o, \lambda) = \frac{A_n(\gamma_o, \lambda^2)}{B_m(\gamma_o, \lambda^2)} \qquad 1, (IV-1. 4)$$

with $n \leqslant m < \infty$ because of the boundedness of the distribution function.

In general, $B_m(\gamma_o, \lambda^2)$ may have poles of multiplicity higher than one. Let us denote the p poles of by

$$\left.\begin{array}{l} B_m(\gamma_o, \lambda^2) = 0 \\ \\ \mu_1^2 \text{ occurs } \alpha_1 \text{ times} \\ \\ \mu_2^2 \text{ occurs } \alpha_2 \text{ times} \\ \\ \dots\dots\dots\dots\dots \\ \\ \mu_p^2 \text{ occurs } \alpha_p \text{ times} \\ \\ \alpha_1 + \alpha_2 + \dots + \alpha_p = m \end{array}\right\} \qquad 2, (IV\text{-}1.4)$$

subjected to the restriction

By partial fraction expansion, expression 1, (IV-1.4) can be written as

$$\left.\begin{array}{l} \dfrac{A_n(\gamma_o, \lambda^2)}{B_m(\gamma_o, \lambda^2)} = \displaystyle\sum_{k=1}^{k=p} \sum_{j=1}^{j=\alpha_k} \frac{K_{k,j}}{(\lambda^2 - \mu_k^2)^{\alpha_k - j+1}} + K_o \\ \\ \text{where} \\ \\ K_{k,j} = \dfrac{1}{(j-1)!}\left[\dfrac{d^{j-1}}{d(\lambda^2)^{j-1}}(\lambda^2 - \mu_k^2)^{\alpha_k}\dfrac{A_n(\gamma_o, \lambda^2)}{B_m(\gamma_o, \lambda^2)}\right] \\ \\ K_o = \dfrac{A_n(\gamma_o, \infty)}{B_m(\gamma_o, \infty)} \end{array}\right\} \qquad 3, (IV\text{-}1.4)$$

As in section IV-1.3, 3 we use the Encke roots of 2, (IV-1.4), so that finally we write

$$U(\gamma_o, \lambda^2) = \frac{A_n(\gamma_o, \lambda^2)}{B_m(\gamma_o, \lambda^2)} = K_o + \sum_{k=1}^{k=p} \sum_{j=1}^{j=\alpha_k} \frac{K_{k,j}}{(\lambda^2 + \lambda_k^2)^{\alpha_k - j+1}} \qquad 4, (IV\text{-}1.4)$$

The corresponding transfer function F(s) generated by this density distribution function is given by

$$F(s) = K_o + \sum_{k=1}^{k=p} \sum_{j=1}^{j=\alpha_k} K_{k,j} \frac{2s}{\pi} \int_0^\infty \frac{d\lambda}{(\lambda^2 + \lambda_k^2)^{\alpha_k - j+1} (\lambda^2 + s^2)} \qquad 5, (IV\text{-}1.4)$$

Now the sufficient condition of the theorem will be proved, if we can show that the integral

$$I(\lambda_k, q) = \frac{2s}{\pi} \int_0^\infty \frac{d\lambda}{(\lambda^2 + \lambda_k^2)^q (\lambda^2 + s^2)} \qquad 6, (IV\text{-}1.4)$$

is a rational function of s for all finite values of q as

$$0 < q = \alpha_k - j + 1 < \infty \tag{7, (IV-1.4)}$$

To show that this is so, we start with the integral

$$I(\lambda_k, 1) = \frac{2s}{\pi}\int_0^\infty \frac{d\lambda}{(\lambda^2 + \lambda_k^2)(\lambda^2 + s^2)} = \frac{1/\lambda_k}{s + \lambda_k} \tag{8, (IV-1.4)}$$

which was computed in 7, (IV-1.3, 2).

Now let us consider two cases:

(a) If $U(\gamma_o, \lambda)$ is bounded for every point on Γ_o, then the integral 8, (IV-1.4) defines a uniform function of λ_k^2. Hence, by differentiation of 8, (IV-1.4) with respect to λ_k, one gets

$$I(\lambda_k, 2) = \frac{2s}{\pi}\int_0^\infty \frac{d\lambda}{(\lambda^2 + \lambda_k^2)^2\,(\lambda^2 + s^2)} = \frac{s + 2\lambda_k}{\lambda_k^3(s + \lambda_k)^2} \tag{9, (IV-1.4)}$$

By successive differentiation with respect to λ_k, and by momentarily setting $\lambda_k = a$ in order to simplify the writing, one gets

$$I(\lambda_k, 3) = \frac{3s^2 + 9as + 8a^2}{2! \times 2^2 \times a^5(s+a)^3}$$

$$I(\lambda_k, 4) = \frac{15s^3 + 60s^2a + 87sa^2 + 48a^3}{3! \times 2^3 \times a^7(s+a)^4}$$

$$I(\lambda_k, 5) = \frac{105s^4 + 525s^3a + 103sa^2 + 975sa^3 + 384a^4}{4! \times 2^4 \times a^9(s+a)^5}$$

$$I(\lambda_k, 6) = \frac{945s^5 + 5670s^4a + 1407s^3a^2 + 18270s^2a^3 + 12645sa^4 + 3840a^5}{5! \times 2^5 \times a^{11}(s+a)^6} \tag{10, (IV-1.4)}$$

$$\cdots$$

$$I(\lambda_k, q) = \frac{\left[3\cdot5\cdot7\cdot9\ldots(2q-3)\right]s^{q-1} + \ldots + \left[2\cdot4\cdot6\ldots(2q-2)\right]a^{q-1}}{(q-1)! \times 2^{(q-1)} \times a^{(2q-1)}(s+a)^q}$$

where $a = \lambda_k$, $s = s - \gamma_o$.

Expression 10, (IV-1.4) shows that the integral 6, (IV-1.4) represents a proper rational fraction for every positive integer value of q, having a denominator of q degree and a numerator of degree (q-1). The substitution of expressions 10, (IV-1.4) for each integral in 5, (IV-1.4) finally shows that F(s) is a rational function having a denominator of degree m, $(m = \alpha_1 + \alpha_2 + \ldots + \alpha_p)$ and a numerator whose degree is n such that $n \leqslant m$. The equality sign applies when $K_o = 0$. Consequently, the sufficient condition of the theorem, corresponding to case (a), stated above, has been proved.

(b) In this case we consider that the rational function $U(\gamma_o, \lambda)$ may have one or several sets of poles, at most equal to m in number, along the contour Γ_o.

Under this assumption the integral 2, (IV-1.1) exists in the Stieltjes sense, as we have shown in reference 1. To prove the theorem we first proceed to write down the rational function $U(\gamma_o, \lambda)$, Eq. 1, (IV-1.4) in two parts: the one containing the poles located on the contour Γ_o, and the one whose poles lie outside Γ_o. This last part is regular, and we have considered it under case (a). Consequently, we must now show that the remaining group of terms of $U(\gamma_o, \lambda)$ which have poles on Γ_o also contribute rational terms for F(s) in Eq. 4, (IV-1.4).

The corresponding Stieltjes integrals can then be evaluated in the manner explained in Parts I and II (references 1 and 2), where one finds the corresponding rational expressions for these cases.

This concludes the proof of case (b) and therefore of the sufficient part of the existence theorem 1, (IV-1.4).

We close this section with the following remark. In the previous discussion we have generated rational transfer functions by giving an arbitrary rational density distribution function along a contour Γ_o in the s plane. When the contour Γ_o is selected to the right of the imaginary axis of the s plane, then the given density distribution function must be regular and bounded for every point of the contour Γ_o.

Section IV-2

On Transmission of Impulses. Basic Ideas.

Definitions and nomenclature. Window functions and window distributions. The density distribution function $U(0, \lambda)$ associated with impulses. Heuristic approach. Polygonal skeleton. Corner theorems.

IV-2.0 Introduction. Sections IV-2, IV-3, IV-4, and IV-5 of this report deal with a formal study of a fundamental problem of network synthesis: the transmission of impulses through a finite, passive, discrete, and linear network.

The reader will have a concrete idea of the problem in question from the following explicit formulations.

Let: $u_o(t, 0)$ denote the unit impulse of time at $t = 0$;

$f(t)$ be a real bounded single-valued function of time having the following properties: given the constants t_o and δ such that $0 < t_o < +\infty$, $0 < \delta < t_o$, then

$$f(t) = \left\{ \begin{array}{ll} \equiv 0 & \text{for } t < t_o - \delta \\ \not\equiv 0 & \text{for } t_o - \delta < t < t_o + \delta \\ \equiv 0 & \text{for } t_o + \delta < t < +\infty \end{array} \right\} \qquad 1, (IV\text{-}2.0)$$

Hence $f(t)$ is a function which lasts 2δ units of time. We will consider that δ is small in comparison to t_o, so $f(t)$ can be looked upon as a narrow impulse train which is delayed $t_o - \delta$ units from the origin. We are particularly interested in single narrow pulses.

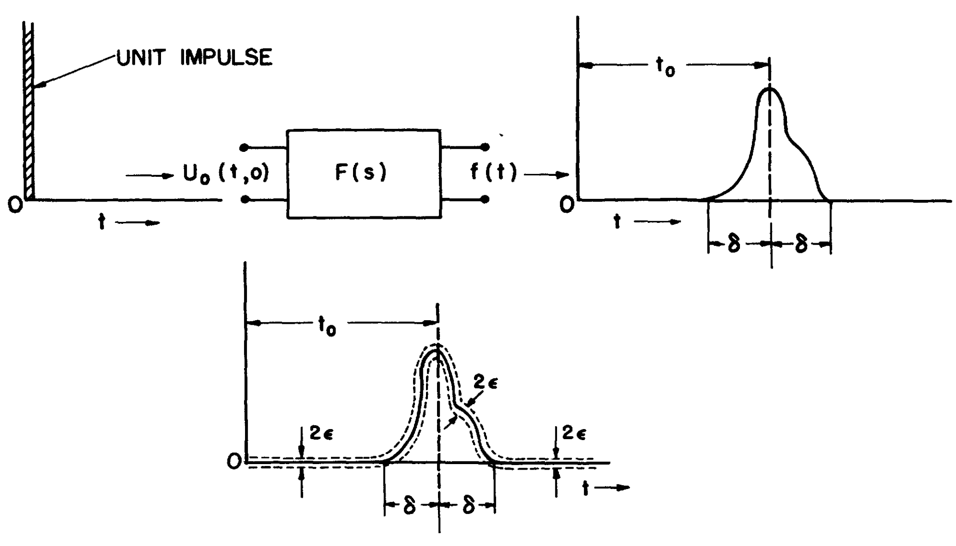

Fig. 1, (IV-2.0)
The proposed problem and the vicinal tolerance ϵ.

Now, let us investigate the possibility of finding a finite, discrete, passive, and linear four-terminal network, whose transfer function will be denoted by F(s), such that its output under the excitation of $u_0(t, 0)$ reproduces a prescribed function f(t), given above, inside a prescribed vicinal tolerance ϵ.

By vicinal tolerance ϵ we mean that the graph of the output function of such a network must be completely contained in the vicinity of ϵ along the graph of the function f(t) and the time axis. Figure 1, (IV-2.0) illustrates the problem in question and also gives the graphic interpretation of the ϵ vicinity of the graph of f(t).

IV-2.0, 1 The solution of the problem above has both formal and practical importance.

It is of formal importance because, particularly in the case when f(t) has the shape of a single, all-positive, large spike of short duration, as in Fig. 1, (IV-2.0, 1), it shows the existence of window functions and window function distributions. These distributions constitute a convenient mathematical tool which has been introduced in network analysis in order to show the existence of finite, discrete, passive networks which are capable of transforming a prescribed excitation function into a prescribed output function, inside a prescribed vicinal tolerance ϵ. Very little will be said in Part IV about window functions and their distribution. Here, we shall lay the foundation of certain principles which will be used in Part V in the theory of window functions.

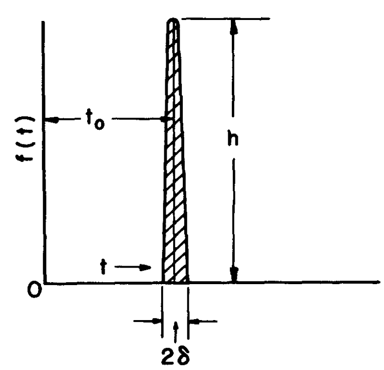

Fig. 1, (IV-2.0, 1)
An isolated single-spike response.

The practical importance of the solution of our problem comes from the increasing use of pulses in the transmission of intelligence. Past experience has taught us that transmission of impulses through networks is generally perturbed by the phenomenon of "ringing," the term used to describe the presence of highly oscillatory spurious oscillations which may almost completely wipe out the presence of the transmitted pulse. The problem of controlling this oscillation is discussed in section IV-4 of this report.

IV-2.0, 2 The main objective of section IV-2 is the formal and heuristic evaluation of the density distribution function $U(0, \lambda)$ associated with narrow pulses. A generalization of the study of the $U(0, \lambda)$ will be carried out in section IV-3, under the light of the corner theorems.

IV-2.1 Pulses and pulse trains. We begin with some definitions.

The term "pulse" will be used in its ordinary connotation: an almost unidirectional current of voltage of relatively short duration.

"Pulse train" will mean the recurrent succession of a finite number of similar pulses.

IV-2.1,1 Symmetric and antisymmetric pulse components. Let g(t) be the function represented by a narrow pulse of duration 2δ and delayed $t_o - \delta$ units of time. In general, the function g(t) is not symmetric with respect to the point $t = t_o$. See Fig. 1,(IV-2.1,1). The pulse function g(t) can always be decomposed into the symmetric pulse components $g_2(t)$ and the antisymmetric pulse component $g_1(t)$. Both functions, $g_2(t)$ and $g_1(t)$, have the same delay t_o and width 2δ as the original pulse.

Let us introduce the auxiliary variable

$$x = t - t_o \qquad 1,(\text{IV-2.1,1})$$

We have the relations

with

$$\left.\begin{aligned} g(t_o + x) &= g_1(t_o + x) + g_2(t_o + x) \\ g_2(t_o + x) &= \frac{g(t_o + x) + g(t_o - x)}{2} \\ g_1(t_o + x) &= \frac{g(t_o + x) - g(t_o - x)}{2} \end{aligned}\right\} \qquad 2,(\text{IV-2.1,1})$$

IV-2.1,2 Window pulse and window function. Definition 1,(IV-2.1,2): "A single spike-like, symmetric, tailless pulse of finite duration $2\delta = a$, which is always positive, or always negative, will be called a window type of pulse. The function which represents this pulse will be called a window function."

The important elements of a window function are its duration, called "aperture;" its mean delay t_o; its height H, and its enclosed area A. Note that the definition does not specify a definite form of the spike. Figure 1,(IV-2.1.2) is a graphical illustration of the definition. The enclosed area A of a pulse is considered here as an intrinsic invariant element of a window function.

If one lets the aperture of a pulse tend to zero, then the height H of the pulse increases without limit in such a way as to keep an invariant enclosed area. Therefore, the limit of such a process transforms the pulse into an impulse. Conversely, a window function may be considered as the finite spreading, or dispersion, of an impulse.

The reader may note that window functions are defined as "symmetric pulses." This restriction is necessary because we wish to keep such windows as functions which are closed under the operation of the sum. Nonsymmetric pulses, for example, can be decomposed into the symmetric and antisymmetric components. By definition, the antisymmetric pulse is not a "window" because it attains positive and negative values. Thus, if we define a nonsymmetric, single, all-positive spike as a window function, it would be the sum of one which is a window and one which is not a window function.

Finally, we should add a few words about the term "window." The term is temporarily adopted for the lack of a better name, and because of the scanning effect of this function in certain operations of integration in which we can "see" the result of such integrals through these pulses.

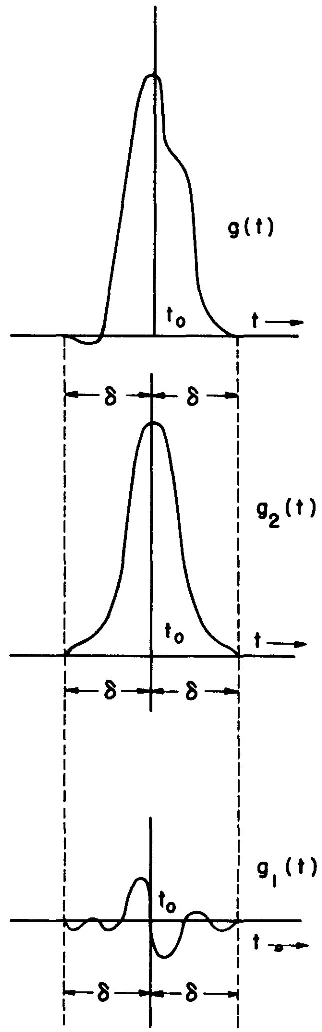

Fig. 1, (IV-2.1, 1)
The symmetric and antisymmetric pulse components.

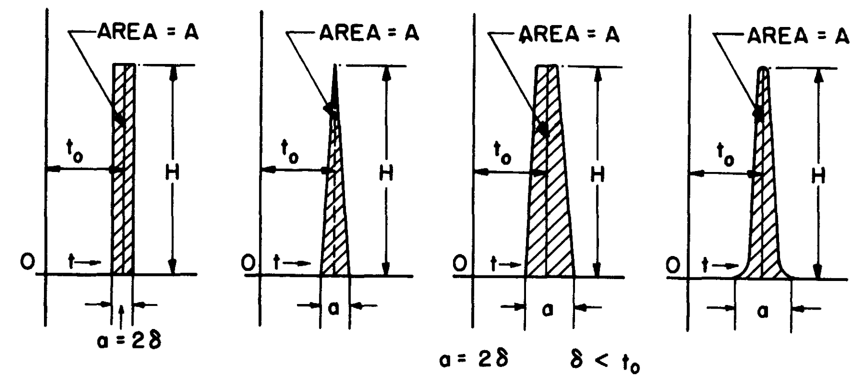

Fig. 1, (IV-2.1, 2)
Examples of window functions.

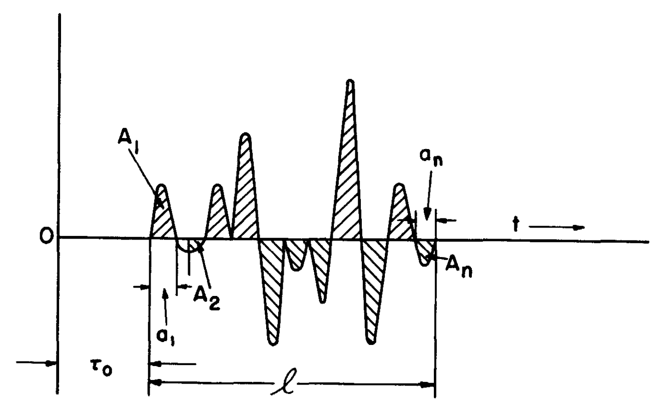

Fig. 1, (IV-2.1, 3)
Illustration of a window distribution.

IV-2.1,3 Window distributions. Definition 1,(IV-2.1,3): "A successive time distribution of a finite number of nonoverlapping and nondisjointed window functions is called a 'window distribution'." The individual pulses may have different signs and enclosed areas. Figure 1,(IV-2.1,3) gives a graphical illustration of a window distribution. The "coverage" of the distribution is evidently

$$\ell = \sum_{k=1}^{n} a_n$$

The reader may note that pulses with lateral oscillations and nondisjointed pulse trains enter into the category of "window function distributions."

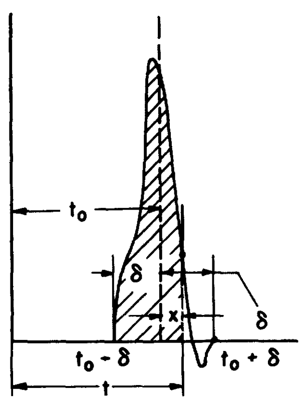

Fig. 1,(IV-2.2,1)
Example of f(t).

The present report does not cover more ground in the theory of window functions and its distributions. Part V and the following parts will cover the corresponding theories and application of these concepts. Part IV deals almost exclusively with the study of the possibility of finding a solution of the problem presented in section IV-2, so as to prepare the ground for future parts.

We shall consider, here, pulses in general, but particular emphasis will be given to symmetric pulses, because of the definition of window function, and not because these last pulses have a particular important application.

IV-2.2 The density distribution function $U(\gamma_o, \lambda)$. We shall begin with a formal study of the determination of density distribution functions which correspond to pulses, in particular to symmetric ones. We shall compute density distribution functions, $U(\gamma_o, \lambda)$, along the contour Γ_o. Continuous use will be made of the case $\gamma_o = 0$, which corresponds to density distributions along the imaginary axis of the s plane.

In computing the function $U(\gamma_o, \lambda)$ we shall use expressions 2,(IV-1.1) and 6,(IV-1.1). In section IV-2 we shall consider that the pulses produce a function T(t), see Eq. 5, (IV-1.1), uniform and bounded for all values of t. Hence 6,(IV-1.1) and related equations exist in a Riemann sense. The computation of $U(\gamma_o, \lambda)$ when these integrals exist in a Stieltjes sense will be made in section IV-3.

IV-2.2,1 The function $U(\gamma_o, \lambda)$; its amplitude and phase-deviation associated functions. Let f(t) be a nonsymmetric pulse of length 2δ and mean delay t_o, as shown in Fig. 1,(IV-2.2,1). The area of the pulse is A. Let $f_1(t)$ and $f_2(t)$ be, respectively, its antisymmetric and symmetric components. The function f(t) may admit positive and negative values.

Under the assumptions of the last subsection we can write

$$\left.\begin{aligned} U(\gamma_o, \lambda) &= \int_0^{\infty} e^{-\gamma_o t} f(t) \cos \lambda t \, dt \\ &= \int_{t_o-\delta}^{t_o+\delta} e^{-\gamma_o(t)} f(t) \cos \lambda t \, dt \end{aligned}\right\} \qquad 1, (IV-2.2, 1)$$

Let us introduce the auxiliary variable

$$t - t_o = x \qquad 2, (IV-2.2, 1)$$

After routine algebraic operations we get

$$U(\gamma_o, \lambda) = e^{-\gamma_o t_o} \left\{ \alpha(\gamma_o, \lambda) \cos t_o \lambda + \beta(\gamma_o, \lambda) \sin t_o \lambda \right\} \qquad 3, (IV-2.2, 1)$$

where

$$\left.\begin{aligned} \alpha(\gamma_o, \lambda) &= 2 \int_0^{\delta} \left[f_2(x) \cosh(\gamma_o x) - f_1(x) \sinh(\gamma_o x) \right] \cos \lambda x \, dx \\ \beta(\gamma_o, \lambda) &= 2 \int_0^{\delta} \left[f_2(x) \sinh(\gamma_o x) - f_1(x) \cosh(\gamma_o x) \right] \sin \lambda x \, dx \end{aligned}\right\} \qquad 4, (IV-2.2, 1)$$

The functions α and β can be recognized at once in 3,(IV-2.2,1) as the partial envelopes, respectively, of the oscillations $\cos \lambda t_o$ and $\sin \lambda t_o$.

By simple trigonometric steps, the function $U(\gamma_o, \lambda)$ can also be written as

$$U(\gamma_o, \lambda) = \left\{ e^{-\gamma_o t_o} \sqrt{\left[\alpha(\gamma_o, \lambda)\right]^2 + \left[\beta(\gamma_o, \lambda)\right]^2} \right\} \cos \left[t_o \lambda - \theta(\gamma_o, \lambda) \right]$$

where

$$\tan \theta(\gamma_o, \lambda) = \frac{\beta(\gamma_o, \lambda)}{\alpha(\gamma_o, \lambda)} \qquad 5, (IV-2.2, 1)$$

Hence, we are able to state the following theorem:

Theorem 1, (IV-2.2, 1): "The density distribution function of a pulse is formed by a damped oscillation (γ_o = damping) whose amplitude is

$$\sqrt{\alpha^2 + \beta^2}$$

whose frequency, in λ, is t_o, and whose phase deviation function is equal to $\theta(\gamma_o, \lambda)$."

IV-2.2,2 Most of the future discussion is concerned with the function U, above, for $\gamma_o = 0$ (Γ_o along the imaginary axis of the s plane). Here, expressions 3, (IV-2.2,1), 4, (IV-2.2,1), and 5, (IV-2.2,1) reduce to simpler forms.

$$\left.\begin{aligned} U(0,\lambda) &= \left\{\alpha(0,\lambda)\cos\lambda t_o + \beta(0,\lambda)\sin\lambda t_o\right\} \\ &= \sqrt{\left[\alpha(0,\lambda)\right]^2 + \left[\beta(0,\lambda)\right]^2}\cos\left[\lambda t_o - \theta(0,\lambda)\right] \end{aligned}\right\} \qquad 1, (IV-2.2,2)$$

and

$$\left.\begin{aligned} \alpha(0,\lambda) &= 2\int_0^\delta f_2(x)\cos\lambda x\,dx \\ \beta(0,\lambda) &= -2\int_0^\delta f_1(x)\sin\lambda x\,dx \end{aligned}\right\} \qquad 2, (IV-2.2,2)$$

IV-2.2,3 The following particular cases have an illuminating meaning.

Case A. Symmetric pulse $\left[f_1(x) \equiv 0\right]$

$$\left.\begin{aligned} U(0,\lambda) &= \alpha(0,\lambda)\cos\lambda t_o \\ \alpha(0,\lambda) &= 2\int_0^\delta f(x)\cos\lambda x\,dx \end{aligned}\right\} \qquad 1, (IV-2.2,3)$$

Case B. Antisymmetric pulse $\left[f_2(x) \equiv 0\right]$

$$\left.\begin{aligned} U(0,\lambda) &= \beta(0,\lambda)\sin\lambda t_o \\ \beta(0,\lambda) &= -2\int_0^\delta f(x)\sin\lambda x\,dx \end{aligned}\right\} \qquad 2, (IV-2.2,3)$$

The reader must note the fundamental property of symmetric and antisymmetric pulses.

(a) The phase function $\theta(0,\lambda)$ disappears.

(b) The oscillations in both cases have a constant frequency which depends only on the delay and not on the width of the pulse.

IV-2.3 Particular examples. A further discussion on the functions $U(0,\lambda)$ will be greatly facilitated for the reader by the actual computation of the functions which correspond to particular pulse shapes. The aim of these examples is to make the reader acquainted with several characteristic elements of such functions. These elements play a basic role in the evaluation of tolerances and in the introductory discussion to the questions of unicity, and other important properties. The discussion of these aspects will be given in later sections.

A set of typical examples of pulse-wave shapes has been selected. Details of the computation of the corresponding $U(0, \lambda)$ are omitted for brevity. Only the final results are given. The graphs of the functions $U(0, \lambda)$ are not scaled plots of the actual graphs of the functions. Here, the graphs are intentionally exaggerated in order to emphasize the characteristic elements of the functions in which we are particularly interested. Scaled plots of these functions can be found, for example, in reference 4.

IV-2.3,1 Examples, symmetric pulses.

A. Rectangular pulse. See Fig. 1, (IV-2.3, 1). After simple computations we obtain the density distribution function as

$$U(0, \lambda) = A \frac{\sin \lambda\delta}{\lambda\delta} \cos \lambda t_o \qquad 1, (IV-2.3, 1)$$

The important characteristic features of the graph of this function are:

1) The function oscillates with constant frequency and variable amplitude $\alpha(0, \lambda)$. The constant frequency, which is equal to $2\pi/t_o$, depends only on the delay time t_o, and it is independent of the width pulse, provided $t_o > \delta$.

2) The position of the zeros of the amplitude function depends only on the pulse width and is independent of the delay t_o. The envelope zeros are given by

$$\lambda_{0,K} = \frac{\pi}{\delta} K; \quad K = \text{integer}$$

3) The envelope function shows a decay, as measured with the decay line, which decreases inversely with the first power of λ, as $A/\lambda\delta$.

B. Symmetric triangular pulse. See Fig. 2, (IV-2.3, 1). The density distribution function is given by

$$U(0, \lambda) = A \frac{2(1 - \cos \lambda\delta)}{(\delta\lambda)^2} \cos \lambda t_o \qquad 2, (IV-2.3, 1)$$

Here

1) The oscillations have a constant frequency $2\pi/t_o$.

2) The position of the zeros of $\alpha(0, \lambda)$ depends on δ. They are given by

$$\lambda_{0,K} = \frac{2\pi}{\delta} K; \quad K = \text{integer}$$

3) The function $\alpha(0, \lambda)$ shows a decay line decreasing with the second power of λ as $2A/(\delta\lambda)^2$.

C. Symmetric trapezoidal pulse. See Fig. 3, (IV-2.3, 1). Here we get

$$U(0, \lambda) = A \frac{4}{(\delta^2 - \delta_1^2)\lambda^2} \left\{ \sin \lambda \left(\frac{\delta + \delta_1}{2}\right) \sin \lambda \left(\frac{\delta - \delta_1}{2}\right) \right\} \cos \lambda t_o \qquad 3, (IV-2.3, 1)$$

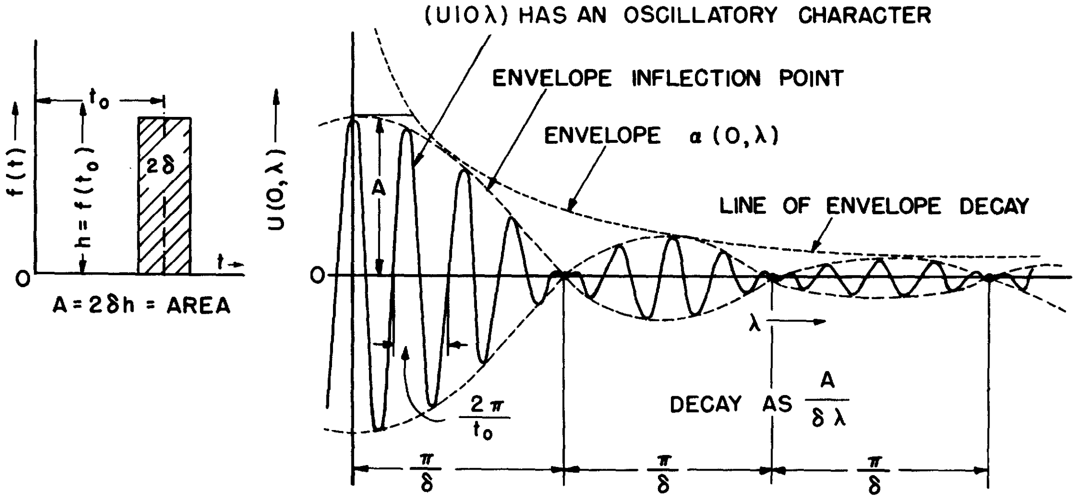

Fig. 1, (IV-2.3, 1)
Rectangular pulse.

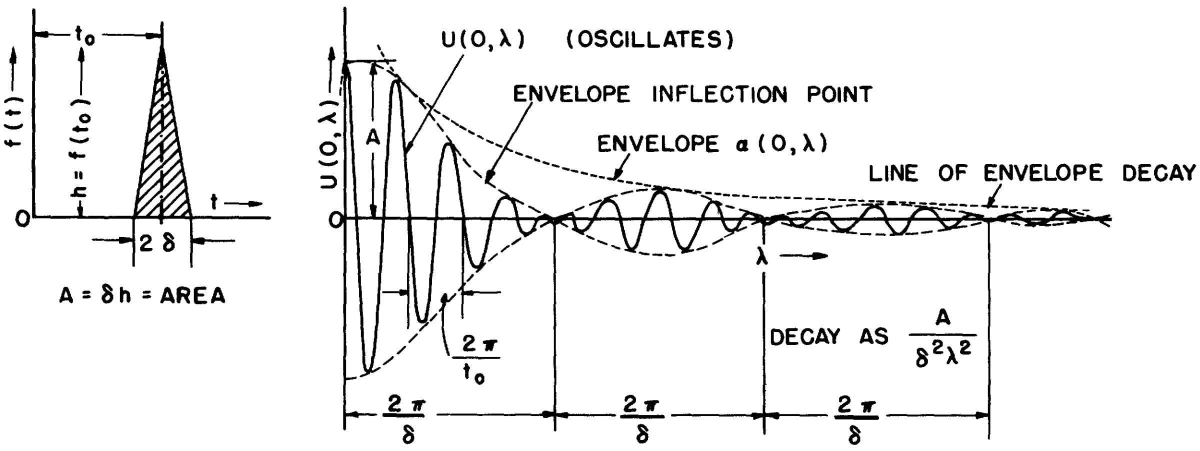

Fig. 2, (IV-2.3, 1)
Triangular pulse.

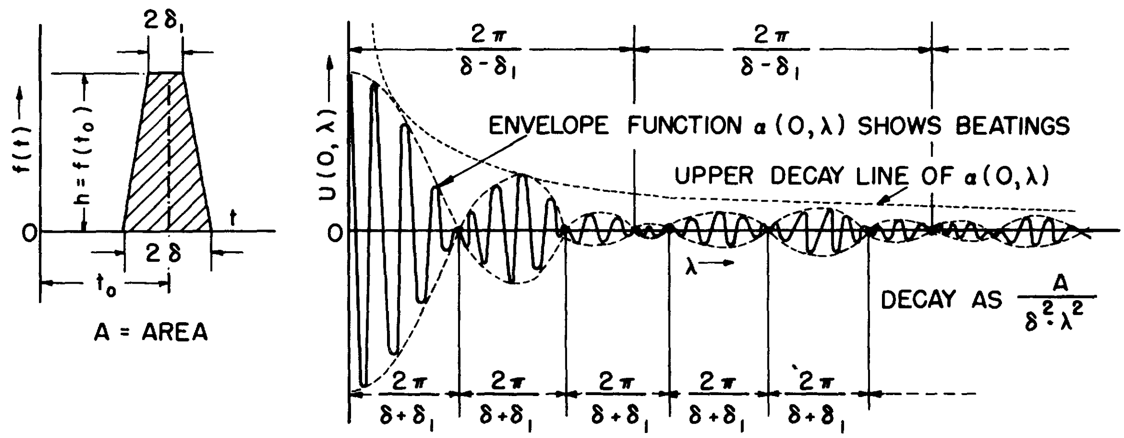

Fig. 3, (IV-2.3, 1)
Trapezoidal pulse.

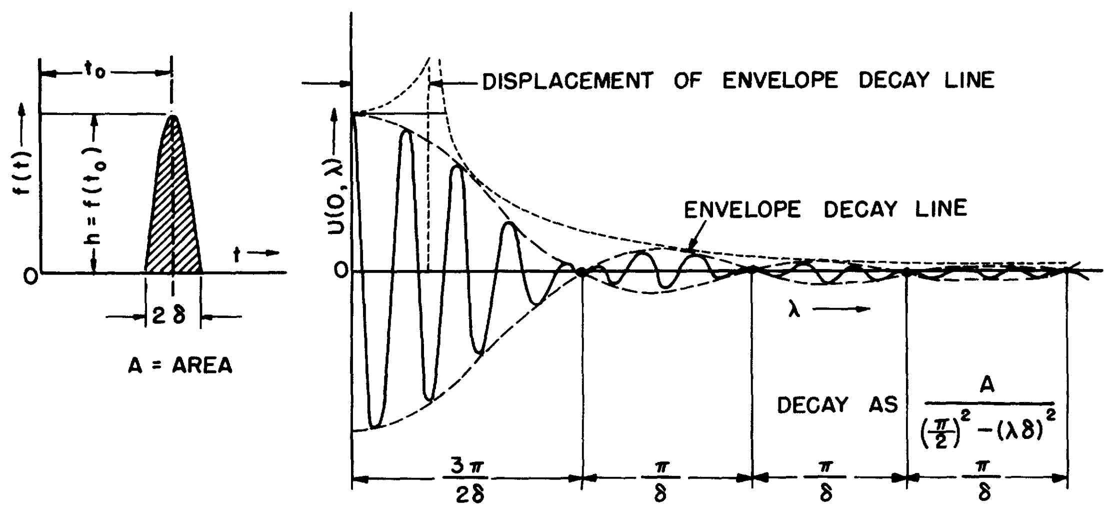

Fig. 4, (IV-2.3, 1)
Half-cosine pulse.

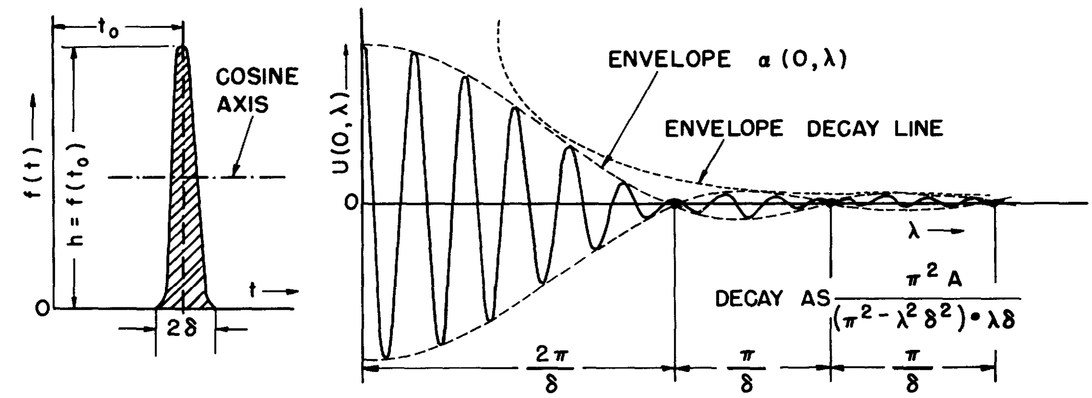

Fig. 5, (IV-2. 3, 1)
Complete cosine pulse.

A new characteristic feature of the envelope function is shown in this example. The function $\alpha(0, \lambda)$ presents a beating phenomena. There are two different sets of zeros of the envelope function

$$\left.\begin{aligned} \lambda_{0, K_1} &= \frac{2\pi}{\delta + \delta_1} K_1; \\ & \qquad\qquad K_1,\ K_2 = \text{integers} \\ \lambda_{0, K_2} &= \frac{2\pi}{\delta - \delta_1} K_2; \end{aligned}\right\} \qquad 4, (IV-2. 3, 1)$$

The characteristic features of 3, (IV-2. 3, 1) are:

1) Oscillations of constant frequency $2\pi/t_o$

2) Beatings in the envelope function; note that the position of the first zero of $\alpha(0, \lambda)$ in this example lies somewhere in between the first zeros of a square and a triangular pulse of the same aperture 2δ, respectively

3) The function $\alpha(0, \lambda)$ shows a general decay as

$$\frac{4A}{(\delta^2 - \delta_1^2)\,\lambda^2}$$

This line of decay does not touch all partial maxima of $\alpha(0, \lambda)$. Different relative maxima show a cyclical line of decay. For example, the smaller partial maxima show a line of decay which is a sort of displacement of the line given above.

D. Cosine (half) pulse. See Fig. 4, (IV-2. 3, 1). Here

$$U(0, \lambda) = A \frac{4 \cos \lambda\delta}{\left[\left(\frac{\pi}{2}\right)^2 - (\lambda\delta)^2\right]} \cos \lambda t_o \qquad 5, (IV-2. 3, 1)$$

The reader may observe a similar over-all behavior as in the first two cases except for a new characteristic feature: the displacement of the line of decay. Here, the line of decay is given by

$$A \frac{4}{\left[\left(\frac{\pi}{2}\right)^2 - (\lambda\delta)^2\right]}$$

It shows two vertical asymptotes at $\lambda\delta = \pm\pi/2$. Note that the function $\alpha(0, \lambda)$ remains regular at these two points. For large values of λ, the decay line decreases as λ^2.

E. Cosine (complete) pulse. See Fig. 5, (IV-2.3, 1). Here we have

$$U(0, \lambda) = \frac{A\pi^2 \sin(\lambda\delta)}{\left[\pi^2 - (\delta\lambda)^2\right] (\lambda\delta)} \cos \lambda t_o \qquad 6, (IV\text{-}2.3, 1)$$

The corresponding graph shows some familiar characteristics of the previous examples. The reader may note

1) Displacement of the decay line at $\lambda\delta = \pm\pi$
2) The decay lines are attenuated as $1/\lambda^3$ for large values of λ.

We close this subsection IV-2.3, 1 by remarking that the characteristic analytical elements, which were individualized in the discussion above, will play a basic role in the syntheses of networks which transmit impulses.

IV-2.3, 2 In this subsection we will produce a few examples of antisymmetric pulses. Antisymmetric pulses of the type here presented are frequently called "doublets." The letter A here represents the area of each lobe. (Total area = 0.)

In a condensed form we have for the density distribution function

$$U(0, \lambda) = -\beta(0, \lambda) \sin \lambda t_o \qquad 1, (IV\text{-}2.3, 2)$$

the values:

A. Rectangular doublet. See Fig. 1, (IV-2.3, 2).

$$U(0, \lambda) = A \frac{(1 - \cos \lambda\delta)}{\lambda\delta} \sin \lambda t_o \qquad 2, (IV\text{-}2.3, 2)$$

B. Triangular doublet. See Fig. 2, (IV-2.3, 2).

$$U(0, \lambda) = \frac{16A}{(\delta\lambda)^2} \left\{\sin \frac{\lambda\delta}{2} \left(1 - \cos \frac{\lambda\delta}{2}\right)\right\} \sin \lambda t_o \qquad 3, (IV\text{-}2.3, 2)$$

C. Sinusoidal doublet. See Fig. 3, (IV-2.3, 2).

$$U(0, \lambda) = \frac{\pi^2}{2} \frac{A}{\pi^2 - (\delta\lambda)^2} \sin \lambda\delta \times \sin \lambda t_o \qquad 4, (IV\text{-}2.3, 2)$$

D. Sine-squared doublet. See Fig. 4, (IV-2.3, 2).

$$U(0, \lambda) = A \frac{4\pi^2 (1 - \cos \lambda\delta)}{\left[4\pi^2 - (\lambda\delta)^2\right] (\lambda\delta)} \sin \lambda t_o \qquad 5, (IV\text{-}2.3, 2)$$

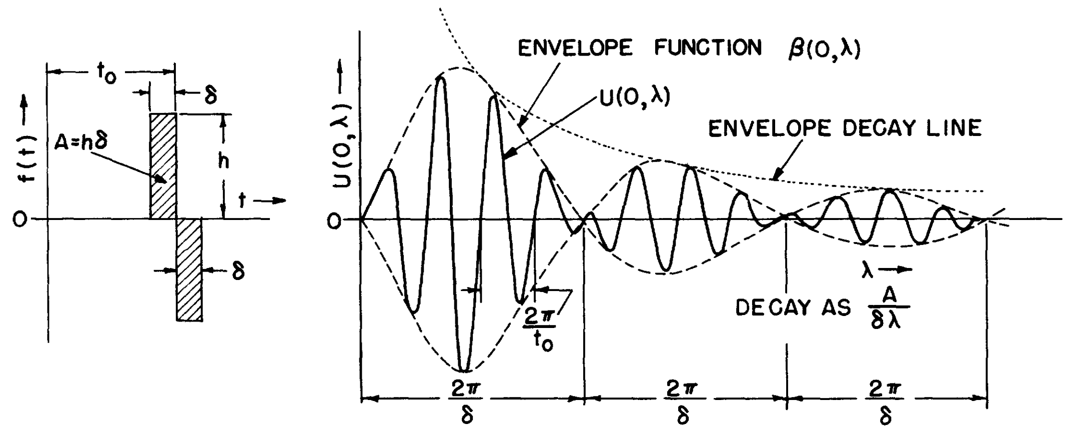

Fig. 1, (IV-2. 3, 2)
Rectangular doublet.

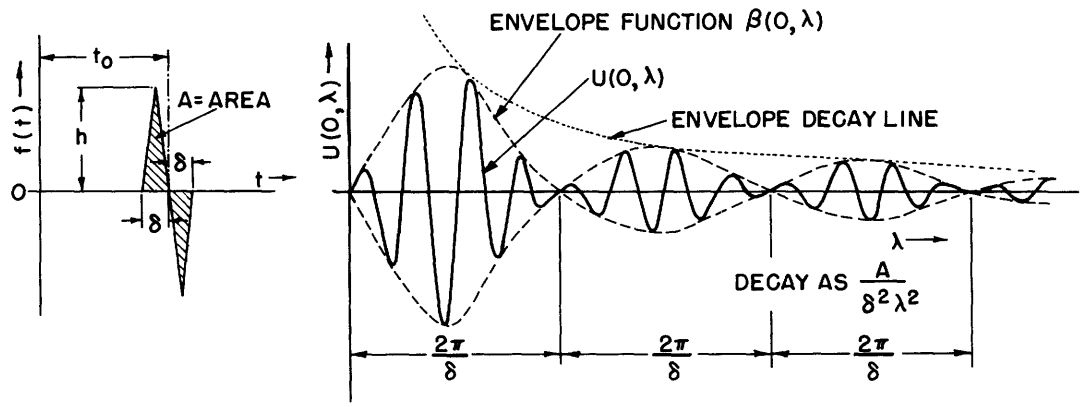

Fig. 2, (IV-2. 3, 2)
Triangular doublet.

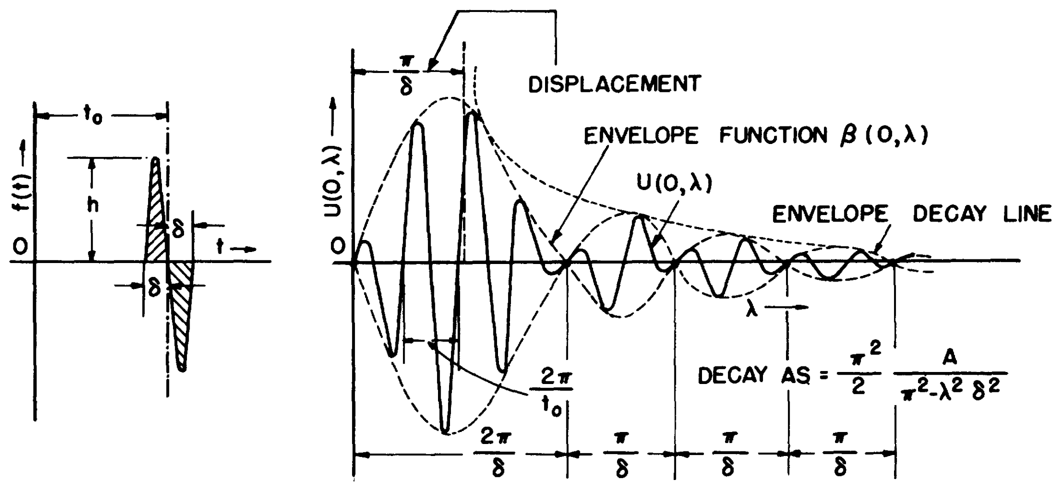

Fig. 3, (IV-2.3, 2)
Sinusoidal doublet.

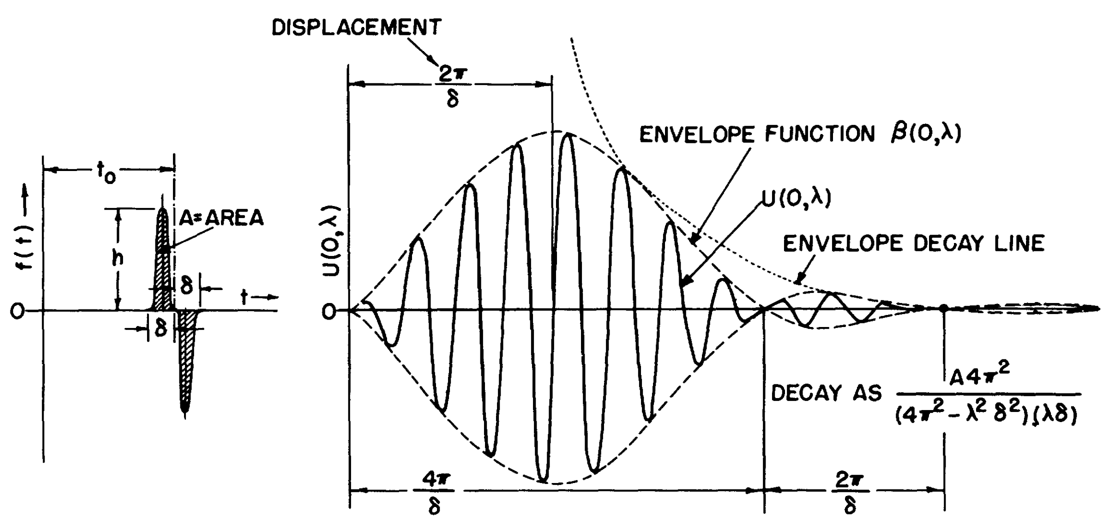

Fig. 4, (IV-2.3, 2)
Sine-squared doublet.

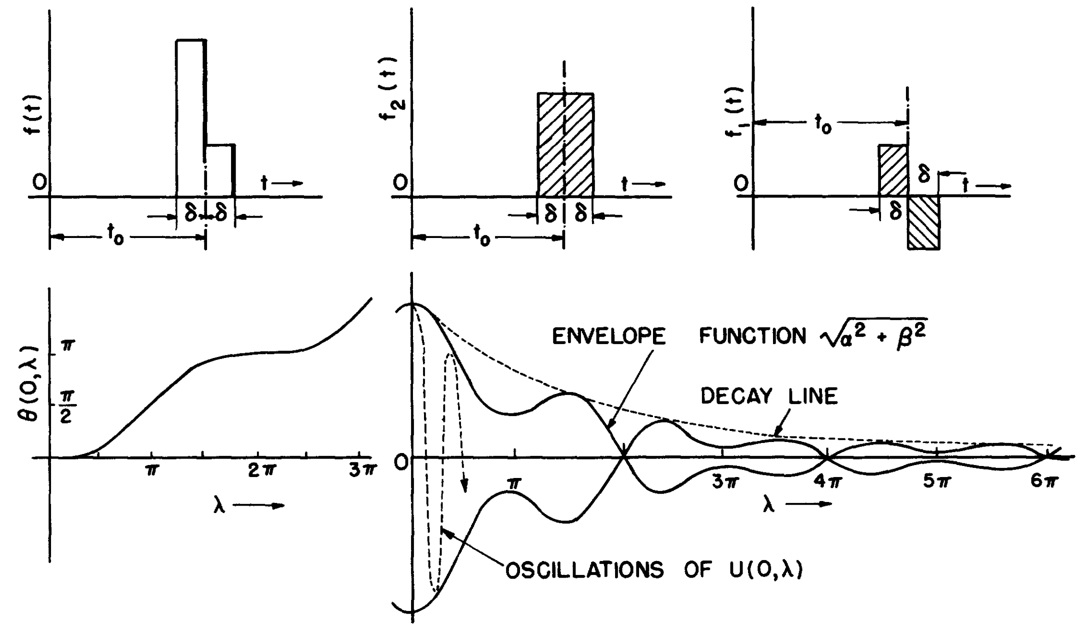

Fig. 1, (IV-2. 3, 3)
Pulse with rectangular components.

