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AN INTER-TEMPORAL MODEL FOR INVESTMENT MANAGEMENT

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This paper is based on my doctoral dissertation "An Adaptive Model for Investment Management" submitted to the Graduate School of Industrial Administration, Carnegie-Mellon University in May 1967. I am indebted for advice and guidance to my dissertation committee, Professors Kalman J. Cohen, Michael Thomson, and Norman Starr.

I. INTRODUCTION AND SUMMARY

The purpose of this paper is to present a portfolio management model which would be applicable to the portfolio management problem faced by institutional investors. The model developed extends previous efforts of the author toward removing a number of limitations of existing models which restrict their usefulness to institutional investors.(1)

The major limitations of existing models which restrict their applicability to practical investment problems are: first, the single-period nature of most portfolio selection models; second, neglect of the securities transactions' costs involved in modifying an existing portfolio which is no longer optimal in terms of revised expectations about security performance; third, neglect of a number of additional considerations, such as differential tax effects and the possibilities for portfolio leverage, which may have considerable significance in realistic portfolio decision situations.

The model developed in this paper is inter-temporal as it considers the portfolio management process as a multi-stage decision problem rather than a series of single-stage unrelated decision problems. As such it allows for the planned switching of funds among securities at various decision points within a multiple period investment horizon. Explicit consideration is given to the investor's expectations about the transactions costs involved in moving from an existing non-optimal

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1. See, Gerald A. Pogue, "An Extension of the Markowitz Portfolio Selection Model to Include Variable Transactions Costs, Short Sales, Leverage Policies and Taxes," The Journal of Finance (forthcoming).

portfolio to one which is efficient in terms of the investor's revised expectations regarding security performance. The model generates an efficient set of portfolios which trade off between return over the investment time horizon (after transactions' costs) and the risk associated with the level of return. Additionally, operational risk measures are defined for use in ex ante decision making and ex post evaluation which depend upon the information channels by which data were collected and estimates formed.[1]

Before proceeding with the development of the model, an overall framework for the investment management process is developed and its relationship to the model presented is discussed.

II. THE PORTFOLIO MANAGEMENT PROCESS

Portfolio management is a continuous process, rather than a single "once and for all" decision problem. It can be structured as a cyclical evaluation-action-reaction-reevaluation feedback control process (see Fig. 1). The process begins with the selection (or revision) of a universe of securities which are suitable for investment -- suitable being defined in conjunction with the investor's restrictions and preferences. Given the universe of securities, historical data are collected for the set of corporate and economic variables which are hypothesized to be relevant for explaining observed security yields. Expectations based on these historical data are combined with the analyst's subjective expectations about future corporate performance