The density distribution functions $U(0, \lambda)$ corresponding to antisymmetric pulses differ from the ones corresponding to symmetric pulses in these main points:

1) They oscillate as $\sin \lambda t_o$ instead of $\cos \lambda t_o$.

2) The function $-\beta(0, \lambda)$ is zero at $\lambda = 0$ and its successive zeros are displaced with respect to $\alpha(0, \lambda)$.

Otherwise, the reader will find in the graphs of this last set of examples the familiar characteristic features discussed in the last subsection.

IV-2. 3, 3 In this subsection we produce two illustrative examples of nonsymmetric pulses. The corresponding expression for $U(0, \lambda)$ is given in Eq. 1, (IV-2. 2, 2). Here, the oscillations show a variable frequency deviation $\theta(0, \lambda)$. The envelope function contains both $\alpha(0, \lambda)$ and $\beta(0, \lambda)$. The reader may note that if the zeros of $\alpha(0, \lambda)$ and $\beta(0, \lambda)$ are not coincident, then the corresponding envelope function may not present zeros. This is often the case.

A. Pulse with rectangular components. See Fig. 1,(IV-2.3,3). After simple routine computations we get

$$\left.\begin{aligned} U(0, \lambda) &= \frac{1}{\lambda\delta} \sqrt{A_1^2 (1 - \cos \lambda\delta) + A_2^2 \sin^2 \lambda\delta}\, \cos \left[\lambda t_o + \theta(0, \lambda)\right] \\ \text{where} \quad & \\ \tan \theta(0, \lambda) &= \frac{A_1 (1 - \cos \lambda\delta)}{A_2 \sin \lambda\delta} \end{aligned}\right\} \quad 1, (\text{IV-2. 3, 3})$$

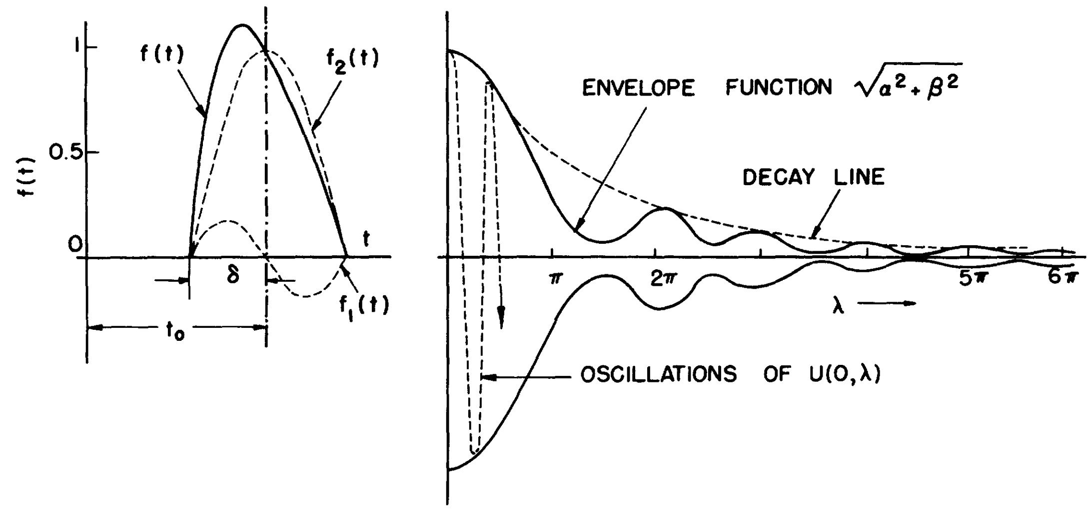

Fig. 2, (IV-2.3, 3)
Pulse with sine-cosine components.

The corresponding functions $f_1(t)$, $f_2(t)$, $\sqrt{\alpha^2 + \beta^2}$, and $\tan\theta(0,\lambda)$ are shown in Fig. 1, (IV-2.3, 3). The oscillations corresponding to $\cos\left[\lambda t_o - \theta(0,\lambda)\right]$ are not shown in this figure.

B. Nonsymmetric pulse with cosine and sine components. See Fig. 2, (IV-2.3, 3). Here

$$\left.\begin{aligned} U(0,\lambda) &= \sqrt{\left(\frac{A_1}{2}\right)^2 \frac{\pi^4 \sin^2 \lambda\delta}{\left[\pi^2 - (\lambda\delta)^2\right]^2} + \frac{(4A_2 \cos\lambda\delta)^2}{\left[\left(\frac{\pi}{2}\right)^2 - (\lambda\delta)^2\right]^2}}\, \cos\left[\lambda t_o + \theta(0,\lambda)\right] \\ \tan\theta(0,\lambda) &= \frac{\pi^2 A_1 \left[\left(\frac{\pi}{2}\right)^2 - (\lambda\delta)^2\right]}{8A_2\left[\pi^2 - (\lambda\delta)^2\right]} \tan\lambda\delta \end{aligned}\right\} \quad 2, (\text{IV-2.3, 3})$$

IV-2.4 Pulses of exponential type. In the last discussion we have considered pulses of small finite duration. For completeness we will consider two typical pulses which show a predominant high spike, but possess infinitely long tails of negligible height so that they may be looked upon as practically finite pulses. The examples considered here are the exponential "gaussian" pulse and the exponential pulse whose sides are given by $e^{-a|t-t_o|}$. These types of pulses do not show a uniquely definite aperture 2δ. A conventional definition for their apertures must be introduced.

The $U(0,\lambda)$ function associated with these pulses shows an envelope function which shows no oscillations and possesses a decay line of fast attenuation.

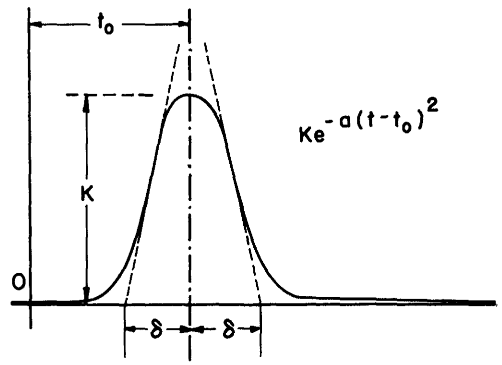

Fig. 1, (IV-2. 4, 1)
Conventional main aperture of a "gaussian" pulse.

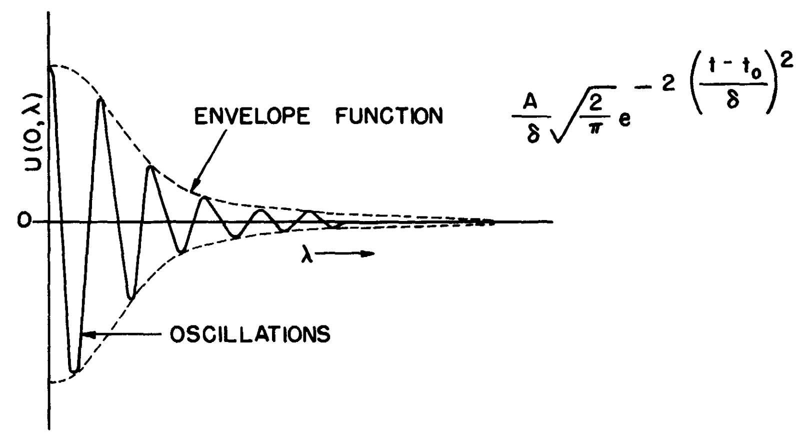

Fig. 2, (IV-2. 4, 1)
The function $U(0, \lambda)$ for a gaussian pulse.

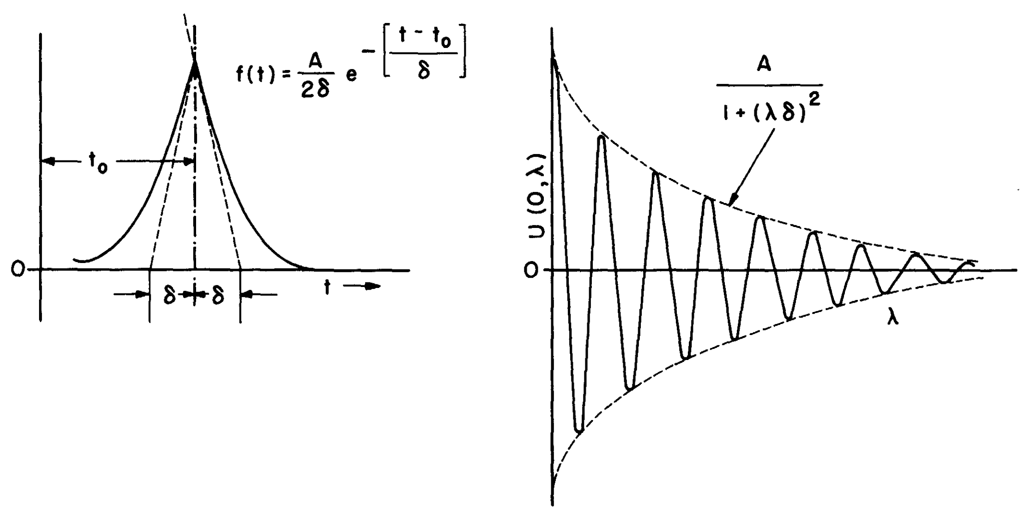

Fig. 1, (IV-2. 4, 2)
The exponential pulse.

IV-2.4, 1 The gaussian pulse is formed by the function

$$f(t) = Ke^{-(t-t_o)^2 a} \; ; \quad K, a = \text{constants} \qquad 1, (IV-2.4, 1)$$

We can define its aperture conventionally by the distance between the crossing points of its maximum tangents with the time axis. See Fig. 1, (IV-2.4, 1). The constants K and a can be determined in terms of the pulse area A and its conventional aperture 2δ as defined above. Elementary computations yield the analytical expression of a gaussian pulse of area A and aperture 2δ.

The corresponding expression for the pulse function is

$$f(t) = \frac{A}{\delta}\sqrt{\frac{2}{\pi}}\, e^{-2\left(\frac{t-t_o}{\delta}\right)^2} \qquad 2, (IV-2.4, 1)$$

It must be noted that the pulse as defined above is not zero for $t < 0$. Under the assumption that δ is small, and $t_o > \delta$, then we can neglect the value of 2, (IV-2.4, 1) for $t < 0$. Using this approximation, it can be shown that the associated density distribution function is given by

$$U(0, \lambda) \approx A\, e^{-\frac{\lambda^2 \delta^2}{8}} \cos \lambda t_o \qquad 3, (IV-2.4, 1)$$

This type of pulse has a monotonic decaying envelope. See Fig. 2, (IV-2.4, 1).

IV-2.4, 2 The second example of the exponential type of pulse having an area A and a conventional aperture 2δ is given by the expression

$$f(t) = \frac{A}{2\delta}\, e^{-\frac{|t-t_o|}{\delta}} \qquad 1, (IV-2.4, 2)$$

The corresponding density distribution function is given by

$$U(0, \lambda) = \frac{A}{1 + (\lambda\delta)^2} \cos \lambda t_o \qquad 2, (IV-2.4, 2)$$

The conventional aperture 2δ is also defined by the maximum tangents of the pulse. See Fig. 1, (IV-2.4, 2).

Section IV-3

The Corner Theorems

Stieltjes integral aspects of the density distribution functions. Discontinuities in the function f(t), in its first, second, and so forth, derivatives. Polygonal pulse skeleton.

IV-3.0 Introduction. In the previous sections we have developed the integral representation of the density distribution functions $U(0, \lambda)$ in a general case, and in particular for the case of time functions which represent pulses. We have associated with the function f(t) two forms of time distribution functions. They are given by

or alternately

$$\left.\begin{aligned} t(t) &= \int_0^t e^{-\gamma_0 \mu} f(\mu)\, d\mu \\ T(t) &= \int_0^t f(\mu)\, d\mu \end{aligned}\right\} \qquad 1, (IV\text{-}3.0)$$

For the cases when t(t) or T(t) happens to possess uniform continuity for every value of t in the interval $0 < t < \infty$, then the integral expression 3, (IV-1.1) for the function $U(0, \lambda)$ exists in the Riemann sense. The examples worked out in section IV-2 for particular pulses show the application of these integrals as Riemann's integrals.

We will now consider the possible existence of discontinuities and other singularities of the function f(t) and in its derivatives, in which case the integrals 2, (IV-1.1) for $U(\gamma_0, \lambda)$ would exist in the Stieltjes sense. The study of the Stieltjes integrals above is of canonic importance in the foundation of the network theory, particularly in the time-domain synthesis aspects. The integrals 1,(IV-2.2,1) yield a series of "corner" theorems which establish a direct analytical link between certain analytical entities of the function f(t) and the nature and position of the generating singularities of the function $U(\gamma, \lambda)$. The denomination corner in connection with these theorems will be justified during the first part of this discussion.

IV-3.0,1 The scope and analytical detail of the corner theorems developed in Part IV are limited to the study of pulses. This is, in fact, not a severe restriction. The ideas and methods presented here can be extended with extreme ease to other, more general, situations.

In the study of pulse transmission, corner theorems are needed mainly to suit the following two purposes:

(a) To reveal the character of the singularities of $U(0, \lambda)$ from the pulse time function

(b) To provide a method of rapid and accurate computation of the function $U(0, \lambda)$, particularly in determining the simple characteristic elements of $U(0, \lambda)$, which were emphasized in the particular examples given in section IV-2.

IV-3.0, 2 In spite of the analytical simplicity of the integrals 4, (IV-2.2, 1) which yield the amplitude functions $\alpha(0, \lambda)$ and $\beta(0, \lambda)$, the evaluation of these integrals for arbitrary narrow pulses is far from being a simple problem. In fact, it is very hard, in general, to get an exact value of those integrals. Approximate methods of integration must be used. In the case of narrow arbitrary pulses, we must proceed with utmost care in obtaining approximate expressions for the function $U(0, \lambda)$. It has been found that negligibly small deviations from the exact value of $U(0, \lambda)$ may produce an intolerably strong convergence phenomenon in the solutions of certain time-domain synthesis problems of pulse transmissions.

Corner theorems are useful in showing the correct way in which the function $U(0, \lambda)$ has to be approximated.

A heuristic discussion of the methods of approximate evaluation of $U(0, \lambda)$ can be based on corner theorems, as will be shown in section IV-4.

IV-3.1 A convenient way to introduce the reader to corner theorems is by means of simple examples. Let us take a pulse, as in Fig. 1, (IV-3.1), a, for which we try to compute the function $U(0, \lambda)$. As before, we will use the auxiliary variable x defined by

$$t - t_o = x \qquad 1, (IV\text{-}3.1)$$

In computing the $U(0, \lambda)$ by means of the integrals 3, (IV-1.1) or 2, (IV-2.2, 2), we will naturally be tempted to obtain a quick approximate evaluation by approximating the pulse shape by a series of superimposed rectangles, as in Fig. 1, (IV-3.1), b, or better by a polygonal approximation as in Fig. 1, (IV-3.1), c. The selection of such polygonal approximation methods may be a sound procedure, or may be a complete mistake, depending on whether we want to approximate the function $U(0, \lambda)$ for small, medium, or

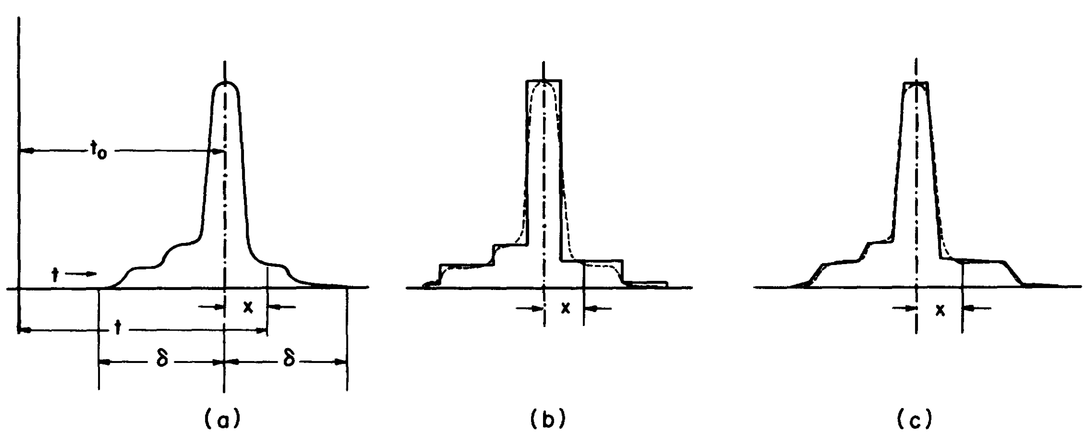

Fig. 1, (IV-3.1)
A pulse and its polygonal approximation.

very large values of λ. For the purpose of explaining the basic ideas concerning corner theorems, we will momentarily assume that we are interested in the ranges of λ in which the polygonal pulse approximation in Fig. 1,(IV-3.1),b, or in Fig. 1,(IV-3.1),c, is valid. The term corner theorem is used because the density distribution function $U(0, \lambda)$, in the case of polygonal pulses, depends exclusively on the corners of the polygon. The objective of the following three subsections is to show that this is truly the case.

IV-3.1, 1 For completeness we will start the discussion with the case in which the pulse is reduced to an impulse of area A, delayed t_o units of time. As was already shown in Part I, reference 1, of these notes, the integrals 2, (IV-1.1) and 2, (IV-2.2, 2) exist as Stieltjes integrals. The function $T(t)$ is here a step function of height H at $t = t_o$. Hence

$$U(0, \lambda) = \int_0^\infty \cos \lambda t \, dT(t) = A \cos \lambda t_o \qquad 1, (IV-3.1, 1)$$

The envelope function 2, (IV-2.2, 2) is therefore also a Stieltjes integral

$$\alpha(0, \lambda) = \int_{-\delta}^{\delta} \cos \lambda x \, dT(x) = 1 \qquad 2, (IV-3.1, 1)$$

since $T(x)$ is an impulse at $x = 0$.

The function $U(0, \lambda) = A \cos \lambda t_o$ obviously corresponds to the transfer function

$$F(s) = Ae^{-st_o} \qquad 3, (IV-3.1, 1)$$

as was shown in one example in reference 1. Hence, impulses in the function $f(t)$ correspond to a transcendental transfer function generated at $s = \infty$ by an essential singularity of the exponential type. These results are, of course, very well known.

IV-3.1, 2 Let us consider in this subsection the continuous pulse and its rectangular skeleton which are given by Fig. 1, (IV-3.1), a and b.

1) First, take the continuous pulse. The time function $f(t)$ representing this pulse is, by construction, at least of class C^I. (Derivative of $f(t)$ is finite and continuous for all values of t, $t_o - x_e < t < t_o + x_u$.) Hence, the integral expression for $U(0, \lambda)$ can be written as

$$\left.\begin{aligned} U(0, \lambda) &= \cos \lambda t_o \int_{-\delta}^{\delta} f(x) \cos \lambda x \, dx + \sin \lambda t_o \int_{-\delta}^{\delta} f(x) \sin \lambda x \, dx \\ &= \left\{ \frac{2}{\lambda} \sin \lambda\delta \, f_2(\delta) - \frac{2}{\lambda} \int_0^\delta \sin \lambda x \, f_2'(x) \, dx \right\} \cos \lambda t_o \\ &\quad + \left\{ \frac{2}{\lambda} (1 - \cos \lambda\delta) \, f_1(\delta) - \frac{2}{\lambda} \int_0^\delta (1 - \cos \lambda x) \, f_1'(x) \, dx \right\} \sin \lambda t_o \end{aligned}\right\} \qquad 1, (IV-3.1, 2)$$

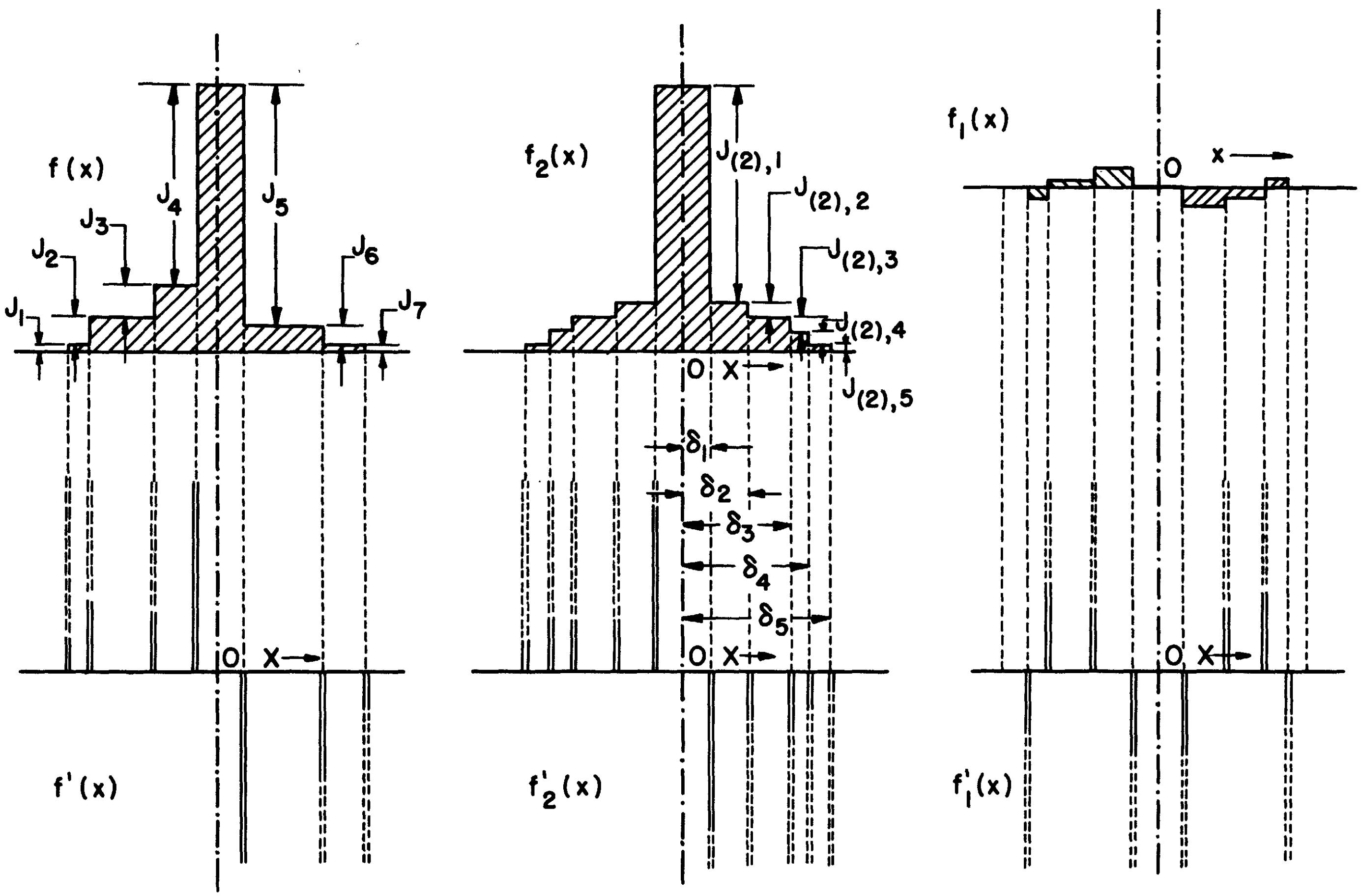

Fig. 1, (IV-3. 1, 2)
Impulses associated with a rectangular skeleton.

where

$$\left.\begin{aligned}
f_1(x) &= \frac{f(x) - f(-x)}{2} \quad \text{(odd)}\\
f_2(x) &= \frac{f(x) + f(-x)}{2} \quad \text{(even)}\\
&\text{Hence}\\
f'_1(x) &= \frac{f'(x) + f'(-x)}{2}\\
f'_2(x) &= \frac{f'(x) - f'(-x)}{2}
\end{aligned}\right\} \qquad 2,\text{(IV-3. 1, 2)}$$

2) Let us suppose now that $f_1(x)$ is not of class C^I and assume that there is a finite set of isolated points in which the first derivative of the pulse function shows impulses, as for example, in the case of the rectangular skeleton in Fig. 1, (IV-3. 1), b.

Extending previous ideas, let us introduce the time distribution function associated with the derivative

$$\left.\begin{aligned}
T_1^{(I)}(x) &= \int_0^x f'_1(\mu)\, d\mu\\
T_2^{(I)}(x) &= \int_0^x f'_2(\mu)\, d\mu
\end{aligned}\right\} \qquad 3,\text{(IV-3. 1, 2)}$$

The functions above show a finite jump at the points of impulse behavior of $f'_1(x)$ or $f'_2(x)$.

Hence, Eq. 1, (IV-3. 1, 2) has a Stieltjes extension as

$$\left.\begin{aligned}
U(0, \lambda) &= \alpha(0, \lambda) \cos \lambda t_o + \beta(0, \lambda) \sin \lambda t_o\\
&\text{where}\\
\alpha(0, \lambda) &= -\frac{2}{\lambda} \int_0^\delta \sin \lambda x \; dT_2^{(I)}(x)\\
\beta(0, \lambda) &= -\frac{2}{\lambda} \int_0^\delta (1 - \cos \lambda x)\; dT_1^{(I)}(x)
\end{aligned}\right\} \qquad 4,\text{(IV-3. 1, 2)}$$

These expressions lead to a simple computation of the envelope functions corresponding to pulses which have a contour formed by a rectangular skeleton. For example, for the pulse in Fig. 1,(IV-3.1),b, we have for the function $f'(x)$, $f'_2(x)$, and $f'_1(x)$ distributions of impulses whose areas are equal to the heights of the jumps. Here, the Stieltjes integrals 4, (IV-3. 1, 2) are given, in general, by

$$\left.\begin{aligned} \alpha(0, \lambda) &= -\frac{2}{\lambda} \sum_{k=1}^{k=n} J_{(2), k} \sin \lambda\delta_k \\ \beta(0, \lambda) &= -\frac{2}{\lambda} \sum_{k=1}^{k=n} J_{(1), k} (1 - \cos \lambda\delta_k) \end{aligned}\right\} \qquad 5, (\text{IV-3. 1, 2})$$

where $J_{(2), k}$ and $J_{(1), k}$ are the resulting jumps of the function $f_2(x)$ and $f_1(x)$, respectively. In the particular case of the example, $n = 5$ and all $J_{(2), k}$ are negative. See Fig. 1, (IV-3. 1, 2).

IV-3. 1, 3 Let us consider again in this subsection a uniform continuous pulse and its polygonal skeleton as in the example of Fig. 1, (IV-3. 1), a and c.

Take the continuous pulse first. Here, let us assume that the pulse function f(t) is of class C^{II}. Hence, a new integration by parts is permitted.

The functions $\alpha(0, \lambda)$ and $\beta(0, \lambda)$ given by the bracket parenthesis in 1, (IV-3. 1, 2) become

$$\left.\begin{aligned} \alpha(0, \lambda) &= \frac{2}{\lambda} \sin \lambda\delta\, f_2(\delta) - \frac{2}{\lambda^2} (1 - \cos \lambda\delta)\, f_2'(\delta) \\ &\quad + \frac{2}{\lambda^2} \int_0^{\delta} (1 - \cos \lambda x)\, f_2''(x)\, dx \\ \beta(0, \lambda) &= \frac{2}{\lambda} (1 - \cos \lambda\delta)\, f_1(\delta) - \frac{2}{\lambda} \left(1 + \delta - \frac{\sin \lambda\delta}{\lambda}\right) f_1'(\delta) \\ &\quad + \frac{2}{\lambda} \int_0^{\delta} \left(1 + x - \frac{\sin x\lambda}{\lambda}\right) f_1''(x)\, dx \end{aligned}\right\} \qquad 1, (\text{IV-3. 1, 3})$$

Let us now suppose that f(x) is not of class C^{II} and assume that there is a finite set of isolated points in which the second derivative shows an impulsive character, as for example, in the polygonal pulse case in Fig. 1, (IV-3. 1), c.

Equations 1,(IV-3.1,3) admit a Stieltjes extension when we introduce the time distribution functions associated with the second derivative as

$$\left.\begin{aligned} T_1^{(II)} &= \int_0^{x} f_1''(\mu)\, d\mu \\ T_2^{(II)} &= \int_0^{x} f_2''(\mu)\, d\mu \end{aligned}\right\} \qquad 2, (\text{IV-3. 1, 3})$$

The functions 1, (IV-3. 1, 3) can then be written as

$$\left.\begin{aligned} \alpha(0,\lambda) &= \frac{2}{\lambda}\sin\lambda\delta\, f_2(\delta) + \frac{2}{\lambda}\int_0^{\delta} \frac{(1-\cos\lambda\delta)}{\lambda}\, dT_2^{(II)}(x) \\ \beta(0,\lambda) &= \frac{2}{\lambda}(1-\cos\lambda\delta)\, f_1(\delta) + \frac{2}{\lambda}\int_0^{\delta}\left(1 + x - \frac{\sin x\lambda}{\lambda}\right) dT_1^{(II)}(x) \end{aligned}\right\} \quad 3,(\text{IV-3.1, 3})$$

These last integrals render a simple solution in the case of polygonal skeleton pulses. Figure 1, (IV-3. 1, 3) shows the even and odd pulse components which correspond to the polygonal pulse in Fig. 1, (IV-3. 1), c. The procedure to be followed is given in Fig. 1, (IV-3. 1, 3) in a self-explanatory manner. In determining the magnitude and sign of the pulses of the second derivative, the graphs must be continuously followed in the direction a to c, d to f.

Let $\delta_{(2),k}$, $\delta_{(1),j}$ be the set of points of derivative discontinuity in $f_2(x)$ and $f_1(x)$ respectively, and $\theta_1, \ldots, \theta_p$ and $\gamma_1, \ldots, \gamma_q$ be the angle jump at each corner of the polygon. The angles must be counted as shown in the figures.

If one uses the notation

$$\left.\begin{aligned} \tan\theta_{k+1} - \tan\theta_k &= I_{(2),k}; \quad k = 1, 2, \ldots, p \\ \tan\gamma_{j+1} - \tan\gamma_j &= I_{(1),j}; \quad j = 1, 2, \ldots, q \end{aligned}\right\} \quad 4,(\text{IV-3.1, 3})$$

(the integers p and q are not necessarily equal)

then, after performing a simple algebraic operation, the Stieltjes integrals 3,(IV-3.1,3) immediately yield

$$\left.\begin{aligned} \alpha(0,\lambda) &= \frac{2}{\lambda}\sin\lambda\delta\, f_2(\delta) + \frac{2}{\lambda^2}\sum_{k=1}^{k=p} I_{(2),k}\left(1 - \cos\lambda\delta_{(2),k}\right) \\ \beta(0,\lambda) &= \frac{2}{\lambda}(1-\cos\lambda\delta)\, f_1(\delta) + \frac{2}{\lambda}\sum_{j=1}^{j=q} I_{(1),j}\left(1 + \delta_{(1),j} - \frac{\sin\lambda\delta_{(1),j}}{\lambda}\right) \\ &\text{with} \\ &\sum_{k=1}^{k=p} I_{(2),k} = 0; \quad \sum_{j=1}^{j=q} I_{(1),j} = 0 \end{aligned}\right\} \quad 5,(\text{IV-3.1, 3})$$

IV-3.2 We proceed in this section to state the first corner theorem. The theorem follows at once from all previous discussions. We will explain the notation. Suppose we have a pulse f(t) which is of the general polygonal type. This pulse may contain isolated pulses of finite area, rectangular partial components, and ordinary polygonal pulses. The pulse function can be written as

$$f(x) = f_{(0)}(x) + f_{(I)}(x) + f_{(II)}(x) \qquad 1, (IV-3.2)$$

in which $f_{(0)}(x)$ contains all the impulses, $f_{(I)}(x)$ represents the rectangular parts of the pulse, and $f_{(II)}(x)$ represents the remaining polygonal parts of the pulse. Let us divide in odd and even parts in the following way:

$$\left.\begin{aligned} f_{(0)}(x) &= f_{(0),1}(x) + f_{(0),2}(x) \\ f_{(I)}(x) &= f_{(I),1}(x) + f_{(I),2}(x) \\ f_{(II)}(x) &= f_{(II),1}(x) + f_{(II),2}(x) \end{aligned}\right\} \qquad 2, (IV-3.2)$$

Now let us introduce the notation for the points corresponding to impulses, jumps, and the like, of the last function as follows:

$$\left.\begin{array}{llll} \delta_{1,i}; & \delta_{2,j}; & i = 1, 2, \ldots, h; & j = 1, 2, \ldots, g \\ \delta_{1,k}; & \delta_{2,\ell}; & k = 1, 2, \ldots, m; & \ell = 1, 2, \ldots, n \\ \delta_{1,r}; & \delta_{2,s}; & r = 1, 2, \ldots, p; & s = 1, 2, \ldots, q \end{array}\right\} \qquad 3, (IV-3.2)$$

They are respectively the set of points where:
First, the functions

$$f_{(0),1}(x); \quad f_{(0),2}(x) \quad \text{have impulses}$$

whose areas are respectively denoted by

$$\left.\begin{aligned} D_{1,i} &= f_{(0),1}\left[\delta_{1,i}(+)\right] - f_{(0),1}\left[\delta_{1,i}(-)\right] \\ D_{2,j} &= f_{(0),2}\left[\delta_{2,j}(+)\right] - f_{(0),2}\left[\delta_{2,j}(-)\right] \end{aligned}\right\} \qquad 4, (IV-3.2)$$

The symbols, $\delta_{1,i}(+)$ and $\delta_{1,i}(-)$, etc., mean approaching from the right and left, respectively.
Second, the functions

$$f_{(I),1}(x); \quad f_{(I),2}(x) \quad \text{have jumps denoted by}$$

$$\left.\begin{aligned} J_{1,k} &= f_{(I),1}\left[\delta_{1,k}(+)\right] - f_{(I),1}\left[\delta_{1,k}(-)\right] \\ J_{2,\ell} &= f_{(I),2}\left[\delta_{1,\ell}(+)\right] - f_{(I),2}\left[\delta_{1,\ell}(-)\right] \end{aligned}\right\} \qquad 5, (IV-3.2)$$

Third, the functions

$$f'_{(II),1}(x); \quad f'_{(II),2}(x) \quad \text{have jumps denoted by}$$

$$\left.\begin{aligned} I_{1,r} &= f'_{(II),1}\left[\delta_{1,r}(+)\right] - f'_{(II),1}\left[\delta_{1,r}(-)\right] \\ I_{2,s} &= f'_{(II),2}\left[\delta_{1,s}(+)\right] - f'_{(II),2}\left[\delta_{2,s}(-)\right] \end{aligned}\right\} \qquad 6, (IV-3.2)$$

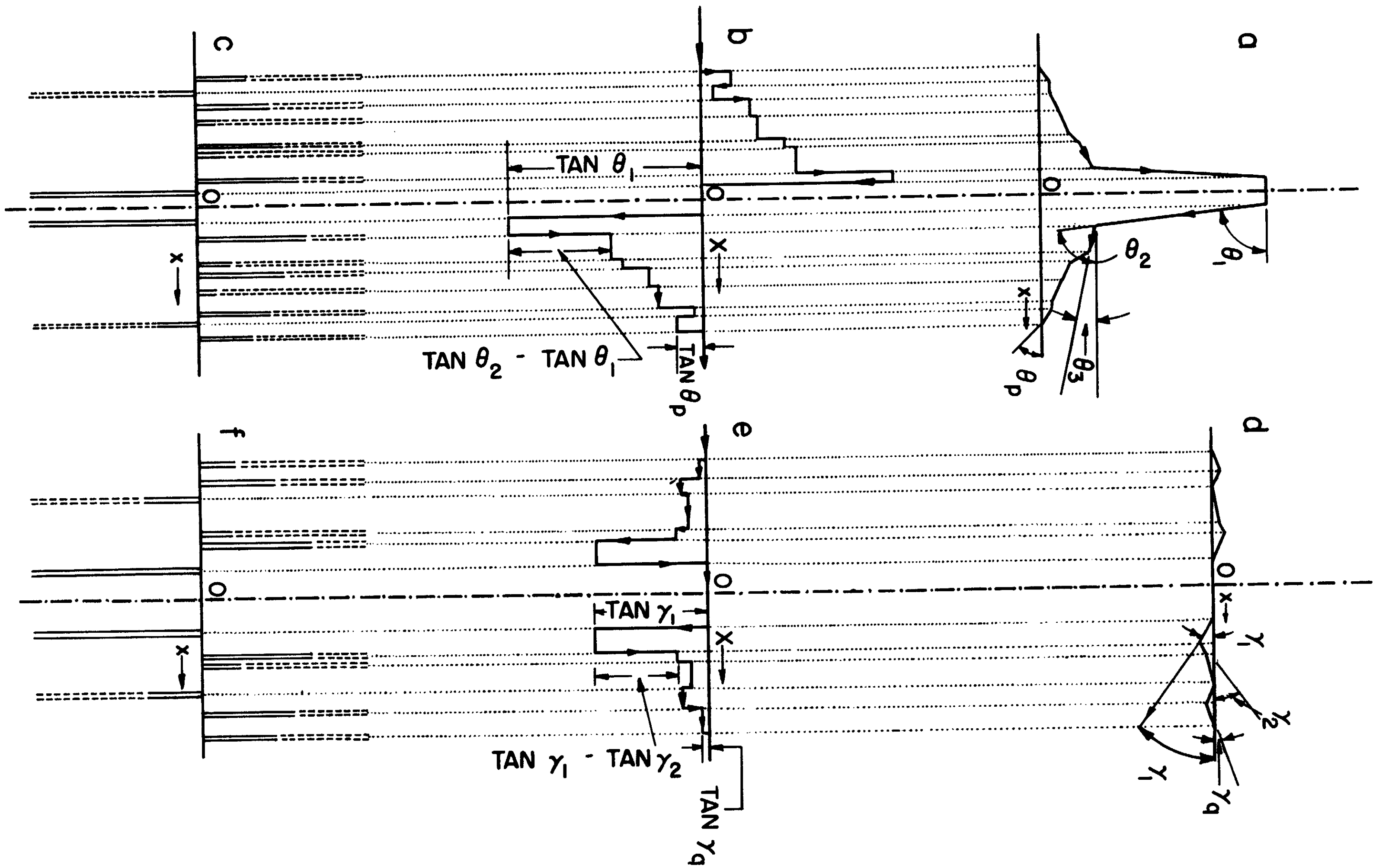

Fig. 1, (IV-3. 1, 3)
Polygonal pulses and their first and second derivatives.

By applying the results of the last subsections to the functions 2, (IV-3.2), we obtain a theorem which here is given as a condensation of several particular cases.

Corner theorem. Let f(t) be a general polygonal pulse function which satisfies the requisites of the previous sections ($t - t_o = x$). The density distribution function $U(0, \lambda)$ associated with this function is given by

$$U(0, \lambda) = \alpha(0, \lambda) \cos \lambda t_o + \beta(0, \lambda) \sin \lambda t_o$$

where

$$\left.\begin{aligned} \alpha(0, \lambda) &= \sum_{j=1}^{j=g} D_{2,j} - \frac{2}{\lambda} \sum_{\ell=1}^{\ell=n} J_{2,\ell} \sin \lambda\delta_{2,\ell} \\ &\quad + \frac{2}{\lambda^2} \sum_{s=1}^{s=q} I_{2,s} \left[1 - \cos \lambda\delta_{2,s}\right] \\ \beta(0, \lambda) &= \sum_{i=1}^{i=h} D_{1,i} - \frac{2}{\lambda} \sum_{k=1}^{k=m} J_{1,k} (1 - \cos \lambda\delta_{1,k}) \\ &\quad + \frac{2}{\lambda} \sum_{r=1}^{r=p} I_{1,r} \left(1 + \delta_{1,r} - \frac{\sin \lambda\delta_{1,r}}{\lambda}\right) \end{aligned}\right\} \qquad 7, (IV\text{-}3.2)$$

<u>IV-3.3</u> The extension of the results and theorems for cases, in which the second, third, etc., derivatives have discontinuities can be carried out immediately. For the purpose of Part IV, we shall make an explicit extension of the corner theorem to the case in which the second derivative is discontinuous. Pulses whose skeletons are formed by appropriate quadratic curves may produce examples of this situation. A simple example of a quadratic skeleton approximation of a symmetric pulse is illustrated in Fig. 1,(IV-3.3).

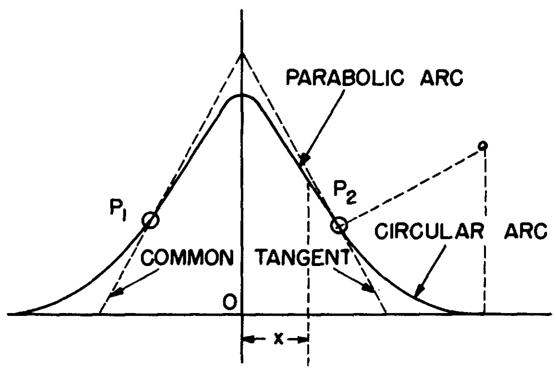

Fig. 1, (IV-3.3)
Pulse with discontinuous second derivative.

IV-3. 3, 1 Let us introduce the time distribution functions associated with the third derivative:

$$\left.\begin{aligned} T_1^{(III)} &= \int_0^x f_1'''(\mu)\, d\mu \\ T_2^{(III)} &= \int_0^x f_2'''(\mu)\, d\mu \end{aligned}\right\} \qquad 1, (IV-3.3, 1)$$

The reader will find that for a function of class C^{III}, one gets

$$\left.\begin{aligned} \alpha(0,\lambda) &= \frac{2}{\lambda} \sin \lambda\delta\, f_2(\delta) - \frac{2}{\lambda^2}(1 - \cos \lambda\delta)\, f_2'(\delta) + \frac{2}{\lambda^2}\left(1 + \delta - \frac{\sin \lambda\delta}{\lambda}\right) f_2''(\delta) \\ &\quad - \frac{2}{\lambda^2}\int_0^\delta \left(1 + x - \frac{\sin \lambda x}{\lambda}\right) f_2'''(x)\, dx \\ \beta(0,\lambda) &= \frac{2}{\lambda}(1 - \cos \lambda\delta)\, f_1(\delta) - \frac{2}{\lambda}\left(1 + \delta - \frac{\sin \lambda\delta}{\lambda}\right) f_1'(\delta) \\ &\quad + \frac{2}{\lambda}\left[\delta + \frac{1}{2}\delta^2 - \frac{1}{\lambda^2}(1 - \cos \lambda\delta)\right] f_1''(\delta) \\ &\quad - \frac{2}{\lambda}\int_0^\delta \left[x + \frac{1}{2}x^2 - \frac{1}{\lambda^2}(1 - \cos \lambda x)\right] f_1'''(x)\, dx \end{aligned}\right\} \qquad 2, (IV-3.3, 1)$$

If the pulse function possesses a discontinuous second derivative, then the expressions above have a Stieltjes integral representation of class C^I.

$$\left.\begin{aligned} \alpha(0,\lambda) &= \frac{2}{\lambda} \sin \lambda\delta\, f_2(\delta) - \frac{2}{\lambda^2}(1 - \cos \lambda\delta)\, f_2'(\delta) \\ &\quad - \frac{2}{\lambda^2}\int_0^\delta \left(1 + x - \frac{\sin \lambda x}{\lambda}\right) dT_2^{(III)}(x) \\ \beta(0,\lambda) &= \frac{2}{\lambda}(1 - \cos \lambda\delta)\, f_1(\delta) - \frac{2}{\lambda}\left(1 + \delta - \frac{\sin \lambda\delta}{\lambda}\right) f_1'(\delta) \\ &\quad - \frac{2}{\lambda}\int_0^\delta \left(x + \frac{1}{2}x^2 - \frac{1 - \cos \lambda x}{\lambda^2}\right) dT_1^{(III)}(x) \end{aligned}\right\} \qquad 3, (IV-3.3, 1)$$

IV-3. 3, 2 The Stieltjes integral representation 3, (IV-3. 3, 1) produces simple direct solutions in case of quadratic skeleton pulses.

Let

$$\delta_{1,\mu};\quad \delta_{2,\nu};\quad \mu = 1, 2, \ldots, \alpha;\quad \nu = 1, 2, \ldots, \beta \qquad 1, (IV-3.3, 2)$$

be the set of points at which the pulse function components $f_1(x)$ and $f_2(x)$ have discontinuous second derivatives. At these points, the second derivative has jumps denoted respectively by

$$\left.\begin{aligned} R_{1,\mu} &= f_1''\left[\delta_{1,\mu}(+)\right] - f_1''\left[\delta_{1,\mu}(-)\right] \\ R_{2,\nu} &= f_2''\left[\delta_{2,\nu}(+)\right] - f_2''\left[\delta_{2,\nu}(-)\right] \end{aligned}\right\} \qquad 2, (IV\text{-}3.3, 2)$$

For quadratic skeleton pulses the Stieltjes integrals in 3, (IV-3. 3, 1) become

$$\left.\begin{aligned} -\frac{2}{\lambda^2}\int_0^{\delta}\left(1 + x - \frac{\sin \lambda x}{\lambda}\right) dT_2^{(III)}(x) &= -\frac{2}{\lambda^2}\sum_{\nu=1}^{\nu=\beta}\left(1 + \delta_{2,\nu} - \frac{\sin \lambda\delta_{2,\nu}}{\lambda}\right) R_{2,\nu} \\ -\frac{2}{\lambda^2}\int_0^{\delta}\left[x + \frac{1}{2}x^2 - \frac{1 - \cos \lambda x}{\lambda}\right] dT_1^{(III)}(x) & \\ = -\frac{2}{\lambda^2}\sum_{\mu=1}^{\mu=\alpha}\left(\delta_{1,\mu} + \frac{1}{2}\delta_{1,\mu}^2 - \frac{1 - \cos \lambda\delta_{1,\mu}}{\lambda}\right) R_{1,\mu} & \end{aligned}\right\} \qquad 3, (IV\text{-}3.3, 2)$$

The addition of these expressions to the corner theorem 7, (IV-3. 2) produces a more general theorem which covers pulse skeletons containing rectilinear and quadratic arcs.

Section 4

Heuristic Analysis of Pulses, with Particular Attention to Symmetric Pulses

Simple basic ideas of approximation. The double-trapezoidal frame. Maximum tangent theorem. Beatings and displacement. Exponent of decay.

IV-4.0 Introduction. The principal objective of this section is concerned with the direct evaluation of the characteristic analytical elements of the function $U(0,\lambda)$ which are associated with arbitrary pulses, particularly the symmetrical ones. Approximate solutions will be obtained for $U(0,\lambda)$.

The evaluation of the functions $\alpha(0, \lambda)$ and $\beta(0, \lambda)$ is, generally speaking, a hard and delicate problem. In sections 2 and 3 we have given several formulas which permit the analytical computation of the functions $\alpha(0, \lambda)$ and $\beta(0, \lambda)$. Now, we want to use those expressions in designing simple and reliable approximate methods of computation.

The reader may notice, from the examples given in section 2, that the shape of the graph of $U(0, \lambda)$ is changed considerably by minor changes in the pulse shape. This effect is particularly strong for values of λ following the first zero of the envelope function. In the process of network synthesis for the reproduction of pulses, the actual form of the function $U(0, \lambda)$ is of primary importance, if one wishes to avoid pulse distortion and intolerable network ringing.

The basic characteristic elements of the $U(0, \lambda)$ function we want to determine with accuracy are:

1) The position of the first and a few subsequent zeros of the envelope function

2) The actual shape of the envelope function in its first few cycles

3) The presence of beatings and the actual shape of the envelope function in the first beating group

4) The line, or lines, of envelope decay; the law of decay

5) The displacement of the lines of decay.

In the following discussion, we develop heuristic methods which directly produce these characteristic elements without performing the direct integration for the determination of $U(0, \lambda)$.

IV-4.1 The behavior of $U(0,\lambda)$.

IV-4.1,1 The interval $0 < \lambda < \pi/2\delta$. For simplicity we shall consider symmetric pulses. The extension of these results to nonsymmetric pulses can be done with ease by following methods similar to the one presented in this section.

Let us consider the integral 1, (IV-2.2, 3) which produces the value of the envelope function $\alpha(0, \lambda)$ for symmetrical pulses.

$$\alpha(0, \lambda) = 2 \int_0^{\delta} f(x) \cos \lambda x \, dx \qquad 1,(IV\text{-}4.1)$$

For a fixed value of λ, the cosine term has a period equal to $2\pi/\lambda$. Hence, this period decreases as λ increases. Figure 1,(IV-4.1) shows half of the function f(x) and a set of cosine curves, which correspond to the particular values of λ given as $\pi/8\delta$, $\pi/4\delta$, and $\pi/2\delta$. All of these values are contained in the interval $0 \leq \lambda \leq \pi/2\delta$.

Our aim is to discuss the effect of the cosine term, for $0 \leq \lambda \leq \pi/2\delta$, in relation to the factor f(x) in 1,(IV-4.1). We will do that by considering pertinent values of λ.

(a) $0 < \lambda < \pi/8\delta$. We can see at once that in this interval the cosine term varies slowly in the interval $0 < x < \delta$. The cosine term for $\lambda = \pi/8\delta$ is shown in the figure. In this interval the function $\alpha(0, \lambda)$ is practically independent of the pulse wave shape of pulses of equal area. Hence, for all spike-like pulses we can write

$$\alpha(0, \lambda) \approx A \frac{\sin \lambda\delta}{\lambda\delta}; \quad 0 < \lambda < \pi/8\delta$$

because we can use a square pulse approximation.

(b) $\pi/8\delta < \lambda < \pi/2\delta$. For a fixed value of λ in this interval, the term $\cos \lambda x$ starts changing rather fast for values of x in $0 < x < \delta$. The waveform of the pulse starts having effect on the integral. The reader can immediately see that for the pulse used in the figure, a double trapezoidal pulse skeleton renders a high degree of approximation. See Fig. 2,(IV-4.1) and insert the trapezoidal skeleton in Fig. 1,(IV-4.1). In subsection IV-4.2 we shall give the corresponding expression of the function $\alpha(0, \lambda)$ derived from a double trapezoidal skeleton.

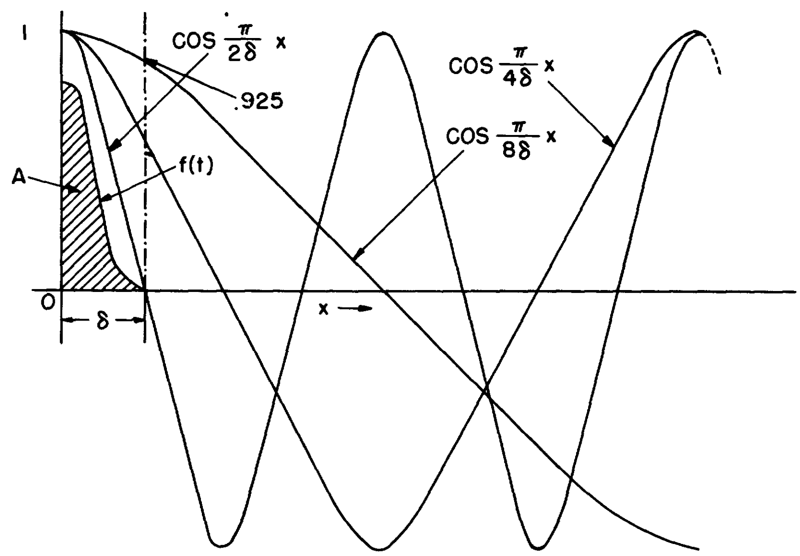

Fig. 1,(IV-4.1)
The interval $0 < \lambda < \pi/2\delta$.

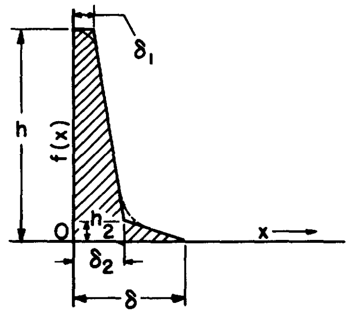

Fig. 2,(IV-4.1)
Trapezoidal approximation.

The reader can immediately see that no zero of the envelope $a(0, \lambda)$ can occur in the interval $0 < \lambda < \pi/2\delta$.

IV-4.1,2 Interval $\pi/2\delta < \lambda < 2\pi/\delta$. This interval is of interest because it contains the first zero of the envelope function $a(0, \lambda)$ corresponding to spike-like pulses. Figure 1,(IV-4.1,1) suggests at once the existence of the first zero and the means of computation of its approximate position. From this figure we see that when λ exceeds the value $\pi/2\delta$, then the term cos λx starts chopping up the function f(x) into a positive and a negative part. The variation of the function $a(0, \lambda)$ is now attributed both to the decrease of cos λx in $0 < \lambda < x$ and to the chopping effect of cos λx. This situation suggests that in the vicinity of $\lambda = \pi/2\delta$ the function $a(0, \lambda)$ has an inflection point. We shall show later that this is actually the case. As the period of cos λx decreases in our interval, we see from the figure that there exists a position of cos λx, such that the function f(x) is chopped into two parts of opposite sign. Then the first zero of the interval 1,(IV-4.1) takes place. Using a double trapezoidal pulse approximation, we can obtain the position of the zero with good accuracy. See subsection IV-4.2 and subsequent subsections.

IV-4.1,3 We will consider here values of $\lambda > 2\pi/\delta$. From Fig. 1,(IV-4.1,1) we see that the chopping effect of the term cos λx becomes increasingly stronger. New zeros of the envelope function must take place, although their exact positions are not easy to visualize.

The reader may note that we can still use the double trapezoidal pulse as an approximation of the pulse chosen as an example. In general, however, we must use a polygonal skeleton containing as many corners as there are points of rapid tangent variation. The use of the corner theorem expressions for $a(0, \lambda)$, which were developed in section IV-3.1, will tell us at once whether or not beatings are present in the envelope function $a(0, \lambda)$ after its first zero. The reason is that each corner produces a damped trigonometric term of different frequency.

IV-4.2 Double trapezoidal pulse approximation. The shape of pulses in which we are most interested in the study of window functions admits a double trapezoidal skeleton as its approximation. For this reason, we will produce some discussions on such pulses.

Let us consider a double trapezoidal pulse of the kind indicated in Fig. 1,(IV-4.2).

The application of the corner theorem, plus some algebraic manipulations renders

$$\left.\begin{aligned} U(0,\lambda) &= a(0,\lambda)\cos\lambda t_o \\ a(0,\lambda) &= \frac{4}{\lambda^2}\left\{\frac{h_1}{\delta_2-\delta_1}\left[\sin\left(\lambda\,\frac{\delta_1+\delta_2}{2}\right)\sin\left(\lambda\,\frac{\delta_2-\delta_1}{2}\right)\right]\right. \\ &\quad \left. + \frac{h_2}{\delta-\delta_2}\left[\sin\left(\lambda\,\frac{\delta+\delta_2}{2}\right)\sin\left(\lambda\,\frac{\delta-\delta_2}{2}\right)\right]\right\} \end{aligned}\right\} \qquad 1, (IV-4.2)$$

For convenience, let us introduce the auxiliary variables

$$\left.\begin{aligned} \tan\mu(\lambda) &= \frac{\dfrac{h_2}{\delta-\delta_2}\sin\lambda\left(\dfrac{\delta-\delta_2}{2}\right)}{\dfrac{h_1}{\delta_2-\delta_1}\sin\lambda\left(\dfrac{\delta_2-\delta_1}{2}\right)} \\ \tan\nu(\lambda) &= \frac{\sin\lambda\,\dfrac{\delta_2+\delta_1}{2}}{\sin\lambda\,\dfrac{\delta_2+\delta}{2}} \\ M &= h_2^2\left(\frac{\sin\left(\lambda\,\dfrac{\delta-\delta_2}{2}\right)}{\lambda\,\dfrac{\delta-\delta_2}{2}}\right)^2 + h_1^2\left(\frac{\sin\left(\lambda\,\dfrac{\delta_2-\delta_1}{2}\right)}{\lambda\,\dfrac{\delta_2-\delta_1}{2}}\right)^2 \\ N &= (\delta+\delta_2)^2\left(\frac{\sin\left(\lambda\,\dfrac{\delta+\delta_2}{2}\right)}{\lambda\,\dfrac{\delta+\delta_2}{2}}\right)^2 + (\delta_2+\delta_1)^2\left(\frac{\sin\left(\lambda\,\dfrac{\delta_2+\delta_1}{2}\right)}{\lambda\,\dfrac{\delta_2+\delta_1}{2}}\right)^2 \end{aligned}\right\} \qquad 2, (IV-4.2)$$

By means of these introduced quantities we finally get

$$a(0,\lambda) = \left\{\sqrt{M}\ \sqrt{N}\right\}\sin\left[\mu(\lambda)+\nu(\lambda)\right] \qquad 3, (IV-4.2)$$

The quantities M, N, $\mu(\lambda)$, and $\nu(\lambda)$ are called the long-beating amplitude, the short-beating amplitude, the long-beating phase function, and the short-beating phase function, respectively.

For a given pulse, the functions M, N, $\mu(\lambda)$, and $\nu(\lambda)$ can be computed with ease.

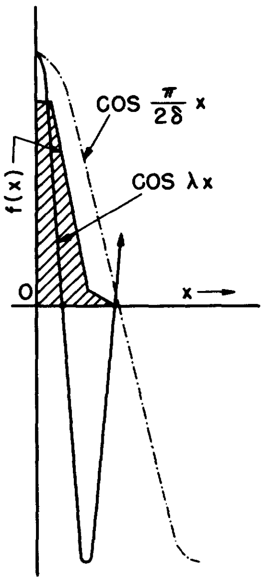

Fig. 1, (IV-4. 1, 1)
The interval $\pi/2\delta < \lambda < \pi/\delta$.

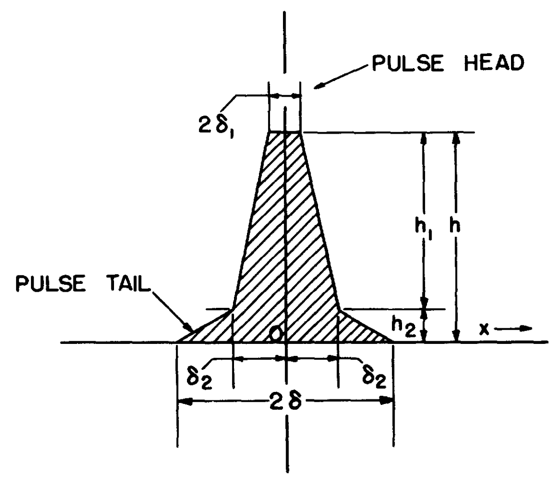

Fig. 1, (IV-4. 2)
Trapezoidal pulse.

IV-4. 2, 1 Consider now the interval $0 < \lambda < \pi/8\delta$. Here the quantities

$$\lambda \frac{\delta_2 - \delta_1}{2} \text{ and } \lambda \frac{\delta - \delta_2}{2}$$

are small, so we may write

$$\left.\begin{aligned} a(0,\lambda) &\approx h_1(\delta_1 + \delta_2) \frac{\sin \lambda \frac{\delta_1 + \delta_2}{2}}{\lambda \frac{\delta_1 + \delta_2}{2}} + h_2(\delta + \delta_2) \frac{\sin \lambda \frac{\delta + \delta_2}{2}}{\lambda \frac{\delta + \delta_2}{2}} \\ &\approx A \frac{\sin \lambda \frac{\delta + \delta_2}{2}}{\lambda \frac{\delta + \delta_2}{2}}; \quad A = \text{area of the pulse} \end{aligned}\right\} \quad 1, (\text{IV-4. 2, 1})$$

since in this interval

$$\frac{\sin \lambda \frac{\delta_1 + \delta_2}{2}}{\lambda \frac{\delta_1 + \delta_2}{2}} \approx \frac{\sin \lambda \frac{\delta_1 + \delta_2}{2}}{\lambda \frac{\delta + \delta_2}{2}}$$

as we have shown in subsection 4. 1.

IV-4.2,2 The "tangent theorem" in the vicinity of $\lambda = \pi/2\delta$. The intermediate member of 1, (IV-4. 2, 1) shows that for small values of λ the double trapezoidal pulse is equivalent to a double rectangular pulse. This is also true for spike-like polygonal pulses. A quick glance at Fig. 1,(IV-4.1) tells us that there is not much difference if we replace the double trapezoidal pulse by a double rectangular pulse of equal area. By using the corner theorem in Eq. 5(IV-3.1,2)

$$\alpha(0,\lambda) = -\frac{2}{\lambda}\sum_{k=1}^{k=n} J_{2,k} \sin\lambda\delta_{2,k} \quad \text{for } \lambda \approx \frac{\pi}{2\delta} \qquad 1,(\text{IV-4.2,2})$$

Let us introduce the auxiliary variable y defined by

$$\delta\lambda = \frac{\pi}{2}(1 + y\delta) \qquad 2,(\text{IV-4.2,2})$$

We then have

$$\sin\lambda\delta_{2,k} = \sin\left[\frac{\pi}{2}\frac{\delta_{2,k}}{\delta} + \frac{\pi}{2}y\delta_{2,k}\right]$$

$$\approx \frac{\pi}{2}\left(\frac{\delta_{2,k}}{\delta} + y\delta_{2,k}\right)$$

since the hypothesis makes the areas small. Hence

$$\alpha(0,\lambda) \approx -\frac{2}{\lambda\delta}\frac{\pi}{2}\left\{\sum_{k=1}^{k=n} J_{2,k}\,\delta_{2,k}\right\}(1 + \delta y) \approx -\frac{A}{2}(1 + \delta y) \qquad 3,(\text{IV-4.2,2})$$

since the pulse area

$$A = \sum_{k=1}^{k=n} J_{2,k}\,\delta_{2,k}$$

Hence, the property follows:

Theorem 1,(IV-4.2,2) "In the vicinity of the point $\lambda = \pi/2\delta$ the envelope function $\alpha(0,\lambda)$ behaves almost linearly, having a tangent

$$\frac{d\alpha(0,\lambda)}{d\lambda} = \frac{d\alpha}{dy}\cdot\frac{dy}{d\lambda} = -\frac{2}{\pi}\,\delta A = \text{const"} \qquad 4,(\text{IV-4.2,2})$$

Sometimes it is convenient to use normalized units in drawing the plots of the function $\alpha(0,\lambda)$. If we normalize our units

$$\sigma = \frac{\lambda}{2\pi}, \quad 2\delta = 1 \text{ and } A = 1$$

as is frequently done (see ref. 4), then the envelope $\alpha(0,\sigma)$ for all pulses has the same maximum tangent, $d\alpha(0,\sigma)/d\sigma_1 = -2$, corresponding to an angle = 120° at the point $\sigma = 1/2$. The value of the function $\alpha(0,\lambda)$, at the point $\lambda = \pi/2\delta$, which is equivalent to $\sigma = 1/2$, is given by

$$\alpha(0, \lambda) = \frac{A}{2}; \quad \text{(in normalized units)}$$

The expression 3, (IV-4.2, 2) tells us that the inflection point of $\alpha(0, \lambda)$, in the first cycle of the envelope, lies approximately in the vicinity of the point $\lambda = \pi/2\delta$.

Theorem 1,(IV-4.2,2) has important practical application in the plotting of the function $\alpha(0, \lambda)$. Inspection of the envelope curves in reference 4 tells us that the linear part of this envelope function is rather large for all spike-like pulses.

IV-4.2, 3 Positions of the first zeros of $\alpha(0, \lambda)$. The first zeros of the envelope function $\alpha(0, \lambda)$ can be located with ease directly from the pulse shape without performing the integration 1, (IV-4.1). For spike-like pulses the double trapezoidal skeleton approximation renders accurate enough results.

To make a simple explanation, we will begin by considering tailless spike-like pulses. These pulses, like the half cosine, square, parabolic, triangular, and other such pulses, can be approximated very closely, for the purposes of this subsection, by means of a simple trapezoidal pulse; that is, $\delta_2 = \delta$, $h_2 = 0$, and $h_1 = h$. Hence, Eq. 1, (IV-4.2) can be reduced to

$$\alpha(0, \lambda) = \frac{1}{\lambda^2} \frac{4h}{\delta - \delta_1} \sin\left(\lambda \frac{\delta + \delta_1}{2}\right) \sin\left(\lambda \frac{\delta - \delta_1}{2}\right) \qquad 1, (IV\text{-}4.2, 3)$$

The zeros of $\alpha(0, \lambda)$ are thus given by the set of points

$$\left.\begin{aligned} \lambda_{K_1} &= \frac{2\pi}{\delta + \delta_1} K_1; \quad K_1 = 1, 2, \ldots \\ \lambda_{K_2} &= \frac{2\pi}{\delta - \delta_1} K_2; \quad K_2 = 1, 2, \ldots \end{aligned}\right\} \qquad 2, (IV\text{-}4.2, 3)$$

from which it follows that the first zero is located at $(K_1 = 1)$

$$\lambda_1 = \frac{2\pi}{\delta + \delta_1} \qquad 3, (IV\text{-}4.2, 3)$$

Some examples are now given.

1) Square pulse. $\delta_1 = \delta$, $\lambda_1 = \pi/\delta$ (norm. $\sigma_1 = 1.0$).

2) Triangular pulse. $\delta_1 = 0$, $\lambda_1 = 2\pi/\delta$ (norm. $\sigma_1 = 2.0$).

3) Half-cosine pulse. To find the zero, we first draw the pulse and then the trapezoidal skeleton and measure δ_1 from the graph. For example, if we draw the cosine pulse such that $\delta = 0.5$ and $A = 1$, we find, by graph measuring, $\delta_1 = 0.16$. Hence

$$\lambda_1 \approx 3.02\pi \qquad (\sigma_1 = 1.51)$$

In section 2 we found that the exact value is $\lambda_1 = 3\pi$ $(\sigma_1 = 1.5)$.

Formula 3, (IV-4.2, 3) cannot be applied to pulses with tails. Double trapezoidal approximation is then to be used. We proceed as follows: From the graph of the trapezoidal skeleton, we measure the quantities h, h_1, h_2; $\delta, \delta_1, \delta_2$. By means of Eq. 2, (IV-4.2), we calculate the functions $\mu(\lambda)$ and $\nu(\lambda)$ and make the plot of

$$\mu(\lambda) + \nu(\lambda) = \theta(\lambda) \qquad 4, \text{(IV-4.2, 3)}$$

Then the intersection of $\theta(\lambda)$ with the lines at π, 2π, 3π, ... gives in order the zeros of the envelope function.

<u>IV-4.2, 4 Envelope beatings and displacement.</u> The first few cycles of the envelope functions $a(0, \lambda)$ can be computed from its polygonal skeleton. For spike-like pulses, the double trapezoidal frame is quite adequate. By means of measuring the quantities h, h_1, h_2; $\delta, \delta_1, \delta_2$, from the graph of the pulse we can easily compute the graph of $a(0, \lambda)$ for the first group beating. The zeros of $a(0, \lambda)$ can be computed as in subsection IV-4.2, 3. The values of the maxima are given by $\sqrt{MN}$; see Eq. 2, (IV-4.2).

The graph of the function $a(0, \lambda)$ produces an important analytical entity: the displacement λ_d. This quantity enters in the computation of the tolerance ϵ to which we referred in section 2. Examples of functions $U(0, \lambda)$ showing displacement have already been given.

For arbitrary pulses we can estimate the displacement λ_d as follows. Compute the first and second group beating of the envelope function $a(0, \lambda)$ as was indicated above. Plot the lines of decay corresponding to the greatest and smallest amplitude beatings in each group. Then choose a decay line between these two decay lines. This middle line is conventionally called the average decay line. Fig. 1, (IV-4.2, 4) shows graphically the corresponding procedure. The quantity λ_d is estimated as one-half the horizontal displacement of the decay lines L_1 and L_2.

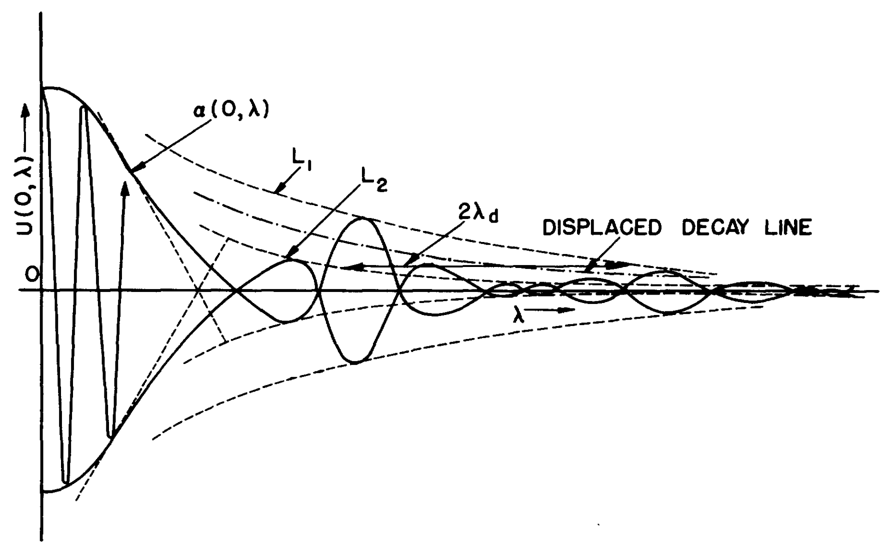

Fig. 1, (IV-4.2, 4)
The "displacement" decay line.

IV-4.3 The tail of the function $\alpha(0, \lambda)$ and its exponential decay. The polygonal frame approximation of a pulse fails to produce reliable functions $\alpha(0, \lambda)$ for relatively large values of λ. The graphical construction in Fig. 1, (IV-4.3) clearly shows that the values of λ after which the polygonal representation starts to fail are those at which the $\cos \lambda x$ have a period of the order of magnitude equal to the horizontal projection of the roundness of the pulse at the corners. Let this projection of the roundness of the fast-changing corner, as measured from the graph of the pulse, be denoted by Δx_{C1}, etc. Then, the value of λ, say, λ_{th}, which constitutes the threshold of failure, can be estimated by

$$\lambda_{th} = \frac{\pi}{\Delta x_{C1}} \qquad 1, (\text{IV-4.3})$$

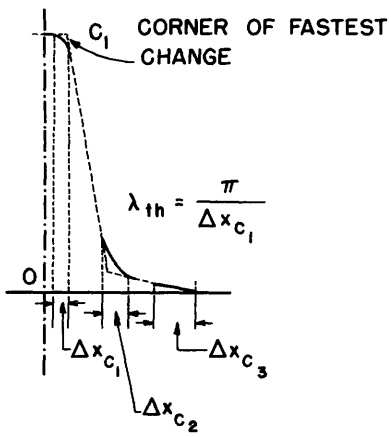

Fig. 1, (IV-4.3)
The threshold value λ_{th}.

The justification of this threshold value selection depends on the rapid chopping effect of $\cos \lambda x$ in the slow variation intervals of the pulse. The contribution of these intervals to the integral 1, (IV-4.1) is practically negligible. Only at the fast variation intervals, the corners, the chopping effect does not make the integral expression for $\alpha(0, \lambda)$ equal to zero.

We will agree to call the interval beyond the point λ_{th}, that is, $\lambda_{th} < \lambda < \infty$, the "tail" of the function $\alpha(0, \lambda)$ of the function $U(0, \lambda)$.

IV-4.3, 1 In the beginning of the tail part, the polygonal frame of the pulse has to be abandoned in order to compute the function $\alpha(0, \lambda)$, or $U(0, \lambda)$. In this part the quadratic skeleton frame can be used. This quadratic frame has a constant second derivative in the vicinity of the corners. Consequently, the function $\alpha(0, \lambda)$ or $U(0, \lambda)$, can be approximated by using the integrals given in 1, (IV-3.1, 3) assuming that the second derivative is zero except in the vicinity of the corners. Fig. 1, (IV-4.3, 1) shows a simple symmetric pulse. The corresponding $\alpha(0, \lambda)$ is given, when we use the notation indicated in this figure, by

$$\left.\begin{aligned} \alpha(0, \lambda) \approx H_1 \frac{2}{\lambda^2}(\delta'_1 - \delta_1) + H_1 \frac{2}{\lambda^3}(\sin \lambda\delta'_1 - \sin \lambda\delta_1) \\ - H_2 \frac{2}{\lambda^2}(\delta'_2 - \delta_2) - H_2 \frac{2}{\lambda^3}(\sin \lambda\delta'_2 - \sin \lambda\delta_2) \\ - H_3 \frac{2}{\lambda^2}(\delta'_3 - \delta_3) - H_2 \frac{2}{\lambda^3}(\sin \lambda\delta'_3 - \sin \lambda\delta_3) \end{aligned}\right\} \qquad 1, (\text{IV-4.3, 1})$$

for the first part of the tail.

In this expression, particularly the second term in the first line is of numerical importance. The terms in $1/\lambda^2$ practically cancel to zero. This means that the tail decays practically as λ^{-3}.

IV-4. 3, 2 If one wants to go further into the tail of $\alpha(0, \lambda)$, we must consider also the variation of the third, fourth, etc., derivatives. From the point of view of network synthesis, the actual shape of the tail of the pulse is quite unimportant. The only important analytical element in which we are interested is the exponent of decay of the envelope decay line. That is, in the computation of the pulse tolerances which were mentioned in section 2, we must know the exponent "r" of decay (decay as λ^{-r}).

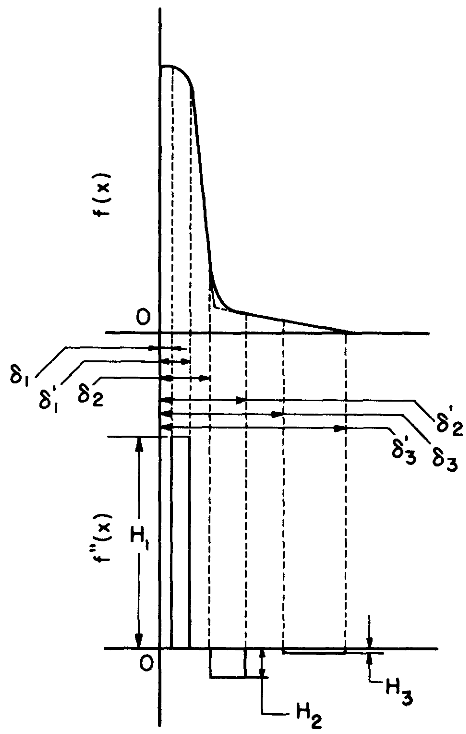

Fig. 1, (IV-4. 3, 1)
Quadratic frame approximation of a pulse.

We have seen, from the corner theorems and from the examples in section 2, that the exponent of envelope decay is related to the highest discontinuous derivative of the pulse function. In a future discussion we shall assume that for a given pulse we know the class, say, C^I, C^{II}, etc., of the function which represents the pulse. For this class the exponent r can be immediately found.

We close section 4 by remarking that from the heuristic analysis of pulses in this section, and from the results of the corner theorems, we can construct directly, with a good approximation, the graph of the function $\alpha(0, \lambda)$ in its first cyclic group. The zeros, displacement, maximum tangent, and exponent of decay can also be extracted directly from a given pulse. This situation ends the analysis of arbitrary pulses from the point of view of the present investigations.

In this section we have considered symmetrical pulses. The analysis can easily be appropriately extended to unsymmetrical pulses.

Section 5

The Analytical Character of $U(0, \lambda)$

The transcendental analytical character of the functions $U(0, \lambda)$ and $F(s)$, associated with pulses of finite duration. The rational approximation $U_N(0, \lambda)$ for such pulses. Two basic questions.

IV-5.0 The first part of this section contains the discussion of the nonrational character of the functions $U(0, \lambda)$ and $F(s)$, which are associated with a pulse of finite duration. The transcendental analytical character of $U(0,\lambda)$ or $F(s)$ can be brought into evidence by different simple procedures.

For example, one well-known method is to show directly the transcendental character of the Laplace transform $F(s)$ of a given pulse $f(t)$ of finite duration. As a convenient means of illustration we will follow this procedure.

The transcendental character of $F(s)$ is due to two main reasons:

(a) The delay t_o introduces a factor e^{-st_o} in the transform.

(b) The pulse itself can be expressed as the difference of two displaced time functions, which identically cancel out after the pulse length δ.

For convenience several examples are given here.

$f(t)$	$F(s)$
Square pulse	$H \times \frac{2}{s} e^{-st_o} \sinh \frac{s\delta}{2}$
Triangular pulse	$H \times \frac{a}{s^2} e^{-st_o} \left(1 - 4 \sinh \frac{s\delta}{2}\right)$
Half cosine	$H \times \dfrac{2 \frac{\pi}{\delta} e^{-st_o}}{s^2 + \left(\frac{\pi}{\delta}\right)^2} \sinh \frac{s\delta}{2}$

where H is the pulse height.

The corresponding density distribution function $U(0, \lambda)$ can be evaluated by setting $s = +i\omega$, $\omega = \lambda$ and then taking the real part. The function $U(0, \lambda)$ is obviously transcendental. A table of Laplace transforms will provide numerous examples.

IV-5.0,1 The above procedure is defective from the point of view of our approach to network synthesis. We have in Parts I, II, and IV of these notes given general theorems of the existence of transfer functions. These theorems establish the necessary and sufficient condition for a function to be "transfer." From the sufficient part of the theorem we know that given, except for certain requirements, an arbitrary density distribution function $U(0, \lambda)$, we can find the corresponding transfer function. Hence, from this point of view it is preferable to show directly the transcendental character of the function $U(0, \lambda)$. That is exactly what we intend to do.

The function $U(0, \lambda)$ is a bounded real function of the real variable λ. The proof of the transcendental character of the density distribution function $U(0, \lambda)$ must be based on these premises only. The following proposition settles the situation.

"Let $\phi(0, \lambda)$ be a real, single-valued, bounded function of the real variable λ. The number of maxima and minima must necessarily be finite if $\phi(0, \lambda)$ is rational." Of course, this is not a sufficient condition. This proposition shows the nonrational character of $U(0, \lambda)$ associated with a delayed pulse of finite duration because the function $U(0, \lambda)$ has an infinite number of maxima and minima for such pulses. Hence the nonrational character of such a $U(0, \lambda)$ is proved.

IV-5.1 In the construction of finite, discrete networks the corresponding density distribution function must be rational because of the basic existence theorem given in section 1, Part IV. Therefore, we must approach the function $U(0, \lambda)$ by some function $U^*(0, \lambda)$ which is rational, in such a way that the corresponding time function $f^*(t)$ approaches $f(t)$ inside the tolerances, as is indicated in section 2.

In accordance with the theorem in the last subsection, a first necessary step in obtaining our goal is to cut the function $U(0, \lambda)$ in such a way as to leave a function $U^*(0, \lambda)$ which has a finite number of maxima and minima. The form of the graph immediately suggests that we must keep the lobes which are contained in the first few cycles of the envelope function of $U(0, \lambda)$ and disregard the remaining part of the function. The syncopated (mutilated) $U(0, \lambda)$ is indicated in Fig. 1, (IV-5.1).

It is obvious that such a syncopated $U(0, \lambda)$, say $U^*(0, \lambda)$, is not, in general, an algebraic function, in spite of the fact that $U^*(0, \lambda)$ contains a finite number of points of maxima and minima.

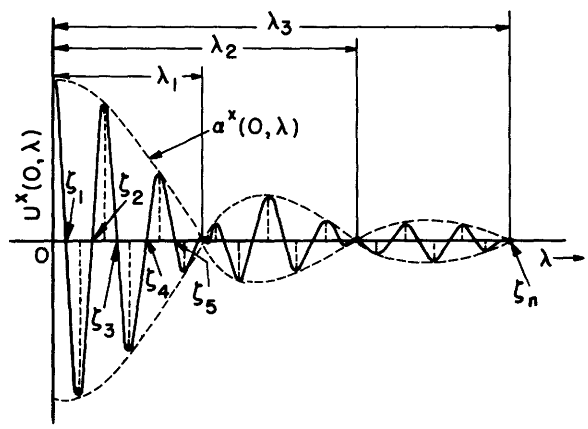

Fig. 1, (IV-5.1)
The syncopated $U(0, \lambda)$ function.

IV-5.1,1 In connection with the suggestion of using $U^*(0, \lambda)$ in the synthesis process, we are automatically confronted with two basic questions:

A. Let $f^*(t)$ be the inverse time function which corresponds to $U^*(0, \lambda)$ and which can be expressed by one of our basic integrals

$$f^*(t) = \frac{2}{\pi} \int_0^\infty \cos \lambda t \left[U^*(0, \lambda)\, d\lambda\right] \qquad 1, (\text{IV-5.1,1})$$

Now, it remains to prove that: Given a vicinal tolerance ϵ, it is always possible to find a zero of the function $\alpha(0,\lambda)$, say, at $\lambda = \lambda_N$, such that if one cuts the function $U(0,\lambda)$ after this point, the resulting syncopated function $U_N^*(0, \lambda)$, where N represents the point of syncopation, is such that the following condition is satisfied:

$$\epsilon = f(t) - f_N^*(t) \leq \frac{2}{\pi} \int_0^\infty \cos \lambda t \left[U(0, \lambda) - U^*(0, \lambda)\right] d\lambda \qquad 2, (\text{IV-5.1,1})$$

for every value of t in the interval $0 < t < \infty$.

B. Let $U_N^*(0, \lambda)$ be the syncopated function as above.

Now, it also remains to show, that for every value of N it is possible to find an algebraic rational function, say $U_{N,R}^*(0, \lambda)$, having the same zero-point distribution and the same distribution of extremal points as $U^*(0, \lambda)$ in the interval $0 < \lambda < \lambda_N$, and approximately the same maxima and minima values as $U_N^*(0, \lambda)$, so that the substitution of $U_N^*(0, \lambda)$ by $U_{N,R}^*(0, \lambda)$ in 2, (IV-5.1,1) still satisfies this condition.

The existence of such a rational function as $U_{N,R}^*(0, \lambda)$ is required by the fundamental theorem of Part IV, a theorem which states the necessary and sufficient conditions required to obtain a realizable rational transfer function $F^*(s)$.

IV-5.1,2 The process of syncopation of $U(0, \lambda)$ into $U^*(0, \lambda)$ makes sense if, and only if, the two basic questions A and B above are answered in the affirmative. We will show that this is so.

In fact, the corresponding theorems connected with these questions are of fundamental importance in the theory of finite networks. They are so basically important because these properties constitute two of the canonic theorems of physical network existence of the so-called window function. A report on such functions will be published in the future.

Section 6 of the present report is dedicated to the discussion of question A. Section 7 is concerned with question B.

A Basic Theorem Leading to the Evaluation of Errors and Vicinal Tolerances

Characteristic waveform factors. Relative errors. Tolerances. Riemann's zeta function. Typical examples.

IV-6.0 Introduction. The main objective of section IV-6 is to prove the affirmative answer to question A, which was raised in the last section.

A basic product of the corresponding discussion is the establishment of the analytical relations which connect the analytical elements of the given pulse and the prescribed vicinal tolerance ϵ with the index N of syncopation in such a way that the conditional relation

$$\epsilon = f(t) - f_N^*(t) \leqslant \frac{2}{\pi} \int_0^\infty \cos \lambda t \left[U(0, \lambda) - U_N^*(0, \lambda)\right] d\lambda \qquad 1, (\text{IV-6.0})$$

is satisfied for every value of N greater than a certain fixed one, say, for $N \geqslant N_o$.

The final objective of this section has a synthetic character. That is, first, we must find such a number as N_o. Second, we must construct the syncopated function $U_N^*(0, \lambda)$, for $N \geqslant N_o$, which satisfies 1, (IV-6.0).

IV-6.0, 1 An exact evaluation of the error committed by means of the integral representation 1, (IV-6.0) is very hard to perform and perhaps the result is irrelevant in connection with arbitrary pulses. It is irrelevant because it would come out, generally speaking, as a complicated mathematical expression, which would not show an explicit practical structure connecting the error committed with the analytical elements of the prescribed pulse. Several examples have shown this practical inconvenience.

If, however, we are satisfied with the evaluation of certain bounds of the error committed, bounds which are not necessarily the least upper bound of ϵ, but fairly close to it, then we can obtain, as it is shown in this discussion, a very simple and practical expression of the error. This simple expression allows us to produce a definite answer to the question of the establishment of prescribed tolerances. The present discussion will be directed towards obtaining the bound expressions as we have described them above.

Our main interest resides with symmetric narrow pulses because of their relation to window functions. The present discussion is concerned with such symmetric pulses. There is no particular difficulty in extending the results of section IV-6 to nonsymmetric pulses.

IV-6.1 The average decay line. The difference $U(0,\lambda) - U^*(0,\lambda)$ represents a function which is equal to the tail of the density distribution function $U(0,\lambda)$. A typical tail is illustrated in Fig. 1, (IV-6. 1).

In order to produce a simple expression for the error committed, we will assume that the tail of the density distribution function decays along a certain average line of envelope decay. This average line is placed in the middle of the upper and lower decay lines. This average line is therefore a displaced line. The equation of this average decay line is given by a general expression

$$\frac{K}{(\lambda + \lambda_d)^r} \qquad 1, (IV\text{-}6.1)$$

where K = constant, r = decay exponent, λ_d = displacement as measured from the upper decay line. The modified decay line is shown in Fig. 2, (IV-6. 1). The reader will find the justification of this substitution in the following evaluation of the errors.

IV-6. 1, 1 We will compute first the contribution to the integral 1, (IV-6. 0) from one arbitrarily placed lobe of the envelope. The lobe is located between

$$\lambda_\nu \leqslant \lambda \leqslant \lambda_{\nu+1}; \quad \nu > N \qquad 1, (IV\text{-}6.1, 1)$$

In doing this computation we will introduce a set of constants, k_1, k_2, k_3, s, and r, which are associated with an arbitrary symmetric pulse. These parameters are associated with and characterize a given pulse.

Fig. 1, (IV-6. 1, 1) shows one lobe of the envelope function placed between λ_ν and $\lambda_{\nu+1}$, $\nu > N$. The needed notation is indicated in this figure. To fix the position of this lobe we will introduce the parameter k_1, as follows

$$k_1 \delta\lambda_\nu = \pi\nu; \quad \nu = \text{integer} \qquad 2, (IV\text{-}6.1, 1)$$

The value of k_1 can evidently be found from the position of the first envelope lobe by setting $\nu = 1$. Hence

$$k_1 = \frac{\pi}{\delta\lambda_1} \qquad 3, (IV\text{-}6.1, 1)$$

The value of k_1 can be expressed in terms of the pulse elements. For example, a trapezoidal approximation of a pulse, as in Fig. 2, (IV-6. 1, 1), leads to the following expression for λ_1

$$\lambda_1 = \frac{2\pi}{\delta + \delta_1} \qquad 4, (IV\text{-}6.1, 1)$$

See Eq. 4, (IV-2. 3, 1). This last expression yields the following value of k_1

$$k_1 \approx \frac{1}{2}\left(1 + \frac{\delta_1}{\delta}\right) \qquad 5, (IV\text{-}6.1, 1)$$

More accurate expressions for k_1 can be obtained by using double trapezoidal pulse frames in the determination of λ_1. For spike-like pulses the expression 5, (IV-6. 1, 1) has enough accuracy. We now proceed to the evaluation of h_ν, the amplitude of an

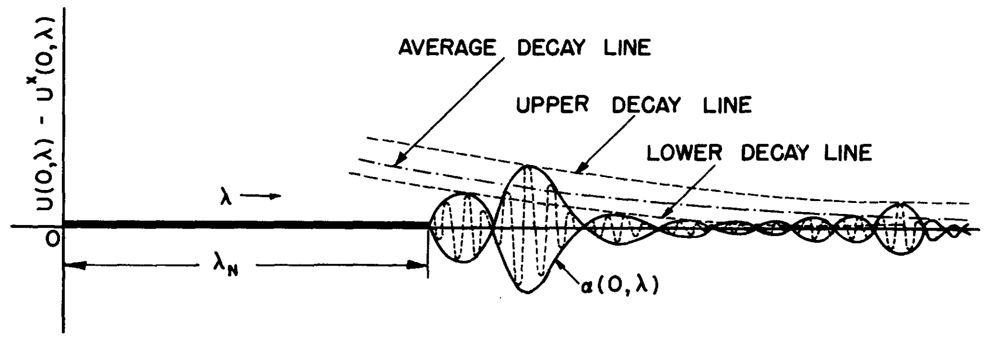

Fig. 1, (IV-6. 1)
The typical graph of the function U(0, λ) - U*(0, λ).

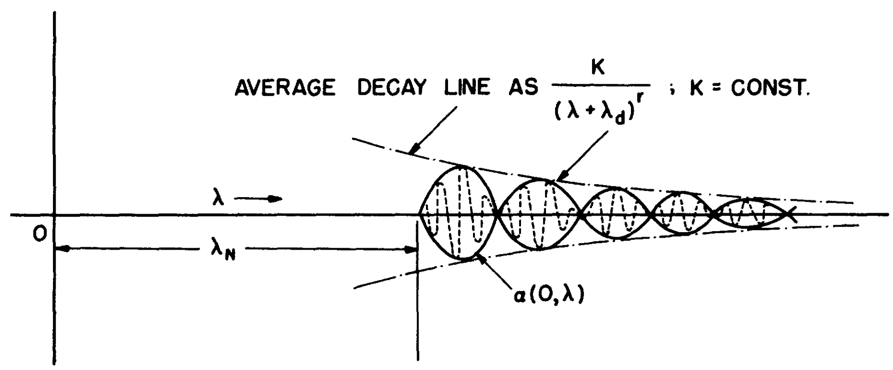

Fig. 2, (IV-6. 1)
Modified tail for purpose of computation.

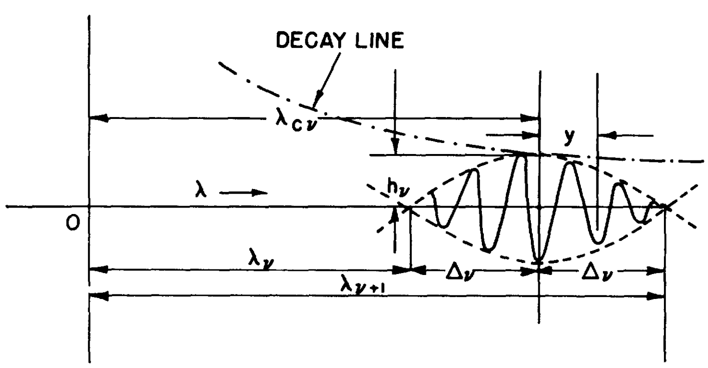

Fig. 1, (IV-6. 1, 1)
A lobe of the envelope function.

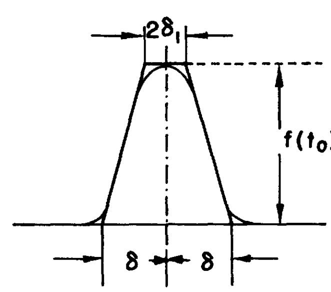

Fig. 2, (IV-6. 1, 1)
Trapezoidal frame.

envelope lobe. See Fig. 1, (IV-6. 1, 1).

By using a simple trapezoidal pulse frame, we immediately obtain, from Eq. 3, (IV-2. 3, 1), the corresponding expression

$$h_\nu = \frac{4f(t_o)}{\lambda^2_{c,\nu}(\delta - \delta_1)} \cdot \sin \frac{\lambda_{c,\nu}(\delta - \delta_1)}{2}$$

$$= \frac{4f(t_o)(\delta + \delta_1)}{\lambda^2_{c,\nu}(\delta^2 - \delta_1^2)} \cdot \sin \frac{\lambda_{c,\nu}(\delta - \delta_1)}{2} \ ; \quad \delta > \delta_1 \qquad 6, (IV\text{-}6.1, 1)$$

when $f(t_o)$ is the pulse height. $\lambda_{c,\nu}$ is defined in Fig. 1, (IV-6. 1, 1).

The expressions 6, (IV-6. 1, 1) suggest at once how to develop a more convenient empirical estimate of the height h_ν, which can be used in connection with pulses leading to an envelope decay of exponent r and a displacement λ_d.

Let us introduce a constant k_3 defined by

$$k_3 = \frac{\text{Area of pulse}}{f(t_o)\,\delta} \qquad 7, (IV\text{-}6.1, 1)$$

We now propose for h_ν a more general empirical expression

$$h_\nu = \frac{f(t_o)\, k_3}{(\lambda_{c,\nu} + \lambda_d)^r\, \delta}\; k_2 \qquad 8, (IV\text{-}6.1, 1)$$

in which k_2 remains to be determined. $\lambda_{c,\nu}$ is the value of λ at which the maximum of the lobe takes place; λ_d is the displacement and r the decay exponent.

For example, for a simple trapezoidal pulse frame

$$k_3 = \frac{f(t_o) \times (\delta_1 + \delta)}{f(t_o)\,\delta} = 1 + \frac{\delta_1}{\delta} \qquad 9, (IV\text{-}6.1, 1)$$

By direct comparison of Eqs. 6, (IV-6. 1, 1), 8, (IV-6. 1, 1), and 9, (IV-6. 1, 1) one gets for k_2: $\lambda_d = 0$, $r = 2$.

$$k_2 = \frac{4}{1 - \frac{\delta_1^2}{\delta^2}} \times \sin\left[\frac{\lambda_{c,\nu}(\delta - \delta_1)}{2}\right] = \frac{4}{1 - \frac{\delta_1^2}{\delta^2}} \ ; \quad \delta > \delta_1$$

since $\sin \lambda_{c,N} (\delta - \delta_1)/2 = 1$ at the center of the lobe; see Fig. 1, (IV-6. 1, 1).

If one lets $\delta_1 \rightarrow \delta$, as in a square pulse, then the limiting value of k_2 is given by

$$k_2 = \frac{2\delta^2}{\delta + \delta_1} \cdot \lambda_{c,\nu} = \delta\lambda_{c,\nu}$$

Hence

$$\left.\begin{aligned} k_2 &= \frac{4}{1 - \dfrac{\delta_1^2}{\delta^2}} \quad \text{for } \delta > \delta_1 \\ k_2 &= \delta\lambda_{c,\nu} \quad \text{for } \delta = \delta_1 \end{aligned}\right\} \qquad 10,(\text{IV-6.1,1})$$

Now, we proceed to compute $\lambda_{c,\nu}$.

By using 2, (IV-6. 1, 1) together with the notation of Fig. 1, (IV-6. 1, 1) one gets

$$\lambda_{c,\nu} = \lambda_\nu + \Delta_\nu = \frac{\pi\nu}{k_1\delta} + \frac{\pi}{2k_1\delta} = \frac{\pi}{k_1\delta}\left(\nu + \frac{1}{2}\right) \qquad 11,(\text{IV-6.1,1})$$

$\lambda_{c,\nu}$ represents the center of each envelope lobe, which corresponds to the average line of decay in Fig. 2, (IV-6. 1).

The displacement is measured from the graph of the function $U(0, \lambda)$ when this function is constructed from the given pulse following the heuristic procedure given in section IV-4.

For convenience of mathematical notation we shall introduce a constant s defined by

$$s = \frac{2\lambda_d k_1 \delta}{\pi} + 1 \qquad 12,(\text{IV-6.1,1})$$

hence

$$(\lambda_{c,\nu} + \lambda_d) = \frac{\pi}{k_1\delta}\left(\nu + \frac{s}{2}\right) \qquad 13,(\text{IV-6.1,1})$$

Finally one gets

$$h_\nu = \frac{2^r k_1^r f(t_o)\,\delta^{r-1} k_2 k_3}{\pi^r(2\nu + s)^r}$$

By means of this expression we can immediately write the expression for the ν-th envelope lobe of $\alpha(0, \lambda)$, indicated in Fig. 1, (IV-6. 1, 1). Hence

$$h_\nu \cos\frac{\pi}{2}\frac{y}{\Delta_\nu} = \frac{2^r k_1^r f(t_o)\,\delta^{r-1} k_2 k_3}{\pi^r(2\nu + s)^r} \cos k_1 \delta y \qquad 14,(\text{IV-6.1,1})$$

By using our present notation, the value of λt_o becomes

$$\lambda t_o = \frac{\pi}{k_1\delta}\left(\nu + \frac{1}{2}\right) t_o + y t_o \qquad \lambda_\nu < \lambda < \lambda_{\nu+1} \qquad 15,(\text{IV-6.1,1})$$

Thus, a general expression for the ν-th envelope lobe of the density distribution function becomes

$$U_\nu(0, \lambda) \approx \left\{ \frac{f(t_o)\, k_2 k_3 2^r k_1^r \delta^{r-1}}{\pi^r(2\nu + s)^r} \cos k_1\delta(\lambda - \lambda_{c,\nu}) \right\} \cos\left[\frac{\pi}{k_1\delta}\left(\nu + \frac{1}{2}\right) t_o + y t_o\right]$$

16, (IV-6. 1, 1)

IV-6.1,2 We propose here to compute the contribution to the integral 1,(IV-6.0) of the ν-th, $\nu > N$, lobe of the function $U_\nu(0, \lambda)$, which is shown in Fig. 1,(IV-6.1,1). Our principal aim is to establish an upper bound, not necessarily the least upper bound, of the contribution of the ν-th lobe to the integral 1,(IV-6.1,1).

For $\nu > N$, see Fig. 2,(IV-6.1), we have

$$\epsilon_\nu \approx \frac{2^{r+1}}{\pi^{r+1}} \frac{f(t_o)\, k_1^r\, k_2\, k_3\, \delta^{r-1}}{(2\nu + s)^r} \times I_\nu \qquad 1,(IV-6.1,2)$$

where

$$I_\nu = \int_{\lambda_\nu}^{\lambda_{\nu+1}} \cos k_1\, \delta(\lambda - \lambda_{c,\nu}) \cos \left[\frac{\pi}{k_1 \delta}\left(\nu + \frac{1}{2}\right) t_o + y t_o\right] \cos \lambda t\, d\lambda \qquad 2,(IV-6.1,2)$$

It can be seen at once, by using Fig. 1,(IV-6.1,1), that the first cosine factor above always remains positive for $\lambda_\nu < \lambda < \lambda_{\nu+1}$. Now, on account of 15,(IV-6.1,1) the integral above is simply equal to

$$I_\nu = \int_{\lambda_\nu}^{\lambda_{\nu+1}} \cos k_1 \delta(\lambda - \lambda_{c,\nu}) \cos \lambda t_o \cos \lambda t\, dt \qquad 3,(IV-6.1,2)$$

Hence, I_ν attains an upper bound at $t = t_o$, because the integral is necessarily positive. Using

$$2 \cos^2 \lambda t_o = (1 + \cos 2\lambda t_o)$$

we get

$$I_\nu < \frac{1}{2} \int_{\lambda_\nu}^{\lambda_{\nu+1}} \cos \left[k_1 \delta(\lambda - \lambda_{c,\nu})\right] d\lambda + \frac{1}{2} \int_{\lambda_\nu}^{\lambda_{\nu+1}} \cos \left[k_1 \delta(\lambda - \lambda_{c,\nu})\right] \cos 2\lambda t_o\, d\lambda$$

The last integral becomes negligibly small on account of the cancelling chopping interaction effect of $\cos 2\lambda t_o$ and the slow variation term, $\cos\left[k_1 \delta(\lambda - \lambda_{c,\nu})\right]$.

Consequently, by direct integration of the first integral, one gets for the bound

$$I_\nu < \text{Bound } I_\nu = \frac{1}{k_1 \delta} \qquad 4,(IV-6.1,2)$$

Hence, we finally get from 1,(IV-6.1,2)

$$\xi_\nu = \frac{\epsilon_\nu}{f(t_o)} = \frac{2^{r+1}}{\pi^{r+1}} \frac{k_1^{r-1}\, k_2\, k_3\, \delta^{r-2}}{(2\nu + s)^r}\,; \quad \nu > N \qquad 5,(IV-6.1,2)$$

IV-6.1,3 The expression 5,(IV-6.1,2) leads at once to the evaluation of the relative error committed when we substitute $U(0, \lambda)$ by $U^*(0, \lambda)$. The use of Eq. 1,(IV-6.0) renders the required expression of the relative bound of this integral.

The bound is given by

$$\xi = \sum_{\nu=N}^{\nu=\infty} \frac{\epsilon_\nu}{f(t_o)} = \frac{2}{\pi^{r+1}} k_1^{r-1} k_2 k_3 \delta^{r-2} \sum_{\nu=N}^{\nu=\infty} \frac{1}{\left(\nu + \frac{s}{2}\right)^r} \qquad 1, (IV\text{-}6.1, 3)$$

These results suggest immediately that this relative error can be expressed in terms of the generalized zeta function of Riemann.

IV-6.2 The zeta function of Riemann. Using the notation of these notes, the generalized zeta function of Riemann is defined by

$$\zeta(r, a) = \sum_{\nu=0}^{\infty} \frac{1}{(a+\nu)^r}; \quad a = \frac{s}{2} \qquad 1, (IV\text{-}6.2)$$

The ordinary zeta function of Riemann is attained for the particular value of a = 1. That is

$$\zeta(r, 1) = \zeta(r) = \sum_{n=1}^{\infty} \frac{1}{n^r}; \quad n = \nu + 1 \qquad 2, (IV\text{-}6.2)$$

The use of the letter n is immaterial in the last expression. The discussion of the analytical character and properties of the above function is outside the scope of these notes. The reader is referred to a regular text on analysis, in particular, to reference 5, where a formal presentation of the subject can be found. For the purpose of these notes, it is quite unnecessary to consult such a treatise.