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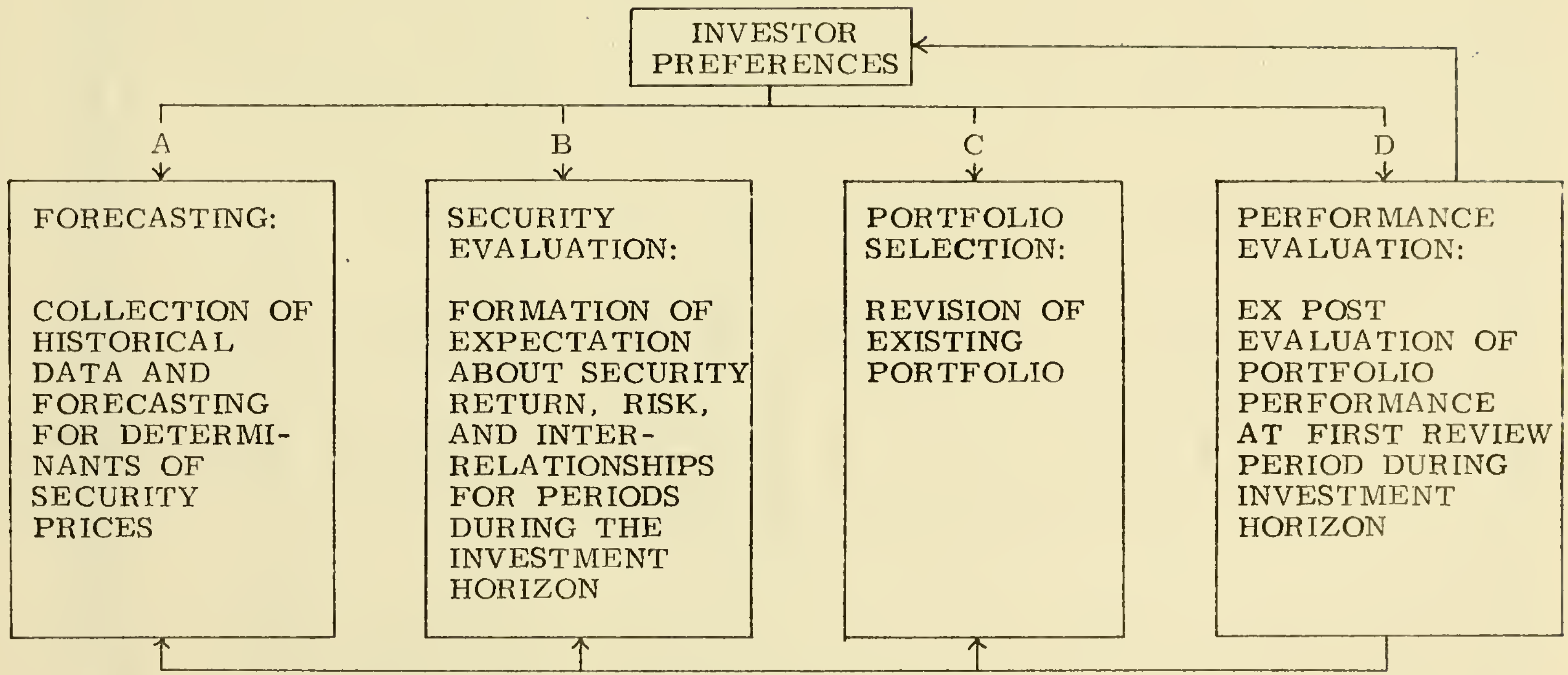
1. Consideration of short sales, portfolio leverage and differential taxes within a single-period framework has been presented elsewhere (op.cit.) The extension of the features to an inter-temporal framework is straightforward and for reasons of brevity will not be presented here.

FIGURE 1

A FRAMEWORK FOR THE PORTFOLIO MANAGEMENT PROCESS



PERFORMANCE FEEDBACK FOR ADAPTIVE MECHANISMS

and economic conditions to prepare forecasts of the explanatory variables. Expectations for these variables are then transformed into estimates of the future return and risk for each security in the universe. Given estimates of the future performance of individual securities, the existing portfolio is then revised to one which is more optimal in terms of the current expectations. The criterion for optimality is an expression of the investor's risk-return preferences.

At the end of the first review period, the existing portfolio is evaluated in terms of its fulfillment of the investor's expectations. The difference between the expected and actual performance now provides a data base for reevaluating one's concepts about the securities in the universe and of the mechanism by which future expectations are formed. With the new information obtained, new estimates are adaptively generated, and the portfolio revision process is repeated to produce a portfolio with optimal expectations over the investment time horizon.

The model presented in the author's thesis attempts to deal with this entire process.[1] The model presented in this paper deals with a limited subset of this process, primarily Part C — Portfolio Selection. In addition, a development of risk measures for use in Part B (Security Evaluation) and Part D (Performance Evaluation) has also been included.

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1. "An Adaptive Model for Investment Management," Unpublished Doctoral Dissertations, Carnegie-Mellon University, June 1967.

III. LIMITATIONS OF EXISTING PORTFOLIO SELECTION MODELS

The basic Markowitz Model for portfolio selection[1] (from which the current model derives) and variants, such as the index models of Sharpe[2] and Cohen and Pogue[3] are only partially adequate for dealing with the portfolio revision part of the process described above. Their primary deficiencies involve the neglect of transactions costs and inter-temporal effects.