IV-6.2, 1 The series 1, (IV-6.2) converges uniformly in any domain in which $r = \sigma + i\tau$; $\sigma > 1$. In our case r is real. Hence our formulas can be used for $r \geq 2$.

A numerical tabulation of the function

$$\zeta(r) = \sum_{n=1}^{\infty} \frac{1}{n^r}$$

can be found, for example, in reference 6. For the application of these formulas to some examples we will use a few values of the function above. Certain numerical values are here reproduced for convenience

r	$\zeta(r)$	
2	1.645...	
3	1.202...	1, (IV-6.2, 1)
4	1.0823...	
5	1.036...	

IV-6.2, 2 The functions $\zeta(r, 0)$ and $\zeta(r)$ show simple analytical connections for particular values of $a = s/2$. For example, $a = 1/2$; $(s=1)$,

$$\zeta\left(r, \frac{1}{2}\right) = 2^r \sum_{\nu=0}^{\nu=\infty} \frac{1}{(2\nu + 1)^r} = 2^r \left(1 + \frac{1}{3^r} + \frac{1}{5^r} + \ldots\right) = (2^r - 1)\, \zeta(r)$$

IV-6.3 The relative bound of the error. The relative bound of the error 1, (IV-6.1, 3) can be expressed in terms of the zeta function of Riemann simply by setting

$$\zeta\left(r, \frac{s}{2}\right) = \sum_{\nu=0}^{\nu=N-1} \frac{1}{\left(\nu + \frac{s}{2}\right)^r} + \sum_{\nu=N}^{\nu=\infty} \frac{1}{\left(\nu + \frac{s}{2}\right)^r}$$

Let us introduce the notation

$$\tau_r\left(N, \frac{s}{2}\right) = \frac{2}{\pi^{r+1}} \left\{ \zeta\left(r, \frac{s}{2}\right) - \sum_{\nu=0}^{\nu=N-1} \frac{1}{\left(\nu + \frac{s}{2}\right)^r} \right\} \qquad 1, (\text{IV-6.3})$$

Then we finally get

$$\xi = \frac{\epsilon}{f(t_o)} \approx k_1^{r-1}\, k_2\, k_3\, \delta^{r-2}\, \tau_r\left(N, \frac{s}{2}\right) \qquad 2, (\text{IV-6.3})$$

IV-6.3, 1 The expression 2, (IV-6.3) is the relative error committed when we use the syncopated function $U_N^*(0, \lambda)$ instead of $U(0, \lambda)$. This expression is clearly equal to the tolerance defined in section 2 of Part IV. It establishes a relationship between the error committed and the number of lobes, N, of the function $U(0, \lambda)$ which we must preserve. This tolerance is expressed in 2, (IV-6.3) in terms of simple elements of the pulse.

Now, we can produce an affirmative answer to question A of section IV-5. By observing the bracket parenthesis in 1, (IV-6.3), it can be seen at once that the function $\tau_r[N, (s/2)]$ tends to zero, and rather fast too, for $r \geqslant 2$, when N increases without limit. Hence, it is possible to find a value N such that for a given pulse the tolerance can be made smaller than a prescribed positive quantity.

IV-6.3, 2 We proceed here to compute the function $\tau_r[N, (s/2)]$ using N as a variable. The quantities s and r are parameters. For the purpose of these notes, we shall choose the following set of values, r = 2, 3 and s = 1, 5/3, 2, because these values are more frequently found in the examples given in section 3 of Part IV.

We will compute as an illustration the value of the parameter s, which corresponds to several typical pulses.

A. Consider a pulse which shows a zero displacement in the corresponding envelope decay line of the density distribution function $U(0, \lambda)$. For example, for the triangular pulse, one gets s = 1 from 12, (IV-6.1, 1), since $\lambda_d = 0$.

B. Half-cosine pulse. See subsection IV-2.3, 1, example D. By considering Eqs. 5, (IV-2.3, 1), 3, (IV-6.1, 1), and 6, (IV-6.1, 1), one gets

$$\lambda_1 \delta = \pi\left(1 + \frac{1}{2}\right)$$

$$r = 2$$

$$k_1 = \frac{2}{3}$$

$$\lambda_d \delta = \frac{\pi}{2}$$

from which we obtain s = 5/3.

C. Complete cosine pulse. See subsection IV-2.3, 1, example E. By using a similar procedure as in B, above, one finds s = 2, and r = 3.

We now compute a few important members of the family of functions $\tau_r[N, (s/2)]$, taking N as an independent variable. We assign to r and s values from the previous computation.

Figure 1, (IV-6.3, 2) shows several curves representing the function $\tau_r[N, (s/2)]$. The reader may notice, that (a) for a given constant value of r, all the members of the function almost coincide for all large s and N values; (b) the numerical values of the functions decrease as r increases for the same values of N and s.

IV-6.4 Error computations. The set of curves $\tau_r[N, (s/2)]$ and the formulas developed in section IV-6 allow us to make a rapid computation of the relative error which is committed when the function $U(0, \lambda)$ is syncopated after the N-th lobe of the envelope function. See Eqs. 1, (IV-6.3), 5, (IV-6.1, 1), 9, (IV-6.1, 1), 10, (IV-6.1, 1), and 12, (IV-6.1, 1). We shall apply these formulas to several different pulse shapes. Results are given in the next subsections.

IV-6.4, 1 The following pulse constant values are computed for a pulse having a ratio $\delta/t_o = 10$.

Pulse Constants

Pulse shape	k_1	k_2	k_3	r	s
Triangular	1/2	4	1	2	1
Half cosine	2/3	4.4	$4/\pi$	2	5/3
Complete cosine	1	4.1	2	3	2

The reader will not have any difficulty in extending this table for other pulse shapes.

IV-6.4, 2 The evaluation of the committed errors can be computed at once for the pulse shapes which were used as examples. The results are given by the graphs in Fig. 1, (IV-6.4, 2).

IV-6.4, 3 Figure 1, (IV-6.4, 2) shows that the error committed by using the function $U_N^*(0, \lambda)$ instead of $U(0, \lambda)$ decreases very rapidly for a syncopation after the first few envelope cycles. For large values of N, say, after N = 6, the error decreases rather slowly.

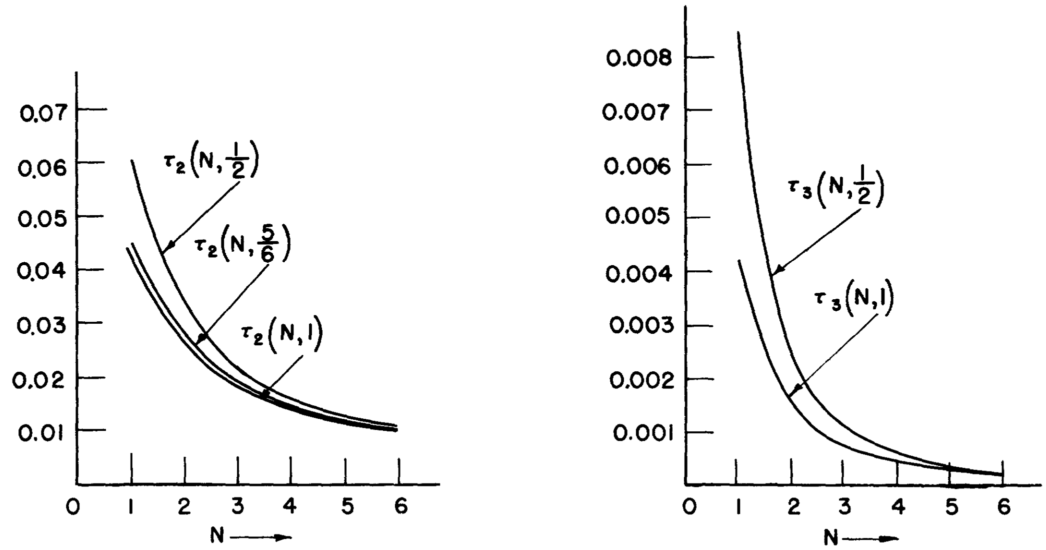

Fig. 1, (IV-6. 3, 2)
A few members of the family $\tau_r[N, (s/2)]$.

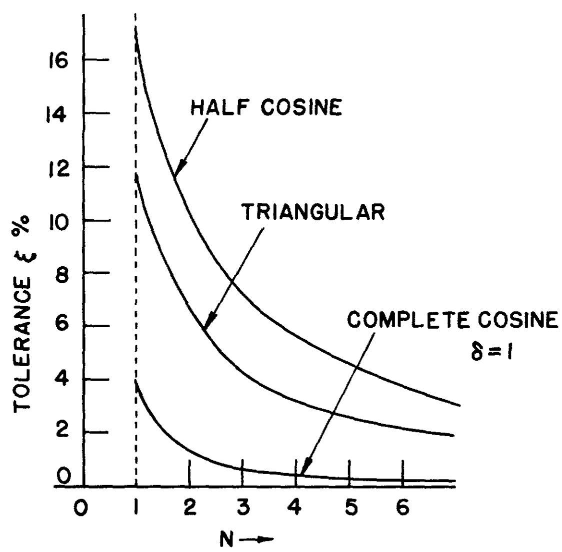

Fig. 1, (IV-6. 4, 2)
Typical tolerances for a few different pulse shapes.

The exponent r of envelope decay has a pronounced effect on the magnitude of the committed error. This is apparent from the set of curves shown in Fig. 1, (IV-6. 4, 2). The curve at the bottom has an $r = 3$. For practical pulses the parameter r does not exceed the value 3.

The reader must notice that the formula 1, (IV-6. 3) cannot be applied for $r = 1$ because of the divergence of the function $\zeta(r, s)$ for $r = 1$. However, an expression for the bound of the error committed when $r = 1$ has been developed. Such a formula makes use of a still more generalized type of Riemann's zeta function: a function of the Lerch type

$$\phi(x, a, r) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(a+n)^r}$$

See, for instance, reference 5, page 280.

Unfortunately, the corresponding error expression is quite impractical for the purpose of numerical computation. Therefore, we omit such formulas in the present notes.

IV-6. 4, 4 The analysis in this section shows that question A, which was raised in section IV-5, has an affirmative answer. That is,

(a) We can replace $U_N^*(0, \lambda)$ by $U(0, \lambda)$.

(b) The error committed in the time domain by this substitution tends to zero as $N \to \infty$.

(c) The rate of the error decrease is fast enough for practical purposes of network design.

Section IV-7

First Consideration of the Rational Approximation of the Density Distribution Function $U_N^*(0, \lambda)$

The rational transfer function $F^*_{N,R}(s)$. The correlation between the number of extremal points of $U(\gamma_0, \lambda)$ and the number of poles of a rational transfer function. Construction of the function $U^*_{N,R}(0, \lambda)$.

IV-7.0 Introduction. The direct objective of this section is to prove the affirmative answer to question B in subsection IV-5.1, 1. For convenience, we will repeat here the principal aspects of this question.

Let $U_N^*(0, \lambda)$ be a syncopated $U(0, \lambda)$ function such that the index N is large enough to satisfy the tolerances prescribed in the previous section.

Now we shall show that there is a rational distribution function, say $U^*_{N,R}(0, \lambda)$ which has: (a) the same zero-distribution, (b) a similar distribution of extremal points, and (c) approximately the same values of the maxima and minima as $U_N^*(0, \lambda)$. The function $U^*_{N,R}(0, \lambda)$ is such that the substitution of $U_N^*(0, \lambda)$ by $U^*_{N,R}(0, \lambda)$ in Eq. 2, (IV-5.1, 1) still satisfies this condition.

IV-7.0, 1 The rational transfer function $F^*_{N,R}(s)$. Once such a function as $U^*_{N,R}(0, \lambda)$ is constructed, we can produce the rational transfer function, say $F^*_{N,R}(s)$, which characterizes a network, capable of transforming the unit impulse into a prescribed output pulse, as was indicated in the basic network synthesis problem of section IV-2.

The function $F^*_{N,R}(s)$ is constructed from $U^*_{N,R}(0, \lambda)$ by using the fundamental existence theorems on the rational transfer function already discussed in section IV-1. This fundamental theorem states that if $U^*_{N,R}(0, \lambda)$ is an arbitrary rational bounded function satisfying a set of simple conditions, the associated transfer function, which is given by

$$F^*_{N,R}(s) = \frac{2}{\pi}\int_0^{\infty} \frac{U^*_{N,R}(0, \lambda)}{s^2 + \lambda^2}\, d\lambda \qquad 1, (IV\text{-}7.0, 1)$$

is rational and physically realizable as a four-terminal network. (The realizability of $F^*_{N,R}(s)$ is a consequence of the general basic theorems discussed in reference 1.)

IV-7.0, 2 The principal aim of section IV-7. The principal aim of this section is therefore to show how we can constructed the function $U^*_{N,R}(0, \lambda)$ and $F^*_{N,R}(s)$. During the discussion tending to obtain these functions, we will obtain some important side products. These side products will serve as a basis for two important questions.

1) The unicity of the function $F^*_{N,R}(s)$

2) The question of the minimum number of elements needed for the realization of a network having $F^*_{N,R}(s)$ as a transfer function.

In section IV-8 of this report we shall deal specifically with the clarification of these questions.

IV-7.1 The correlation between the number of extremal points of $U(\gamma_o, \lambda)$ and the number of poles of a rational transfer function. In section IV-5 we have shown the transcendental analytical character of the density distribution function, which is associated with a finite pulse of finite duration. If one wants to construct finite passive linear networks for the transmission of such pulses, then it is necessary to produce rational density distribution functions. A first step was taken toward the production of rational density distribution functions in sections IV-5 and IV-6. This first step was the syncopation of $U(0, \lambda)$ into $U_N^*(0, \lambda)$ in order to obtain a finite number of points of maxima and minima. This is a necessary condition for $U_N^*(0, \lambda)$ to be rational.

We shall open the discussion of this section by showing the connection between the points of maxima and minima of $U_{N,R}^*(0, \lambda)$ and the number of poles of the function $F_{N,R}^*(0, \lambda)$.

IV-7.1,1 The rational transfer function R(s). Let us assume, for simplicity of notation, that R(s) is a rational transfer function, like $F_{N,R}^*(s)$, which has $m < \infty$ poles at the left of the imaginary axis. Multiple poles of R(s) are counted in accordance with their multiplicity. The assumption that R(s) has poles only to the left of the imaginary axis is made because of the bounded character of the density distribution function $U(0,\lambda)$, which is associated with a spike-like pulse. However, this assumption does not change the situation of the basic properties which are discussed in this section.

The position of the poles of R(s) implies (see ref. 1): (a) $c_o < 0$, c_o = abscissa of convergence and (b) the boundedness of $U(0, \lambda)$.

IV-7.1,2 Isometric plots of the function $U(0, \lambda)$. Let us now consider the s-plane associated with a function such as R(s). In this plane let us place a contour Γ_o in the right half-plane far from the imaginary axis. We are going to displace this contour Γ_o from the right half- to the left half-plane and observe, in each particular position, the graph of the function $U(\gamma_o, \lambda)$. An illustration is provided by Fig. 1, (IV-7.1,2), where a sort of isometric plot of the s-plane is shown for two simple poles of R(s). The magnitude and direction of the residues at these poles are indicated by the arrows emanating from each pole. The figure shows the graph of the density distribution function $U(\gamma_o, \lambda)$ for six different positions of the contour Γ_o. We observe three extremal finite points. As we get closer and closer to the poles, the positions of two of the extremal points move respectively toward each of the poles and the extremal values increase without limit. When the contour Γ_o is aligned with the line of the poles, then the value of $U(\gamma_o,\lambda)$ becomes infinite at the poles. The third extremal point of the graph of $U(\gamma_o, \lambda)$ lies continuously along the real axis of the s-plane. Fig. 2, (IV-7.1,2) shows the graph of $U(\gamma_o, \lambda)$, $\gamma_o = 0$, for five simple poles, having real residues with sign alternation. We observe here an alternation of the extremal values of $U(0, \lambda)$. The real pole produces an extremal value at $\lambda = 0$. The complex poles also produce extremal values at this point in such a way that these extremal values coincide at $\lambda = 0$.

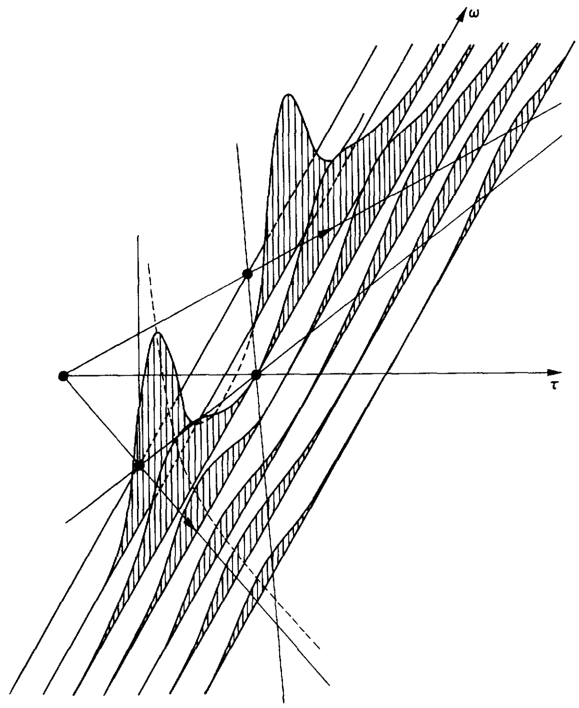

Fig. 1, (IV-7.1, 2)
Isometric plot for two simple poles of R(s).

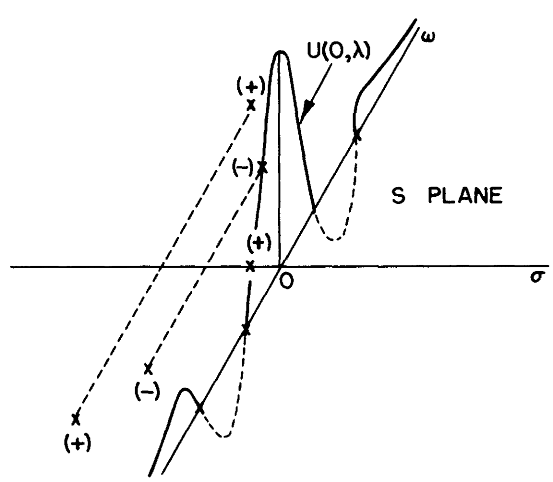

Fig. 2, (IV-7.1, 2)
The function U(0, λ).

IV-7.1,3 Upper and lower bounds to the number of extremal points (heuristic proofs).

The analytical correlation between the extremals of $U(\gamma_o, \lambda)$ as γ_o sweeps the s-plane and the number of poles of the rational function $R(s)$ can easily be established.

Consider, for example, that the function $R(s)$ has m simple poles. Let p = number of real poles and 2q = number of conjugate poles. Hence

$$p + 2q = m$$

The function $R(s)$ can be written as

$$R(s) = \sum_{1}^{m} \frac{A_k/\lambda_k}{s + \lambda_k} \qquad 1, (\text{IV-7.1,3})$$

See section IV-1.

A straightforward computation renders the following result

$$\left.\begin{aligned}
\operatorname{Re}[R(s)] = U(\gamma_o,\lambda) &= \sum_{j=1}^{j=p} \frac{A_j}{\lambda_j} \frac{(\gamma_o + \lambda_j)}{\lambda^2 + (\gamma_o + \lambda_j)^2} + \sum_{k=1}^{k=q} 2 \, \frac{\lambda^2 K_k + L_k}{\lambda^4 + \lambda^2 M_k + N_k} \\
&\text{where} \\
K_k &= \frac{1}{2}\left(\frac{A_k}{\lambda_k} + \frac{\bar{A}_k}{\bar{\lambda}_k}\right)\gamma_o + (A_k + \bar{A}_k) \\
L_k &= \gamma_o^3 \frac{1}{2}\left(\frac{A_k}{\lambda_k} + \frac{\bar{A}_k}{\bar{\lambda}_k}\right) + \gamma_o^2 \frac{1}{2}\left[(A_k + \bar{A}_k) + 2\left(\frac{A_k}{\lambda_k}\bar{\lambda}_k + \frac{\bar{A}_k}{\bar{\lambda}_k}\lambda_k\right)\right] \\
&\quad + \gamma_o \frac{1}{2}\left[2(A_k\bar{\lambda}_k + \bar{A}_k\lambda_k) + \left(\frac{A_k}{\lambda_k}\bar{\lambda}_k^2 + \frac{\bar{A}_k}{\bar{\lambda}_k}\lambda_k^2\right)\right] + \frac{1}{2}(A_k\bar{\lambda}^2 + \bar{A}_k\lambda_k^2) \\
M_k &= (\gamma_o + \lambda_k)^2 + (\gamma_o + \bar{\lambda}_k)^2 \\
N_k &= (\gamma_o + \lambda_k)^2 (\gamma_o + \bar{\lambda}_k)^2
\end{aligned}\right\} \qquad 2, (\text{IV-7.1,3})$$

The degree of the common denominator of the function $U(\gamma_o, \lambda)$ is equal to 2m, as one can easily check. Hence, the total number of poles of $U(\gamma_o, \lambda)$ is equal to 2m. The function $R(s)$ contains only 2m poles, since it is formed only by the poles of $U(\gamma_o, \lambda)$ having negative or zero real parts.

The derivative of Eq. 2, (IV-7. 1, 3) is given by

$$\left.\begin{aligned} \frac{dU(\gamma_o, \lambda)}{d\lambda} = {} & -2\lambda \sum_{j=1}^{j=p} \frac{A_j}{\lambda_j} \frac{(\gamma_o + \lambda_j)^2}{\left[\lambda^2 + (\gamma_o + \lambda_j)^2\right]^2} \\ & + 4\lambda \sum_{k=1}^{k=j} \frac{(\lambda^4 + M_k\lambda^2 + N_k)K_k - (\lambda^2 K_k + L_k)(2\lambda^2 + M_k)}{(\lambda^4 + M_k\lambda^2 + N_k)^2} \end{aligned}\right\} \quad 3, (IV-7.1, 3)$$

A simple computation renders the value 4m-1 for the degree of the numerator of Eq. 3, (IV-7. 1, 3). Consequently, the maximum possible number of extremal points of the derivative would be equal to 4m-1. The number of zeros of the numerator of Eq. 3, (IV-7. 1, 3) depends on the position of γ_o, since the coefficients K_k, L_k, M_k, and N_k depend upon γ_o (see Eq. 2, (IV-7. 1, 3). We may observe from Eq. 3, (IV-7. 1, 3) that the point $\lambda = 0$ on Γ_o is always an extremal point of $U(\gamma_o, \lambda)$ for every position of the contour Γ_o. Consequently, the number of lateral extremal points of $U(\gamma_o, \lambda)$ on Γ_o cannot exceed 2m-1. This number constitutes the upper bound of the number of extremal points.

The actual number of extremal points of $U(\gamma_o, \lambda)$ depends on the particular position of Γ_o because the coefficients K_k, L_k, M_k, and N_k depend on γ_o. Besides, some of the roots of the final numerator of Eq. 3, (IV-7. 1, 3) may become complex as Γ_o sweeps the plane. Some other roots may disappear because the coefficient of the highest power of λ in the final numerator of Eq. 3, (IV-7. 1, 3) may become zero for certain positions of the contour Γ_o.

A lower bound to the number of extremal points of the function $U(\gamma_o, \lambda)$ can be obtained by the following simple method of reasoning. The function R(s) in Eq. 1, (IV-7. 1, 3) behaves as

$$R(s) \approx \frac{A_k/\lambda_k}{s + \lambda_k}$$

in the immediate vicinity of the pole at λ_k. Hence, when the contour Γ_o moves in the immediate vicinity of such a pole, we can observe in this vicinity, at most, two extremal points, (one maximum and one minimum), or at least one extremal point, depending upon the direction of the complex quantity A_k/λ_k. By considering all the poles of R(s) we can sum up these results as follows: Let Γ_o sweep the s-plane as before. Then the number of extremal points of $U(\gamma_o, \lambda)$ cannot be less than p + q when A_k/λ_k are all real quantities, and not less than p + 2q when A_k/λ_k are all nonreal quantities.

These heuristic results on the number of points of maxima and minima of $U(\gamma_o, \lambda)$ on Γ_o are enough to support our future investigation. However, we must remember that this discussion on extremal points is not quite complete. A general study of the critical points of density distribution functions are outside the scope of this report.

IV-7.2 Construction of the function $U^*_{N,R}(0, \lambda)$. The discussion in the last subsection illustrates the correlation between the number of extremal points of $U(\gamma_o, \lambda)$ and the number of poles of a rational transfer function. We are now going to apply the above discussion in the solution of the following problem.

Let us assume that $U^*_N(0, \lambda)$ is the syncopated function from $U(0,\lambda)$ such that the index N is large enough to satisfy the tolerances prescribed in the sense of section IV-6.

Now, we want to construct the function $U^*_{N,R}(0,\lambda)$, which is associated to the rational transfer function $F^*_{N,R}(s)$, such that $U^*_{N,R}(0, \lambda)$ approximates $U^*_N(0, \lambda)$. See subsection IV-7.0.

IV-7.2, 1 Conditions for $U^*_{N,R}(0, \lambda)$. The method of extraction of $U^*_{N,R}(0, \lambda)$ from $U^*_N(0, \lambda)$ consists of a procedure of interpolation. A graphical illustration of the method is convenient. Fig. 1, (IV-7.2, 1) shows the graph of the functions $U(0, \lambda)$ and $U^*_N(0, \lambda)$.

The function $U^*_N(0, \lambda)$ is constructed as follows:

(a) $U^*_N(0, \lambda) \equiv U(0, \lambda)$; for $0 \leq \lambda \leq \lambda_N$ (N = 3 in Fig. 1, (IV-7.2, 1))

(b) $U^*_N(0, \lambda)$ has one extremal point for $\lambda_N < \lambda < \infty$

(c) $U^*_N(0, \lambda)$ shows monotonic behavior after this last extremal point, and finally

(d) $U^*_N(0, \lambda) \underset{\lambda \to \infty}{\to} \dfrac{\text{constant}}{\lambda^r}$, where r is the exponent of envelope decay.

The function $U^*_{N,R}(0, \lambda)$ is extracted, by construction, by the rational interpolation of the function $U^*_N(0, \lambda)$ at the zero, extremal, and infinite value points. This is graphically illustrated in Fig. 2, (IV-7.2, 1).

Let us use the notation

$$\overset{o}{\lambda}_k, \quad k = 1, 2, \ldots, n$$

to indicate the zero points of $U^*_N(0, \lambda)$ and

$$\overset{e}{\lambda}_j, \quad j = 0, 1, \ldots, m$$

to indicate the extremal points of $U^*_N(0, \lambda)$.

Then, by construction

$$\left.\begin{array}{lll}
\text{(a)} & U^*_{N,R}(0, \overset{o}{\lambda}_k) = U^*_N(0, \overset{o}{\lambda}_k) = 0; & k = 1, 2, \ldots, n \\
\text{(b)} & U^*_{N,R}(0, \overset{e}{\lambda}_j) = U^*_N(0, \overset{e}{\lambda}_j); & j = 0, 1, \ldots, m \\
\text{(c)} & U^*_{N,R}(0, \lambda) \underset{\lambda \to \infty}{\to} \dfrac{\text{constant}}{\lambda^r} &
\end{array}\right\} \qquad 1, (IV\text{-}7.2, 1)$$

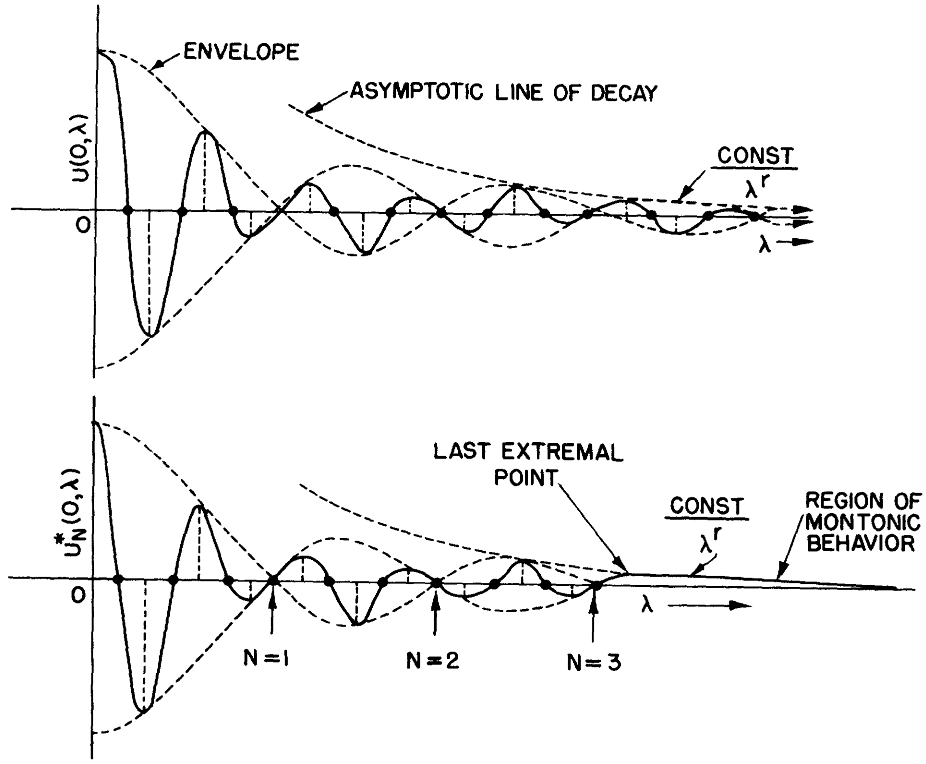

Fig. 1, (IV-7.2, 1)
The functions $U(0, \lambda)$ and $U_N^*(0, \lambda)$.

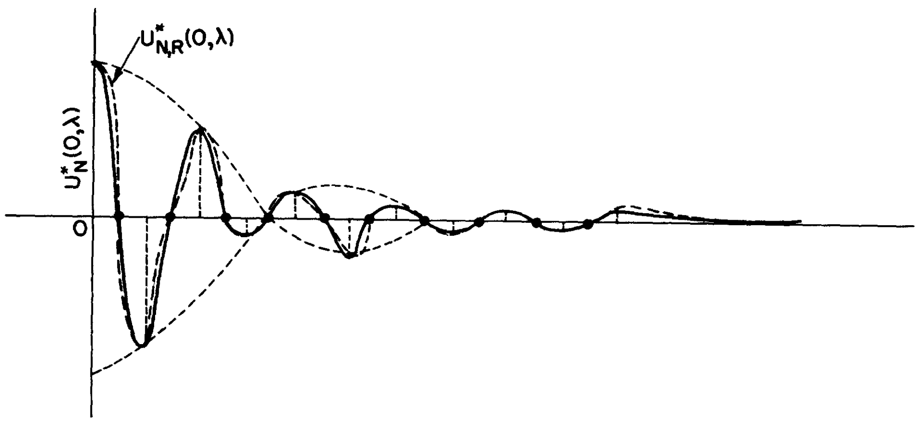

Fig. 2, (IV-7.2, 1)
$U_N^*(0, \lambda) = U_{N, R}^*(0, \lambda)$ at the zeros and extremal points of $U_N^*(0, \lambda)$.

IV-7.2,2 The interpolation procedure. We shall now begin with the description of the procedure of interpolation. The function $U(0, \lambda)$ is an even function of λ. Hence, it can be written in terms of λ^2. The function $U^*_{N,R}(0, \lambda)$ must be a function of λ^2 also. In accordance with the general existence theorems of section IV-1 we shall write

$$U^*_{N,R}(0, \lambda) = \frac{P_n(\lambda^2)}{Q_m(\lambda^2)} \qquad 1, (IV-7.2, 2)$$

where P_n and Q_m are both real polynomials in λ^2 of degree n and m, respectively.

The polynomial $P_n(\lambda^2)$ is constructed uniquely at once as follows:

$$P_n(\lambda^2) = (-1)^n (\lambda^2 - \mathring{\lambda}_1^2)(\lambda^2 - \mathring{\lambda}_2^2) \ldots (\lambda^2 - \mathring{\lambda}_n^2) \qquad 2, (IV-7.2, 2)$$

Two properties follow:

A. $P_n(\lambda^2)$ and $U^*_N(0, \lambda)$ have, by construction, the same set of zero points.

B. $P_n(\lambda^2)$ and $U^*_N(0, \lambda)$ have the same sign for every value of λ.

As a consequence, the expression

$$\frac{P_n(\lambda^2)}{U^*_N(0, \lambda)} \qquad 3, (IV-7.2, 2)$$

remains positive and different from zero for all finite values of λ.

The value of the expression 3,(IV-7.2, 2) can be obtained by direct computation. The graph of this function shows, for the class of $U(0, \lambda)$ in which we are interested, a slow varying curve with a definite monotonic increase for large values of λ. Figure 1, (IV-7.2, 2) shows a typical behavior of the graph of the function 3, (IV-7.2, 2).

The asymptotic behavior of the function 3, (IV-7.2, 2) can be found at once. By considering Eqs. 1, (IV-7.2, 1) and 2, (IV-7.2, 1) one gets

$$\left(\frac{P_n(\lambda^2)}{U^*_N(0, \lambda)}\right)_{\lambda \to \infty} = \frac{(-1)^n \lambda^{2n+r}}{\text{constant}} \qquad 4, (IV-7.2, 2)$$

where r is the exponent of envelope decay (r = 2 for most cases).

IV-7.2, 3 Determination of a linear equation system for the unknown quantities q_{m-1}, q_{m-2}, ..., q_1. The evaluation of the function $U^*_{N,R}(0, \lambda)$ (see Eq. 1, (IV-7.2, 2)) is now reduced to the approximation of the function 3, (IV-7.2, 2), Fig. 1, (IV-7.2, 2), by a polynomial of degree 2n + r. This polynomial can be constructed by a process of interpolation at a convenient set of points, as suggested in Fig. 1, (IV-7.2, 2). The close relationship between the maxima and minima of the function $U^*_N(0, \lambda)$ and poles of the associated transfer function suggests that the set of points of interpolation can be selected as the set of extremal points of $U^*_N(0, \lambda)$. This selection is convenient, but it is not necessary as we shall see later on. The selection of the set of points

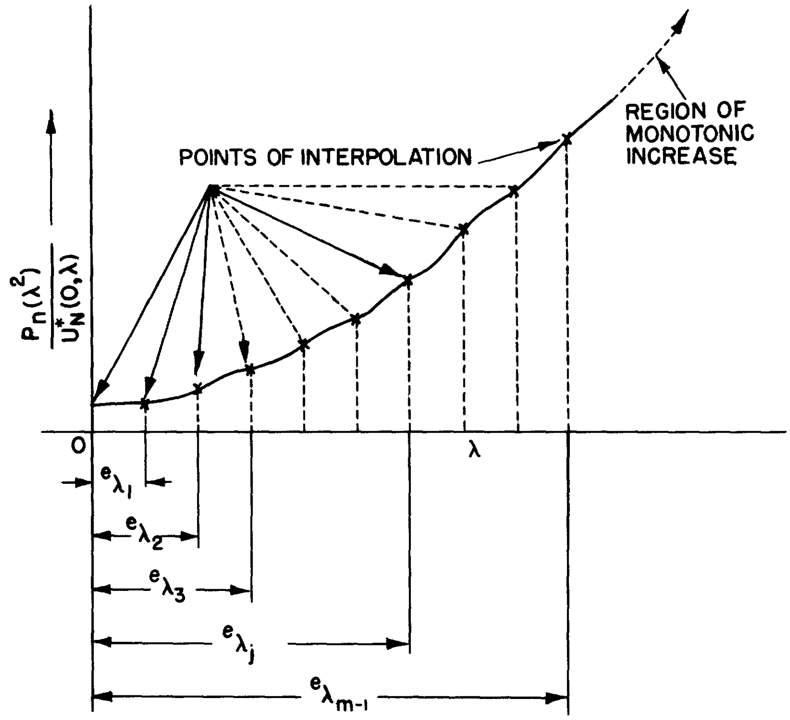

Fig. 1, (IV-7.2, 2)
Typical graph of the function $P_n(\lambda^2)/U_N^*(0, \lambda)$ for symmetric pulses. For this particular case the points of interpolation $\overset{e}{\lambda}_0, \overset{e}{\lambda}_1, \ldots, \overset{e}{\lambda}_{m-1}$ corresponding to the extremal points of $U_N(0, \lambda)$, are equally spaced. In the general case the set of points of interpolation must not necessarily be equally spaced.

$\overset{e}{\lambda}_j$, $j = 0, 1, \ldots, m$ leads to the establishment of some minimal properties of networks. These minimal properties will be studied in the next sections.

Let us use the set of points $\overset{e}{\lambda}_j$, $j = 0, 1, \ldots, m$ as the points of interpolation.

Because of the asymptotic behavior of the function $U_N^*(0, \lambda)$, we must set

$$2n + r = 2m \qquad 1, (IV\text{-}7.2, 3)$$

This implies that r must be an even number. Since we are dealing with a problem of approximation, this is not an objectionable limitation.

Let us write

$$Q_m(\lambda^2) = q_m\lambda^{2m} + q_{m-1}\lambda^{2(m-1)} + \ldots + q_1\lambda^2 + q_0 \qquad 2, (IV\text{-}7.2, 3)$$

We proceed to the determination of the m+1 coefficients $q_0, \ldots, q_m$ by means of the interpolation condition

$$Q_m(\overset{e}{\lambda}{}_j^2) = \frac{(-1)^n (\overset{e}{\lambda}{}_j^2 - \overset{o}{\lambda}{}_1^2) \ldots (\overset{e}{\lambda}{}_j^2 - \overset{o}{\lambda}{}_n^2)}{U_N(0, \overset{e}{\lambda}{}_j^n)} \qquad 3, (IV\text{-}7.2, 3)$$

The coefficients q_0 and q_m are computed at once since

$$\left.\begin{aligned} q_0 &= \frac{\overset{\circ}{\lambda}{}_1^2\,\overset{\circ}{\lambda}{}_2^2 \ldots \overset{\circ}{\lambda}{}_n^2}{U_N(0,0)} \\ &\text{and} \\ q_m &= \frac{(-1)^n}{\text{constant}} = \frac{(-1)^n}{\lim\limits_{\lambda\to\infty} \left[U_N(0,\lambda)\right] \lambda^{2m-2n}} \end{aligned}\right\} \qquad 4,\text{(IV-7.2, 3)}$$

The coefficient q_m is, of course, a positive quantity because the constant indicated in Eq. 4, (IV-7.2, 3) has the same sign as $(-1)^n$.

The rest of the coefficients $q_1, q_2, \ldots, q_{m-1}$ are determined by the following set of linear equations.

$$q_{m-1}\overset{e}{\lambda}{}_j^{2(m-1)} + q_{m-2}\overset{e}{\lambda}{}_j^{2(m-2)} + \ldots + q_1\overset{e}{\lambda}{}_j^2 = \left[\frac{P_n(\overset{e}{\lambda}{}_j^2)}{U_N^*(0,\overset{e}{\lambda}_j)} - (q_0 + q_m\overset{e}{\lambda}{}_j^{2m})\right] \qquad 5,\text{(IV-7.2, 3)}$$

for $j = 1, 2, \ldots, (m-1)$.

The system shown in Eq. 5, (IV-7.2, 3) is potentially equivalent to the determination of the unknown quantities $q_{m-1}, q_{m-2}, \ldots, q_1$, provided the determinant of the system does not vanish. We shall show that this is the case in the next subsection.

IV-7.2, 4 Proof that the system determinant does not vanish. We now proceed to show that the determinant of the system of Eq. 5, (IV-7.2, 3) does not vanish. The proof is somewhat cumbersome but otherwise straightforward.

However, it is rather involved to carry through this proof when we use the complicated notations which are used in Eq. 5, (IV-7.2, 3). We therefore momentarily change some of the notations for simpler ones as follows.

Set

$$\left.\begin{aligned} m - 1 &= h \\ \overset{e}{\lambda}{}_j^2 &= \mu_j, \quad j = 1, 2, \ldots, h \\ y_j &= \frac{P_n(\overset{e}{\lambda}{}_j^2)}{U_N^*(0,\overset{e}{\lambda}_j)} - (q_0 + q_m\overset{e}{\lambda}{}_j^{2m}) \end{aligned}\right\}$$

The system 5, (IV-7.2, 3) becomes

$$\left.\begin{array}{l} q_h\mu_1^h + q_{h-1}\mu_1^{h-1} + q_{h-2}\mu_1^{h-2} + \ldots + q_1\mu_1 = y_1 \\ q_h\mu_2^h + q_{h-1}\mu_2^{h-1} + q_{h-2}\mu_2^{h-2} + \ldots + q_1\mu_2 = y_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ q_h\mu_h^h + q_{h-1}\mu_h^{h-1} + q_{h-2}\mu_h^{h-2} + \ldots + q_1\mu_h = y_h \end{array}\right\} \qquad 1, (IV-7.2, 4)$$

The system determinant can be written as

$$\Delta = \begin{vmatrix} \mu_1^h & \mu_1^{h-1} & \cdots & \mu_1 \\ \mu_2^h & \mu_2^{h-1} & \cdots & \mu_2 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_h^h & \mu_h^{h-1} & \cdots & \mu_h \end{vmatrix} = \mu_1\,\mu_2 \cdots \mu_h \begin{vmatrix} \mu_1^{h-1} & \mu_1^{h-2} & \cdots & 1 \\ \mu_2^{h-1} & \mu_2^{h-2} & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_h^{h-1} & \mu_h^{h-2} & \cdots & 1 \end{vmatrix} \qquad 2, (IV-7.2, 4)$$

Now, take the set formed by the (h-1) quantities

$$\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \ldots,\ \mu_{h-1} \qquad 3, (IV-7.2, 4)$$

where we have omitted the μ_h terms. With the above set let us form the symmetric functions

$$\left.\begin{array}{l} a_{h-1} = 1 \\ a_{h-2} = (-1)(\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_{h-1}) \\ a_{h-3} = (-1)^2 (\mu_1\mu_2 + \mu_1\mu_3 + \ldots + \mu_1\mu_{h-1} + \\ \qquad\qquad + \mu_2\mu_3 + \ldots + \mu_2\mu_{h-1} \\ \qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \qquad\qquad\qquad + \mu_{h-2}\mu_{h-1}) \\ a_{h-4} = (-1)^3 (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \ldots + \mu_1\mu_2\mu_{h-1} + \\ \qquad\qquad + \mu_1\mu_3\mu_4 + \ldots + \mu_1\mu_3\mu_{h-1} + \\ \qquad\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \\ \qquad\qquad\qquad + \mu_{h-3}\mu_{h-2}\mu_{h-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_0 = (-1)^{h-1}\,\mu_1\mu_2\mu_3, \ldots, \mu_{h-1} \end{array}\right\} \qquad 4, (IV-7.2, 4)$$

The equation of h-1 degree formed by these symmetric functions as coefficients

$$\mu^{h-1} + a_{h-2}\mu^{h-2} + a_{h-3}\mu^{h-3} + \ldots + a_1\mu + a_0 = F_1(\mu) \qquad 5, (IV-7.2, 4)$$

has the numbers $\mu_1, \mu_2, \ldots, \mu_{h-1}$ as roots. The quantity μ_h, on the contrary, is not a root of the above equation.

Hence

$$F_1(\mu_k) = \mu_k^{h-1} + a_{h-2}\mu_k^{h-2} + a_{h-3}\mu_k^{h-3} + \ldots + a_1\mu_k + a_0 \begin{cases} = 0 & \text{for } k = 1, 2, \ldots, h-1 \\ \neq 0 & \text{for } k = h \end{cases} \qquad 6, (IV-7.2, 4)$$

Now, consider the last determinant in Eq. 2, (IV-7.2, 4). By simple multiplication and addition of columns we obtain

$$\left.\begin{aligned} J_1(\mu) &= \begin{vmatrix} \mu_1^{h-1} & \mu_1^{h-2} & \ldots & 1 \\ \mu_2^{h-1} & \mu_2^{h-2} & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_h^{h-1} & \mu_h^{h-2} & \ldots & 1 \end{vmatrix} = \begin{vmatrix} F_1(\mu_1) & \mu_1^{h-2} & \ldots & 1 \\ F_1(\mu_2) & \mu_2^{h-2} & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ F_1(\mu_{h-1}) & \mu_{h-1}^{h-2} & \ldots & 1 \\ F_1(\mu_h) & \mu_h^{h-2} & \ldots & 1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & \mu_1^{h-2} & \ldots & 1 \\ 0 & \mu_2^{h-2} & \ldots & 1 \\ 0 & \mu_3^{h-2} & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \mu_{h-1}^{h-2} & \ldots & 1 \\ F_1(\mu_h) & \mu_h^{h-2} & \ldots & 1 \end{vmatrix} = (-1)^{h-1} F_1(\mu_h) \begin{vmatrix} \mu_1^{h-2} & \mu_1^{h-3} & \ldots & 1 \\ \mu_2^{h-2} & \mu_2^{h-3} & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{h-1}^{h-2} & \mu_{h-1}^{h-3} & \ldots & 1 \end{vmatrix} \end{aligned}\right\} \qquad 7, (IV-7.2, 4)$$

where $F_1(\mu_h) \neq 0$ because of Eq. 6, (IV-7.2, 4).

We will now expand, in a similar way, the last determinant in Eq. 7, (IV-7.2, 4).

Take the set of quantities

$$\mu_1, \mu_2, \ldots, \mu_{h-2} \qquad 8, (IV-7.2, 4)$$

Here we also omit μ_{h-1}. Let us form the symmetric functions associated with the set 8, (IV-7.2, 4). One gets

$$\left.\begin{aligned}
b_{h-2} &= 1 \\
b_{h-3} &= (-1)(\mu_1 + \mu_2 + \ldots + \mu_{h-2}) \\
b_{h-3} &= (-1)^2 (\mu_1\mu_2 + \ldots + \mu_1\mu_{h-2} \\
&\qquad + . \quad . \quad . + \\
&\qquad + \mu_{h-3}\mu_{h-2}) \\
b_{h-4} &= (-1)^3 (\mu_1\mu_2\mu_3 + \ldots + \mu_1\mu_2\mu_{h-2} + \\
&\qquad + \ldots + \mu_{h-4}\mu_{h-3}\mu_{h-2}) \\
&. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
b_0 &= (-1)^{h-2} \mu_1\mu_2\mu_3 \cdots \mu_{h-2}
\end{aligned}\right\} \qquad 9, (IV-7.2, 4)$$

Let us form a polynomial of h-2 degree, whose coefficients are the symmetric functions above

$$F_2(\mu) = \mu^{h-2} + b_{h-3}\mu^{h-3} + \ldots + b_1\mu + b_0 \qquad 10, (IV-7.2, 4)$$

The quantities $\mu_1, \mu_2, \ldots, \mu_{h-2}$ are, by construction, roots of the polynomial above. But μ_{k-1} is not a root of such a polynomial. Consequently,

$$F_2(\mu_k) = \mu_k^{h-2} + b_{h-3}\mu_k^{h-3} + \ldots + b_1\mu_k + b_0 \begin{cases} = 0; & k = 1, 2, \ldots, h-2 \\ \neq 0; & k = h-1 \end{cases}$$

11, (IV-7.2, 4)

By a similar operation of multiplication of columns by the set of symmetric functions 9, (IV-7.2, 4) and appropriate addition of columns, one obtains

$$J_2(\mu) = \begin{vmatrix} \mu_1^{h-2} & \mu_1^{h-3} & \ldots & 1 \\ \mu_2^{h-2} & \mu_2^{h-3} & \ldots & 1 \\ . & . & . & . \\ \mu_{h-1}^{h-2} & \mu_{h-1}^{h-3} & \ldots & 1 \end{vmatrix} = (-1)^{h-2} F_2(\mu_{h-1}) \begin{vmatrix} \mu_1^{h-3} & \mu_1^{h-4} & \ldots & 1 \\ \mu_2^{h-3} & \mu_2^{h-4} & \ldots & 1 \\ . & . & . & . \\ \mu_{h-2}^{h-3} & \mu_{h-2}^{h-4} & \ldots & 1 \end{vmatrix} \qquad 12, (IV-7.2, 4)$$

Now, we can repeat the procedure once more to expand the last determinant in Eq. 12, (IV-7.2, 4). We form here the set of symmetric functions with the numbers

$$\mu_1, \mu_2, \mu_3, \ldots, \mu_{h-3} \qquad 13, (IV-7.2, 4)$$

Let us call the symmetric functions

$$\left.\begin{aligned} c_{h-3} &= 1 \\ c_{h-4} &= (-1)(\mu_1 + \mu_2 + \mu_3 + \dots + \mu_{h-3}) \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_0 &= (-1)^{h-3}\, \mu_1\mu_2\mu_3 \cdots \mu_{h-3} \end{aligned}\right\} \qquad 14, (IV\text{-}7.2, 4)$$

We then form the polynomial of h-3 degree

$$F_3(\mu) = \mu^{h-3} + c_{h-4}\mu^{h-4} + \dots + c_1\mu + c_0 \qquad 15, (IV\text{-}7.2, 4)$$

Obviously

$$F_3(\mu_k) = \mu_k^{h-3} + c_{h-4}\mu_k^{h-4} + \dots + c_1\mu_k + c_0 \begin{cases} = 0; & k = 1, 2, \dots, h-3 \\ \neq 0; & k = h-2 \end{cases} \qquad 16, (IV\text{-}7.2, 4)$$

from which we obtain

$$J_3(\mu) = \begin{vmatrix} \mu_1^{h-3} & \mu_1^{h-4} & \dots & 1 \\ \mu_2^{h-3} & \mu_2^{h-4} & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{h-2}^{h-3} & \mu_{h-2}^{h-4} & \dots & 1 \end{vmatrix} = (-1)^{h-3} F_3(\mu_{h-2}) \begin{vmatrix} \mu_1^{h-4} & \mu_1^{h-5} & \dots & 1 \\ \mu_2^{h-4} & \mu_2^{h-5} & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{h-3}^{h-4} & \mu_{h-3}^{h-5} & \dots & 1 \end{vmatrix} \qquad 17, (IV\text{-}7.2, 4)$$

The same process of expansion can be used with the last determinant of Eq. 17,(IV-7.2,4), resulting in successive new determinants. The process will terminate with the expansion of a second-order determinant

$$\begin{vmatrix} \mu_1 & 1 \\ \mu_2 & 1 \end{vmatrix} \qquad 18, (IV\text{-}7.2, 4)$$

To be consistent we will use the same method used in handling the last determinant. Here, the symmetric functions have to be formed only by the quantity μ_1. The symmetric functions are 1 and $-\mu_1$.

The resulting polynomial is

$$F_h(\mu) = \mu - \mu_1$$

Hence

$$\begin{vmatrix} \mu_1 & 1 \\ \mu_2 & 1 \end{vmatrix} = (-1)(\mu_2 - \mu_1)$$

The determinant of Eq. 2, (IV-7.2, 4) can therefore be expanded as

$$\Delta = (-1)^{h(h-1)/2}\, \mu_1\mu_2 \dots \mu_h\, F_1(\mu_h)\, F_2(\mu_{h-1}) \dots F_h(\mu_1) \qquad 19, (IV\text{-}7.2, 4)$$

Now, since

$$\mu_k \neq 0, \quad k = 1, 2, \ldots, h$$

(See Fig. 1, (IV-7. 2, 2)) and every factor

$$F_1(\mu_h), \ldots, F_h(\mu_1)$$

is, by construction, different from zero, we finally conclude that

$$\Delta \neq 0 \qquad 20, (IV\text{-}7.2, 4)$$

which was to be proved.

Hence, the property: No matter how we select the set of intermediate points of interpolation, say $\overset{e}{\lambda}_1, \overset{e}{\lambda}_2, \ldots, \overset{e}{\lambda}_{m-1}$, the system 5, (IV-7. 2, 3) always has a solution.

IV-7. 2, 5 Determination of the unknown quantities $q_{m-1}, q_{m-2}, \ldots, q_1$. Explicit solutions for the unknown quantities $q_1, \ldots, q_{m-1}$, Eq. 5, (IV-7 .2, 3) or Eq. 1, (IV-7 .2, 4), can easily be obtained.

We will consider the solution of the system of Eq. 1, (IV-7. 2, 4) instead of that of Eq. 5, (IV-7. 2, 3) because of its simpler notation. The procedure of solution of 1, (IV-7. 2, 4) will be facilitated if we exchange this system for an equivalent one whose coefficient matrix is the transposed matrix of Eq. 1, (IV-7. 2, 4). The transposition is performed by means of a linear transformation. Several steps are required in the solution of the system 1, (IV-7. 2, 4) for the unknowns $q_1, q_2, \ldots, q_h$. We will denote, for convenience, each step by means of a Roman numeral.

I. Auxiliary matrix to obtain the transposed system.

Let

$$B = [b_{g,k}] = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,h} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,h} \\ \cdot & \cdot & \cdot & \cdot \\ b_{h,1} & b_{h,2} & \cdots & b_{h,h} \end{bmatrix} \qquad h = m - 1 \qquad 1, (IV\text{-}7.2, 5)$$

The elements of this matrix are going to be determined.

II. The auxiliary system for $b_{g,k}$.

Let us multiply the first equation of 1, (IV-7. 2, 4) by $b_{1,k}$, the second equation by $b_{2,k}$, and so on. Let us then add these equations. One gets

$$\left.\begin{aligned} \sum_{g=1}^{h} y_g b_{g,k} = q_h\left(\sum_{g=1}^{h} \mu_g^h b_{g,k}\right) + q_{h-1}\left(\sum_{g=1}^{h} \mu_g^{h-1} b_{g,k}\right) + \ldots \\ + q_k\left(\sum_{g=1}^{h} \mu_g^k b_{g,k}\right) + \ldots + q_1\left(\sum_{g=1}^{h} \mu_g b_{g,k}\right) \end{aligned}\right\} \qquad 2, (IV\text{-}7.2, 5)$$

The coefficients $b_{g,k}$, $g = 1, 2, \ldots, h$; k = integer, will be determined by the following conditions: (a) All the coefficients of $q_h, q_{h-1}, \ldots, q_{k-1}, q_{k+1}, \ldots, q_1$ must be zero in Eq. 2,(IV-7.2,5). (b) The coefficient of q_k must be equal to one.

That is

$$\left.\begin{aligned} 0 &= \sum_{g=1}^{h} \mu_g^h b_{g,k} \\ 0 &= \sum_{g=1}^{h} \mu_g^{h-1} b_{g,k} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1 &= \sum_{g=1}^{h} \mu_g^k b_{g,k} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 &= \sum_{g=1}^{h} \mu_g b_{g,k} \end{aligned}\right\} \qquad 3,(\text{IV-7.2,5})$$

The system of equations 3,(IV-7.2,5) is potentially sufficient to determine the h unknowns $b_{1,k}, b_{2,k}, \ldots, b_{h,k}$, provided the determinant of the coefficients is different from zero. The two following properties of 3,(IV-7.2,5) can be seen at once.

1. The system determinant of 3,(IV-7.2,5) is exactly equal to the determinant Δ of the original system 1,(IV-7.2,4). Hence, system 3,(IV-7.2,5) has a nonzero unique solution for the unknowns $b_{1,k}, b_{2,k}, \ldots, b_{h,k}$.

2. The coefficient matrix in Eq. 3,(IV-7.2,5) is equal to the transposed matrix of the original system 1,(IV-7.2,4).

Now, Eq. 2,(IV-7.2,5) renders directly the value of the unknown q_k by virtue of conditions (a) and (b).

Hence

$$q_k = \sum_{g=1}^{h} y_g b_{g,k} \qquad 4,(\text{IV-7.2,5})$$

Consequently, we have transferred our main problem to the problem of solving the system 3,(IV-7.2,5) for $b_{1,k}, b_{2,k}, \ldots, b_{h,k}$.

The only reason for forming the auxiliary system 3,(IV-7.2,5) is that this auxiliary system facilitates the required algebraic manipulations.

III. The determinant expression for the unknown quantities $b_{j,k}$, k fixed.

The determinant expression for one unknown, say $b_{j,k}$, is, after a simple cancellation of common factors, given by

$$b_{j,k} = \begin{vmatrix} \mu_1^{h-1} & \cdots & \mu_{j-1}^{h-1} & 0 & \mu_{j+1}^{h-1} & \cdots & \mu_h^{h-1} \\ \mu_1^{h-2} & \cdots & \mu_{j-1}^{h-2} & 0 & \mu_{j+1}^{h-2} & \cdots & \mu_h^{h-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1^{k-1} & \cdots & \mu_{j-1}^{k-1} & 1 & \mu_{j+1}^{k-1} & \cdots & \mu_h^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1 & \cdots & \mu_{j-1} & 0 & \mu_{j+1} & \cdots & \mu_h \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \end{vmatrix} \div \mu_j \begin{vmatrix} \mu_1^{h-1} & \cdots & \mu_{j-1}^{h-1} & \mu_j^{h-1} & \mu_{j+1}^{h-1} & \cdots & \mu_h^{h-1} \\ \mu_1^{h-2} & \cdots & \mu_{j-1}^{h-2} & \mu_j^{h-2} & \mu_{j+1}^{h-2} & \cdots & \mu_h^{h-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1^{k-1} & \cdots & \mu_{j-1}^{k-1} & \mu_j^{k-1} & \mu_{j+1}^{k-1} & \cdots & \mu_h^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1 & \cdots & \mu_{j-1} & \mu_j & \mu_{j+1} & \cdots & \mu_h \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{vmatrix} \qquad 5,(IV\text{-}7.2,5)$$

IV. The symmetric function and auxiliary polynomial.

Consider the set of (h-1) quantities

$$\mu_1,\ \mu_2,\ \ldots,\ \mu_{j-1},\ \mu_{j+1},\ \ldots,\ \mu_h \qquad 6,(IV\text{-}7.2,5)$$

Note that the quantity μ_j is omitted in this set.

Now, construct the h symmetric functions associated with the set above. They are

$$\left.\begin{aligned} \overset{j}{S}_{h-1} &= 1 \\ \overset{j}{S}_{h-2} &= (-1)\,(\mu_1 + \mu_2 + \ldots + \mu_{j-1} + \mu_{j+1} + \ldots + \mu_h) \\ \overset{j}{S}_{h-3} &= (-1)^2\,(\mu_1\mu_2 + \ldots + \mu_1\mu_{j-1} + \mu_1\mu_{j+1} + \ldots + \mu_1\mu_h + \ldots + \mu_{h-1}\mu_h) \\ \overset{j}{S}_{h-4} &= (-1)^3\,(\mu_1\mu_2\mu_3 + \ldots + \mu_1\mu_2\mu_{j-1} + \mu_1\mu_2\mu_{j+1} + \ldots + \mu_1\mu_2\mu_h + \ldots + \mu_{h-2}\mu_{h-1}\mu_h) \\ &\cdots \\ \overset{j}{S}_0 &= (-1)^{h-1}\,(\mu_1\mu_2\mu_3 \times \ldots \times \mu_{j-1}\,\mu_{j+1} \times \ldots \mu_h) \end{aligned}\right\} \qquad 7,(IV\text{-}7.2,5)$$

The upper index j indicates that the quantity μ_j has been omitted.

Now, let us introduce a polynomial of (h-1) degree, which is constructed as follows

$$\overset{j}{G}(\mu) = \mu^{h-1} + \overset{j}{S}_{h-2}\mu^{h-2} + \overset{j}{S}_{h-3}\mu^{h-3} + \ldots + \overset{j}{S}_1\mu + \overset{j}{S}_0 \qquad 8,(IV\text{-}7.2,5)$$

The polynomial in Eq. 8,(IV-7.2,5) has, by construction, the property

$$\overset{j}{G}(\mu_i) \begin{cases} = 0 & \text{for } i = 1,\ 2,\ \ldots,\ (j-1),\ (j+1),\ \ldots,\ h \\ \neq 0 & \text{for } i = j \end{cases} \qquad 9,(IV\text{-}7.2,5)$$

V. The solution for $b_{j,k}$.

Consider here the determinant expression 5, (IV-7.2, 5) for $b_{j,k}$. Now, we multiply the first line of each determinant in the numerator and denominator by $\overset{j}{S}_{h-1}$, the second line of both determinants by $\overset{j}{S}_{h-2}$, the third line of both determinants by $\overset{j}{S}_{h-3}$, and so on. The last line of each determinant is thus multiplied by $\overset{j}{S}_0$.

We then add each resultant line to the first line of each determinant in the numerator and denominator, and write down the remaining lines to form both determinants. Now, after canceling obvious factors in the remaining lines of both determinants, one gets

$$b_{j,k} = \left.\frac{\begin{vmatrix} \overset{j}{G}(\mu_1) & \dots & \overset{j}{G}(\mu_{j-1}) & \overset{j}{S}_{k-1} & \overset{j}{G}(\mu_{j+1}) & \dots & \overset{j}{G}(\mu_h) \\ \mu_1^{h} & \dots & \mu_{j-1}^{h-2} & 0 & \mu_{j+1}^{h-2} & \dots & \mu_h^{h-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1^{k-1} & \dots & \mu_{j-1}^{k-1} & 1 & \mu_{j+1}^{k-1} & \dots & \mu_h^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1 & \dots & \mu_{j-1} & 0 & \mu_{j+1} & \dots & \mu_h \\ 1 & \dots & 1 & 0 & 1 & \dots & 1 \end{vmatrix}}{\mu_j \begin{vmatrix} \overset{j}{G}(\mu_1) & \dots & \overset{j}{G}(\mu_{j-1}) & \overset{j}{G}(\mu_j) & \overset{j}{G}(\mu_{j+1}) & \dots & \overset{j}{G}(\mu_h) \\ \mu_1^{h} & \dots & \mu_{j-1}^{h-2} & \mu_j^{h-2} & \mu_{j+1}^{h-2} & \dots & \mu_h^{h-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1^{k-1} & \dots & \mu_{j-1}^{k-1} & \mu_j^{k-1} & \mu_{j+1}^{k-1} & \dots & \mu_h^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_1 & \dots & \mu_{j+1} & \mu_j & \mu_{j+1} & \dots & \mu_h \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{vmatrix}}\right\} \qquad 10, (IV-7.2, 5)$$

Note that every element of the first line in both determinants is equated with zero, except the term $\overset{j}{S}_{k-1}$ in the numerator and the term $\overset{j}{G}(\mu_j)$ in the denominator.

By expanding both determinants by the elements of the first line and canceling out the common minor determinants, we obtain at once the following simple and elegant solution for $b_{j,k}$

$$b_{j,k} = \frac{\overset{j}{S}_{k-1}}{\mu_j \, \overset{j}{G}(\mu_j)} \qquad 11, (IV-7.2, 5)$$

An alternative and more useful form is evidently given by

$$b_{j,k} = \frac{\overset{j}{S}_{k-1}}{\mu_j(\mu_j - \mu_1)(\mu_j - \mu_2), \dots, (\mu_j - \mu_{j-1})(\mu_j - \mu_{j+1}), \dots, (\mu_j - \mu_h)} \qquad 12, (IV-7.2, 5)$$

VI. The solution for the unknown quantities q_i, $i = 1, 2, \ldots, h$.

The h unknown quantities q_k, $k = 1, 2, \ldots, h$ are now obtained at once from Eqs. 4,(IV-7.2, 5) and 12,(IV-7.2, 5).

The explicit formula is

$$q_k = \sum_{j=1}^{h} y_j \frac{\overset{j}{S}_{k-1}}{\mu_j \overset{j}{G}(\mu_j)} \qquad k = 1, 2, \ldots, h \qquad 13,(\text{IV-7.2, 5})$$

because the letter with which we designate the finite index of the summation is immaterial.

VII. The explicit solution for q_k in terms of the notation of the system 5,(IV-7.2, 3) is

$$\left.\begin{aligned}
&\text{(a)} \quad q_k = \sum_{j=1}^{m-1} \left[\frac{P_n(\overset{e}{\lambda}{}_j^2)}{\left[U_N^*(0, \overset{e}{\lambda}_j)\right]} - (q_0 + q_m \overset{e}{\lambda}{}_j^{2m}) \right] \frac{\overset{j}{S}_{k-1}}{\overset{e}{\lambda}{}_j^2 \, \overset{j}{G}(\overset{e}{\lambda}{}_j^2)} \\
&\qquad \text{for } k = 1, 2, \ldots, m-1 \\
&\text{(b)} \quad q_0 = \frac{\overset{o}{\lambda}{}_1^2 \overset{o}{\lambda}{}_2^2 \cdots \overset{o}{\lambda}{}_n^2}{U_N(0, 0)} \\
&\text{(c)} \quad q_m = \frac{(-1)^n}{\lim\limits_{\lambda \to \infty} \left[U_N(0, \lambda)\right] \lambda^r}
\end{aligned}\right\} \qquad 14,(\text{IV-7.2, 5})$$

Remarks: It is important to recall that the quantity $m - 1 = h$, in formula 14,(IV-7.2,5) and others, represents the number of points of interpolation which are different from $\lambda = 0$, and $\lambda = \infty$. The total number of points of interpolation, including those at $\lambda = 0$ and $\lambda = \infty$, is $m + 1$.

The minimum number of points of interpolation includes at least $\lambda = 0$ or $\lambda = \infty$. This makes $m = 1$ a lower bound.

For $m = 1$, formula 14,(IV-7.2,5 (a)) has no meaning. We only need to apply formulas 14,(IV-7.2,5 (b) and (c)) to determine q_0 and q_m. Examples of the application of formula 14,(IV-7.2, 5) will be given in a following subsection.

IV-7.3 Illustrative examples. Before proceeding further with the theoretical aspect of the approximation of the function $U(0, \lambda)$ by rational functions, it is convenient to recapitulate the previous part with the help of some simple illustrative examples.

The examples will be divided into two groups. The first group illustrates the application of the general theorem on rational transfer functions which was considered in section IV-1. The second group illustrates the application of the above interpolation procedure to some illuminating cases.

IV-7.3,1 Group I.

Example I.

Let the density distribution function along the imaginary axis, $U(0, \lambda)$, be given as

$$U(0, \lambda) = \frac{(\lambda^2 - 1)(\lambda^2 - 9)}{0.809\lambda^4 + 0.0625\lambda^2 + 1.8} \qquad 1, (IV-7.3, 1)$$

(See Fig. 1, (IV-7.3, 1).) The problem is to find $F(s)$.

Following the indications given in section IV-1, one gets

(a) Denominator roots

$$\mu_{1,2} = -0.03861 \pm i1.4905$$

(b) The Encke roots produce

$$\lambda_1 = -1.229e^{-45°}$$

$$\lambda_2 = -1.229e^{+45°}$$

(c) The partial fraction coefficients are

$$A_0 = 1.235$$

$$A_1 = 6.81 \ \underline{/204°}$$

$$A_2 = 6.81 \ \underline{/155.7°}$$

(d) The transfer function is then

$$F(s) = 1.235 + \frac{5.56e^{-i110°}}{s + 1.229e^{-i45°}} + \frac{5.56e^{i110°}}{s + 1.229e^{i45°}} \qquad 2, (IV-7.3, 1)$$

Example II.

Given

$$U(0, \lambda) = \frac{-(\lambda^2 - 1)(\lambda^2 - 9)(\lambda^2 - 25)}{1.498\lambda^6 - 25.98\lambda^4 + 147.4\lambda^2 + 45} \qquad 3, (IV-7.3, 1)$$

(See Fig. 2, (IV-7.3, 1).)

After routine computations one gets

$\lambda_1 = -0.537$ $\qquad K_0 = -0.6675$

$\lambda_2 = -3.187e^{-i75°}$ $\qquad K_2 = 1.8607$

$\lambda_3 = -3.187e^{+i75°}$ $\qquad K_3 = 5.09e^{-i13.82°}$

$\qquad K_4 = 5.09e^{+i13.82°}$

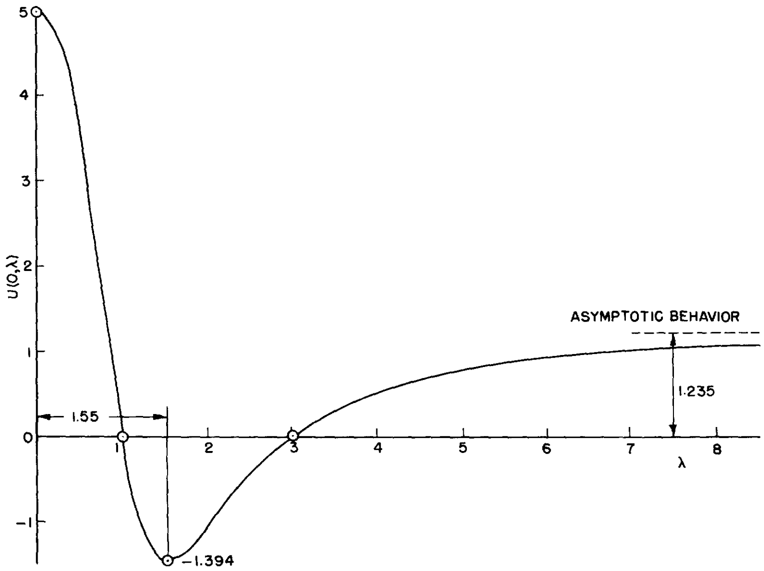

Fig. 1, (IV-7.3, 1)
The graph of U(0, λ), Eq. 1, (IV-7.3, 1).

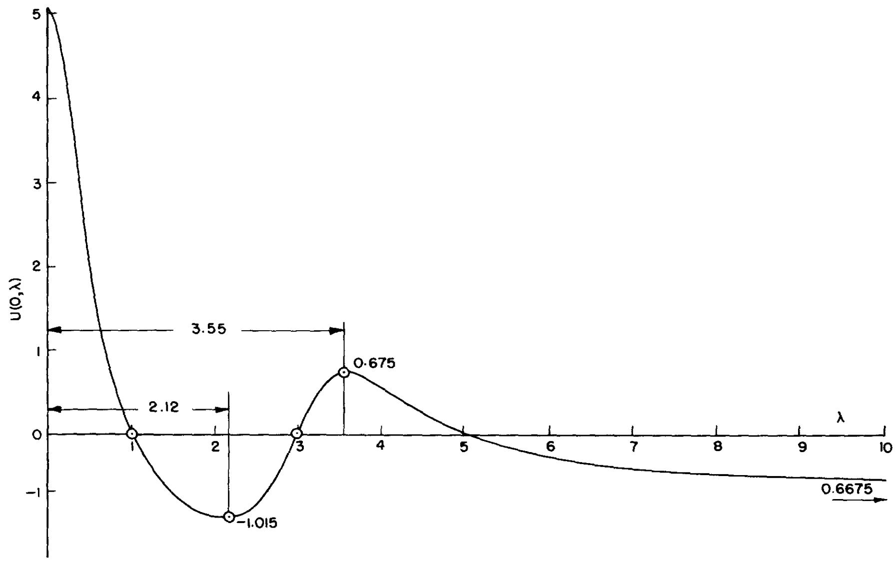

Fig. 2, (IV-7.3, 1)
The graph of U(0, λ), Eq. 3, (IV-7.3, 1).

From which

$$F(s) = -0.6675 + \frac{3.46}{s + 0.538} + \frac{1.595e^{i61.12°}}{s + 3.185e^{-i75°}} + \frac{1.595e^{-i61.12°}}{s + 3.185e^{+i75°}} \qquad 4, (IV-7.3, 1)$$

IV-7.3, 2 Group II.

This group of examples is intended to illustrate the use of formulas 14, (IV-7.2, 5). We will consider some simple cases. The objective of the examples is to construct a rational approximant to $U(0, \lambda)$ as described in this and previous sections. Since we want to illustrate only the manipulation of the formulas we may as well use the two previous examples as follows.

1. From the function $U(0, \lambda)$, or its graph, we extract:

$$\begin{aligned} &(a) \quad \text{The set of zero points, say } \overset{\circ}{\lambda}_k \\ &(b) \quad \text{The set of maximum points, say } \overset{e}{\lambda}_j \\ &(c) \quad \text{The values of } U(0, 0) \\ &(d) \quad U(0, \overset{e}{\lambda}_j) \text{ and } \lim_{\lambda \to \infty} \left[U(0, \lambda)\, \lambda^r \right]. \end{aligned} \qquad 1, (IV-7.3, 2)$$

From these data, and only these data, we will reconstruct the original function $U(0, \lambda)$ by using the formulas 14, (IV-7.2, 5).

Example III.

Take the density distribution function 1, (IV-7.3, 1). The data required by expression 1, (IV-7.3, 2) are

$$\overset{\circ}{\lambda}_1 = 1; \quad \overset{\circ}{\lambda}_2 = 3; \quad \overset{e}{\lambda}_1 = 1.55$$

$$U(0, 0) = 5; \quad U(0, \overset{e}{\lambda}_1) = -1.397. \quad \underset{\lambda \to \infty}{U(0, \lambda)} \to \frac{1}{0.809}; \quad m = 2$$

A. We have

$$r = 2m - 2n = 0$$

$$h = m - 1 = 1$$

Then we have the index $j = 1$, so that the summation reduces to its first term.

B. Construction of $P_2(\lambda^2)$

$$P_2(\lambda^2) = (-1)^2 (\lambda^2 - 1)(\lambda^2 - 9)$$

C. Construction of $Q_2(\lambda^2)$

$$(a) \qquad q_0 = \frac{\overset{\circ}{\lambda}{}_1^2 \cdot \lambda_2^2}{U(0, 0)} = 1.8$$

$$(b) \qquad q_3 = \lim_{\lambda \to \infty} \frac{(-1)^2}{U(0, \lambda)} = 0.809$$

(c) Computation of y_1 at $\overset{e}{\lambda}_1$

$$y_1 = \frac{P_2(\overset{e}{\lambda}{}_1^2)}{U(0, \overset{e}{\lambda}_1)} - (q_0 + q_2 \overset{e}{\lambda}{}_1^4) = +0.15$$

(d) The symmetric function $\overset{1}{S}_{h-1} = \overset{1}{S}_0 = 1$

(e) $\overset{j}{G}(\mu_1) = 1$

Hence we have

$$q_1 = \frac{y_1}{\overset{e}{\lambda}{}_1^2} = 0.0625$$

as it should be.

Example IV.

Take the density distribution function, 3,(IV-7.3, 1), of example II. The data required in 1,(IV-7.3, 2) are

$$m = 3; \quad n = 3$$

$$\overset{o}{\lambda}_1 = 1, \quad \overset{o}{\lambda}_2 = 3, \quad \overset{o}{\lambda}_3 = 5; \quad \overset{e}{\lambda}_1 = 2.12, \quad \overset{e}{\lambda}_2 = 3.55$$

$$U(0, 0) = 5; \quad U(0, \overset{e}{\lambda}_1) = -1.015; \quad U(0, \overset{e}{\lambda}_2) = 0.675$$

$$U(0, \lambda)_{\lambda \to \infty} \to -0.6675$$

The following results are obtained in order:

A. $r = 2m - 2n = 0; \quad h = m - 1 = 2; \quad$ hence $j = 1, 2$

B. Construction of $P_n(\lambda^2)$

$$P_3(\lambda^2) = -(\lambda^2 - 1)(\lambda^2 - 9)(\lambda^2 - 25)$$

C. Construction of $Q_m(\lambda^2)$

(a) $q_0 = \dfrac{\overset{o}{\lambda}{}_1^2 \overset{o}{\lambda}{}_2^2 \overset{o}{\lambda}{}_3^2}{U(0, 0)} = 45$

(b) $q_3 = 1.498$

(c) $y_1 = 137.5; \quad y_2 = -2249$

(d) $\overset{1}{S}_1 = 1, \quad \overset{1}{S}_0 = -12.5; \quad \overset{2}{S}_1 = 1, \quad \overset{2}{S}_0 = -4.5$

(e) $\overset{1}{G}(\mu_1) = -8.1 \qquad \overset{2}{G}(\mu_2) = +8.1$

$$q_1 = y_1 \frac{\overset{1}{S}_0}{\overset{e}{\lambda}^2 \overset{1}{G}(\mu_1)} + y_2 \frac{\overset{2}{S}_0}{\overset{e}{\lambda}_2^2 \overset{2}{G}(\mu_2)} = 147.4$$

$$q_2 = \quad . \quad . \quad . \quad . \quad . \quad . \quad . = -25.9$$

Hence we have

$$Q_3(\lambda^2) = 1.498\lambda^6 - 25.9\lambda^4 + 147.4\lambda^2 + 45$$

as it should be.

Example V.*

This example is intended to illustrate one case in which the possible number of extremal points obtains the lower bound described in the beginning of subsection (IV-7.1, 3).

The selected density distribution function is

$$U(0, \lambda) = \frac{1 - 2\lambda^2}{1 + \lambda^6} \qquad 2, (IV-7.3, 2)$$

The graph of this function is represented in Fig. 1, (IV-7.3, 2).

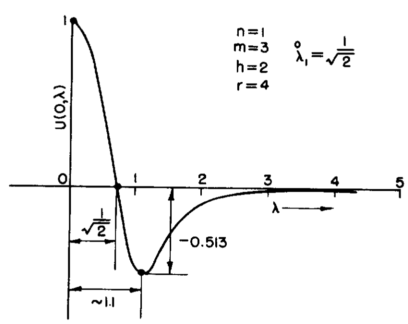

Fig. 1, (IV-7.3, 2)
The graph of U(0, λ), Eq. 2, (IV-7.3, 2).

In this case h = 2 and hence j = 1,2. However, we have one extremal point at $\overset{e}{\lambda}_1 = 1.1$ at our disposal. By direct computation we find at once that

$$P_1(\lambda) = -1\left(\lambda^2 - \frac{1}{2}\right)$$

$$q_0 = \frac{1}{2} \qquad q_3 = \frac{1}{2}$$

*This example was worked out by Mr. N. DeClaris.

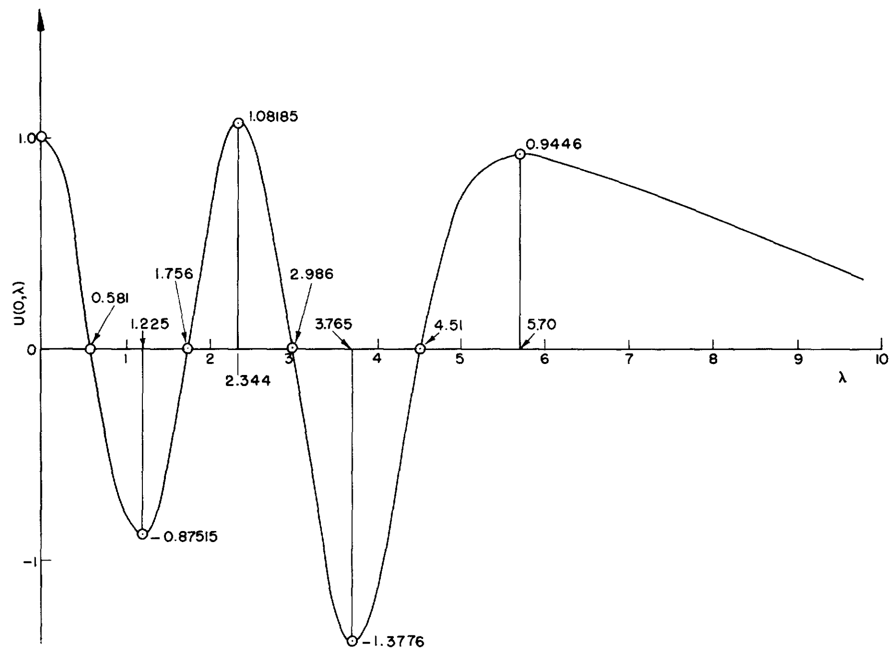

Fig. 2, (IV-7.3, 2)
The graph of $U(0, \lambda)$, Eq. 4, (IV-7.3, 2).

In order to show that q_1 and q_2 are both zero, independently of the points we choose for interpolation, one will observe that $y_1 \equiv 0$ and $y_2 \equiv 0$. Hence, the interpolation formulas 14, (IV-7.2, 5) still are potentially sufficient for determining the rational function approximation.

Example VI.

As a final example let us select a transfer function having its 5 poles and 4 zeros distributed in the complex plane in a transmission-like manner.

$$F(s) = \frac{(s^2 - 2s + 2)(s^2 - 2s + 10)}{(s+1)(s^2 + 2s + 5)(s^2 + 2s + 17)} \qquad 3, (IV-7.3, 2)$$

The real part is

$$U(0, \lambda) = \frac{\lambda^8 - 32.667\lambda^6 + 282.33\lambda^4 - 650.67\lambda^2 + 188.89}{0.02614\lambda^{10} - 0.9150\lambda^8 + 11.974\lambda^6 - 52.026\lambda^4 + 123.95\lambda^2 + 188.89}$$

4, (IV-7.3, 2)

See Fig. 2, (IV-7.3, 2).

Following the procedure indicated above, we get:

$$m = 5, \quad n = 4, \quad r = 2m - 2n = 2, \quad h = 4. \quad \lambda^2 = \mu$$

$$\text{zeros: } \mathring{\mu}_1 = 0.338 \qquad \text{zeros: } \mathring{\mu}_3 = 8.912$$

$$\mathring{\mu}_2 = 3.085 \qquad \mathring{\mu}_4 = 20.335$$

extremal points: $\overset{e}{\mu}_1 = 1.5$ $\quad U(0, \overset{e}{\mu}_1) = -0.87515$

$\overset{e}{\mu}_2 = 5.5$ $\quad U(0, \overset{e}{\mu}_2) = 1.08185$

$\overset{e}{\mu}_3 = 14.2$ $\quad U(0, \overset{e}{\mu}_3) = -1.3776$

$\overset{e}{\mu}_4 = 32.5$ $\quad U(0, \overset{e}{\mu}_4) = 0.9446$

A. Construction of $P_4(\mu)$

$$P_4(\mu) = (\mu - 0.338)(\mu - 3.085)(\mu - 8.912)(\mu - 20.335)$$

B. Construction of $Q_5(\mu)$

(a) $$q_0 = \frac{0.338 \cdot 3.085 \cdot 8.912 \cdot 20.335}{1} = 188.97$$

(b) $$q_5 = \frac{(-1)^4}{\left(\frac{38.25}{\mu} \ \mu\right)} = 0.02614$$

(c) $y_1 = 104.71$

$y_2 = 262.79$

$y_3 = -11652$

$y_4 = -660614$

(d) Symmetric functions

	$\overset{1}{S}$	$\overset{2}{S}$	$\overset{3}{S}$	$\overset{4}{S}$
S_0	-2538.25	-692.25	-268.125	-117.15
S_1	718.35	531.55	235.75	107.65
S_2	-52.2	-48.2	-39.5	-21.2
S_3	1	1	1	1

(e) Coefficient values obtained

Coefficients	Exact Values	Approximate Values	Errors	
			Difference	%
q_0	188.89	188.96	+0.07	+0.037
q_1	123.95	123.88	-0.07	-0.057
q_2	-52.026	-51.995	-0.031	-0.060
q_3	11.974	11.966	-0.008	-0.067
q_4	-0.9150	-0.9147	-0.0003	-0.033
q_5	0.02614	0.02614	-	-

IV-7.4 Concluding remarks. We shall give a brief condensation of the results obtained in section IV-7. At the same time we shall analyze the standing of the problem of rational representation of $U(0, \lambda)$ in the light of these results.

The discussion will reveal that the analysis carried on in section IV-7 does not completely cover the solution of the problem of rationalization. Specifically, the limitations of the analysis are mainly concerned with the analytical character of the polynomial $Q_m(\lambda^2)$, which is obtained by the process of interpolation already described.

The complete clarification of the problem of rational representation of $U(0, \lambda)$ will be carried on in the last section, IV-9. The starting point of the discussion of section IV-9 is the set of conclusions at which we are arriving in this subsection, IV-7.4. The material presented in the next section, IV-8, will produce some results that are needed in the clarification of the question of tolerances.

IV-7.4, 1 Let us start the discussion by reviewing the examples of the last subsection. All of these examples show that it is possible to construct both $U(0, \lambda)$ and $F(s)$ from the given data required by the expression 1, (IV-7.3, 2). By inspecting these examples we note that in all cases we have selected functions which originally were rational and which satisfy the conditions of network realizability.

Now a question still open is the following: If we have given an arbitrary set of quantities and points for the data required by 1, (IV-7.3, 2) and construct from them the rational approximation in accordance with the method described in section IV-7, what can we say about the realizability of the rational functions obtained?

We shall now illustrate the possibilities of a failure. Figure 2, (IV-7.2, 1) shows that the function

$$\frac{P_n(\lambda^2)}{U_N(0, \lambda)} \qquad 1, (\text{IV-7.4, 1})$$

from which the polynomial $Q_m(\lambda^2)$ was constructed, is necessarily a positive nonzero function of λ^2. The polynomial $Q_m(\lambda^2)$ is constructed by a process of interpolation through the points corresponding to the extremal values of $U_N(0, \lambda)$.

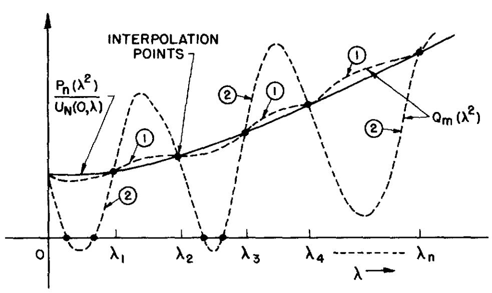

Fig. 1, (IV-7.4, 1)
The function 1, (IV-7.4, 1) and possible behaviors of $Q_m(\lambda^2)$.

By the process of interpolation described in subsection IV-7.2, 5 we can be sure that the graph of $Q_m(\lambda^2)$ coincides with 1, (IV-7.4, 1) at the points of interpolation. Nothing else, however, can be said about the behavior of $Q_m(\lambda^2)$ for other points. Some possible ways in which $Q_m(\lambda^2)$ may run are indicated in Fig. 1,(IV-7.4,1). It can be seen at once that if $Q_m(\lambda^2)$ has the graph indicated as curve 1 in the figure, then such a polynomial may be quite acceptable, provided the magnitude of the oscillations around $P_n(\lambda^2)/U_N(0, \lambda)$ are small enough. On the other hand, curve 2 must be repudiated because it implies a considerable distortion of the envelope of the function $U(0, \lambda)$ with the corresponding serious distortion of the time pulse, which is the origin of $U(0, \lambda)$. In the case in which $Q_m(\lambda^2)$ crosses the λ axis, the situation becomes still worse because it means that the rational approximation of $U(0, \lambda)$ has one or several poles along the λ axis. Such poles are, as was shown in section IV-2, also poles of $F(s)$. Since these poles are situated along the imaginary axis, the corresponding time response possesses periodic components which are not at all present in the original pulse. Thus this condition is not acceptable for the problem in question. Consequently the process of interpolation fails in such cases to produce constructive solutions for $Q_m(\lambda^2)$.

IV-7.4, 2 The failure described above is not intrinsic to the process of interpolation itself. The examples with rational functions presented above indicate that it may produce completely correct solutions. Let us then trace the cause of possible failures.

The fundamental cause of failure resides in the selection of the data required by 1, (IV-7.3, 2). An arbitrary selection is, in general, not potentially sufficient for determining a polynomial $Q_m(\lambda^2)$ which approaches $P_n(\lambda^2)/U_N(0, \lambda)$ with minor oscillations, as in curve 1, Fig. 1, (IV-7.4, 1). The correct determination of the degrees of the polynomials $P_n(\lambda^2)$ and $Q_m(\lambda^2)$, particularly the latter one, are of primary importance for the solution of the problem in question.

The criterion we need for fixing the degree of $P_n(\lambda^2)$ rests on the number of zeros of the syncopated function $U_N(0, \lambda)$. The degree of $Q_m(\lambda^2)$ is determined by the number of maxima and minima of $U_N(0, \lambda)$ and its behavior when $\lambda \to \infty$.

The selection of the extremal points was used because of its correlation to the number of poles of $U_N(0, \lambda)$. In subsection IV-7.1, 3 we have established certain bounds relating the number of poles to the number of extremal points of $U_N(0, \lambda)$. The criterion for establishing such bounds and particularly the actual number of poles needed in $U_N(0, \lambda)$ requires, however, more information in the s plane than is obtained from a set of isolated points along the imaginary axis.

Section IV-9 is devoted to the correct establishment of all of these questions. Besides, it introduces a set of important theorems which provide a more rigorous approach to the study of the analytical character of $U_N(0, \lambda)$.

Section IV-8

The Removal of a Lobe of $U_N(0, \lambda)$ and Its Effect on the Tolerance. Ringing.

The time response associated with a lobe. The effect of the suppression of consecutive lobes of $U_N(0, \lambda)$. Ringing.

IV-8.0 Introduction. The rational approximation of the density distribution function $U(0, \lambda)$ requires the syncopation of this function by the removal of its last part, starting from a prescribed zero of the envelope function $\alpha(0, \lambda)$. In section IV-6 we have computed the error which is committed by this syncopation. On the basis of the bounds for this error we have established a relation between the tolerance and the number of zeros of the remaining envelope function $\alpha(0, \lambda)$. In other words, for a prescribed tolerance we can find the zero point of $\alpha(0, \lambda)$ at which the cutting must take place.

The process of syncopation of $U(0, \lambda)$ produces the function $U_N(0, \lambda)$, where N indicates the last number of the zero ordinate that we must preserve in order to remain inside the prescribed tolerance.

In this section, IV-8, we are confronted with a new and important question. Its statement is as follows:

Consider the function $U_N(0, \lambda)$. Suppose that we remove one of its lobes. By lobe we mean a half-cycle of oscillation, leaving a function as is illustrated in Figs. 1, (IV-8.0) and 2, (IV-8.0).

Now, we are going to show that by the removal of such a lobe the remaining density distribution function

$$\left\{U_N(0, \lambda) - L(n, \lambda)\right\}$$

produces a time response which is necessarily outside the prescribed tolerances.

Since we are primarily interested in the computation of the bounds of the error committed by the removal of a lobe, say the nth one, we shall introduce some approximations that will simplify the computations.

IV-8.1 The time response associated with a lobe. We shall now compute the error committed by the suppression of the nth lobe. The notation of sections IV-3 and IV-4 will be used.

Let f(t) be the time function which represents the original pulse. Let t_0 be the delay and 2δ be the pulsewidth. The error committed in the time domain is given by

$$\epsilon_n(t) = \frac{2}{\pi} \int_0^{\infty} L(n, \lambda) \cos \lambda t \, d\lambda \qquad 1, (IV\text{-}8.1)$$

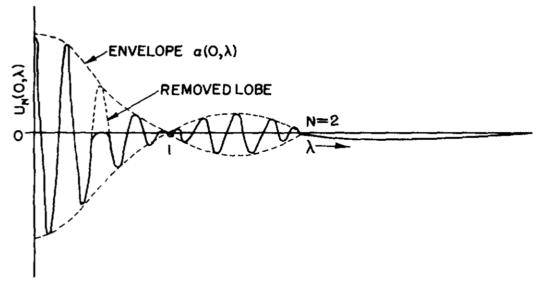

Fig. 1, (IV-8.0)
The function $U_N(0, \lambda)$ with a lobe removed.

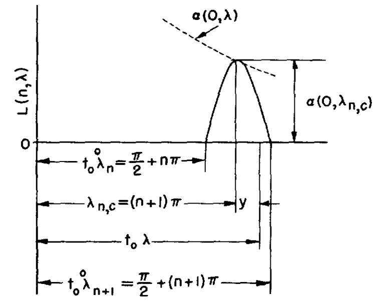

Fig. 2, (IV-8.0)
The function $L(n, \lambda)$ representing the removed nth lobe.

This error can be expressed directly in terms of the function $U(0, \lambda)$ and $f(t)$ because

$$f(t) = \frac{2}{\pi} \int_0^\infty U(0, \lambda) \cos \lambda t \, d\lambda$$

The contribution to this last integral between λ_n and λ_{n+1} is also equal to the error. Hence

$$\epsilon_n(t) = \frac{2}{\pi} \int_{\lambda_n}^{\lambda_{n+1}} U(0, \lambda) \cos \lambda t \, d\lambda \qquad 2, (IV\text{-}8.1)$$

We shall also use the basic expression

$$\left.\begin{aligned} U(0, \lambda) &= 2 \cos \lambda t_o \int_0^\delta f(x) \cos \lambda x \, dx \\ &= \alpha(0, \lambda) \cos \lambda t_o \end{aligned}\right\} \qquad 3, (IV\text{-}8.1)$$

The starting point in evaluating the error is to find the positions of the zeros of $U(0, \lambda)$. The zeros of $U(0, \lambda)$ originating from $\cos \lambda t_o$, see Eq. 3, (IV-8.1), are given by

$$\lambda_n t_o = \frac{\pi}{2} + n\pi; \quad n = 0, 1, 2, \ldots \qquad 4, (IV\text{-}8.1)$$

Hence, the nth lobe is placed between λ_n and λ_{n+1}.

Let us introduce a new variable defined by

$$\lambda t_o = (n+1)\pi + y \qquad 5, (IV\text{-}8.1)$$

See Fig. 2, (IV-8.0). Combining 2, (IV-8.1) and 3, (IV-8.1) with 5, (IV-8.1), we obtain the following expression for the error:

$$\left.\begin{aligned}
\epsilon_n(t) &= \frac{4}{\pi}\int_{\lambda_n}^{\lambda_{n+1}} \cos\left(\lambda t_o \frac{t}{t_o}\right) \cos \lambda t_o \left\{\int_0^{\delta} f(x) \cos\left(\lambda t_o \frac{x}{t_o}\right) dx\right\} d\lambda \\
&= \frac{4}{\pi}\int_{\frac{\pi}{2}+n\pi}^{\frac{\pi}{2}+(n+1)\pi} \cos\left(\lambda t_o \frac{t}{t_o}\right) \cos \lambda t_o \left\{\int_0^{\delta/t_o} f(x) \cos\left(\lambda t_o \frac{x}{t_o}\right) d\left(\frac{x}{t_o}\right)\right\} d(\lambda t_o) \\
&= \frac{4}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\left\{\frac{t}{t_o}\left[(n+1)\pi + y\right]\right\} \cos\left[(n+1)\pi + y\right] \left\{\int_0^{\delta/t_o} f(u) \cos\left[(n+1)\pi + y\right] u\, du\right\} dy
\end{aligned}\right\} \quad 6,(\text{IV-8.1})$$

where

$$u = \frac{x}{t_o}$$

Let us now consider the integral

$$I_1 = \int_0^{\delta/t_o} f(u) \cos\left[(n+1)\pi + y\right] u\, du$$

This integral depends, of course, on the particular shape of the pulse. Since we are only interested in a bound of $\epsilon_n(t)$, it is convenient to evaluate this integral in an approximate way, preferably independent of the actual pulse shape. This can easily be done in the case in which δ/t_o, the ratio of the pulsewidth to its delay, is small. We are particularly interested in such cases. We may assume, then, that f(u) suffers small changes in the interval $0 < u < \delta/t_o$. From the discussion of sections IV-4 and IV-5 we know that we can set $f(u) \approx f(u)_{u=0}$ for the interval 0 to δ/t_o.

$$I_1 \approx f(t_o) \frac{\sin\left[(n+1)\pi + y\right] \frac{\delta}{t_o}}{(n+1)\pi + y} \qquad 7,(\text{IV-8.1})$$

since

$$f(u)_{u=0} = f(u)_{x=0} = f(t_o)$$

The expression 6,(IV-8.1) now takes the form

$$\left.\begin{aligned}\epsilon_n(t) &= \frac{4}{\pi} f(t_o) \int_{-\pi/2}^{+\pi/2} \cos\left\{\frac{t}{t_o}\left[(n+1)\pi + y\right]\right\} \cos\left[(n+1)\pi + y\right] \frac{\sin\left[(n+1)\pi + y\right]\frac{\delta}{t_o}}{(n+1)\pi + y} dy \\ &\approx \frac{4}{\pi} f(t_o) (-1)^{n+1} \int_{-\pi/2}^{+\pi/2} \cos\left\{\frac{t}{t_o}\left[(n+1)\pi + y\right]\right\} \cos y \\ &\qquad \times \frac{\sin\left[(n+1)\pi \frac{\delta}{t_o}\right] + \frac{\delta}{t_o} y \cos\left[(n+1)\pi \frac{\delta}{t_o}\right]}{(n+1)\pi} dy\end{aligned}\right\}$$

8, (IV-8.1)

since δ/t_o is small and $(n+1)\pi \gg y$; $-\pi/2 < y < \pi/2$.

By elementary trigonometric manipulation and the cancellation of terms containing odd integrand functions one gets

$$\epsilon_n(t) \approx \frac{\frac{4}{\pi} f(t_o) (-1)^{n+1}}{(n+1)\pi} \left\{\sin\left[(n+1)\pi \frac{\delta}{t_o}\right] \cos\left[\frac{t}{t_o}(n+1)\pi\right] 2 \int_0^{+\pi/2} \cos\left(\frac{t}{t_o} y\right) \cos y \, dy\right.$$

$$\left. + \cos\left[(n+1)\pi \frac{\delta}{t_o}\right] \sin\left[\frac{t}{t_o}(n+1)\pi\right] \frac{\delta}{t_o} 2 \int_0^{+\pi/2} y \sin\left(\frac{t}{t_o} y\right) \cos y \, dy\right\}$$

From a table of integrals one gets

$$2 \int_0^{\pi/2} \cos\left(\frac{t}{t_o} y\right) \cos y \, dy = \frac{\sin\left(\frac{t}{t_o} - 1\right)\frac{\pi}{2}}{\frac{t}{t_o} - 1} + \frac{\sin\left(\frac{t}{t_o} + 1\right)\frac{\pi}{2}}{\frac{t}{t_o} + 1}$$

$$\begin{aligned}2 \int_0^{\pi/2} y \sin\left(\frac{t}{t_o} y\right) \cos y \, dy &= \frac{\sin\left[\left(\frac{t}{t_o} - 1\right)\frac{\pi}{2}\right] - \left(\frac{t}{t_o} - 1\right)\frac{\pi}{2} \cos\left[\left(\frac{t}{t_o} - 1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o} - 1\right)^2} \\ &\quad + \frac{\sin\left[\left(\frac{t}{t_o} + 1\right)\frac{\pi}{2}\right] - \left(\frac{t}{t_o} + 1\right)\frac{\pi}{2} \cos\left[\left(\frac{t}{t_o} + 1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o} + 1\right)^2}\end{aligned}$$

In order to express the error in terms of the envelope of the oscillation we shall introduce the functions

$$\left.\begin{aligned}
\phi_n(t) &= \sin\left[(n+1)\pi\frac{\delta}{t_o}\right]\left\{\frac{\sin\left[\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}\right]}{\frac{t}{t_o}-1}+\frac{\sin\left[\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}\right]}{\frac{t}{t_o}+1}\right\}\\
\psi_n(t) &= \cos\left[(n+1)\pi\frac{\delta}{t_o}\right]\left\{\frac{\sin\left[\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}\right]-\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}\cos\left[\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o}-1\right)^2}\right.\\
&\qquad\left.+\frac{\sin\left[\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}\right]-\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}\cos\left[\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o}+1\right)^2}\right\}\\
\tan\mu_n &= \frac{\frac{\delta}{t_o}\psi_n(t)}{\phi_n(t)}
\end{aligned}\right\}\quad 9,(\text{IV-8.1})$$

Using these functions we finally get the following expression for the relative error:

$$\left.\begin{aligned}
\frac{\epsilon_n(t)}{f_o(t)} &\approx \frac{4}{\pi}\frac{(-1)^{n+1}}{(n+1)\pi}\left\{\phi_n(t)\cos\left[\frac{t}{t_o}(n+1)\pi\right]-\frac{\delta}{t_o}\psi_n(t)\sin\left[\frac{t}{t_o}(n+1)\pi\right]\right\}\\
&= \frac{4}{\pi}\frac{(-1)^{n+1}}{(n+1)\pi}\sqrt{\phi_n^2(t)+\left(\frac{\delta}{t_o}\right)^2\psi_n^2(t)}\;\cos\left[(n+1)\pi\frac{t}{t_o}+\mu_n\right]
\end{aligned}\right\}\quad 10,(\text{IV-8.1})$$

and

$$\left|\frac{\epsilon_n(t)}{f_o(t)}\right| \approx \frac{4}{\pi}\frac{1}{(n+1)\pi}\sqrt{\phi_n^2(t)+\left(\frac{\delta}{t_o}\right)^2\psi_n^2(t)}\qquad 11,(\text{IV-8.1})$$

IV-8.2 The effect of the suppression of consecutive lobes. Our next concern is to observe the time response of a density distribution function of the shape indicated in Fig. 1,(IV-8.0). A graphical illustration will well serve our main objective.

We are going to remove, in succession, the lobes at $n = 0, 1, 2, \ldots$. In particular, the index values $n = 0, 1, 3, 5, 7, 15$ have been chosen as suggestive examples.

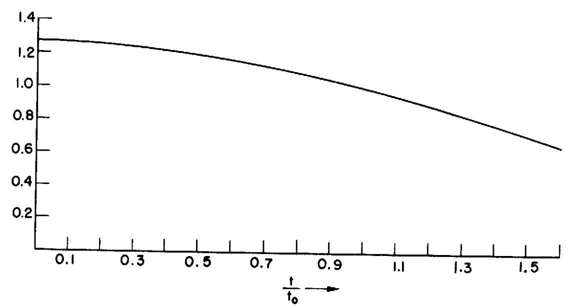

Fig. 1, (IV-8.2)
The graph of Eq. 3, (IV-8.2).