(1) Transactions Costs

By not considering transactions costs, existing portfolio management models implicitly assume that it is costless to modify an existing portfolio to obtain another portfolio which is efficient in terms of revised expectations about security performance. In practical situations, since transactions costs are significant, the investor's initial portfolio must be taken into consideration by operational portfolio selection models.

Security transactions costs consist of two major parts: brokerage fees which depend mainly on the price of the shares traded, and marketability related costs which are primarily dependent on the amount of stock purchased or sold relative to the amount available or normally traded.

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1. Harry M. Markowitz, Portfolio Selection: Efficient Diversification of Investment, Cowles Foundation Monograph Number 16, (New York: Wiley, 1959).

2. W. F. Sharpe, "A Simplified Model for Portfolio Analysis," Management Science, Vol. 9, (January 1963), pp. 277-293.

3. K. J. Cohen and J. A. Pogue, "An Empirical Evaluation of Alternate Portfolio Selection Models," The Journal of Business, Vol. 40, (April 1967), pp. 166-193.

The brokerage fees on a 100-share transaction for securities at various price levels are summarized in Table 1.

Table 1

New York Stock Exchange Brokerage Fee Schedules[1]

Share Price	Value of Stock	Total Trading Costs (in and out)	Percent of Value of Stock
$ 200	$ 20,000	$123.40	0.62%
100	10,000	103.20	1.03%
50	5,000	93.10	1.86%
10	1,000	37.37	3.78%
3	300	20.01	6.67%

Source: Cohen and Zinbarg, Investment Analysis and Portfolio Management, Irwin 1967, p.74.

At the end of 1969 the average price of a share of stock listed on the New York Stock Exchange was approximately $50.[2] The average purchase and sale transaction would therefore cause the investor's capital to be reduced by about 2% of the assets involved. Thus, small changes in the outlook for particular stocks, which could result in large portfolio changes when transactions costs are ignored, may lead to little portfolio revision when the cost of revising the portfolio is traded off against the expected gains.

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1. This tabulation assumes a purchase and sale. It takes into consideration brokerage commissions, New York State transfer tax and SEC transfer tax. It ignores fee differentials associated with odd lots (less than 100-share transactions) and volume discounts (more than 1000-share transactions).

2. Source: New York Stock Exchange 1970 Fact Book.

The second source of transactions costs is primarily of interest to large institutional investors. These costs are associated with the purchase or sale of large quantities of stock. The costs take the form of either unfavorable price discounts or premiums that the investor may have to pay (for example, when acquiring or disposing of a large block of stock in the auction market) or additional fees that must be paid to an intermediary to complete the desired transaction (for example, in the case of a secondary distribution or acquisition).

The difficulty in purchasing or selling a given quantity of stock in a specified period is generally considered to be related to the liquidity of the auction market which, for a specific security, can be measured in terms of the "normal" trading volume of the stock. A particular transaction which represents 10-20% of the average trading volume in a given period can, in most cases, be more easily transacted than a trade which represents many times the normal auction market volume. The additional expense results from the costs of informing additional purchasers or sellers about the current unusual opportunities that exist and offering them inducements to rebalance their portfolios, which can consist of favorable price spreads and/or payment of any brokerage fees resulting from the trade. In addition, in relation to purchases of large blocks of stock, some additional incentive may be required to induce individuals with capital gains liabilities to provide their shares.

Special institutions[1] and arrangements have evolved for dealing with large block transactions. The New York Stock Exchange, for example,

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(1) Such as broker dealers who specialize in block positioning.

provides several special distribution methods. These are listed in Table 2 along with data illustrating seller's cost experiences. The techniques tend to occur in descending order as the size of the block relative to normal trading volume in the stock increases. The cost figures represent the sum of commissions paid plus the difference between the previous auction market bid and the bid for the block of stock.