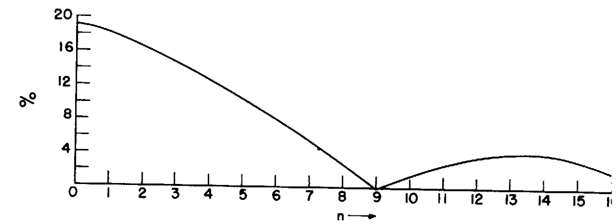

Fig. 2, (IV-8.2)
The graph of Eq. 4, (IV-8.2), $\delta/t_o = 0.1$.

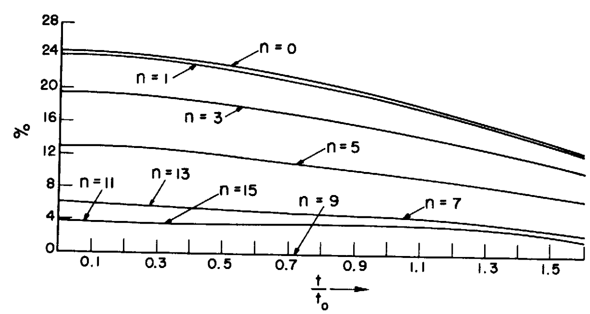

Fig. 3, (IV-8.2)
The graph of Eq. 5, (IV-8.2).

Let us take a pulse of aperture $2\delta = 0.2$, delayed the normalized time interval, $t_o = 1$. Hence

$$\frac{\delta}{t_o} = 0.1 \qquad 1,(IV\text{-}8.2)$$

Eq. 9,(IV-8.1) shows that the functions $\phi_n(t)$ and $\psi_n(t)$ have the same order of magnitude. Consequently, for $\delta/t_o = 0.1$, the term $(\psi_n(t)\,\delta/t_o)^2$ can be neglected. Hence, we can set

$$\frac{\epsilon_n(t)}{f_o(t)} \approx \frac{4}{\pi}\,\frac{(-1)^{n+1}}{(n+1)\pi}\,\phi_n(t)\cos\left[(n+1)\pi\frac{t}{t_o}\right] \qquad 2,(IV\text{-}8.2)$$

Figures 1,(IV-8.2), 2,(IV-8.2) and 3,(IV-8.2) show the graph of the functions

$$\frac{\sin\left[\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o}-1\right)\frac{\pi}{2}} + \frac{\sin\left[\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}\right]}{\left(\frac{t}{t_o}+1\right)\frac{\pi}{2}} \qquad 3,(IV\text{-}8.2)$$

$$2\,\frac{\sin\left[(n+1)\pi\frac{\delta}{t_o}\right]}{(n+1)\pi} \qquad 4,(IV\text{-}8.2)$$

and the envelope

$$\frac{4}{\pi(n+1)}\,\phi_n(t) \qquad 5,(IV\text{-}8.2)$$

for the selected values of n.

Figure 4,(IV-8.2) shows graphically the effect of the removal of successive lobes from the function $U_N(0,\lambda)$. The original pulse, which is represented by dashed lines, serves as a reference. The solid line represents the time response of the syncopated $U_N(0,\lambda)$ function. Analytically, the original pulse has the representation

$$f(t) = \frac{2}{\pi}\int_0^\infty U(0,\lambda)\cos\lambda t\,d\lambda$$

while the response corresponding to the syncopated function corresponds to

$$f(t) - \epsilon_n(t) = \frac{2}{\pi}\int_0^\infty \left[U(0,\lambda) - L(n,\lambda)\right]\cos\lambda t\,d\lambda$$

<u>IV-8.3 The effect of the suppression of one single lobe.</u> The curves of Fig. 4,(IV-8.2) have an important theoretical significance. Their practical value is perhaps rather unimportant, except in their interpretation in connection with the so-called ringing effect. In this subsection we shall concentrate on the theoretical significance of the curves in the problem of rational approximation of a density distribution function $U(0,\lambda)$.

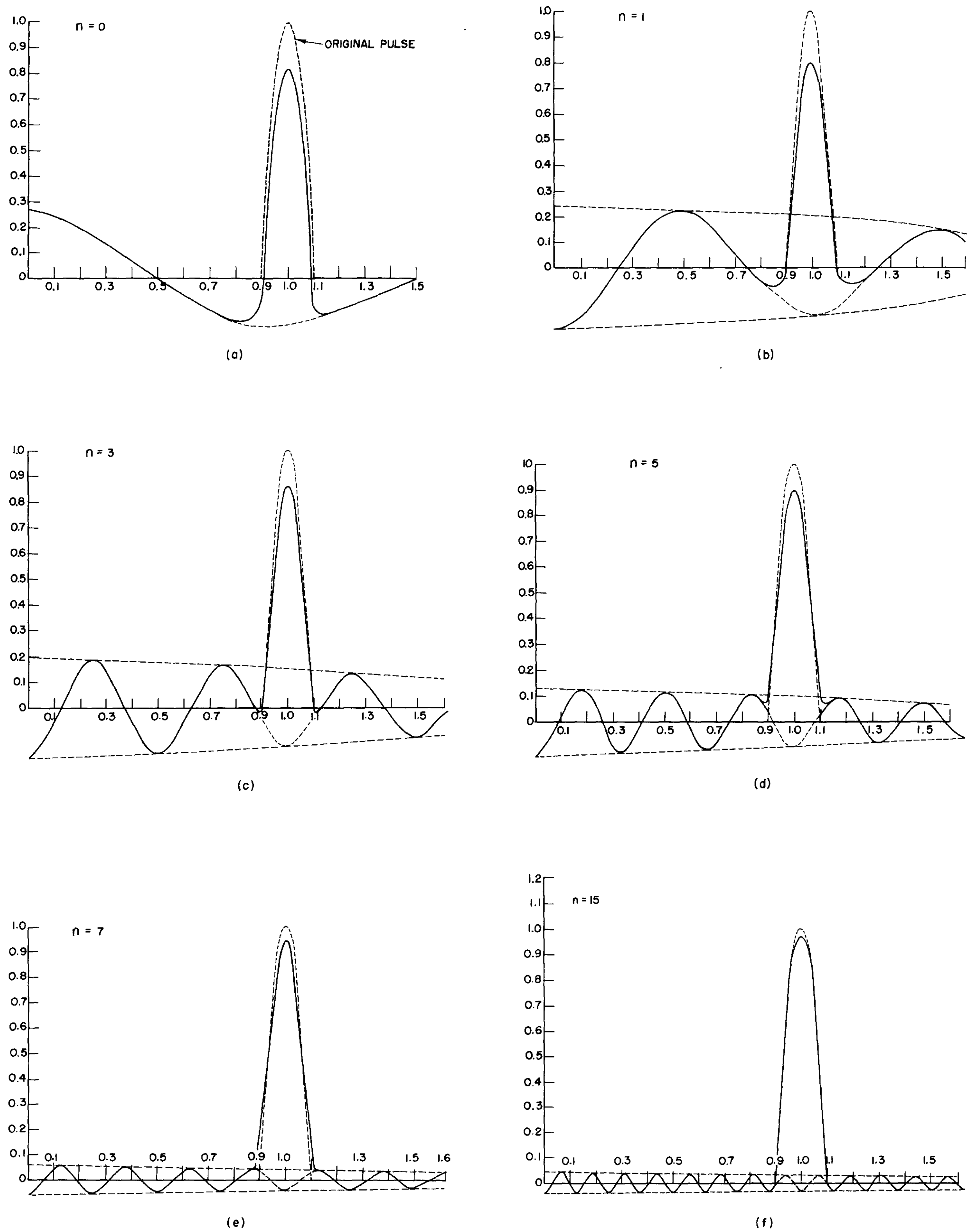

Fig. 4, (IV-8.2)
Effect of the removal of a lobe of $U_N(0, \lambda)$ in the transmission of pulses.

The different graphs in Fig. 4, (IV-8.2) show the strong effect of the "main" lobes of $U(0, \lambda)$ in the transmission of pulses. The omission of one of the lobes produces a strong oscillation of relatively large amplitude. The oscillation appears at the expense of the energy of the pulse, as is quite lucidly shown by the figures. In other words, the removal of a main lobe tends to spread out the pulse for all values of t, particularly in the interval $0 < t < t_o$, which corresponds to the delay of the original pulse.

The strong effect on the tolerance of the suppression of a lobe is self-explanatory from the last set of figures. For, suppose we have selected a tolerance for the transmission of a pulse in the sense indicated in sections IV-1 and IV-6. This tolerance determines a number N, which is equal to the number of zeros in the first group of zeros of the envelope function $a(0, \lambda)$. With N we fix $U_N(0, \lambda)$, which is the part of $U(0, \lambda)$ which must be preserved in order for the response to stay within the prescribed tolerance.

Now, the curves in Fig. 4, (IV-8.2) show clearly that the suppression of the lobe of $U_N(0, \lambda)$ is accompanied by the presence of a strong oscillation, "ringing." The smaller the number n of the suppressed lobe, the greater is the oscillation which appears in the time domain. This oscillation has a large amplitude, which is capable of overpassing the original tolerance lines. Therefore, the solution would be inacceptable as far as the tolerance goes.

The results given above indicate that:

(a) It is basically important to keep the whole $U_N(0, \lambda)$.

(b) The lobes of $U_N(0, \lambda)$ are basically important. (That is why they are sometimes referred to as "main" lobes.)

(c) The rational approximation of $U_N(0, \lambda)$ by $U_{N,R}(0, \lambda)$ must be such that it preserves all points of maxima and minima of $U_N(0, \lambda)$.

IV-8.3, 1 Note on the least number of network elements. Conclusion (c) is of fundamental importance in the question concerning the "least" number of network elements which is required to transmit a given pulse within a prescribed tolerance.

In the first part of section IV-7 we presented heuristic reasoning which shows the connection between the poles of the required transfer function associated with $U_{N,R}(0,\lambda)$ and the number of extremal points of $U_N(0, \lambda)$.

This relation was not, however, established as a one-to-one correspondence. We have only established certain bounds for the required number of poles. We have also shown the possibility of determining a rational approximation of $U_N(0, \lambda)$ based on its set of zeros and its sets of extremal points.

A more accurate estimation of the "minimum" number of required elements for a given tolerance will be given in section IV-9. There we shall bring together the lower and upper bounds discussed in section IV-7; and in doing so, the minimum number will be found. The mathematical procedure followed in section IV-9 is straightforward and does not use the methods and results which were obtained in section IV-7. However, the mathematical ideas and results of section IV-9, although complete by themselves, are difficult to interpret and evaluate without the results of section IV-7.

IV-8.4 Ringing. Introduction. We shall now turn our attention to a problem of practical importance. The results of the previous subsections have thrown a certain light on the understanding of the so-called ringing effect. By "ringing" we shall understand the presence of small or large undesired time oscillations which may appear as part of the response. The ringing tends to be almost ubiquitous in the transmission of impulses and it also appears at sharp discontinuities in the transmitted signal.

We have to distinguish between two main sources of ringing:

(a) It appears as a result of a mathematical phenomenon of convergence which usually is inherent in a certain synthesis procedure. One example is the ringing produced at the points of sharp discontinuity of the time function. It is typical for almost all direct mean-square methods ("Gibbs phenomenon").

(b) It appears as a result of the presence of spurious humps in the function $U_{N,R}(0,\lambda)$.

There may exist, of course, other sources of ringing. In this subsection, IV-8.4, we shall present a heuristic study of ringing belonging to this class (b). Ringing as a convergence phenomenon at sharp discontinuities has been considered in references 7 and 8.

IV-8.4, 1 The difference function $D(0, \lambda)$. For simplicity in the explanation we are going to assume that the function $U(0, \lambda)$ has a slow average behavior. The case of fast oscillations of $U(0, \lambda)$ has already been treated in a previous subsection, where the transient response of a single lobe was found.

Let us assume that a density distribution function $U(0, \lambda)$ is given, as indicated in Fig. 1, (IV-8.4, 1), curve a. In the rational approximation $U_R(0, \lambda)$ of $U(0, \lambda)$, humps may often appear. They correspond to the deviation between $U(0, \lambda)$ and $U_R(0, \lambda)$.

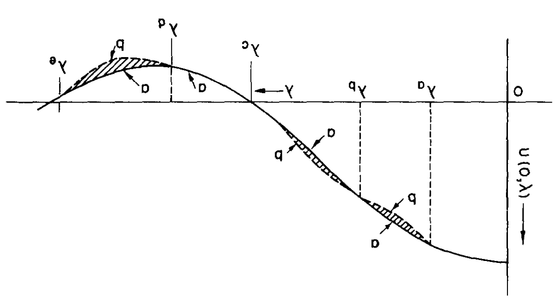

Fig. 1, (IV-8.4, 1)
The appearance of humps in the rational approximation of $U(0,\lambda)$, $U_R(0, \lambda)$.

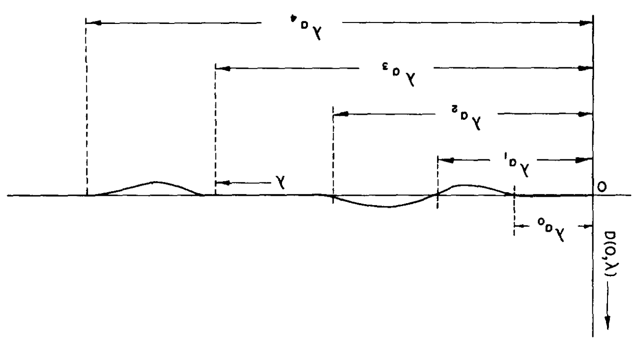

Fig. 2, (IV-8.4, 1)
The difference function $D(0, \lambda)$.

Let us introduce the notation

$$D(0, \lambda) = U(0, \lambda) - U_R(0, \lambda) \qquad 1, (IV\text{-}8.4, 1)$$

This function is indicated in Fig. 2, (IV-8.4, 1). The difference in the time responses obtained by applying a unit impulse to system functions generated by $U(0, \lambda)$ and $U_R(0, \lambda)$ is given by

$$\delta(t) = \sum_{k=0}^{k=n} \delta_{j,\, j+1}(t) \qquad 2, (IV\text{-}8.4, 1)$$

where n is the number of humps and

$$\delta_{j,\, j+1}(t) = \frac{2}{\pi} \int_{\lambda_{a_j}}^{\lambda_{a_{j+1}}} D(0, \lambda) \cos \lambda t \, d\lambda \qquad 3, (IV\text{-}8.4, 1)$$

The following discussion has two objectives:

(a) the approximate evaluation of the integral 3, (IV-8.4, 1);

(b) the formal procedure of network correction to avoid the corresponding ringing.

We start with an approximate evaluation of the integral 3, (IV-8.4, 1). For definiteness we shall assume n = 3, as in Fig. 2, (IV-8.4, 1). Also, it is convenient to combine the integrals as follows:

$$\left.\begin{aligned} \delta_{3,\, 4}(t) &= \int_{\lambda_{a_3}}^{\lambda_{a_4}} D(0, \lambda) \cos \lambda t \, d\lambda \\ \text{and} \qquad & \\ \delta_{0,\, 2}(t) &= \int_{\lambda_{a_0}}^{\lambda_{a_2}} D(0, \lambda) \cos \lambda t \, d\lambda \end{aligned}\right\} \qquad 4, (IV\text{-}8.4, 1)$$

<u>IV-8.4, 2 An approximate solution for the ringing originating from one single hump of $D(0, \lambda)$.</u> Let us consider the first integral in 4, (IV-8.4, 1). An approximate solution will now be given.

Figure 1, (IV-8.4, 2) shows the part of $D(0, \lambda)$, say $D_{3,\, 4}(0, \lambda)$, between λ_{a_3} and λ_{a_4}. The vertical scale has been exaggerated intentionally.

Let us decompose the function $D_{3,\, 4}(0, \lambda)$ into its even and odd parts, denoted respectively by $D_{e_{3,4}}(0,\lambda)$ and $D_{o_{3,4}}(0,\lambda)$. The following notations will be introduced

$$\left.\begin{aligned} \lambda_{a_4} + \lambda_{a_3} &= 2\lambda_{3,\, 4} \\ \lambda_{a_4} - \lambda_{a_3} &= 2\Delta \\ \lambda - \lambda_{3,\, 4} &= x \end{aligned}\right\} \qquad 1, (IV\text{-}8.4, 2)$$

One gets the solution

$$
\left.
\begin{aligned}
\delta_{3,4}(t) &= M_{3,4}(t)\cos\lambda_{3,4}t - N_{3,4}(t)\sin\lambda_{3,4}t \\
&= \sqrt{M^2_{3,4}(t) + N^2_{3,4}(t)}\,\cos\left[\lambda_{3,4}t + \theta_{3,4}(t)\right] \\
\text{where} \qquad & \\
M_{3,4}(t) &= \frac{4}{\pi}\int_0^{\Delta} D_{e_{3,4}}(0,x)\cos xt\,dx \\
N_{3,4}(t) &= \frac{4}{\pi}\int_0^{\Delta} D_{o_{3,4}}(0,x)\sin xt\,dx \\
\tan\theta_{3,4}(t) &= \frac{N(t)}{M(t)}
\end{aligned}
\right\} \qquad 2,(\text{IV-8.4, 2})
$$

The integrals for M(t) and N(t) have analytical structures analogous to the integrals for $\alpha(0,\lambda)$ and $\beta(0,\lambda)$ in sections IV-2 and IV-4, except for the interchange of t and λ and the factor $(2/\pi)$. Hence, the application of the heuristic procedures and methods of section IV-4, enables us to find the values of the envelope components M(t) and N(t). We can, therefore, assume that the solution of the first integral in 4,(IV-8.4, 1) is completed.

Thus the ringing function $\delta_{3,4}(t)$ is the same in shape as the density distribution function $U(0,\lambda)$ which corresponds to a time impulse of equal shape $D_{3,4}(0,\lambda)$, except, of course, for a constant factor and corresponding change of independent variable. When the hump is symmetric with respect to $\lambda_{3,4}$, the function N(t), as well as $\theta(t)$, vanishes. Then the ringing has an instantaneous oscillation of M(t) amplitude and constant frequency.

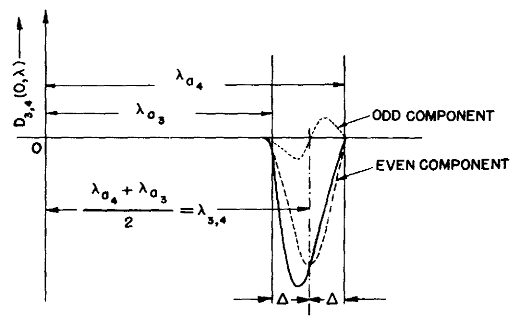

Fig. 1, (IV-8.4, 2)
The function $D_{3,4}(0,\lambda)$ and its even and odd components.

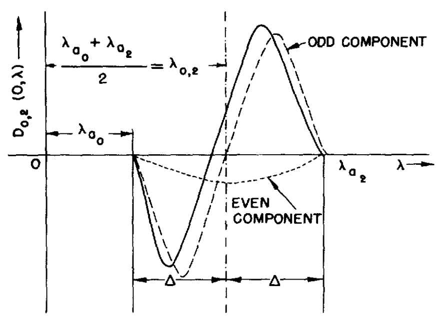

Fig. 1, (IV-8.4, 3)
The function $D_{0,2}(0,\lambda)$ and its even and odd components.

IV-8.4, 3 An approximate solution for the ringing originating from two consecutive humps of $D(0, \lambda)$. Let us now consider the second integral in 4, (IV-8.4, 1). Figure 1,(IV-8.4, 3) shows the part of $D(0,\lambda)$, say $D_{0,2}(0,\lambda)$, between λ_{a_0} and λ_{a_2}. The vertical scale has intentionally been exaggerated. The function $D_{0,2}(0, \lambda)$ contains two humps of opposite signs. The objective of this subsection is to obtain an approximate solution for the corresponding ringing.

Let us decompose the function $D_{0,2}(0, \lambda)$ in its even and odd components, say $D_{e_{0,2}}(0, \lambda)$ and $D_{o_{0,2}}(0, \lambda)$.

The following notations will be introduced

$$\left.\begin{aligned} \lambda_{a_0} + \lambda_{a_2} &= 2\lambda_{0,2} \\ \lambda_{a_2} - \lambda_{a_0} &= 2\Delta \\ \lambda - \lambda_{0,2} &= x \end{aligned}\right\} \qquad 1, (\text{IV-8.4, 3})$$

Once we make the decomposition of the double hump into its odd and even components, the solution of the second integral 4, (IV-8.4, 1) becomes the same as 2, (IV-8.4, 2), except for the notations. Hence

$$\left.\begin{aligned} \delta_{0,2}(t) &= M_{0,2}(t) \cos \lambda_{0,2} t - N_{0,2}(t) \sin \lambda_{0,2} t \\ &= \sqrt{M_{0,2}^2(t) + N_{0,2}^2(t)} \cos \left[\lambda_{0,2} t + \theta_{0,2}(t)\right] \end{aligned}\right\} \qquad 2, (\text{IV-8.4, 3})$$

where the values of $M_{0,2}$, $N_{0,2}$ and $\theta_{0,2}$ are the same as those in 2, (IV-8.4, 2), except for the notations.

Similarly as in the previous subsection, the values of the envelope components can easily be obtained by using the heuristic procedures and methods of section IV-4. Consequently, we have obtained formally the solution of the function 2, (IV-8.4, 3). By the application of the simple rules of section IV-4 we can find the actual shape of the envelope functions $M_{0,2}(t)$ and $N_{0,2}(t)$.

IV-8.4, 4 Remarks. A few remarks will be added here concerning the functions $\delta_{3,4}(t)$ and $\delta_{0,2}(t)$.

The response $\delta_{3,4}(t)$ results from an isolated hump. The odd component is small. Hence, the main effect is produced by the term

$$M_{3,4}(t) \cos \lambda_{3,4} t$$

The function $M_{3,4}(t)$ has a maximum value at $t = 0$. It changes slowly and its behavior can easily be predicted from the heuristic methods of section IV-4.

The response $\delta_{0,2}$ combines in one the effect of two consecutive humps. Since the even component is small for the shape used in Fig. 1, (IV-8.4, 3), the main effect is produced by the term

$$-N_{0,2}(t) \sin \lambda_{0,2} t$$

The function N(t) is zero at t = 0, but attains a maximum between t_o and its first zero. The general behavior of $N_{0,2}(t)$ can easily be predicted from the heuristic methods of section IV-4.

IV-8.5 Ringing suppression. Lattice structures. This last subsection of section IV-8 is concerned with the formal aspect of the problem of the suppression of undesired ringing, which is due to humps in the density distribution function.

The method of correction is based on the additive property of the density distribution function and on its lattice realization.

Let us assume that we have a certain density distribution function which is equal to the algebraic sum of several components. This relationship can be written

$$U(0, \lambda) = \sum_{k=1}^{k=N} U_k(0, \lambda) \qquad 1, (IV\text{-}8.5)$$

We shall show that we can construct a four-terminal lattice structure in which the network elements generated by each component are individually represented.

Consider the kth density distribution function $U_k(0, \lambda)$. Decompose it in its nonnegative and nonpositive components, $U_k^{(+)}(0, \lambda)$ and $U_k^{(-)}(0, \lambda)$, as is indicated in the basic theorems given in references 1 and 9.

The lattice elements corresponding to each index k are given respectively by

$$\left.\begin{aligned} Z_{(1),k}(s) &= \frac{2s}{\pi} \int_0^\infty \frac{d\phi_k^{(+)}(\lambda)}{s^2 + \lambda^2} \\ Z_{(2),k}(s) &= \frac{2s}{\pi} \int_0^\infty \frac{d\phi_k^{(-)}(\lambda)}{s^2 + \lambda^2} \end{aligned}\right\} \qquad 2, (IV\text{-}8.5)$$

where

$$\left.\begin{aligned} \phi_k^{(+)}(\lambda) &= \int_0^\lambda U_k^{(+)}(0, \lambda)\, d\lambda \\ \phi_k^{(-)}(\lambda) &= \int_0^\lambda U_k^{(-)}(0, \lambda)\, d\lambda \end{aligned}\right\} \qquad 3, (IV\text{-}8.5)$$

Due to the additive property of the density distribution function we obtain the structure shown in Fig. 1, (IV-8.5). This lattice structure is equivalent to the one formed with the complete components $U^{(+)}(0, \lambda)$ and $U^{(-)}(0, \lambda)$.

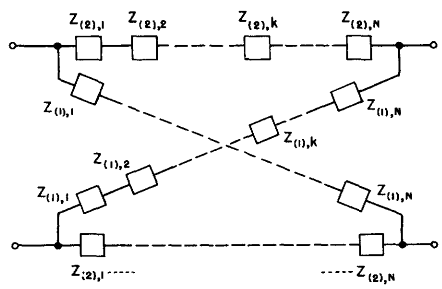

Fig. 1,(IV-8.5)

Lattice structure showing the individual components.

IV-8.5, 1 The procedure for suppressing ringing. The suppression of undesired ringing of the hump type follows immediately from the last results. Let us assume that we want to realize a lattice network corresponding to a given density distribution function, as, for instance, $U(0, \lambda)$, shown by the solid curve in Fig. 1,(IV-8.4, 1).

Suppose that in the process of rational approximation we obtain for $U_R(0,\lambda)$ the dotted curve in the same figure. The humps which produce some spurious ringing are shown. If we want to suppress the ringing we take the difference between the two curves, as shown in Fig. 2,(IV-8.4, 1). By reversing the sign of this last graph and by separating the nonnegative and the nonpositive components we can find the corresponding impedances which will correct the lattice structure associated with the function $U_R(0, \lambda)$.

Rational Approximants to Transfer Functions Obtained by a Nonlinear Process of Summability.

Absolute monotonic components associated with a transfer function. Power series expansion of Heaviside type. Kronecker theorems. Analytic continuation of $F(\sigma)$. Nonlinear summability and Padé tables.

IV-9.0 Introduction. In this section, we shall continue with the discussion of the rational approximation and character of transfer functions, particularly of those associated with the transmission of impulses.

There are three main objectives of this section:

A. The establishment of a set of additional theorems that can serve as criteria for testing whether a transfer function, when it is expressed by its integral representation, is or is not a rational function;

B. The application of these theorems and other results to the rational approximation of a given transfer function;

C. Further elucidation of the question of "minimum" number of elements.

Section IV-9 is, in fact, a continuation of the study of sections IV-2, IV-7, and IV-8. The discussion under objective A is an extension of the material presented in section IV-2. Objectives B and C are extensions of the results of sections IV-7 and IV-8.

IV-9.0, 1 The positive part of the real axis of the s plane as the contour Γ. Consistent with the previous study, we shall again base the future discussion on the density distribution function which characterizes uniquely a given transfer function. As we have previously shown, this density distribution function is equal to the real part of the transfer function taken along a contour line Γ. The contour Γ can be chosen in an arbitrary way in the s plane, except for a set of conditions which have already been given.

In section IV-9 we are interested in density distributionfunctions along the positive part of the real axis of the s plane. This selection has no particular meaning. It is used just to facilitate the derivation of the results; not because of any intrinsic property of this distribution.

IV-9.0, 2 Mathematical tools used. The main mathematical tools that are used in this section are:

(a) introduction of absolute monotonic components associated with a transfer function;

(b) power series expansions of density distribution functions along the positive real axis of the s plane;

(c) introduction of a nonlinear summability that transforms these power series into rational functions. This nonlinear summability contains, as a particular case, Padé table expansions.

IV-9.0, 3 A summary of section IV-9. The main results of the investigation of this section can be summarized as follows:

(a) development of "test" theorems which permit us to find whether a given transfer function, when expressed in integral representations, is or is not a rational function;

(b) methods of "filtering" out the rational part of a transfer function if this rational part exists;

(c) construction of a sequence of rational approximations of given transfer functions;

(d) proof of the existence of certain "supports" for the rational expansions and their implications on the question of the minimum number of elements.

IV-9.1 Complete monotonic components of a transfer function. We have shown in references 1 and 2 that if the function

$$f(t) \begin{cases} = 0 & t < 0 \\ \not\equiv 0 & t > 0 \end{cases}$$

is single-valued and bounded in the interval $0 < t < \infty$, then its Laplace transform $F(s)$ is necessarily a transfer function.

The well-known integral representation of $F(s)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad 1, (IV-9.1)$$

Let us introduce the time density distribution function

$$T(t) = \int_0^t f(\tau)\, d\tau \qquad 2, (IV-9.1)$$

Then $F(s)$ can be expressed by the Stieltjes integral

$$F(s) = \int_0^\infty e^{-st}\, dT(t) \qquad 3, (IV-9.1)$$

as was shown in references 1 and 2.

If we consider the function $F(s)$ along the positive real axis of the s plane, $s = \sigma$, we have

$$F(\sigma) = \int_0^\infty e^{-\sigma t}\, dT(t) \qquad 4, (IV-9.1)$$

Since this integral is real, we have

$$F(\sigma) = \text{Real } F(s)\Big]_{s=\sigma} \qquad 5, (IV-9.1)$$

Consequently, the integral 4, (IV-9.1) produces the density distribution function of F(s) when Γ attains the limiting position along the positive real axis.

IV-9.1, 1 The function f(t). In general, f(t) may take positive and negative values when t varies in the interval $0 < t < \infty$. Let us construct the nonnegative functions

$$\left.\begin{aligned} f^{(+)}(t) &\begin{cases} \equiv f(t) & \text{when } f(t) > 0 \\ \equiv 0 & \text{when } f(t) < 0 \end{cases} \\ f^{(-)}(t) &\begin{cases} \equiv -f(t) & \text{when } f(t) < 0 \\ \equiv 0 & \text{when } f(t) > 0 \end{cases} \end{aligned}\right\} \qquad 1, (IV\text{-}9.1, 1)$$

so that

$$f(t) = f^{(+)}(t) - f^{(-)}(t) \qquad 2, (IV\text{-}9.1, 1)$$

Associated with the above nonnegative functions we can construct the nonnegative, non-decreasing distribution functions of time, defined as

$$\left.\begin{aligned} T^{(+)}(t) &= \int_0^t f^{(+)}(\tau)\, d\tau \\ T^{(-)}(t) &= \int_0^t f^{(-)}(\tau)\, d\tau \end{aligned}\right\} \qquad 3, (IV\text{-}9.1, 1)$$

With the aid of these functions the integral 4, (IV-9.1) can be written

$$F(\sigma) = F^{(+)}(\sigma) - F^{(-)}(\sigma)$$

where

$$\left.\begin{aligned} F^{(+)}(\sigma) &= \int_0^\infty e^{-\sigma t}\, dT^{(+)}(t) \\ F^{(-)}(\sigma) &= \int_0^\infty e^{-\sigma t}\, dT^{(-)}(t) \end{aligned}\right\} \qquad 4, (IV\text{-}9.1, 1)$$

Both functions, $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$, are absolutely monotonic functions, as we shall see in the next subsection.

IV-9.1, 2 Some definitions. The complete monotonic character of $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$ is a result of a basic theorem on the theory of such functions. In the remainder of this section we need only a few properties of absolutely monotonic functions. A detailed discussion of this subject is, therefore, not justified. In what follows, we shall assume that the reader is acquainted, or can become acquainted, with the general properties of these functions. Reference 9 is suggested as a good textbook on the subject.

For convenience, we shall present a few definitions and needed properties. The definition adopted here for completely monotonic sequences is the one that is analogous to the Hausdorff definition of completely monotonic sequences. These sequences will play an important role in the window-function theories of reference 8.

Definition 1, (IV-9.1, 2) A function $G(\sigma)$ is completely monotonic in the interval $0 \leqslant \sigma < \infty$ if it satisfies the condition

$$(-1)^k \, G^{(k)}(\sigma) \geqslant 0$$

where $G^{(k)}(\sigma)$ stands for the k derivative of $G(\sigma)$.

Definition 2, (IV-9.1, 2) A function $H(\sigma)$ is absolutely monotonic in the interval $0 \leqslant \sigma < \infty$ if it has nonnegative derivatives of all orders such that

$$H^{(k)}(\sigma) \geqslant 0, \quad k = 0, 1, 2, \ldots$$

Examples of complete and absolute monotonic functions in the interval $0 \leqslant \sigma < \infty$ are given, respectively, by

$$\left.\begin{aligned} G(\sigma) &= \int_0^\infty e^{-\sigma t} \, d\alpha(t) \\ H(\sigma) &= \int_0^\infty e^{\sigma t} \, d\beta(t) \end{aligned}\right\} \qquad 1, (\text{IV-9.1, 2})$$

where $\alpha(t)$ and $\beta(t)$ are nondecreasing functions in the interval $0 \leqslant t < \infty$.

A simple computation of the derivatives shows that

$$(-1)^k \, G^{(k)}(\sigma) = \int_0^\infty t^k \, e^{-\sigma t} \, d\alpha(t) \geqslant 0$$

$$H^{(k)}(\sigma) = \int_0^\infty t^k \, e^{+\sigma t} \, d\beta(t) \geqslant 0$$

since both integrals are nonnegative by the definitions of $\alpha(t)$ and $\beta(t)$.

Absolutely and completely monotonic functions have certain simple relations. Examples 1, (IV-9.1, 2) illustrate this situation. The function $G(\sigma)$, which is completely monotonic in the interval $0 \leqslant \sigma < \infty$, is absolutely monotonic in the interval $-\infty < x \leqslant 0$. A converse situation exists for $H(\sigma)$.

<u>IV-9.1, 3 A basic theorem on complete monotonic functions.</u> The complete monotonic character of $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$ is a consequence of a basic theorem on such functions. We have the

Theorem 1, (IV-9.1, 3) (Bernstein) "A necessary and sufficient condition that the function $G(\sigma)$ is completely monotonic in the interval $0 \leqslant \sigma < \infty$ is that $G(\sigma)$ admits the integral representation

$$G(\sigma) = \int_0^\infty e^{-\sigma t} \, d\alpha(t)$$

where $\alpha(t)$ is a nondecreasing function in $0 \leqslant t < \infty$, and the integral converges in $0 \leqslant \sigma < \infty$."

The sufficient condition follows at once by the successive derivation of $G(\sigma)$ as performed in the last examples. The proof of the necessary condition is not simple and it is omitted here.

Now the proof that $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$ are completely monotonic sequences follows at once from the fact that both $T^{(+)}(t)$ and $T^{(-)}(t)$ are, by construction, nondecreasing functions. These results lead to the following theorem and corollary of transfer functions.

Theorem 2, (IV-9.1, 3) "Let $F(s)$ be a transfer function. Then $F(\sigma)$ is always expressible as the difference of completely monotonic functions."

Corollary 1, (IV-9.1, 3) "Let $f(t)$ represent a nonnegative pulse (as considered in previous sections) whose associated transfer function is $F(s)$. The $F(\sigma)$ is necessarily a completely monotonic function."

Theorem 2, (IV-9.1, 3) and particularly corollary 1, (IV-9.1, 3) have a significant theoretical importance in a future discussion.

We shall close this subsection by reminding the reader that the transfer function of an impulse of finite area and finite delay is a completely monotonic function. This simply follows from the expression

$$A \, e^{-\sigma t_o}$$

where A is the impulse area, and t_o is its delay.

IV-9.1, 4 A connection between complete monotonic functions and complete monotonic sequences. In the study of impulse transmission we shall make use of a few properties of completely monotonic functions. In this subsection we consider a connection between complete monotonic functions and complete monotonic sequences. We begin with the definition of such sequences. Let

$$\left\{a_n\right\}_0^\infty \quad n = 1, 2, \ldots$$

represent a given sequence of numbers. Let us construct with its elements the differences defined by the well-known expression

$$\Delta^k a_n = \sum_{k=0}^{\infty} (-1)^m \binom{k}{m} a_{(n+k-m)} \qquad 1, (IV-9.1, 4)$$

where $\binom{k}{m}$ stands for the binomial coefficients and $k = 0, 1, 2, \ldots$.

Now we give the following

Definition 1, (IV-9.1, 4) The sequence $\left\{a_n\right\}_0^{\infty}$ is completely monotonic if its elements are nonnegative and its successive differences satisfy the condition

$$(-1)^k \Delta^k a_n \geq 0 \qquad 1', (IV-9.1, 4)$$

(See reference 9, page 108.)

Complete monotonic sequences can always be represented by the finite moment integral

$$a_n = \int_0^1 x^n \, d\mathcal{K}(x) \qquad 2, (IV-9.1, 4)$$

where $\mathcal{K}(x)$ is a nondecreasing bounded function in the interval $0 \leq x \leq 1$. (See reference 9, page 108, theorem 4a.)

Among the complete monotonic sequences we are interested in a particular class, a "minimal" complete monotonic sequence, which is represented by 2, (IV-9.1, 4) with the additional condition that $\mathcal{K}(x)$ is continuous at $x = 0$ (no jumps are allowed at x). (See reference 9, pages 163, 164.)

Now we propose the following construction. Let us find a completely monotonic function $F(\sigma)$ such that at $\sigma = 0, 1, 2, \ldots, n, \ldots,$ $F(n)$ takes the values

$$F(n) = a_n \qquad 3, (IV-9.1, 4)$$

$$\left\{a_n\right\}_0^{\infty} = \text{minimal completely monotonic sequence}$$

We find such a function in the following manner.

Let

$$\left.\begin{aligned} x &= e^{-t} \\ \alpha(t) &= -\mathcal{K}(x) \end{aligned}\right\} \qquad 4, (IV-9.1, 4)$$

Substituting these expressions in 2, (IV-9.1, 4), we get

$$a_n = \int_0^{\infty} e^{-nt} \, d\alpha(t)$$

Hence, the complete monotonic function

$$F(\sigma) = \int_0^\infty e^{-\sigma t}\, d\alpha(t)$$

takes the values of 3, (IV-9.1, 4) at $\sigma = 0, 1, \ldots, n, \ldots$. These results lead to the Theorem 1, (IV-9.1, 4) "Let $\left\{a_n\right\}_0^\infty$ be a minimal completely monotonic sequence of numbers. Then there exists a complete monotonic function $F(\sigma)$ which takes the values

$$F(n) = a_n$$

at $\sigma = 0, 1, 2, \ldots, n, \ldots$."

This theorem provides a method of interpolation of complete monotonic sequences at the points $\sigma = 0, 1, \ldots, n, \ldots$. This property will be used in a subsequent subsection of this report.*

IV-9.1, 5 Hankel's theorem. We shall now discuss a property of completely monotonic functions which plays a basic role in the subsequent investigation. The theorem in question is from Hankel.

The k-th derivative of a completely monotonic function can be written as

$$(-1)^n F^{(n)}(\sigma) = \int_0^\infty t^n e^{-st}\, d\alpha(t)$$

Now let us introduce the set of arbitrary real variables

$$\xi_j; \quad j = 0, 1, 2, \ldots, n \qquad 1, (\text{IV-9.1, 5})$$

With the aid of this variable let us construct the quadratic form

$$Q_1 = \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{(j+k)} F^{(j+k)}(\sigma)\, \xi_j\, \xi_k$$

We shall show that this quadratic form is positive. We can write

$$Q_1 = \int_0^\infty e^{-\sigma t} \left\{ \sum_{j=0}^{n} \sum_{k=0}^{n} \left(t^{j+k}\, \xi_j\, \xi_k \right) \right\} d\alpha(t) = \int_0^\infty e^{-\sigma t} \left\{ \sum_{j=0}^{n} t^j\, \xi_j \left[\sum_{k=0}^{n} t^k\, \xi_k \right] \right\} d\alpha(t)$$

$$= \int_0^\infty e^{-\sigma t} \left\{ \sum_{j=0}^{n} t^j\, \xi_j \right\}^2 d\alpha(t) \geq 0$$

*This property is also used in reference 8, where a complete mathematical discussion can be found.

since

$$\sum_{j=0}^{n} t^j \xi_j = \sum_{k=0}^{n} t^k \xi_k$$

and $\alpha(t)$ is a nonnegative, nondecreasing function.

Similarly, we can show the positive character of the quadratic form

$$Q_2 = \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{(j+k+1)} F^{(j+k+1)}(\sigma)\, \xi_j\, \xi_k = \int_0^{\infty} e^{-\sigma t}\, t \left\{ \sum_{j=0}^{n} \xi_j\, t^j \right\}^2 d\alpha(t) \geq 0$$

It can be noted that the forms Q_1 and Q_2 are positive for every value assigned to n. Then, by a well-known property of the determinant of a quadratic form, it follows that

$$\left.\begin{aligned}
&F(\sigma) \geq 0; \quad \begin{vmatrix} F(\sigma) & (-1)F^{I}(\sigma) \\ (-1)F^{I}(\sigma) & F^{II}(\sigma) \end{vmatrix} \geq 0; \quad \begin{vmatrix} F(\sigma) & -F^{I}(\sigma) & F^{II}(\sigma) \\ -F^{I}(\sigma) & F^{II}(\sigma) & -F^{III}(\sigma) \\ F^{II}(\sigma) & -F^{III}(\sigma) & F^{IV}(\sigma) \end{vmatrix} \geq 0; \ \ldots \\
&-F^{I}(\sigma) \geq 0; \quad \begin{vmatrix} -F^{I}(\sigma) & F^{II}(\sigma) \\ F^{II}(\sigma) & -F^{III}(\sigma) \end{vmatrix} \geq 0; \quad \begin{vmatrix} -F^{I}(\sigma) & F^{II}(\sigma) & -F^{III}(\sigma) \\ F^{II}(\sigma) & -F^{III}(\sigma) & F^{IV}(\sigma) \\ -F^{III}(\sigma) & F^{IV}(\sigma) & -F^{V}(\sigma) \end{vmatrix} \geq 0; \ \ldots
\end{aligned}\right\}$$

2, (IV-9. 1, 5)

These results justify Hankel's theorem.

Theorem 1, (IV-9. 1, 5) "If $F(\sigma)$ is completely monotonic in the interval $0 \leq \sigma < \infty$, then for any $\sigma > 0$ the set of determinants 2, (IV-9. 1, 5) are all nonnegative."

IV-9. 1, 5' The construction of a complete monotonic function which has prescribed derivatives at a certain point. We shall now consider the construction of a completely monotonic function which has prescribed derivatives at a certain point, say $\sigma = \sigma_o$, in the interval $0 \leq \sigma < \infty$.

Let $\left\{c_n\right\}_0^{\infty}$ be a prescribed sequence of numbers, and $F(\sigma)$ be a completely monotonic function such that its successive derivatives take, at $\sigma = \sigma_o$, $0 \leq \sigma_o < \infty$, the values

$$c_n = (-1)^n F^{(n)}(\sigma_o); \quad n = 1, 2, \ldots \qquad 1, (IV-9. 1, 5')$$

In accordance with the Hankel theorem the c's must satisfy the condition

$$\left.\begin{array}{l} c_0 \geqslant 0; \quad \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} \geqslant 0; \quad \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix} \geqslant 0; \ \ldots \\ \\ c_1 \geqslant 0; \quad \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} \geqslant 0; \quad \begin{vmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix} \geqslant 0; \ \ldots \end{array}\right\} \qquad 2, (\text{IV-9.1, 5}')$$

Now, we can proceed to construct the function $F(\sigma)$. We can write

$$F(\sigma) = \int_0^\infty e^{-(\sigma-\sigma_o)t} \left[e^{-\sigma_o t}\, d\alpha(t)\right] = \sum_{k=0}^{\infty} (\sigma - \sigma_o)^k \left\{\frac{(-1)^k}{k!} \int_0^\infty t^k \left[e^{-\sigma_o t}\, d\alpha(t)\right]\right\}$$

$$2', (\text{IV-9.1, 5}')$$

which is obtained by the power series expansion of $\exp\left[-(\sigma - \sigma_o)t\right]$. Expression 2', (IV-9.1, 5') is simply the power series expansion of $F(\sigma)$ around the point $\sigma = \sigma_o$. Hence

$$\left.\begin{array}{l} F(\sigma) = \displaystyle\sum_{k=0}^{\infty} \frac{(\sigma - \sigma_o)^k}{k!} F^{(k)}(\sigma_o) \\ \text{where} \\ F^{(k)}(\sigma_o) = (-1)^k \displaystyle\int_0^\infty t^k \left[e^{-\sigma_o t}\, d\alpha(t)\right] \end{array}\right\} \qquad 3, (\text{IV-9.1, 5}')$$

In accordance with 1, (IV-9.1, 5') we must have

$$\left.\begin{array}{l} c_n = \displaystyle\int_0^\infty t^n \, d\chi(t) \\ \text{where} \\ \chi(t) = \displaystyle\int_0^t e^{-\sigma_o \tau}\, d\alpha(\tau) \end{array}\right\} \qquad 4, (\text{IV-9.1, 5}')$$

with all these integrals taken in the Stieltjes sense.

We must note that the function (distribution) $\chi(t)$ is a positive, nondecreasing function of time, $0 \leqslant t < \infty$. Consequently, expression 4, (IV-9.1, 5') shows that the c_n's must be represented as the moments in the semi-infinite interval (Stieltjes moments) with respect to the nonnegative, nondecreasing, and bounded distribution function $\chi(t)$.

Since the sequence $\left\{c_n\right\}_0^\infty$ is prescribed (or given), we must find the corresponding distribution function $\chi(t)$. If the c_n's satisfy the determinant conditions 2, (IV-9.1, 5') then the Stieltjes moment problem has a solution because of the well-known theorem in the theory of moments, which we repeat for the reader's convenience.

Theorem 1, (IV-9.1, 5') "A necessary and sufficient condition that the set of integrals

$$c_n = \int_0^\infty t^n \, d\chi(t)$$

should have a nonnegative, nondecreasing solution, with infinitely many points of increase, is that the determinants 2, (IV-9.1, 5') should all be nonnegative." (See, for example, reference 9, page 138.)

In the light of these results we can condense all of the above situations under the following

Theorem 2, (IV-9.1, 5') "Let a sequence $\{c_n\}_0^\infty$ of positive numbers be prescribed and such that all of the determinant conditions 2, (IV-9.1, 5') are satisfied, then there exists a completely monotonic function $F(\sigma)$ such that its successive derivatives are given, respectively, by

$$F^{(n)}(\sigma_o) = (-1)^n \, c_n." \qquad 5, (IV\text{-}9.1, 5')$$

Proof. Since the determinant conditions are all satisfied, we can invert the integrals

$$c_n = \int_0^\infty t^n \, d\chi(t)$$

and solve them for $\chi(t)$.

Now, the distribution function $\alpha(t)$ is given by

$$\alpha(t) = \int_0^t e^{\sigma_o \tau} \, d\chi(\tau)$$

and finally

$$F(\sigma) = \int_0^\infty e^{-\sigma t} \, d\alpha(t)$$

IV-9.1, 6 Methods of constructing a function $F(\sigma)$ in the problem of interpolation of a complete monotonic function through a set of numbers which form a complete monotonic sequence. In the two preceding subsections we have considered these problems:

(a) the interpolation of a completely monotonic function through a set of numbers which form a completely monotonic sequence;

(b) the formation of a completely monotonic function whose derivatives attain, at a certain point, prescribed values which are given by a sequence of positive numbers.

Both problems have been discussed in formal terms. The aim of this section is to produce the actual methods of construction of a function $F(\sigma)$ in problem (a), starting from the prescribed sequence $\{a_n\}_0^\infty$ of numbers. The corresponding construction which is required by problem (b) is produced in the next subsection.

The solution of problem (a) is equivalent to the solution of a moment problem in the finite interval, as will be shown in the following procedure.

Let $\{a_n\}_0^\infty$ be a completely monotonic sequence of numbers. By definition, it satisfies condition 1', (IV-9.1, 4). It also admits the integral representation in 2, (IV-9.1, 4)

$$a_n = \int_0^1 x^n \, d\mathcal{K}(x)$$

The procedure of the construction of the function $F(\sigma)$ such that

$$F(n) = a_n$$

involves two main operations. They are described as follows:

I. From $\{a_n\}_0^\infty$ construct the function $\mathcal{K}(x)$.

II. From $\mathcal{K}(x)$ construct $\alpha(t)$.

The function $F(\sigma)$ is then given by

$$F(\sigma) = \int_0^\infty e^{-\sigma t} \, d\alpha(t)$$

We shall now proceed with the successive steps leading to the construction of $\mathcal{K}(x)$.

Step 1. The successive differences of the sequence $\{a_n\}_0^\infty$ are given by Eq. 1,(IV-9.1,4). With the aid of these differences let us construct the auxiliary sequence defined by

$$\Lambda_{n,p} = (-1)^{n-p} \binom{n}{p} \Delta^{(n-p)} a_p \geqslant 0 \qquad 1, \text{(IV-9.1, 6)}$$

The nonnegative character of $\Lambda_{n,p}$ is a consequence of the condition 1', (IV-9.1, 4).

Step 2. Let h represent an auxiliary integer number such that $h > n$, where n is a given number at which we want to have

$$F(n) = a_n$$

The significance of h will become clear in the subsequent discussion.

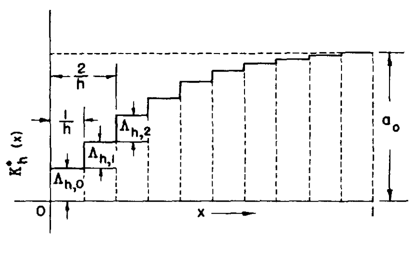

Fig. 1, (IV-9.1, 6)
The graph of the function $\mathcal{K}_h^*(x)$.

Now let us form the nonnegative, nondecreasing, stair-like discontinuous function $\mathcal{K}_h^*(x)$ defined by

$$\mathcal{K}_h^*(x) \left\{ \begin{array}{ll} \equiv 0 & \text{for } x < 0 \\ = \Lambda_{h,0} & \text{for } 0 < x < \frac{1}{h} \\ = \Lambda_{h,0} + \Lambda_{h,1} & \text{for } \frac{1}{h} < x < \frac{2}{h} \\ = \Lambda_{h,0} + \Lambda_{h,1} + \Lambda_{h,2} & \text{for } \frac{2}{h} < x < \frac{3}{h} \\ \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ = \Lambda_{h,0} + \Lambda_{h,1} + \ldots + \Lambda_{h,h} = a_o & \text{for } \frac{h-1}{h} < x < 1 \end{array} \right\} \qquad 2, (IV\text{-}9.1, 6)$$

The support of this function is the interval $0 < x \leqslant 1$, which has been divided into (h+1) parts. The graph of $\mathcal{K}_h^*(x)$ is shown in Fig. 1, (IV-9.1, 6).

Step 3. Now let us suppose that we construct a sequence of functions $\mathcal{K}_h^*(x)$ by giving increasing values to the auxiliary index h.

It can be shown that

$$\lim_{h\to\infty} \mathcal{K}_h^*(x) \to \mathcal{K}^*(x) \text{ uniformly}^* \qquad 3, (IV\text{-}9.1, 6)$$

In other words, the functions $\mathcal{K}_h^*(x)$, for h increasing, constitute a sequence of approximants to the function $\mathcal{K}^*(x)$.

Step 4. Take $\mathcal{K}_h^*(x)$ and construct the following integral expression, when $h > n$

$$\left. \begin{aligned} a_{h,n} &= \int_0^1 x^n \, d\mathcal{K}_h^*(x) \\ &= \sum_{k=0}^{h+1} \left(\frac{k}{h+1}\right)^n \Lambda_{h,k} \end{aligned} \right\} \qquad 4, (IV\text{-}9.1, 6)$$

The sequence of quantities $a_{h,n}$ are also moments with respect to the distribution $\mathcal{K}_h^*(x)$. They have the property

$$\lim_{h\to\infty} a_{h,n} \to a_n \qquad 5, (IV\text{-}9.1, 6)$$

The proof may be found in the last two references mentioned in the footnote.

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*The proof of the limiting value 3, (IV-9.1, 6) is omitted here because it requires a more complete knowledge of the finite moment theory. The insertion of all the required properties would greatly lengthen the presentation of problem (a). The reader is referred to references 9, 8, and 10.

Step 5. The quantities $a_{h,n}$ and a_n differ little when we choose $h \gg n$. Both quantities, $a_{h,n}$ and a_n, are connected by simple linear relations. These relations are as follows.

Let us introduce the double sequence of numbers defined by

$$\alpha_{gj} \left\{ \left. \begin{array}{ll} = (-1)^j \binom{h}{g} \binom{h-g}{j}; & j > g \\ = 1 & j = g \\ = 0 & j < g \end{array} \right\} i, j = 0, 1, 2, \ldots, h \right\} \quad 6, (IV\text{-}9.1, 6)$$

and

$$\beta_{\ell m} \left\{ \begin{array}{lll} = 1 \quad ; & \text{when } \ell = 0; & m = 0, 1, 2, \ldots, h \\ & & \\ = \left(\frac{m}{h}\right)^{\ell}; & \text{when } \ell \neq 0; & m = 0, 1, 2, \ldots, h \end{array} \right\} \quad 7, (IV\text{-}9.1, 6)$$

The linear relationship between the $a_{h,n}$ and a_n is given by the matrix expressions

$$\left. \begin{array}{l} \left(a_{h,g}\right) = \left(\gamma_{gm}\right)\left(a_m\right) \\ \text{where} \\ \left(a_{h,g}\right) = \begin{pmatrix} a_{h,0} \\ a_{h,1} \\ a_{h,2} \\ \vdots \\ a_{h,h} \end{pmatrix}; \quad \left(a_m\right) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \\ \left(\gamma_{gm}\right) = \left(\alpha_{gj}\right)\left(\beta_{jm}\right) \end{array} \right\} \quad 8, (IV\text{-}9.1, 6)$$

The proof of this matrix equation is omitted here. It may be found, for example, in reference 8.

The importance of the linear connection between the quantities $a_{h,n}$ and a_n resides in the fact that we can find a modified density distribution function $K_h(x)$ such that its moments reproduce exactly the quantities $a_0, a_1, \ldots, a_n, \ldots, a_h$.

Let us call

$$\left(\delta_{gm}\right) = \left(\gamma_{gm}\right)^{-1} \quad 9, (IV\text{-}9.1, 6)$$

Hence

$$\left(a_k\right) = \left(\delta_{kj}\right)\left(a_{h,j}\right) \quad 10, (IV\text{-}9.1, 6)$$

By matrix expansion we get, for example,

$$\left.\begin{aligned} a_n &= \sum_{k=0}^{h} \delta_{nk}\, a_{h,j} \\ a_n &= \int_0^1 a_{h,j}\, d\delta_n(x) \end{aligned}\right\} \qquad 11,(\text{IV-9.1, 6})$$

where the function $\delta_n(x)$ is a stair-like discontinuous function defined as

$$\delta_n(x) \left\{\begin{array}{ll} \equiv 0 & \text{when } x < 0 \\ = \sum_{j=0}^{k} \delta_{nj} & \text{when } \frac{k}{h+1} < x < \frac{k+1}{h+1} \end{array}\right\} \qquad 12,(\text{IV-9.1, 6})$$

Finally we can write, when 4,(IV-9.1, 6) is used,

$$a_n = \int_0^1 d\delta_n(x) \int_0^1 x^n\, dK_h^*(x) = \int_0^1 x^n\, d\left\{\int_0^1 \delta_n(x)\, dK_h^*(x)\right\}$$

or

$$a_n = \int_0^1 x^n\, dK_h(x) \qquad 12',(\text{IV-9.1, 6})$$

where

$$K_h(x) = \int_0^1 \delta_n(x)\, dK_h^*(x) \qquad 13,(\text{IV-9.1, 6})$$

Hence, we have arrived at the following

Theorem 1,(IV-9.1, 6) "Given a completely monotonic sequence $\{a_n\}_0^\infty$ we can construct a distribution function $K_h(x)$ such that the integral

$$\int_0^1 x^n\, K_h(x)$$

exactly reproduces the moments $a_0, a_1, \ldots, a_n$."

This result completes the construction of a distribution function $K(x)$, which was required under operation I.

For completeness, two alternate procedures are given in connection with operation I.

First alternate procedure

Let us consider the distribution function 13,(IV-9.1, 6). Since it exactly reproduces the first a_n quantities, then $K_h(x)$ is a nondecreasing function of the interval $0 < x < 1$. The graph of $K_h(x)$ is shown in Fig. 2,(IV-9.1, 6),a. Now, we can take as a distribution

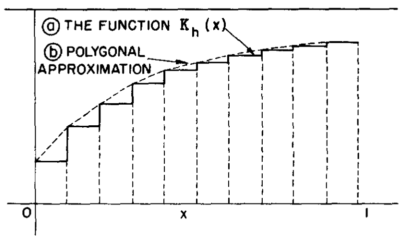


Fig. 2, (IV-9.1, 6)
Polygonal approximation to $\mathcal{K}(x)$.

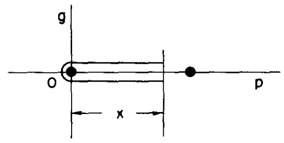


Fig. 3, (IV-9.1, 6)
The contour of integration in 16, (IV-9.1, 6).

function the polygonal curve indicated in graph 2, (IV-9.1, 6),b.

The justification of the polygonal approximation to the function $\mathcal{K}(x)$ rests on the use of corner theorems when

$$F(\sigma) = \int_0^\infty e^{-\sigma t}\, d\alpha(t)$$

$$\alpha(t) = -\mathcal{K}(x)$$

$$x = e^{-t}$$

is to be performed. Corner theorems are discussed in section IV-3 of this report. (Equally well we can use 2^n degree arcs to approximate the graph of $\mathcal{K}(x)$.)

Second alternate procedure

Let us introduce the function defined by

$$g(z) = \int_0^1 \frac{d\mathcal{K}(x)}{z - x}\,; \quad z = p + iq \qquad 14, (IV-9.1, 6)$$

where $\mathcal{K}(x)$ is the same nonnegative, nondecreasing function defined in 4, (IV-9.1, 4).

The integral above is a finite Stieltjes transform; $g(z)$ is an analytic, single-valued function whose support, say D, is the z plane cut along the real axis between the points 0 and 1. This cut represents a singular line of the function $g(z)$.

In a monogenic representation of $g(z)$, this singular line can be interpreted as the location of a pole-zero chain which is everywhere dense at every point of the interval $0 < p < 1$. Let

$$\mathcal{K}(x)_{x=0} = \mathcal{K}(0)$$

and introduce the normalized function, $\hat{\mathcal{K}}(x)$, associated to $\mathcal{K}(x)$. By definition, we have

$$\left.\begin{aligned} \hat{\mathcal{K}}(x) &= \frac{\mathcal{K}(x+) + \mathcal{K}(x-)}{2} \\ \hat{\mathcal{K}}(0) &= \Big[\mathcal{K}(x) - \mathcal{K}(0)\Big]_{x=0} = 0 \end{aligned}\right\} \qquad 15, (IV-9.1, 6)$$

We can now invert the integral 14, (IV-9.1, 6) and solve for $\hat{\mathcal{K}}(x)$. It can be shown that

$$\hat{\mathcal{K}}(x) = \frac{1}{2\pi i} \int_{\substack{\subset\!=\!=\\ 0 \quad x}} g(z)\, dz; \qquad 0 < x < 1 \qquad 16, (IV-9.1, 6)$$

where the contour of integration is taken along the banks of the cut between 0 and 1. See Fig. 3, (IV-9.1, 6).

The proof of this result is omitted here. It can be found in the references given in previous sections. A heuristic proof can be obtained by replacing $\mathcal{K}(x)$ by a stair-like function and observing that $\mathcal{K}(x)$ is equal to the sum of the residues of the poles which form the pole-zero chain between the points 0 and x.

Equation 16, (IV-9.1, 6) expresses the solution of the normalized $\mathcal{K}(x)$ in terms of g(z). The next step is, therefore, to construct the function g(z) starting from the sequence $\{a_n\}_0^\infty$.

By the Taylor expansion of $\{z[1 - (x/z)]\}^{-1}$ one gets

$$\left.\begin{aligned} g(z) &= \int_0^1 \frac{d\mathcal{K}(x)}{z} \sum_{j=0}^{\infty} \frac{x^j}{z^j} \\ &= \frac{1}{z} \sum_{j=0}^{\infty} \frac{1}{z^j} \int_0^1 x^j \, d\mathcal{K}(x) \\ &= \sum_{j=0}^{\infty} \frac{a_j}{z^{j+1}} \end{aligned}\right\} \qquad 17, (IV-9.1, 6)$$

which expresses g(z) asymptotically in terms of the $\{a_n\}_0^\infty$. Finally, by using the last member of 17, (IV-9.1, 6) we can construct a continued fraction whose approximants uniformly approach g(z) in the z plane cut along the real axis, between 0 and 1. The procedures of construction of this continued fraction can be found, for example, in the book by Wall (reference 10).

The normalized function $\hat{\mathcal{K}}(x)$ is equal to $\mathcal{K}(x) - \mathcal{K}(0)$ at every point of continuity of $\mathcal{K}(x)$. At the point of jump, $\hat{\mathcal{K}}(x)$ converges toward the middle point of the jump, as indicated by 15, (IV-9.1, 6). The distribution function will be completely determined by the determination of the points of jump. In the next subsection we shall introduce the jump operators. The construction of g(z) and finally of $\hat{\mathcal{K}}(x)$ ends the second alternate procedure.

Now we are prepared to attack operation II, indicated above. This involves the construction of the distribution function $\alpha(t)$ out of $\mathcal{K}(x)$. This operation does not involve any particular difficulty because it can be attained by a simple change of independent variables. In accordance with 4, (IV-9.1, 4) it is only necessary to set

$$x = e^{-t}$$

and

$$\alpha(t) = -\mathcal{K}(x) = -\mathcal{K}(e^{-t})$$

We can use the approximate solution

$$\alpha_h(t) = -\mathcal{K}_h(e^{-t}) \qquad 18,(\text{IV-9.1,6})$$

and

$$\lim_{h\to\infty} \alpha_h(t) \to \alpha(t) \text{ uniformly}$$

Finally, the required function $F(\sigma)$ can be found as

$$F_h(\sigma) = \int_0^\infty e^{-\sigma t}\, d\alpha_h(t)$$

and, in the limit

$$F(\sigma) = \int_0^\infty e^{-\sigma t}\, d\alpha(t)$$

Consequently, the function $F(\sigma)$ has been constructed in such a way that

$$F(n) = a_n$$

This ends problem (a).

<u>IV-9.1,7 Solution of the problem of constructing a complete monotonic function $F(\sigma)$ which has prescribed derivatives at a certain point.</u> In this subsection we undertake the solution of problem (b). For convenience, we repeat its wording.

"Construct a completely monotonic function $F(\sigma)$ in such a way that at a certain point, say $\sigma = \sigma_0$, its successive derivatives attain the prescribed values given by the sequence $\{c_n\}_0^\infty$ of positive numbers when the sequence $\{c_n\}_0^\infty$ satisfies the determinant conditions 1,(IV-9.1,5')."

The solution of this problem contains two major aspects:

I. The solution of the moment problem in the infinite interval. That is, we must solve the moment integrals for $\chi(t)$

$$c_n = \int_0^\infty t^n\, d\chi(t); \quad n = 0, 1, 2, \ldots$$

II. The construction of the distribution function $\alpha(t)$ given by

$$\alpha(t) = \int_0^t e^{\sigma_0 \tau}\, d\chi(\tau)$$

as we have shown in 4,(IV-9.1,4).

1,(IV-9.1,7)

Aspect I

The existence of a solution for $\chi(t)$ is guaranteed by theorem 1, (IV-9.1, 5'). Let us introduce the function h(z) defined by

$$h(z) = \int_0^\infty \frac{d\chi(t)}{z - t} \qquad 2, (IV\text{-}9.1, 7)$$

which is the Stieltjes transform of $\chi(t)$. The function h(z) is analytic, single-valued in the z plane, $z = p + iq$, cut along the positive real axis, from 0 to $+\infty$. This cut represents a singular line of the function h(z). There is a pole-zero chain everywhere dense along the singular line. The pole-zero chain generates the monogenic function h(z).

Let us denote by $\hat{\chi}(t)$ the normalized function associated with $\chi(t)$. It is defined as

$$\left.\begin{aligned} \hat{\chi}(0) &= \left[\chi(t) - \chi(0)\right]_{t\to+0} = 0 \\ \hat{\chi}(t) &= \frac{\chi(t+) + \chi(t-)}{2} \qquad 0 < t < \infty \end{aligned}\right\} \qquad 3, (IV\text{-}9.1, 7)$$

For the normalized function $\hat{\chi}(t)$ we have

$$\hat{\chi}(t) = \frac{1}{2\pi i} \int_{0 \leftarrow t \rightarrow} h(z)\, dz \qquad 4, (IV\text{-}9.1, 7)$$

where the contour of integration surrounds the banks of the cut in the extension from 0 to t.

We must now construct the function h(z). A Taylor series expansion of the integrand of 2, (IV-9.1, 7) renders

$$h(z) = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_0^\infty t^j \, d\chi(t) = \sum_{j=0}^{\infty} \frac{c_j}{z^{j+1}} \qquad 5, (IV\text{-}9.1, 7)$$

which expresses h(z) as an asymptotic expansion whose coefficients are the elements of $\{c_n\}_0^\infty$.

Since, by hypothesis, the sequence $\{c_n\}_0^\infty$ satisfies both of the determinant sets of conditions 2,(IV-9.1, 5'), then the last series in 5,(IV-9.1, 7) admits a Stieltjes continued fraction representation of the form

$$\left.h(z) = \cfrac{1}{k_1 z - \cfrac{1}{k_2 - \cfrac{1}{k_3 z - \cfrac{1}{k_4 - \ddots}}}}\right\} \qquad 6, (IV\text{-}9.1, 7)$$

(See references 10 and 11.) The coefficients $k_1, k_2, \ldots,$ are functions of the c_n's. These connecting relations can be found, for example, in reference 10, page 202.

We are now in a position to consider an important question in connection with the moment problem in the infinite interval.

The fulfillment of the determinant conditions 2,(IV-9.1,5') guarantees that the moment problem has a solution. The solution may, or may not, be unique. In other words, there may exist two distribution functions, say $\chi(t)$ and $\psi(t)$, of such a character that both produce the same sequence $\left\{c_n\right\}_0^\infty$. That is

$$c_n = \int_0^\infty t^n \, d\chi(t) = \int_0^\infty t^n \, d\psi(t) \qquad 7,(\text{IV-9.1, 7})$$

The continued fraction 6,(IV-9.1, 7) tests whether or not the moment problem is determined. The test is given by the following

Theorem 1,(IV-9.1, 7) "Let $\left\{c_n\right\}_0^\infty$ be a sequence of positive numbers satisfying conditions 2,(IV-9.1, 5'). Then, the Stieltjes moment problem is determinate if, and only if, the positive-term series

$$\sum_{p=0}^{\infty} k_p \qquad 8,(\text{IV-9.1, 7})$$

is divergent. If this series is convergent then the Stieltjes moment problem is indeterminate."

Proofs are omitted here. (See reference 10, page 329, and reference 11, page 410.)

The two following theorems, one from Carleman and the other from M. Riesz (reference 10, page 330, and reference 11, page 416), allow us to test the sequence $\left\{c_n\right\}_0^\infty$ directly, in order to determine whether or not the Stieltjes moment problem is determinate. Both theorems are, for briefness, fused together under the following

Theorem 2,(IV-9.1, 7) "Let $\left\{c_n\right\}_0^\infty$ be as in theorem 1,(IV-9.1, 7). Then, the Stieltjes moment problem is determinate if, and only if, any one of the following conditions is satisfied.

$$\left.\begin{array}{lll} \left.\begin{array}{lll} \text{(a)} & \displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{c_{2k}}\right)^{1/2k} & \text{diverges} \\[2ex] \text{(b)} & \displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{c_k}\right)^{1/2k} & \text{diverges} \end{array}\right\} & \text{Carleman} \\[4ex] \text{(c)} \quad \displaystyle\lim_{k\to\infty} \sqrt[k]{\frac{c_k}{2k}} < \infty." & \text{Riesz} \end{array}\right\} \qquad 9,(\text{IV-9.1, 7})$$

Now, let us denote by

$$h_n(z) = \frac{A_n(z)}{B_n(z)} \qquad 10,(IV\text{-}9.1,7)$$

the approximants of the continued fraction 6, (IV-9.1, 7). By 2, (IV-9.1, 7), the poles of 10, (IV-9.1, 7) are all positive and real. They have positive real residues. We have

$$h_n(z) = \sum_{j=1}^{n} \frac{L_{n,j}}{z - \overset{x}{t}_{n,j}} \qquad 11,(IV\text{-}9.1,7)$$

where $\overset{x}{t}_{n,j}$ is the j-th pole of the n approximant, and $L_{n,j}$ is its residue.

Now consider, first, the determinate case. We have, from 4, (IV-9.1, 7), the series of sequences

$$\left.\begin{aligned} \hat{\chi}_n(t) &= \sum_{j=1}^{n} L_{n,j} \\ \text{and} \qquad \lim_{n\to\infty} \hat{\chi}_n(t) &\to \hat{\chi}(t) \end{aligned}\right\} \qquad 12,(IV\text{-}9.1,7)$$

Consequently, the function $\hat{\chi}(t)$ is uniquely determined.

This result solves aspect I of the problem in the case in which the moment problem associated with 1, (IV-9.1, 7) is determinate.

From the distribution function we have

$$\left.\begin{aligned} \alpha(t) &= \int_0^t e^{\sigma_o \tau}\, d\hat{\chi}(\tau) \\ \text{Hence} \qquad \alpha_n(t) &= \sum_{j=1}^{n} e^{\sigma_o t_{n,j}}\, L_{n,j} \end{aligned}\right\} \qquad 13,(IV\text{-}9.1,7)$$

The substitution of 13, (IV-9.1, 7) produces

$$\left.\begin{aligned} F_n(\sigma) &= \int_0^\infty e^{-\sigma t}\, d\alpha_n(t) = \sum_{j=1}^{n} L_{n,j}\, e^{-(\sigma-\sigma_o)t_{n,j}} \\ \text{and} \qquad F(\sigma) &= \lim_{n\to\infty} F_n(\sigma) \end{aligned}\right\} \qquad 13',(IV\text{-}9.1,7)$$

which is the required solution.

In the indeterminate case, there is more than one solution. Hence, there is more than one function $F(\sigma)$ which satisfies the condition

$$F^{(n)}(\sigma_o) = (-1)^n c_n$$

The normalized function $\hat{\chi}(t)$, Eq. 12,(IV-9.1, 7), is equal to

$$\chi(t) - \chi(0)$$

at the points of continuity of $\chi(t)$. It is possible to find the points of jump of the function $\chi(t)$ by the use of the so-called jump operators. We shall consider the operators $L_{k,t}$ and $\ell_{k,t}$, whose definitions are given in this subsection.

Consider the function h(z), as defined by 2,(IV-9.1, 7), where $z = p + iq$. The operator $L_{k,t}$ is defined as

$$\left.\begin{aligned} L_{k,t}\,[h(p)] &= \frac{(-t)^{k-1}}{k!(k-2)!}\,\frac{d^{2k-1}}{dt^{2k-1}}\left[t^k\ h(t)\right] \quad k = 2, 3, \ldots \\ L_{0,t}\,[h(p)] &= h(t) \\ L_{1,t}\,[h(p)] &= \frac{d}{dt}\,[t\ h(t)] \end{aligned}\right\} \qquad 14,(\text{IV-9.1, 7})$$

The operator $L_{k,t}$ has the property

$$\lim_{k\to\infty} L_{k,t}\,[h(p)] = \frac{\chi(t+) + \chi(t-)}{2} \qquad 15,(\text{IV-9.1, 7})$$

(The proof can be found in reference 9, page 346.) Hence, the operator $L_{k,t}$ produces the distribution function $\chi(t)$ when it is applied to the function $h(z)_{z=p}$. The use of the operator $L_{k,t}$ constitutes, therefore, an alternate method of extracting the distribution function associated with the sequence $\left\{c_n\right\}_0^\infty$.

Now, we shall introduce the jump operator $\ell_{k,t}$. This new operator is defined as

$$\ell_{k,t}\,[h(p)] = 2t\sqrt{\frac{\pi}{k}}\,L_{k,t}\,[h(p)] \qquad 16,(\text{IV-9.1, 7})$$

and it has the property

$$\lim_{k\to\infty} \ell_{k,t}\,[h(p)] = \chi(t+) - \chi(t-), \quad (0<t<\infty) \qquad 17,(\text{IV-9.1, 7})$$

(The proof can be found in reference 9, page 352.) Hence, the operator $\ell_{k,t}$ produces the value of the jump of the distribution function $\chi(t)$ at its points of discontinuity. The operator renders a zero value at the points of continuity.

The use of the operators $L_{k,t}$ and $\ell_{k,t}$ presents great difficulties in practical applications. In the present form, they have only a formal importance in showing the possibility and completeness of the solution of the problem labelled "(b)" in the present subsection.

IV-9.1, 8 An important theorem. We shall close the subsections on completely monotonic functions with the production of a theorem that will be useful in the subsequent discussion.

Let

$$\left.\begin{aligned} F(\sigma) &= \int_0^\infty e^{-\sigma t}\, d\alpha(t) \\ G(\sigma) &= \int_0^\infty e^{-\sigma t}\, d\beta(t) \end{aligned}\right\} . \qquad 1,(IV\text{-}9.1,8)$$

be two completely monotonic functions. This implies that both distributions, $\alpha(t)$ and $\beta(t)$, are nonnegative and nondecreasing functions of time in the interval $0 < t < \infty$. Also, we have

$$\left.\begin{aligned} \alpha(t) &= 0 \\ \beta(t) &= 0 \end{aligned}\right\} \text{ for } t < 0$$

We shall prove the following

Theorem 1,(IV-9.1,8) "Let $F(\sigma)$ and $G(\sigma)$ be two completely monotonic functions in the interval $0 < \sigma < \infty$. Then, the function

$$H(\sigma) = F(\sigma)\, G(\sigma) \qquad 2,(IV\text{-}9.1,8)$$

is also a completely monotonic function."

Proof.

We have to prove that

$$H(\sigma) = \int_0^\infty e^{-\sigma t}\, d\gamma(t)$$

where $\gamma(t)$ is a nonnegative, nondecreasing function of time in $0 < t < \infty$, and $\gamma(t) = 0$ for $t < 0$.

Consider the product

$$H(s) = F(s)\, G(s)$$

which leads to 2,(IV-9.1,8) for $s = \sigma$. Now, if we write

$$f(t) = L^{-1}\, F(s)$$
$$g(t) = L^{-1}\, G(s)$$
$$h(t) = L^{-1}\, H(s)$$

then by the convolution theorem we have

$$h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \geqslant 0$$

since

$$f(t) \geqslant 0$$

$$g(t) \geqslant 0$$

We also have

$$H(s) = \int_0^\infty e^{-st}\, h(t)\, dt$$

Hence

$$\gamma(t) = \int_0^t h(t)\, dt \geqslant 0$$

is a nonnegative, nondecreasing function of time in the interval $0 < t < \infty$. Also $\gamma(t) = 0$ for $t < 0$. Consequently, for $s = \sigma$ we obtain

$$H(\sigma) = \int_0^\infty e^{-\sigma t}\, d\gamma(t)$$

which completes the proof.

IV-9.2 On a power series expansion of transfer functions. In section IV-2 we introduced a theorem concerning the necessary and sufficient condition for a transfer function to be rational. This condition requires that the density distribution function, which generates the transfer function, must itself be a rational function.

This subsection, and most of the remaining discussion of section IV-9, is devoted to finding other tests, which will reveal whether a transfer function, under its integral representation, is, or is not, rational. The tests we are about to develop are intended to be used directly on the time function, which equally defines the transfer function; that is, when F(s) is represented by

$$\left.\begin{aligned} F(s) &= \int_0^\infty e^{-st}\, f(t)\, dt \\ &\text{or} \\ F(s) &= \int_0^\infty e^{-st}\, d\alpha(t) \end{aligned}\right\} \qquad 1, (\text{IV-9.2})$$

rather than the representation given in section IV-2.

The tests presented in the subsequent discussion are very suitable for transfer functions arising from the transmission of impulses.

The tests are of two types. The first type consists of tests that are applied to a certain power series representation of transfer functions. The second can be applied to the power series but its primary use is in connection with sequences extracted from the power series by means of a nonlinear summability process. These last tests are closely associated with theorems that lead to constructive methods for finding the rational approximants of transfer functions in general.

The present subsection, IV-9.2, is primarily devoted to the formulation of tests of the first type. These tests have a rather formal character. Although they are simple, they present certain computational difficulties, at least for the time being. They are, however, of theoretical importance, particularly because of the basic role that they play in the development of tests of the second type.

IV-9.2, 1 Power series expansions of $F(\sigma)$. We gave, in subsection IV-9.1, 1, a theorem that permits the separation of any transfer function $F(s)$ into the difference of two functions $F^{(+)}(s)$ and $F^{(-)}(s)$ which are both completely monotonic for $s = \sigma$, $0 < \sigma < \infty$. By virtue of this theorem, we shall restrict our discussion to one of the components, say $F^{(+)}(s)$, and this is done without loss of generality because $F^{(-)}(s)$ has the same analytic character, and finally because

$$F(\sigma) = F^{(+)}(\sigma) - F^{(-)}(\sigma) \qquad 1, (IV\text{-}9.2, 1)$$

This relation, of course, is identically true for the real parts, since

$$\text{Real } F(s)\Big\}_{s=\sigma} = F(\sigma), \text{ etc.}$$

Consequently, the following theory will be based implicitly on the real part.

From 4, (IV-9.1, 1), we have

$$F^{(+)}(\sigma) = \int_0^\infty e^{-\sigma t}\, dT^{(+)}(t) \qquad 2, (IV\text{-}9.2, 1)$$

For simplicity in the notation we shall drop the (+) sign and use this integral in the form

$$F(\sigma) = \int_0^\infty e^{-\sigma t}\, d\alpha(t) \qquad 3, (IV\text{-}9.2, 1)$$

with the assumption that $\alpha(t)$ is a nonnegative, nondecreasing function of t in the interval $0 < t < \infty$ and $\alpha(t) \equiv 0$, for $t < 0$. We are interested in the power series representation of the integral given above around the point $\sigma = \sigma_0$. Such a power series has already been developed in 2', (IV-9.1, 5') and 3, (IV-9.1, 5').

$$
\left.
\begin{aligned}
F(\sigma) &= \sum_{0}^{\infty} \frac{(\sigma - \sigma_o)^k}{k!} F^{(k)}(\sigma_o) \\
F^{(k)}(\sigma_o) &= (-1)^k \int_0^{\infty} t^k \left[e^{-\sigma_o t} \, d\alpha(t)\right] = (-1)^k c_k \\
c_k &= \int_0^{\infty} t^n \left[e^{-\sigma_o t} \, d\alpha(t)\right] = \int_0^{\infty} t^n \, d\chi(t) \\
\chi(t) &= \int_0^{t} e^{-\sigma_o \tau} \, d\alpha(\tau)
\end{aligned}
\right\}
\qquad 4, (IV\text{-}9.2, 1)
$$

When expansion is made around the point zero, $\sigma = \sigma_o = 0$, we shall use, for convenience, the notation

$$
\left.
\begin{aligned}
F(\sigma) &= \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} F^{(k)}(0) \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{\sigma^k}{k!} \mu_k \\
\mu_k &= \int_0^{t} t^n \, d\alpha(t)
\end{aligned}
\right\}
\qquad 5, (IV\text{-}9.2, 1)
$$

Our aim, now, is to test this power series to see if the functions which it represents are, or are not, rational functions.

IV-9.2, 2 Kronecker's theorem. The starting point of our discussion is a theorem from Kronecker.

Let us write the power series 4, (IV-9.2, 1) under the form

$$
F(\sigma) = \sum_{0}^{\infty} a_n z^n \qquad 1, (IV\text{-}9.2, 2)
$$

where

$$
\sigma - \sigma_o = z
$$

and

$$
\frac{F^{(k)}(\sigma_o)}{k!} = a_k
$$

and form the determinant

$$D_{\lambda,\mu} = \begin{vmatrix} a_\lambda & a_{\lambda+1} & \cdots & a_{\lambda+\mu} \\ a_{\lambda+1} & a_{\lambda+2} & \cdots & a_{\lambda+\mu+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\lambda+\mu} & a_{\lambda+\mu+1} & \cdots & a_{\lambda+2\mu} \end{vmatrix} \quad n = 0, 1, 2, \ldots \qquad 2,(\text{IV-9.2, 2})$$

Theorem 1, (IV-9.2, 2) "The necessary and sufficient condition that the power series 1, (IV-9.2, 2) should represent a rational function is that there exists such an integer number N that for every $n \geqslant N$, say $n = N + \nu$, $\nu = 0, 1, 2, \ldots$, every determinant 2, (IV-9.2, 2) must vanish ($D_{0,n} \equiv 0$)."

The proof of this classical theorem is, for briefness, omitted. The reader interested in its proof may consult, for example, reference 12, chapter X.

Let us consider several simple examples.

I. Take the function

$$\frac{1}{z-p} = \sum_{j=0}^{\infty} \frac{z^j}{p^{j+1}} \; ; \quad a_k = \frac{1}{p^{k+1}}$$

The first determinant is

$$a_0 = \frac{1}{p} \neq 0$$

but every determinant is zero after this. The second is

$$\begin{vmatrix} \frac{1}{p} & \frac{1}{p^2} \\ \frac{1}{p^2} & \frac{1}{p^3} \end{vmatrix} = \frac{1}{p^4} - \frac{1}{p^4} = 0$$

In general, for $n \geqslant 1 = N$

$$\begin{vmatrix} \frac{1}{p} & \frac{1}{p^2} & \frac{1}{p^3} & \cdots & \frac{1}{p^n} \\ \frac{1}{p^2} & \frac{1}{p^3} & \frac{1}{p^4} & \cdots & \frac{1}{p^{n+1}} \\ \frac{1}{p^3} & \frac{1}{p^4} & \frac{1}{p^5} & \cdots & \frac{1}{p^{n+2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{p^n} & \frac{1}{p^{n+1}} & \frac{1}{p^{n+2}} & \cdots & \frac{1}{p^{2n}} \end{vmatrix} = \frac{1}{p}\,\frac{1}{p^2}\,\frac{1}{p^3} \cdots \frac{1}{p^n} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{p} & \frac{1}{p} & \frac{1}{p} & \cdots & \frac{1}{p} \\ \frac{1}{p^2} & \frac{1}{p^2} & \frac{1}{p^2} & \cdots & \frac{1}{p^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{p^{n-1}} & \frac{1}{p^{n-1}} & \frac{1}{p^{n-1}} & \cdots & \frac{1}{p^{n-1}} \end{vmatrix} = 0$$

because the last determinant possesses several equal columns.

II. Now take

$$\frac{3z^2 + 12z + 11}{z^2 + 6s^2 + 11s + 6} = \frac{11}{6} - \frac{49}{36} z + \frac{251}{216} z^2 - \frac{1393}{1296} z^3 + \frac{8051}{7776} z^4$$

$$- \left(1 + \frac{793}{46656}\right) z^5 + \left(1 + \frac{2315}{279936}\right) z^6 \ldots$$

In this example N = 3

$$D_{0,0} = 1.8333 \ldots$$

$$D_{0,1} = 0.2777 \ldots$$

$$D_{0,2} = 0.00051440 \ldots$$

$$D_{0,3} = 0$$

$$D_{0,4} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D_{0,3+\nu} = 0 \quad \nu = 0, 1, 2, \ldots$$

III. Now take

$$\frac{2z + 3}{z^2 + 3z + z} = \frac{3}{2} - \frac{5}{4} z + \frac{9}{8} z^2 - \frac{17}{16} z^3 + \frac{33}{32} z^4 \ldots$$

In this example N = 2

$$D_{0,0} = 1.50$$

$$D_{0,1} = 0.125$$

$$D_{0,2} = 0$$

$$D_{0,2+\nu} = 0 \quad \nu = 0, 1, 2, \ldots$$

IV. The following example is selected for further use. Suppose that a rational function contains simple poles. For simplicity we shall assume that

(a) all poles are positive and $p_1 \neq p_2 \neq p_3 \ldots < 0$

(b) their residues are positive and real.

By partial-fraction expansion we obtain

$$R(z) = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_3}{z - p_3} + \ldots + \frac{A_m}{z - p_m}$$

It is elementary and simple to show that N = m in the above example. To save space we shall produce the proof for m = 3. For any other value of m, the proof is conducted along exactly the same lines. The power series yields

$$a_k = \frac{A_1}{p_1^{k+1}} + \frac{A_2}{p_2^{k+1}} + \frac{A_3}{p_3^{k+1}} \ldots$$

The first determinants are

$$D_{0,0} = \frac{A_1}{p_1} + \frac{A_2}{p_2} + \frac{A_3}{p_3} \neq 0$$

$$D_{0,1} = \begin{vmatrix} \left(\frac{A_1}{p_1} + \frac{A_2}{p_2} + \frac{A_3}{p_3}\right) & \left(\frac{A_1}{p_1^2} + \frac{A_2}{p_2^2} + \frac{A_3}{p_3^2}\right) \\ \left(\frac{A_1}{p_1^2} + \frac{A_2}{p_2^2} + \frac{A_3}{p_3^2}\right) & \left(\frac{A_1}{p_1^3} + \frac{A_2}{p_2^3} + \frac{A_3}{p_3^3}\right) \end{vmatrix}$$

By using a well-known theorem of determinants which allows us to develop $D_{0,1}$ into a sum of determinants whose columns are formed by the combination of columns of each term in the parenthesis, one gets

$$D_{0,1} = \frac{A_1 A_2}{p_1^2 p_2^2}(p_1 - p_2)\,\Delta_{(1,2)} + \frac{A_3 A_1}{p_1^2 p_3^2}\,(p_3 - p_1)\,\Delta_{(1,3)} + \frac{A_2 A_3}{p_2^2 p_3^2}(p_2 - p_3)\,\Delta_{(2,3)}$$

$$\Delta_{(1,2)} = \begin{vmatrix} 1 & 1 \\ \frac{1}{p_1} & \frac{1}{p_2} \end{vmatrix}; \quad \Delta_{(1,3)} = \begin{vmatrix} 1 & 1 \\ \frac{1}{p_1} & \frac{1}{p_3} \end{vmatrix}; \quad \Delta_{(2,3)} = \begin{vmatrix} 1 & 1 \\ \frac{1}{p_1} & \frac{1}{p_2} \end{vmatrix}$$

after the cancellation of the determinants possessing equal columns. The determinant $D_{0,1}$ is not identically zero.

The determinant $D_{0,2}$ can be reduced by following a similar procedure to the form

$$D_{0,2} = \frac{A_1 A_2 A_3}{p_1 p_2 p_3}\,\Delta_{(3)}\left\{\frac{1}{p_1^2 p_2^2}(p_1 - p_2) + \frac{1}{p_2^2 p_3^2}(p_2 - p_3) + \frac{1}{p_3^2 p_1^2}(p_3 - p_1)\right\}$$

$$\Delta_{(3)} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{p_1} & \frac{1}{p_2} & \frac{1}{p_3} \\ \frac{1}{p_1^2} & \frac{1}{p_2^2} & \frac{1}{p_3^2} \end{vmatrix}$$

after the cancellation of determinants possessing two or three identical columns.

The determinant $D_{0,3}$ can be expanded into a sum of determinants as before. After the extraction of the common factor in the columns it will be found that all determinants vanish because they have at least two identical columns. Hence $D_{0,3} = 0$, and the property was shown for $m = 3$. This result shows at once that the theorem is true if the roots are complex, having complex residues, and subjected to the condition that the roots are all different. The procedure for the proof for $m > 3$ is similar to the method used before. It presents the difficulty of involved laborious determinant expansion.

IV-9.2, 3 Hadamard's theorem. We naturally wonder if there is one, or several, tests which allow us to find the number of poles of a rational function represented by a power series when the series satisfies the condition $D_{0,N+\nu} \equiv 0$.

The last example of the previous subsection shows that the number N is equal to the number of poles in a particular case of rational functions. We are interested in the determination of these poles in a general case.

In the classical theory of power series there are several theorems tending to determine this number of poles. Unfortunately such tests prove to be somewhat cumbersome in their practical application to the theory of network synthesis, which is implicitly the objective of this discussion. Among these classical tests there is one, from Hadamard, which possesses some theoretical importance in our research. We reproduce, without proof, this theorem. (See reference 12, chapter X, page 333.)

We shall first introduce some required notations. Let r be the radius of convergence of the power series 1, (IV-9.2, 2). It is given by the well-known expression

$$\frac{1}{r} = \overline{\lim} \left| \sqrt[n]{a_n} \right| \qquad 1,\text{(IV-9.2, 3)}$$

We introduce the expression

$$\left.\begin{aligned} \ell_o &\equiv \frac{1}{r} \\ \ell_p &\equiv \overline{\lim} \left| \sqrt[n]{D_{n,p}} \right| \end{aligned}\right\} \qquad 2,\text{(IV-9.2, 3)}$$

We now have the classical theorem which yields the number of singularities on the circle of convergence.

Theorem 1, (IV-9.2, 3) (Hadamard). "The necessary and sufficient condition that the power series 1, (IV-9.2, 2) should have at most p poles, counted in accordance with their multiplicity, and no other singularities on the circumference of its circle of convergence, is that $\ell_i = 1/r^{i+1}$ for $i = 0, 1, \ldots, p-1$, and $\ell_p < 1/r^{p+1}$." (See reference 12, page 333, from which the theorem was taken.)

Since this test is hard to apply in the theoretical or in the numerical evaluation of the quantities ℓ_p we do not produce the proof of the theorem. In a very few cases it renders the required information. In example I of the last subsection one actually finds $p = 1$, as it should be. Even in this simple case, the application of the test requires unjustified time.

Concerning the total number of poles of a rational function which is represented by the power series when $D_{0,n} \equiv 0$, $n > N$, we have the

Theorem 2, (IV-9.2, 3) "The necessary and sufficient condition that the power series 1, (IV-9.2, 2) should represent a rational function is that there be a number such that

$$\sum D_{\lambda,q}\, z^{\lambda} = \Pi(z)$$

is a polynomial. Then the least value of q is the degree of the denominator, and the degree of $\Pi(z)$ is not less than $m - q$, where m is the degree of the numerator." (See reference 12, page 333, from which the theorem was taken.)

It is equally difficult to apply this test.

IV-9.2,4 A remark concerning the different determinants. The test created by the theorems of the last subsection will formally indicate whether a given power series is, or is not, a rational function. From the network theory point of view, the knowledge expressed above is not enough because we need additional information on the realizability of the transfer function associated with such series.

The corresponding test for this purpose has already been found. The test requires that the determinant conditions 2,(IV-9.1,5') shall also be satisfied if the rational function is completely monotonic.

The Kronecker determinants 2,(IV-9.2,2) are not the same as the determinants in 2,(IV-9.1,5'). We shall illustrate this difference. Take example I, subsection IV-9.2,2. The successive determinants 2,(IV-9.1,5') are given by

$$c_0 = \frac{1}{a} \geqslant 0; \qquad \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & \frac{1}{a^2} \\ \frac{1}{a^2} & \frac{2!}{a^3} \end{vmatrix} = \frac{1}{a^4} \neq 0; \text{ etc.}$$

$$\begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \frac{1}{a^2} & \frac{2!}{a^3} \\ \frac{2!}{a^3} & \frac{3!}{a^4} \end{vmatrix} = \frac{2}{a^6} > 0; \text{ etc.}$$

Note that these determinants do not necessarily vanish for $n > 1$, and that the Kronecker determinants do vanish.

IV-9.2,5 Some remarks concerning the tests performed to find out whether or not a function is rational. The evaluation of the Kronecker determinants $D_{\lambda,\mu}$ and of those in 2,(IV-9.1,5') is hard to perform, so that the use of these tests is completely out of the question. The situation is sad because after a strenuous effort to evaluate these determinants we have accomplished only a fraction of the task of constructing the corresponding rational function.

For quite some time an investigation was directed toward the problem of putting this test into practical terms. Results were negative. Perhaps others may succeed.

We contemplated, for example, studying the possiblity of the rational approximation of transcendental functions by observing the decay in value of the $|D_{0,n}|$ determinants, considering that when the magnitude of these determinants is smaller than a certain specified value, ϵ, then we can form a rational approximation by using the set of non-vanishing determinants and assigning a zero value to those whose magnitude is smaller than ϵ. The main difficulty was that in some cases the magnitude of the determinants goes down so fast that our criterion was meaningless.

We completely abandoned the tests of this subsection when we found other methods of direct construction of the rational function, or of the rational approximants, associated with a given power series.

Integral tests and nonlinear summability methods render quite workable results which are suitable for the synthesis-of-networks point of view. Consequently, we have the impression that power series are incidental and transitory steps in the problem of network synthesis. In other words, power series do not contain explicitly any relevant constructive information from the practical point of view; at least, not for the time being.* More effective methods are discussed in subsection IV-9.4 of this report.

IV-9.3 Analytic continuation of $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$. This subsection deals with a simple but fundamental property of the functions $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$, Eq. 4, (IV-9.1, 1). We consider here the analytic continuation of an element of F(s) which is known only along the positive real axis of the s plane, into a transfer function F(s). This continuation is made over a domain of the s plane which at least covers the open half of the s plane defined by $0 < +\sigma < \infty$.

The main products of this investigation are two theorems. The first theorem states that a transfer function F(s) is completely generated by the knowledge of its real part along the axis $0 < \sigma < \infty$. The second theorem postulates a necessary and sufficient condition for a transfer function to be rational.

IV-9.3, 1 Two theorems on analytic continuation. We propose to prove the

Theorem 1, (IV-9.3, 1) "Let F(s) be a transfer function. Suppose that we know that the real function $U(\sigma)$ satisfies the condition

$$U(\sigma) = \text{Real}\ \Big[F(s)\Big]_{s=+\sigma}, \quad 0 < \sigma < \infty$$

but that no other information of F(s) is known. Now if we set s instead of σ in $U(\sigma)$, we have

$$U(s) = F(s)$$

at least in the open half of the s plane defined by $\sigma > 0$."

Proof.

Since $U(\sigma) = \text{Real}\ \Big[F(s)\Big]_{s=\sigma} = F(\sigma)$; $0 < \sigma < \infty$, then in accordance with theorem 4, (IV-9.1, 1) $U(\sigma)$ is always expressible as the difference of two completely monotonic functions $F^{(+)}(\sigma)$ and $F^{(-)}(\sigma)$. Hence we have

*Power series of the type discussed in this section were first considered by Heaviside. Some of his celebrated theorems are based on them. This type of series has been reconsidered from time to time by several other authors, particularly by R. C. Spencer and W. H. Huggins.

$$U(\sigma) = \int_0^\infty e^{-\sigma t}\, d\left[T^{(+)}(t) - T^{(-)}(t)\right] = \int_0^\infty e^{-\sigma t}\, dT(t) \qquad 1,(IV-9.3,1)$$

By substituting $\sigma = s$ we get

$$U(s) = \int_0^\infty e^{-st}\, dT(t) \qquad 2,(IV-9.3,1)$$

Now, because of the boundedness of T(t) the integral converges uniformly, at least in the half plane defined by $\sigma > 0$. Consequently we have

$$U(s) = F(s) \qquad 3,(IV-9.3,1)$$

as stated by the theorem.

Theorem 2,(IV-9.3,1) "The power series expansion of the function $F^{(+)}(\sigma)$ can be analytically continued into the s plane and defines $F^{(+)}(s)$, which is analytic in the open half of the s plane defined by $\sigma > 0$. A similar situation exists for $F^{(-)}(\sigma)$."

The theorem follows because of the uniform convergence of the functions 4,(IV-9.1,1) in the open half of the plane defined by $\sigma > 0$.

IV-9.3,2 Another theorem. We propose to prove the

Theorem 1,(IV-9.3,2) "The necessary and sufficient condition for a transfer function F(s) to be rational is that its real part along the positive real axis should be rational."

The proof of the necessary condition is trivial. The sufficient condition is far from trivial. This last condition follows from theorems 1,(IV-9.3,1) and 2,(IV-9.3,1) and from the fact that the analytic continuation of a rational function necessarily represents a rational function in the domain of analyticity.

IV-9.4 Power series associated with impulses, pulses, and window functions. The present subsection is devoted to the discussion of a few properties of power series associated with impulses, pulses, and window functions. There are two main objectives of the discussion: (a) the establishment of basic properties of the coefficients of this power series expansion; (b) the extraction of certain transcendental components of F(s) from a power series associated with delayed pulses and window functions.

IV-9.4,1 Definitions of window function and pulse. To keep the discussion under precise terms we recall the accepted definitions of window functions and pulses. (See section IV-1.)

By a pulse we understand a bounded, nonnegative time function, say $f_p(t)$, whose graph shows a predominant hump-like shape of relatively short duration, and has the properties

$$f_p(t)\begin{cases}\equiv 0 & \text{for } t < t_o = \text{constant} \geqslant 0\\[1em] \geqslant 0 \text{ (or always negative)} & \text{for } t_o < t < \infty\end{cases}$$

$$0 < \int_0^\infty f_p(t)\,dt < \infty \qquad 1,\text{(IV-9.4, 1)}$$

By a window function we understand a bounded, nonnegative time function $f_w(t)$, whose graph is composed of a single lobe of relatively short duration, and has the properties

$$f_w(t)\begin{cases}\equiv 0 & \text{for } t < t_o = \text{constant} \geqslant 0\\ \geqslant 0 \text{ (or always negative)} & \text{for } t_o < t < t_1 = \text{constant} < \infty\\ \equiv 0 & \text{for } t_1 < t < \infty\end{cases}$$

$$0 < \int_0^\infty f_w(t)\,dt < \infty \qquad 2,\text{(IV-9.4, 1)}$$

Pulses and window functions may sometimes show small graph differences, mainly in the presence of a tail in pulses. Analytically speaking, pulses and window functions may possess basic mathematical differences in spite of this possible graph resemblance. For example, we have seen in section IV-6 that the transfer function associated with a window function is necessarily transcendental, but that the transfer function associated with a pulse may have a rational character. These basic differences between window functions and the subclass of pulses, whose transfer function is rational, have a fundamental importance in the remainder of this report.

IV-9.4, 2 A property valid for an impulse, a pulse, or a window function. The following property is true for an impulse, a pulse, or a window function.

Theorem 1, (IV-9.4, 2) "Let $F_i(s)$, $F_p(s)$, $F_w(s)$ be, respectively, the transfer function associated with an impulse, a pulse, or a window function. Then $F_i(\sigma)$, $F_p(\sigma)$, $F_w(\sigma)$; $s = +\sigma$ are all completely monotonic functions in the interval $0 < \sigma < \infty$."

This property follows at once from theorem 1, (IV-9.1, 3). Theorem 1, (IV-9.4, 2) can be stated in terms of the real parts of the transfer function along the real axis, since

$$\text{Real}\left[F_{i,p,w}(s)\right]_{s=\sigma} = F_{i,p,w}(\sigma)$$

IV-9.4, 3 A few properties of the coefficients of a power series associated with an impulse, a pulse, or a window function. To reduce wording we adopt in this subsection, IV-9.4, 3, only the simplified notations. We shall drop indices i, p, w. We shall designate the impulse, or the pulse, or the window function by f(t) and the corresponding transfer function by F(s).

For convenience, we repeat here the notation of the power series expansion of $F(\sigma)$ around $\sigma = \sigma_o > 0$. See Eq. 3, (IV-9.1, 5').

$$\left.\begin{aligned} F(\sigma) &= \sum_{k=0}^{\infty} (\sigma - \sigma_o)^k \frac{(-1)^k}{k!} c_k \\ c_k &= \int_0^{\infty} t^k \left[e^{-\sigma_o t} \, d\alpha(t)\right] = \int_0^{\infty} t^k \, d\chi(t) \end{aligned}\right\} \qquad 1, (\text{IV-9.4, 3})$$

The following theorem postulates the existence, boundedness, and positive character of the coefficients c_k.

Theorem 1, (IV-9.4, 3) "Let $f(t)$ be the time function representing a delayed impulse, pulse, or window function. Then, the coefficients c_k exist uniquely, are bounded, and $c_k > 0$ for $k = 0, 1, 2, \ldots$."

The proof follows easily from the preceding theorems.

Corollary 1, (IV-9.4, 3) "The transfer function $F(s)$ associated with an impulse, a pulse, or a window function can never be a (finite) polynomial."

This follows because $c_k > 0$, $k = 0, 1, 2, \ldots$ and by the principle of analytical continuation.*

Theorem 2, (IV-9.4, 3) "The coefficients c_k in 1, (IV-9.4, 3) satisfy the determinant conditions 2, (IV-9.1, 5')."

This property becomes obvious from a preceding theorem.

Theorem 3, (IV-9.4, 3) "Let $c_k(\sigma_o)$ and $c_k(\sigma_o')$; $\sigma_o' > \sigma_o$ be, respectively, the coefficients of the power series 1, (IV-9.3, 2) when the expansion is made around the points σ_o' and σ_o. Then

$$c_k(\sigma_o') < c_k(\sigma_o) < c_k(0), \quad k = 1, 2, \ldots \text{ ."}$$

The proof of the theorem is simple.

Theorem 4, (IV-9.4, 3) "The c_k coefficients associated with an impulse of area equal to A and delayed t_o units of time are given by

$$\left.\begin{aligned} c_k(\sigma_o, t_o) &= A\, t_o^k\, e^{-\sigma_o t_o} \\ c_k(\sigma_o, 0) &= A\, e^{-\sigma_o} = \text{constant} \end{aligned}\right\}\text{"} \qquad 2, (\text{IV-9.4, 3})$$

The proof is elementary.

*It is, therefore, a fundamental mistake to try to approximate a transfer function $F(\sigma)$ by cutting the series at a finite number of terms. Electrically speaking, this syncopation invites one or several oscillations in the approximate time response. We shall consider this situation later on.

The following theorem concerns the absolutely monotonic character of the coefficients c_k for undelayed impulses or window functions.

Theorem 5, (IV-9.4, 3) "The coefficients c_k associated with a delayed window function of duration a (aperture) units of time are linearly expressed in terms of an absolutely monotonic sequence $\{\mu_n\}_0^\infty$."

Proof.

Take a window function of duration a and delayed t_o units of time. We can write

$$c_n = \int_0^\infty t^k \left\{e^{-\sigma_o t}\, d\alpha(t)\right\} = \int_{t_o}^{t_o+a} t^k \left\{e^{-\sigma_o t}\, d\alpha(t)\right\}$$

$$= a^{k+1} e^{-\sigma_o t_o} \int_0^1 \left(\tau + \frac{t_o}{a}\right)^k e^{-\sigma_o a\tau}\, d\alpha(\tau)$$

$$= a^{k+1} e^{-\sigma_o t_o} \sum_{j=1}^{k} \binom{j}{k} \left(\frac{t_o}{a}\right)^{k-j} \mu_j$$

where

$$\mu_j = \int_0^1 \tau^j \left\{e^{-\sigma_o a\tau}\, d\alpha(\tau)\right\} = \int_0^1 \tau^j\, d\chi(\tau) \qquad 3, (IV\text{-}9.4, 3)$$

The sequence $\{\mu_j\}_0^\infty$ is absolutely monotonic because of a well-known theorem of the theory of the finite interval moment. (See references 9 and 8.)

The following theorem refers to the relationship between the coefficients $c_n(\sigma_o)$ and $c_n(0)$ which correspond, respectively, to the power series expansions around the points $\sigma = \sigma_o$ and $\sigma = 0$.

IV-9.4, 4 On the power series expansion of delayed window functions. Several examples will be used to illustrate the power series expansion of a delayed window function. For simplicity in the computations we shall consider

Theorem 1, (IV-9.4, 4) "Let $c_n(\sigma_o)$ and $c_n(0)$ be, respectively, the coefficients of the power series expansions of $F(\sigma)$ around the points $\sigma = \sigma_o$ and $\sigma = 0$. These coefficients are related by the expression

$$c_n(\sigma_o) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sigma_o^k\, c_{n+k}(0)." \qquad 1, (IV\text{-}9.4, 4)$$

Proof.

$$c_n = \int_0^\infty t^n e^{-\sigma_o t}\, d\alpha(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sigma_o^k \int_0^\infty t^{n+k}\, d\alpha(t)$$

from which 1, (IV-9.4, 4) follows.

The following theorem is concerned with the relationship between the coefficients of the power series expansion of a delayed pulse or window function and the undelayed pulse or window function, when both expansions are made around $\sigma = \sigma_o$.

Theorem 2, (IV-9.4, 4) "Let $c_n(\sigma_o, t_o)$ and $c_n(\sigma_o, 0)$ be, respectively, the coefficients of the power series of a delayed pulse or window function and of the same undelayed pulse or window function. Then we have

$$c_n(\sigma_o, t_o) = \sum_{j=1}^{n} \binom{n}{j} c_j(\sigma_o, 0) \qquad 2, (IV-9.4, 4)$$

and conversely, we have

$$\left.\begin{aligned} c_n(\sigma_o, 0) &= \sum_{k=1}^{n} (-1)^k \binom{n}{n-k} c_{n-k}(\sigma_o, t_o) \\ &= \sum_{k=1}^{n} (-1)^k \binom{n}{k} c_{n-k}(\sigma_o, t_o) \end{aligned}\right\} \qquad 3, (IV-9.4, 4)$$

and $c_n(\sigma_o, 0) > 0$."

Proof.

Let F(s) be the transfer function of the undelayed pulse or window function and G(s) be the transfer function corresponding to the delayed pulse or window function. We have

$$G(s) = e^{-st_o} F(s)$$

Because of the absolute convergence of the integral representing each factor in the half s plane, then we can multiply the power series representation of each factor in the right-hand side to obtain the power series representation of the left member. By doing that and by equating the coefficients in both sides we obtain 2, (IV-9.4, 4).

Now 3, (IV-9.4, 4) follows because the matrix whose elements are $\binom{n}{j}$ is not singular. Its inverse matrix has the elements $(-1)^k \binom{n}{n-k}$.

Finally, the expression $c_n(\sigma_o, 0) > 0$ follows because the undelayed pulse is a non-negative function.

Expressions 2, (IV-9.4, 4) and 3, (IV-9.4, 4) have a fundamental formal importance in the theory of pulse transmission and in the theory of window functions presented in Technical Report No. 270. In subsection IV-9.5 we illustrate the significance and importance of expression 2, (IV-9.4, 4).

The discussion is now conducted toward the investigation of a few asymptotic properties of the coefficients c_n when n attains very large values. First consider window functions. Due to the results of the last theorem we shall investigate the asymptotic behavior of undelayed window functions.

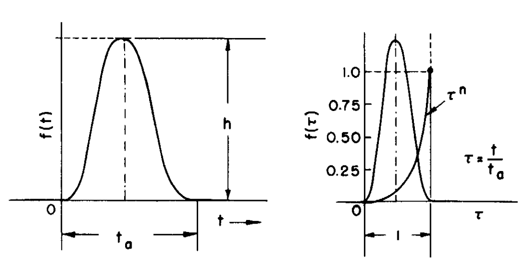


Fig. 1, (IV-9.4, 4)
The graph of $f(\tau)$ and τ^n, n large.

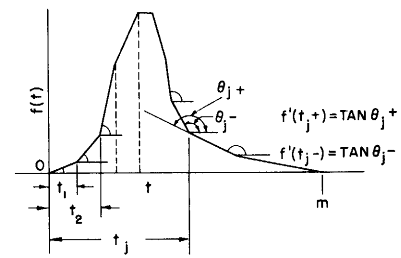


Fig. 2, (IV-9.4, 4)
Polynomial frame of a pulse and the notations used in the corresponding corner theorem.

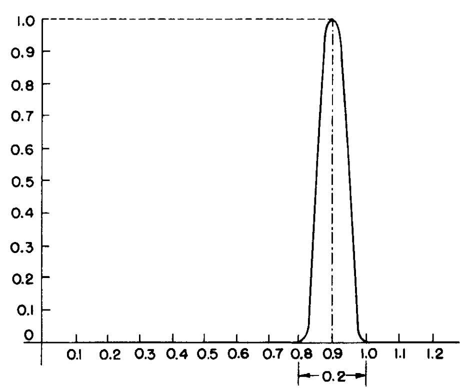


Fig. 1, (IV-9.4, 5)
Delayed complete cosine window function.

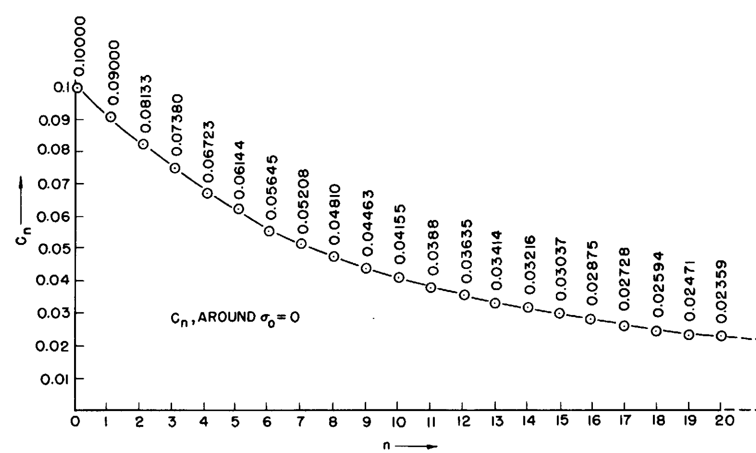


Fig. 2, (IV-9.4, 5)
Graph of the coefficients c_n for the window function 1, (IV-9.4, 5).

Let t_a be the aperture of the window function. See Fig. 1, (IV-9.4, 4). Let us write

$$\tau = \frac{t}{t_a} \qquad 4, (IV-9.4, 4)$$

We obtain

$$c_n = t_a^{n+1} \int_0^1 \tau^n e^{-\sigma_o t_a \tau} d\alpha(\tau) \qquad 5, (IV-9.4, 4)$$

For large values of τ, the function τ^n remains very small in most of the interval 0, 1 but it starts increasing rapidly in the left neighborhood of point one. This property allows us to compute the coefficients c_n for a specific case. For the values of n corresponding to the graph of Fig. 1, (IV-9.4, 4) we can approximate the lagging side of the window function by the tangent. Hence

$$f(\tau) \approx \left|f'(\tau_m)\right| (1-\tau), \quad \tau_o < \tau < 1$$

when $f'(\tau)$ is equal to the maximum tangent of the lagging edge. The value τ_o is taken as

$$\tau_o = 1 - \frac{h}{\left|f'(\tau_m)\right|} \qquad 6, (IV-9.4, 4)$$

Hence we have

$$c_n \approx t_a^{n+1} \left|f'(\tau_m)\right| \int_{\tau_o}^1 \tau^n (1-\tau) e^{-\sigma_o t_a \tau} d\tau \qquad 7, (IV-9.4, 4)$$

The coefficients c_n can be computed approximately in terms of incomplete beta functions for relatively large, but not for extremely large, values of n. For the extremely large values only the right foot of the window function has any effect on the integration. Approximate expression for c_n, when n is extremely large, can be found in terms of higher derivatives. We have not performed this computation because these extremes are not needed for the rational approximation of a transfer function. (See theorem in subsection IV-9.5 in this regard.)

Formula 7, (IV-9.4.4) cannot be applied to the case of pulses because of its tail. A modification is then needed. The procedure is as follows.

Take t_a covering only the last region in which the lagging edge of the pulse is fairly straight. The tail of the pulse, starting at t_a, is approximated by a decreasing exponential. Let the tail of the pulse be approximated by

$$e^{-(\alpha\tau+\beta\tau^2)}$$

We obtain

$$c_n \approx t_a^{n+1} \left|f'(\tau_m)\right| \int_0^1 \tau^n (1-\tau)\, e^{-\sigma_o t_a \tau}\, d\tau + t_a^{n+1} \int_1^\infty \tau^n\, e^{-\gamma\tau-\beta\tau^2}\, d\tau \qquad 8,(IV-9.4,4)$$

where

$$\gamma = \alpha + \sigma_o t_a$$

which holds good for large, but not for extremely large, values of n.

We shall consider the approximation of the coefficients c_n for the first few values of n. This is done by substituting the pulse by its polygonal frame (see section IV-5.0). Corner type theorems can then be used.

Let us introduce the function

$$\psi_n(t) = \frac{1}{\sigma_o^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\sigma_o t)^{n+2}}{(n+1)(n+2)} \qquad 9,(IV-9.4,4)$$

It can be shown that

$$c_n = \int_0^\infty \psi_n(t)\, d\Gamma(t)$$

where

$$\Gamma(t) = \int_0^t f''(t)\, dt$$

Now for the polygonal frame having m + 1 corners

$$c_n \approx \sum_{j=0}^{m} \psi_n(t_j) \left[f'(t_j+) - f'(t_j-)\right] \qquad 10,(IV-9.4,4)$$

For the notation see Fig. 2,(IV-9.4,4).

We close subsection IV-9.4,4 with a theorem that refers to the alternation in sign of the terms in the power series expansions of the transfer functions associated with an impulse, with a pulse, or with a window function.

Theorem 3,(IV-9.4,4) "The terms of the power series expansion, around $\sigma = \sigma_o \geqslant 0$, of $F(\sigma)$ for a delayed or undelayed impulse, or pulse, or window function, alternate in sign for $\sigma \geqslant 0$."

The proof follows at once from 2,(IV-9.1,5') and 6,(IV-9.2,2).

IV-9.4,5 Examples. Several examples will be used to illustrate the power series expansions associated with a delayed or undelayed window function. For simplicity in the computation we consider a "complete cosine" window function. (See sections IV-3 and IV-4 of this report.)

Figure 1, (IV-9.4, 5) shows the graph of a pulse

$$f(t) = \frac{1 - \cos 10\pi t}{2}, \quad 0.800 \leq t \leq 1 \qquad 1,(IV\text{-}9.4,5)$$

delayed 0.800 units of time.

The graph of the c_k, n = 1, 2, ..., $\sigma_o = 0$, are given in Fig. 2, (IV-9.4, 5). The first few terms of the power series expansion around $\sigma_o = 0$ are

$$F(s) = 0.1 - 0.09s + 0.04066s^2 - 0.0123s^3 + 0.002801s^4$$
$$- 0.0005124s^5 + \dots \qquad 2,(IV\text{-}9.4,5)$$

Let us now consider the same complete cosine pulse, but without delay. The first few coefficients c_n are given by

$$c_0 = 10^{-1}; \quad c_1 = 10^{-2}; \quad c_2 = \frac{1}{3} \times 10^{-3}; \quad c_3 = 2 \times 10^{-4}; \quad c_4 = 4 \times 10^{-5}; \quad c_5 = 53 \times 10^{-6}$$

The first few terms of the power series expansion around $\sigma_o = 0$ are

$$F(s) = 0.1 - 0.01s + 0.000666s^2 - 0.0000333s^3$$
$$+ 0.000001333s^4 - 0.0000000444s^5 + \dots$$

IV-9.4, 6 A polygonal frame approximation. The next example consists of the computation of the power series representation of a given pulse when a polygonal frame approximation is used. Take the complete cosine window function of the last example. A rough first approximation is obtained when the given window is approximated by a trapezoidal frame. The selected frame has an area which is equal to the window function area. See Fig. 1, (IV-9.4, 6). The first seven c_n coefficients were computed by means of the corner theorem. By comparison we also reproduce the first c_n coefficient which corresponds to the complete cosine pulse.

n	complete cosine c_n	trapezoidal frame c_n
0	0.10000	0.10000
1	0.09000	0.08999
2	0.08133	0.08112
3	0.07380	0.07322
4	0.06723	0.06618
5	0.06149	0.05991
6	0.05645	0.05431
7	0.05208	0.04927

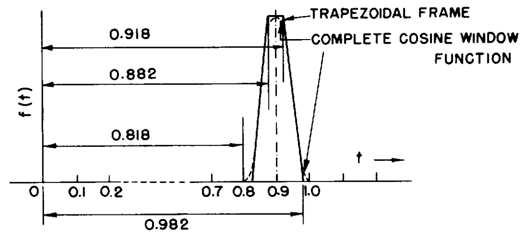


Fig. 1,(IV-9.4, 6)
Trapezoidal frame for the window function 1,(IV-9.4, 4).

IV-9.5 Application of a nonlinear process of summability. This subsection is primarily concerned with the application of a nonlinear process of summability which enables us to construct sequences of rational approximants to a given transfer function.

An introduction of this process has been made in reference 8, Part II. In the present report we are only interested in its application to the problems under discussion. Meaning, properties, and their connection with network theory are discussed in reference 8.

IV-9.5, 1 Particular case (Padé). For clearness in the presentation we shall begin the discussion with a particular case of the summability above which leads to the Padé method of construction of rational approximants.

Let

$$y_0(z),\ y_1(z),\ y_2(z),\ \ldots,\ y_{2n}(z),\ \ldots \qquad 1,(\text{IV-9.5, 1})$$

be a sequence of monogenic functions which converges uniformly toward a function $y(z)$ in a certain domain of the z plane. Let us associate with the sequence another sequence defined as follows

$$r_{\nu,\mu}(z) = \frac{\begin{vmatrix} y_\nu & y_{\nu-1} & y_{\nu-2} & \cdots & y_{\nu-\mu} \\ \Delta_\nu & \Delta_{\nu-1} & \Delta_{\nu-2} & \cdots & \Delta_{\nu-\mu} \\ \Delta_{\nu+1} & \Delta_\nu & \Delta_{\nu-1} & \cdots & \Delta_{\nu-\mu+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta_{\nu+\mu-1} & \Delta_{\nu+\mu-2} & \Delta_{\nu+\mu-3} & \cdots & \Delta_{\nu-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \Delta_\nu & \Delta_{\nu-1} & \Delta_{\nu-2} & \cdots & \Delta_{\nu-\mu} \\ \Delta_{\nu+1} & \Delta_\nu & \Delta_{\nu-1} & \cdots & \Delta_{\nu-\mu+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta_{\nu+\mu-1} & \Delta_{\nu+\mu-2} & \Delta_{\nu+\mu-3} & \cdots & \Delta_{\nu-1} \end{vmatrix}} \qquad 2,(\text{IV-9.5, 1})$$

where

$$\Delta_k = y_{k+1} - y_k, \quad k = \nu - \mu, \quad \nu - \mu + 1, \ldots, \nu + \mu$$

Some interesting properties of the sequence $r_{\nu,\mu}(z)$ can be seen without difficulty.

A. Let the sequence $\{y_k\}$ be all polynomials. Then, $r_{\nu,\mu}$ are rational functions.

B. Let the sequence $\{y_k\}$ be all rational functions. Then, $r_{\nu,\mu}$ are also rational functions.

C. Let all y_ν be rational functions. Then the poles of y_ν, $y_{\nu-1}, \ldots, y_{\nu-\mu}$ are, in general, also poles of the sequence $r_{\nu,\mu}$.

IV-9.5, 2 The case with y_ν as partial sums of a power series. Our particular attention will first be paid to the case in which the sequences y_ν are taken as the partial sums of a power series. This hypothesis will identify the sequence $r_{\nu,\mu}(s)$ with the Padé sequences of rational functions.

Consider the series

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_\nu z^\nu + \ldots \qquad 1,(\text{IV-9.5, 2})$$

and let us define the elements of the sequences $\{y_k\}$ as

$$y_k = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k; \quad k = 0, 1, 2, \ldots \qquad 2,(\text{IV-9.5, 2})$$

After some laborious but simple manipulations we obtain (see reference 8)

$$R_{\nu,\mu}(z) = \frac{\begin{vmatrix} y_\nu & y_{\nu-1}z & y_{\nu-2}z^2 & \ldots & y_{\nu-\mu}z^\mu \\ a_{\nu+1} & a_\nu & a_{\nu-1} & \ldots & a_{\nu-\mu+1} \\ a_{\nu+2} & a_{\nu+1} & a_\nu & \ldots & a_{\nu-\mu+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\nu+\mu} & a_{\nu+\mu-1} & a_{\nu+\mu-2} & \ldots & a_\nu \end{vmatrix}}{\begin{vmatrix} 1 & z & z^2 & \ldots & z^\mu \\ a_{\nu+1} & a_\nu & a_{\nu-1} & \ldots & a_{\nu-\mu+1} \\ a_{\nu+2} & a_{\nu+1} & a_\nu & \ldots & a_{\nu-\mu+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\nu+\mu} & a_{\nu+\mu-1} & a_{\nu+\mu-2} & \ldots & a_\nu \end{vmatrix}} \qquad 3,(\text{IV-9.5, 2})$$

Each term of the sequence $R_{\nu,\mu}(z)$ is the Padé approximant of the series 1,(IV-9.5, 2).

A simple inspection of 3,(IV-9.5,2) shows that it represents a rational function whose numerator and denominator are, respectively, of degree ν and μ, except when there exists a polynomial which is a common factor to the numerator and denominator. For a beautiful presentation of the subject the reader may consult reference 11.

Let us introduce the notation

$$R_{\nu,\mu}(z) = \frac{P_\nu(z)}{Q_\mu(z)} = \frac{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_\nu z^\nu}{\beta_0 + \beta_1 z + \beta_2 z^2 + \ldots + \beta_\mu z^\mu} \qquad 4,(\text{IV-9.5, 2})$$

It can be shown that the coefficients $\alpha_0, \ldots, \alpha_\nu, \beta_0, \ldots, \beta_\mu$ satisfy the linear relations

$$\left.\begin{array}{l} a_0 \ \beta_0 = \alpha_0 \\ a_1 \ \beta_0 + a_0 \ \beta_1 = \alpha_1 \\ a_2 \ \beta_0 + a_1 \ \beta_1 + a_0 \ \beta_2 = \alpha_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_\nu \ \beta_0 + a_{\nu-1} \ \beta_1 + a_{\nu-2} \ \beta_2 + \ldots + a_{\nu-\mu} \ \beta_\mu = \alpha_\nu \end{array}\right\} \qquad 5,(\text{IV-9.5, 2})$$

$$\left.\begin{array}{l} a_{\nu+1} \ \beta_0 + a_\nu \ \beta_1 + a_{\nu-1} \ \beta_2 + \ldots + a_{\nu-\mu+1} \ \beta_\mu = 0 \\ a_{\nu+2} \ \beta_0 + a_{\nu+1} \ \beta_1 + a_\nu \ \beta_2 + \ldots + a_{\nu-\mu+2} \ \beta_\mu = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{\nu+\mu} \ \beta_0 + a_{\nu+\mu-1} \ \beta_1 + a_{\nu+\mu-2} \ \beta_2 + \ldots + a_\nu \ \beta_\mu = 0 \end{array}\right\} \qquad 6,(\text{IV-9.5, 2})$$

The proof follows: Let us multiply the first, second, and third $(\mu+1)^{\text{th}}$ columns of each determinant in the numerator and denominator, respectively, by $\beta_0, \beta_1, \ldots, \beta_\mu$ and add all such columns to the last. Now if by construction, $\beta_0, \ldots, \beta_\mu$ satisfy 6,(IV-9.5,2) then the transformed determinants have a last column with all of its elements equal to zero except the first. By expanding about the last column we get

$$R_{\nu,\mu}(z) = \frac{\beta_0 \, y_\nu + \beta_1 \, y_{\nu-1} z + \beta_2 \, y_{\nu-2} z^2 + \ldots + \beta_\mu \, y_{\nu-\mu} z^\mu}{\beta_0 + \beta_1 z + \beta_2 z^2 + \ldots + \beta_\mu z^\mu}$$

By equating the coefficients in the numerator of this expression and of 4,(IV-9.5, 2) the relations 5,(IV-9.5, 2) follow at once. This result permits us to produce the following stipulation.

This system of homogeneous equations has nonzero solutions for the β coefficients when the rank of the matrix

$$\begin{pmatrix} a_{\nu+1} & a_\nu & \ldots & a_{\nu-\mu+1} \\ a_{\nu+2} & a_{\nu+1} & \ldots & a_{\nu-\mu+2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\nu+\mu} & a_{\nu+\mu-1} & \ldots & a_\nu \end{pmatrix}$$

is not zero. If the rank is μ, then the system has a solution (except for a multiplying factor).

$\mu \diagup \nu$	0	1	2	
0	a_0	$a_0 + a_1 s$	$a_0 + a_1 s + a_2 s^2$	
1	$\dfrac{a_0}{1 - \frac{a_1}{a_0} s}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_2}{a_1^2}\right) s}{1 - \frac{a_2}{a_1} s}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_3}{a_1 a_2}\right) s + a_2 \left(1 - \frac{a_1 a_3}{a_2^2}\right) s^2}{1 - \frac{a_3}{a_2} s}$	$\dfrac{a_0}{\;}$
2	$\dfrac{a_0}{1 - \frac{a_1}{a_0} s + \frac{a_1^2}{a_0^2}\left(1 - \frac{a_0 a_2}{a_1^2}\right) s^2}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_2}{a_1^2} \dfrac{1 - \frac{a_0 a_3}{a_1 a_2}}{1 - \frac{a_0 a_2}{a_1^2}}\right) s}{1 - \frac{a_2}{a_1} \dfrac{1 - \frac{a_0 a_3}{a_1 a_2}}{1 - \frac{a_0 a_2}{a_1^2}} s - \frac{a_2}{a_0} \left(1 - \dfrac{1 - \frac{a_0 a_3}{a_1 a_2}}{1 - \frac{a_0 a_2}{a_1^2}}\right) s^2}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_3}{a_1 a_2} \dfrac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_2^2}}\right) s + a_2 \left\{1 - \frac{a_0 a_3}{a_1 a_2}\left[1 - \left(1 - \frac{a_1^2}{a_0 a_2}\right) \dfrac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_4^2}}\right]\right\} s^2}{1 - \frac{a_3}{a_2} \dfrac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_2^2}} s - \frac{a_3}{a_1} \left(1 - \dfrac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_2^2}}\right) s^2}$	$a_0 + a_1 \left(1 - \frac{a_0 a_4}{a_1 a_3} \dfrac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}}\right) s + a_2$
3	$\dfrac{a_0}{1 - \frac{a_1}{a_0} s + \frac{a_1^2}{a_0^2}\left(1 - \frac{a_0 a_2}{a_1^2}\right) s^2 - \frac{a_1^3}{a_0^3}\left[\left(1 - \frac{a_0 a_2}{a_1^2}\right) - \frac{a_0 a_2}{a_1^2}\left(1 - \frac{a_0 a_3}{a_1 a_2}\right)\right] s^3}$			

Table I, (IV-9.5, 2)

PADÉ TABLE OF RATIONAL FUNCTIONS 4, (IV-9.5, 2) EXPANDED AROUND $\sigma_o = 0$, SO TH...

2	3	ν	$\nu \diagdown \mu$
$- a_1 s + a_2 s^2$	$a_0 + a_1 s + a_2 s^2 + a_3 s^3$	$\sum_{n=0}^{\nu} a_n s^n$	0
$\dfrac{\ldots\Big) s + a_2 \left(1 - \frac{a_1 a_3}{a_2^2}\right) s^2}{1 - \frac{a_3}{a_2} s}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_4}{a_1 a_3}\right) s + a_2 \left(1 - \frac{a_1 a_4}{a_2 a_3}\right) s^2 + a_3 \left(1 - \frac{a_2 a_4}{a_3^2}\right) s^3}{1 - \frac{a_4}{a_3} s}$	$\dfrac{\sum_{n=0}^{\nu} a_n \left(1 - \frac{a_{n-1}}{a_n} \frac{a_{\nu+1}}{a_\nu}\right) s^n}{1 - \frac{a_{\nu+1}}{a_\nu} s}$	1
$\dfrac{\ldots a_2 \left\{1 - \frac{a_0 a_3}{a_1 a_2}\left[1 - \left(1 - \frac{a_1^2}{a_0 a_2}\right) \frac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_4^2}}\right]\right\} s^2}{\ldots - \frac{a_3}{a_1} \left(1 - \frac{1 - \frac{a_1 a_4}{a_2 a_3}}{1 - \frac{a_1 a_3}{a_2^2}}\right) s^2}$	$\dfrac{a_0 + a_1 \left(1 - \frac{a_0 a_4}{a_1 a_3} \frac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}}\right) s + a_2 \left\{1 - \frac{a_0 a_4}{a_2^2}\left[1 - \left(1 - \frac{a_1 a_2}{a_0 a_3}\right) \frac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}}\right]\right\} s^2 + a_3 \left\{1 - \frac{a_1 a_4}{a_2 a_3}\left[1 - \left(1 - \frac{a_2^2}{a_1 a_3}\right) \frac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}}\right]\right\} s^3}{1 - \frac{a_4}{a_3} \frac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}} s - \frac{a_4}{a_2} \left(1 - \frac{1 - \frac{a_2 a_5}{a_3 a_4}}{1 - \frac{a_2 a_4}{a_3^2}}\right) s^2}$	$\dfrac{\sum_{n=0}^{\nu} a_n \left\{1 - \frac{a_{n-2}}{a_n} \frac{a_{\nu+1}}{a_{\nu-1}}\left[1 - \left(1 - \frac{a_{n-1}}{a_{n-2}} \frac{a_{\nu-1}}{a_\nu}\right) \frac{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+2}}{a_{\nu+1}}}{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+1}}{a_\nu}}\right]\right\} s^n}{1 - \frac{a_{\nu+1}}{a_\nu} \frac{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+2}}{a_{\nu+1}}}{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+1}}{a_\nu}} s - \frac{a_{\nu+1}}{a_{\nu-1}} \left(1 - \frac{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+2}}{a_{\nu+1}}}{1 - \frac{a_{\nu-1}}{a_\nu} \frac{a_{\nu+1}}{a_\nu}}\right) s^2}$	2
			3

Table I, (IV-9.5, 2)

ABLE OF RATIONAL FUNCTIONS 4, (IV-9.5, 2) EXPANDED AROUND $\sigma_0 = 0$, SO THAT $z = s$.

The coefficients $\alpha_0, \ldots, \alpha_\nu$ and $\beta_0, \ldots, \beta_\mu$ can be found simply by directly expanding the determinants in 3,(IV-9.5,2) or by solving the systems of linear equations 5,(IV-9.5,2) and 6,(IV-9.5,2), when an arbitrary value, for example, 1, is assigned to β_0. The rational function is insensible to the selection of β_0. During the preparation of the manuscript, Bolinder obtained simple explicit expressions for the coefficients α and β in terms of the a's. He used an ingenious procedure in the expansion of the determinants in 3,(IV-9.5,2). The results are given in Table I,(IV-9.5,2).

Finally, we shall recall a basic property of Padé approximants.

Let $P_\nu(z)$ and $Q_\mu(z)$ be the polynomials defined in 4,(IV-9.5,2). The expression

$$F(z)\,Q_\mu(z) - P_\nu(z) = G(z) \qquad 7,(IV\text{-}9.5,2)$$

is a polynomial which is, in general, of the smaller $\nu+\mu+1$ degree. The polynomial G(z) is, of course, intimately related to the series 1,(IV-9.5,2). When β_μ and α_ν are not zero, then

$$G(z) = a_{\nu+\mu+1} z^{\nu+\mu+1} + a_{\nu+\mu+2} z^{\nu+\mu+2} + \ldots \qquad 8,(IV\text{-}9.5,2)$$

which is the remainder of the series after the $(\nu+\mu)$ term.

It may happen that some of the last β coefficients and some of the last α coefficients in 4,(IV-9.5,2) become zero when the rank of the matrix is smaller than μ. When this happens the degree of $Q_\mu(z)$ is smaller than μ, say equal to q, $(q<\mu)$, and the degree of $P_\nu(z)$ is equal to, or smaller than, ν. Under this assumption the lowest degree in 7,(IV-9.5,2) is $p+q+r+1$, where r is equal to the μ-rank of the matrix of the a coefficients. Expression 7,(IV-9.5,2) is, in fact, the original Padé definition of the rational approximants. We did not start the discussion of Padé tables in this classical approach because in reference 8 we have shown the network significance of a more general procedure of which Padé approximations are a particular case.

IV-9.5,3 Some theorems of the Padé approximation theory. We shall recall here, without proof, some theorems of the theory of Padé approximations which are of significance in this investigation. These properties can be found in references 10 and 11. For convenience we shall follow these references closely.

In the classical theory, the Padé approximants $P_\nu(z)/Q_\mu(z)$ are arranged in a double table whose entrances are the indices ν and μ. In the square whose coordinates are ν,μ we normally place the approximant $P_\nu(z)/Q_\mu(z)$.

Definition 1,(IV-9.5,3) "Take the square whose coordinates are given by (ν,μ), with ν and μ fixed. The sequence ν,μ is normal if the rational approximant $P_\nu(z)/Q_\mu(z)$ is such that $P_\nu(z)$ is exactly of degree ν, $Q_\mu(z)$ is exactly of degree μ, and the fraction $P_\nu(z)/Q_\mu(z)$ is not found in any other square of the table.

"The Padé table is called normal if all its squares are normal.

"When a certain fraction $P_\nu(z)/Q_\mu(z)$ appears in more than one square, then the corresponding square and the complete table are called abnormal."

The following theorems are concerned with the conditions of normality and abnormality of Padé tables.

Let us introduce the determinants

$$\left.\begin{array}{l} \Delta_{\mu,\nu} = \begin{vmatrix} a_\nu & a_{\nu-1} & a_{\nu-2} & \cdots & a_{\nu-\mu} \\ a_{\nu+1} & a_\nu & a_{\nu-1} & \cdots & a_{\nu-\mu+1} \\ a_{\nu+2} & a_{\nu+1} & a_\nu & \cdots & a_{\nu-\mu+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\nu+\mu} & a_{\nu+\mu+1} & a_{\nu+\mu+2} & \cdots & a_\nu \end{vmatrix}; \quad \begin{array}{l} \mu,\nu = 0, 1, 2, \ldots \\ a_k = 0, \text{ when } k < 0 \end{array} \\ \text{with} \\ \qquad \Delta_{-1,\nu} = 1 \end{array}\right\} \qquad 1, (IV\text{-}9.5, 3)$$

Theorem 1, (IV-9.5, 3) "The necessary and sufficient condition for a square $[\mu,\nu]$ to be normal is that all four determinants

$$\Delta_{\mu,\nu}; \quad \Delta_{\mu-1,\nu+1}; \quad \Delta_{\mu-1,\nu}; \quad \Delta_{\mu,\nu+1} \qquad 2, (IV\text{-}9.5, 3)$$

are different from zero." See reference 11, page 429. (The determinants $\Delta_{\mu,\nu}$ are the same as the determinants $D_{\epsilon,\mu}$ in 2, (IV-9.2, 2). By transposing 1, (IV-9.5, 3) and substituting $\nu - \mu = \epsilon$ we obtain 2, (IV-9.2, 2).)

Theorem 2, (IV-9.5, 3) "If every determinant 1, (IV-9.5, 3) for $\mu,\nu = 0, 1, 2, \ldots, \infty$ is different from zero, then the Padé table is normal."

Theorem 3, (IV-9.5, 3) "Let the approximant $r_{\nu,\mu}(z)$ occupying the square $[\nu,\mu]$ have a numerator $P_\nu(z)$ exactly of degree p and a denominator $Q_\mu(z)$ exactly of degree q. When the common zeros in both are reduced, then there exists an integer r, $r \geq 0$ such that the polynomial 6, (IV-9.5, 2) begins with $p+q+r+1$ power, or it is identically zero, in which case we set $r = \infty$.

"Moreover, the squares given by

$$[q + \sigma_1, p + r_2]; \quad r_1, r_2 = 0, 1, 2, \ldots, r$$

which form a block of $(r+1)^2$ squares, are all occupied by the approximant $r_{\nu,\mu}$ and will not appear elsewhere." (See reference 10, page 394, reference 11, page 426.)

Abnormal Padé tables, particularly when $r = \infty$, have a special interest in our investigation. Several illustrative examples follow.

IV-9.5, 4 Examples. Consider the remainder 8, (IV-9.5, 2) which corresponds to the series 1, (IV-9.5, 2). When this remainder vanishes identically it simply means that the power series is a rational function. Hence $r = \infty$ is a condition for series 1, (IV-9.5, 2) to represent a rational function. Several examples are given. All examples are computed for $\sigma_o = 0$; this makes $z = s$.

$\mu \backslash \nu$	0	1	2	3
0	1	$1 - s$	$1 - s + s^2$	$1 - s + s^2 - s^3$
1	$\frac{1}{1+s}$	$\frac{1}{1+s}$	$\frac{1}{1+s}$	$\frac{1}{1+s}$
2	$\frac{1}{1+s}$	$\frac{1}{1+s}$	$\frac{1}{1+s}$	$\frac{1}{1+s}$

Table I, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3	4
0	1	1	$1 - s^2$	$1 - s^2 + s^3$	$1 - s^2 + s^3$
1	1	1	$\frac{1+s-s^2}{1+s}$	$1 - s^2 + s^3$	$1 - s^2 + s^3$
2	$\frac{1}{1+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$
3	$\frac{1}{1+s^2-s^3}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$
4	$\frac{1}{1+s^2-s^3+s^4}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$	$\frac{1+s}{1+s+s^2}$

Table II, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3	4
0	1	1	1	$1 - s^3$	$1 - s^3 + s^4$
1	1	1	1	$\frac{1 + s - s^3}{1 + s}$	$1 - s^3 + s^4$
2	1	1	1	$\frac{1 + s + s^2 - s^3}{1 + s + s^2}$	$1 - s^3 + s^4$
3	$\frac{1}{1 + s^3}$	$\frac{1 + s}{1 + s + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$
4	$\frac{1}{1 + s^3 - s^4}$	$\frac{1 + s}{1 + s + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$
5	$\frac{1}{1 + s^3 - s^4}$	$\frac{1 + s}{1 + s + s^3 - s^5}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$
6	$\frac{1}{1 + s^3 - s^4 + s^6}$	$\frac{1 + s}{1 + s + s^3 - s^5 + s^6}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$
7	$\frac{1}{1 + s^3 - s^4 + s^6 - s^7}$	$\frac{1 + s}{1 + s + s^3 - s^5 + s^6}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$	$\frac{1 + s + s^2}{1 + s + s^2 + s^3}$

Table III, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3	4
0	1	$1 - 3s$	$1 - 3s + 6s^2$	$1 - 3s + 6s^2 - 10s^3$	$1 - 3s + 6s^2 - 10s^3 + 15s^4$
1	$\frac{1}{1 + 3s}$	$\frac{1 - s}{1 + 2s}$	$\frac{1 - \frac{4}{3}s + s^2}{1 + \frac{5}{3}s}$	---	---
2	$\frac{1}{1 + 3s + 3s^2}$	$\frac{1 - \frac{1}{3}s}{1 + \frac{8}{3}s + 2s^2}$	$\frac{1 - \frac{1}{2}s + \frac{1}{6}s^2}{1 + \frac{5}{2}s + \frac{5}{3}s^2}$	---	---
3	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$
4	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$	$\frac{1}{1 + 3s + 3s^2 + s^3}$

Table IV, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3	4
0	2	$2 - 5s$	$2 - 5s + 9s^2$	$2 - 5s + 9s^2 - 14s^3$	$2 - 5s + 9s^2 - 14s^3 + 20s^4$
1	$\frac{2}{1 + \frac{5s}{2}}$	$\frac{2 - \frac{7}{5}s}{1 + \frac{9}{5}s}$	---	---	---
2	$\frac{2}{1 + \frac{5s}{2} + \frac{7s^2}{4}}$	$\frac{2 - \frac{1}{7}s}{1 + \frac{17}{7}s + \frac{11}{7}s^2}$	---	---	---
3	$\frac{2}{1 + \frac{5s}{2} + \frac{7s^2}{4} + \frac{s^3}{8}}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$
4	$\frac{2}{1 + \frac{5s}{2} + \frac{7s^2}{4} + \frac{s^3}{8} - \frac{s^4}{8}}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{2 + s}{1 + 3s + 3s^2 + s^3}$

Table V, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3
0	-2	$-2 + 7s$	$-2 + 7s - 15s^2$	$-2 + 7s - 15s^2 + 26s^3$
1	$\frac{-2}{1 + \frac{7}{2} s}$	$\frac{-2 + \frac{19}{7} s}{1 + \frac{15}{7} s}$	---	---
2	$\frac{-2}{1 + \frac{7}{2} s + \frac{19}{4} s^2}$	$\frac{-2 + \frac{27}{19} s}{1 + \frac{53}{19} s + \frac{43}{19} s^2}$	---	---
3	$\frac{-2}{1 + \frac{7}{2} s + \frac{19}{4} s^2 + \frac{27}{8} s^3}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$
4	$\frac{-2}{1 + \frac{7}{2} s + \frac{19}{4} s^2 + \frac{27}{8} s^3 - \frac{597}{16} s^4}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$	$\frac{-2 + s}{1 + 3s + 3s^2 + s^3}$

Table VI, (IV-9.5, 4)

$\mu \backslash \nu$	0	1	2	3	4
0	6	$6 - 23s$	$6 - 23s + 52s^2$	$6 - 23s + 52s^2 - 93s^3$	---
1	$\frac{6}{1 + \frac{23}{6} s}$	---	---	---	---
2	$\frac{6}{1 + \frac{23}{6} s + \frac{217}{36} s^2}$	---	---	---	---
3	$\frac{6}{1 + \frac{23}{6} s + \frac{217}{36} s^2 + \frac{8819}{216} s^3}$	---	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$
4	---	---	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$	$\frac{6 - 5s + s^2}{1 + 3s + 3s^2 + s^3}$

Table VII, (IV-9.5, 4)

Example I.

$$F(s) = \frac{1}{s + 1} = 1 - s + s^2 - s^3 + s^4 - \ldots$$

The corresponding Padé table is given in Table I, (IV-9.5, 4). The reader must notice that the table becomes immediately irregular with $r = \infty$. The term $1/(1+s)$ is the complete function.

Example II.

$$F(s) = \frac{1 + s}{1 + s + s^2} = 1 - s^2 + s^3 - s^5 + s^6 - s^8 + s^9 - \ldots$$

Note that some powers are missing. Table II, (IV-9.5, 4) shows two types of abnormalities: a block of order $r = 1$ at $\left[0,0\right]$, $\left[0,3\right]$, etc., and a block $r = \infty$ which exactly reproduces the original.

Example III.

$$F(s) = \frac{1 + s + s^2}{1 + s + s^2 + s^3} = 1 - s^3 + s^4 - s^7 + s^2 - 0 + \ldots$$

Note the lacunary character of the series. Table III, (IV-9.5, 4) shows several abnormal finite blocks and also an infinite block whose elements are necessarily the original rational function. (See reference 10, page 399, etc., C-fractions.)

The following examples have a multiple root with zeros at the right or left of the frequency axis.

Example IV. Table IV, (IV-9.5, 4)

$$F(s) = \frac{1}{(s+1)^3} = 1 - 3s + 6s^2 - 10s^3 + 15s^4 - 21s^5 + \ldots$$

Example V. Table V, (IV-9.5, 4)

$$F(s) = \frac{s + 2}{(s+1)^3} = +2 - 5s + 9s^2 - 14s^3 + 20s^4 - 27s^5 + \ldots$$

Example VI. Table VI, (IV-9.5, 4)

$$F(s) = \frac{s - 2}{(s+1)^3} = -2 + 7s - 15s^2 + 26s^3 - 40s^4 + 57s^5 - \ldots$$

Example VII. Table VII, (IV-9.5, 4)

$$F(s) = \frac{(s-2)\,(s-3)}{(s+1)^3} = 6 - 23s + 52s^2 - 93s^3 + 146s^4 - 211s^5 + \ldots$$

The different functions, which appear in all of these examples are intentionally taken as completely monotonic functions along the positive real axis of the s plane. This is not a loss of generality because all transfer functions can be expressed as the difference of two completely monotonic functions.

$\mu \backslash \nu$	0	1	2
0	0.1	$0.1 - 0.01s$	$0.1 - 0.01s + 0.000666\ldots s^2$
1	$\frac{0.1}{1 + 0.1s}$	$\frac{0.1 - 0.00333\ldots s}{1 + 0.0666\ldots s}$	$\frac{0.1 - 0.005s + 0.000166\ldots s^2}{1 + 0.05s}$
2	$\frac{0.1}{1 + 0.1s + 0.00333\ldots s^2}$	$\frac{0.1}{1 + 0.1s + 0.00333\ldots s^2}$	$\frac{0.1 - 0.005s + 0.000166\ldots s^2}{1 + 0.05s}$
3	$\frac{0.1}{1 + 0.1s + 0.00333\ldots s^2}$	$\frac{0.1}{1 + 0.1s + 0.00333\ldots s^2}$	---

Table I, (IV-9.5, 5)
Padé table for the undelayed pulse of Fig. 1, (IV-9.4, 4).

$\mu \backslash \nu$	0	1	2
0	0.1	$0.1 - 0.09s$	$0.1 - 0.09s + 0.04066s^2$
1	$\frac{0.1}{1 + 0.9s}$	$\frac{0.1 - 0.044815s}{1 + 0.45185s}$	$\frac{0.1 - 0.059754s + 0.013466s^2}{1 + 0.30246s}$
2	$\frac{0.1}{1 + 0.9s + 0.40333s^2}$	$\frac{0.1 - 0.029753s}{1 + 0.60247s + 0.15333\ldots s^2}$	$\frac{0.1 - 0.044630s + 0.013501s^2}{1 + 0.45369s + 0.068888s^2}$

Table II, (IV-9.5, 5)
Padé table of the delayed pulse of Fig. 1, (IV-9.4, 5).

It is worthwhile to note, through all of these examples, a group of properties.

I. If a given transfer function is rational, then the Padé procedure recovers the original function from the series. The original rational function always appears as the recurrent approximant of an infinite block.

II. Let us assume that the infinite block appears on and below the line $\mu = \mu_o$ and on or to the right of the column at $\nu = \nu_o$.

Hence

$$\Delta_{\nu_o+\lambda, \mu_o+\epsilon} \equiv 0 \begin{cases} \epsilon = 0, 1, 2, \dots \\ \lambda = 0, 1, 2, \dots \end{cases} \qquad 1, (IV\text{-}9.5, 4)$$

It may happen that one or several determinants

$$\Delta_{\nu,\mu} \begin{cases} \mu < \mu_o \text{ and } & \nu = 0, 1, 2, \dots \\ \text{or } \nu < \nu_o & \mu = 0, 1, 2, \dots \end{cases} \qquad 2, (IV\text{-}9.5, 4)$$

may be zero. Then this produces finite blocks in the outside of the infinite block. The vanishment of isolated determinants does not mean that the elements of the finite block are the required rational function. This situation is found, for example, when several powers of s are missing in the series.

III. The following property is of particular importance.

If F(s) is a rational function, then the complete function F(s) can be constructed by the knowledge of only a finite set of coefficients of the power series.

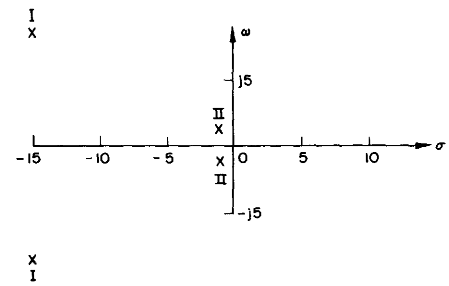
IV-9.5, 5 Pulse approximations. We shall now apply the procedure given above to obtaining the rational approximations for the pulse indicated in Fig. 1, (IV-9.4, 4). The corresponding rational function approximants, for the delayed and undelayed pulse, are given in Tables 1, (IV-9.5, 5) and II, (IV-9.5, 5). We have produced only a few approximants to show specific examples. The reader can increase at will the corresponding tables.

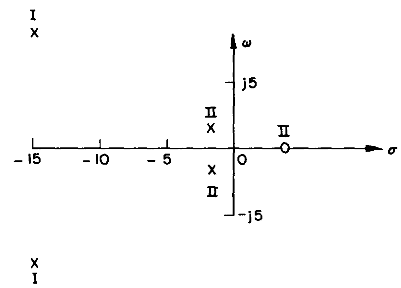
The reader must note that the table becomes rapidly abnormal for the undelayed pulse. The table of the delayed pulse does not show abnormalities. A computation of the position of the poles and zeros of approximants will help to understand the behavior of the tables, particularly for the abnormal squares. Figure 1, (IV-9.5, 5) shows the position of poles and zeros for some squares.

The reader can see that there is a jump of poles and zeros when abnormal squares appear. Steady poles and zeros indicate that they belong to, or form a basic group of, the function represented by Padé tables.

IV-9.5, 6 Final remarks. With regard to the construction of the rational function by means of Padé tables it is necessary to recall to the reader that when dealing with transfer functions the Padé table must be constructed for the completely monotonic components of the transfer function.

Square	I (undelayed pulse)	II (delayed pulse)
(2.0)	$\overset{x}{s}_1 = -15 + j\,8.67$	$\overset{x}{s}_1 = -1.115 + j\,1.115$
	$\overset{x}{s}_2 = -15 - j\,8.67$	$\overset{x}{s}_2 = -1.115 - j\,1.115$
(2.1)	No zero	$\overset{o}{s}_1 = 3.361$
	$\overset{x}{s}_1 = -15 + j\,8.67$	$\overset{x}{s}_1 = -1.965 + j\,1.631$
	$\overset{x}{s}_2 = -15 - j\,8.67$	$\overset{x}{s}_2 = -1.965 - j\,1.631$
(2.2)	$\overset{o}{s}_1 = 15 + j\,19.37$	$\overset{o}{s}_1 = 1.652 + j\,2.16$
	$\overset{o}{s}_2 = 15 - j\,19.37$	$\overset{o}{s}_2 = 1.652 - j\,2.16$
	$\overset{x}{s}_1 = -20$	$\overset{x}{s}_1 = -3.292 + j\,1.917$
		$\overset{x}{s}_2 = -3.292 - j\,1.917$





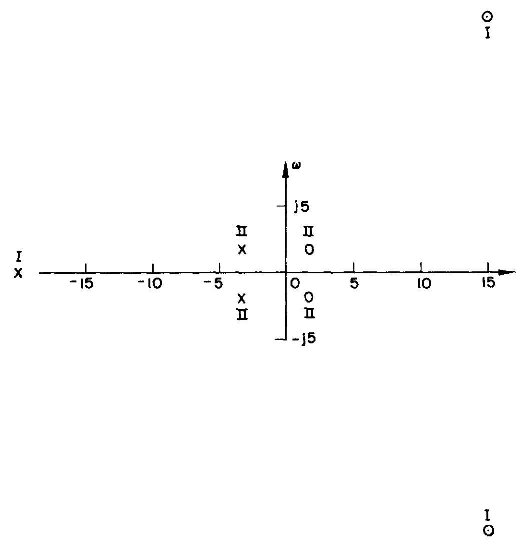


Fig. 1, (IV-9.5, 5)
Poles and zeros of some squares of Tables I, (IV-9.5, 5) and II, (IV-9.5, 5).

As indicated in the introduction of section IV-9, we originally planned to publish additional material in this final section. Due to the length of the report, it has been decided to present that material in Technical Report No. 270, which constitutes Part V of the series of reports on basic existence theorems in network synthesis.

References

1. M. V. Cerrillo, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., Jan. 15, 1952; and "On basic existence theorems in network synthesis, Parts I and II," Technical Report No. 139, Research Laboratory of Electronics, M.I.T. (to be published).
2. M. V. Cerrillo, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1952; and "On basic existence theorems in network synthesis, Parts I and II," Technical Report No. 139, Research Laboratory of Electronics, M.I.T. (to be published).
3. M. V. Cerrillo and E. A. Guillemin, On basic existence theorems in network synthesis, Part III, Technical Report No. 233, Research Laboratory of Electronics, M.I.T., 1952.
4. E. F. Bolinder, Fourier transforms in the theory of inhomogeneous transmission lines, Trans. Roy. Inst. Tech., Stockholm, Sweden, No. 48 (1951).
5. E. T. Whittaker and G. N. Watson, A course of modern analysis (Cambridge University Press, 1927).
6. E. Jahnke and F. Emde, Tables of functions with formulae and curves (Dover Publications, New York, 1945).
7. M. V. Cerrillo, On convergence phenomena and summability, Technical Report No. 269, Research Laboratory of Electronics, M.I.T. (to be published).
8. M. V. Cerrillo, On basic existence theorems in network synthesis, Part V, Technical Report No. 270, Research Laboratory of Electronics, M.I.T. (to be published).
9. D. V. Widder, The Laplace transform (Princeton University Press, 1946).
10. H. S. Wall, Continued fractions (Van Nostrand, New York, 1948).
11. O. Perron, Die Lehre von den Kettenbrüchen (Teubner, Leipzig, 1929).
12. P. Dienes, The Taylor series (Clarendon Press, Oxford, 1931).