Table 2

Composite Cost Ratios of Block Distribution Methods to Minimum Commission

Method	Number of Offerings	Range of Cost Ratios	Average Cost Ratio
1. Auction Market	---	---	1.00
2. Exchange Distribution	196	2.31 - 3.16	2.65
3. Specialist Block Purchase	71	2.80 - 31.59	3.16
4. Special Offering	477	4.29 - 5.81	4.60
5. Secondary Distribution	1338	3.98 - 6.45	4.80

Source: <u>New York Stock Exchange Investment Management Handbook</u>.

Thus a model which is relevant for the portfolio management problem faced by institutional investors should consider not only the brokerage fee component of transactions costs but also any incremental costs that the investor expects will be associated with large transactions.

The relevant problem is one of portfolio revision where the costs associated with revising the portfolio are considered as well as the expected gains.

(2) Inter-temporal Considerations

Given that transactions costs are significant factor for institutional investors, it is no longer an optimal strategy to treat the portfolio management process as a series of single-stage decision problems. An inter-temporal structure is required to deal explicitly with the fact that decisions take place in an inter-related way over a number of periods of time, so that decisions which are made at the current time are not just best for the current period, but are best for the entire planning horizon. This can be contrasted with the static (single-period) models which do not take into consideration the time path of changes over the planning horizon. Thus, static models do not take into account the interactions that result from conscious strategies involving timing considerations based on knowledge, perhaps imperfect, about the future course of prices. This ability to evaluate current actions in terms of longer run effects becomes vital for the large institutional investor for whom transactions costs make the excessive volatility inherent in static models prohibitively expensive.

Inter-temporal models are able to answer the question of how to trade off between longer-term (several review periods) and shorter-run (one period) effects. For example, the long-run expectation for a particular security may be favorable although its short-run outlook may be dim, while another security may be expected to provide handsome short-term profits while its longer-run outlook is bleak. To handle

such situations, the Markowitz analysis must be extended beyond the single-period formulation.

IV. INTER-TEMPORAL PORTFOLIO SELECTION MODEL

Portfolio revision is made necessary not only by changing expectations about future price performance and risk of the securities in the investor's security universe, but also by the need to optimally invest cash inflows or disburse cash outflows from the portfolio. In developing the model, consideration is given to typical constraints on the decision process that would be relevant for institutional investors.

(1) The Investor's Objective Function

The investor's objective function which is to be maximized has the general form

$$u(\tau) = u\left(\widetilde{M}(1/\tau), \ldots, \widetilde{M}(T/\tau)\right)$$

where $\widetilde{M}(t/\tau)$, $t = 1, \ldots, T$ is the market value of the portfolio at the end of the t^{th} period of the investor's planning horizon (i.e., at time $\tau + t$)

T = the number of periods in the planning horizon

τ = the date of the portfolio revision

$u(\tau)$ = the value of the objective function at time τ

One of the following two assumptions will be made to obtain a specific form of the above function.

(a) The investor has an objective function which is quadratic in the variables $\widetilde{M}(t/\tau)$, $t = 1, \ldots, T$.

The investor wishes to maximize the expected value of this objective function.

(b) The distribution of future security market prices is is such that the distribution of the market values of feasible portfolios will be approximately normal. This assumption is potentially less restrictive than (a). It is most applicable in the case of large institutional investors who hold many securities in their portfolios (e.g., a hundred or more), none of which contributes in a major way to the distribution of the total portfolio return.[1]

Given one of the above assumptions, the investor's objective function can be expressed as

$$Z(\tau) = \lambda \sum_{t=1}^{T} w_t^o \; E\left(\widetilde{M}(t/\tau)\right) + \sum_{t=1}^{T} \sum_{t'=1}^{T} w_{tt'} \; \mathrm{Cov}\left(\widetilde{M}(t/\tau) \cdot \widetilde{M}(t'/\tau)\right)$$

where λ, w_t^o, $w_{tt'}$, $t, t' = 1, \ldots, T$ are parameters of the objective function.

In the following development a specialized form of the general quadratic objective function will be used. This was done to provide the parameters of the inter-temporal objective function with an intuitive meaning to the institutional investor. The model developed, however, is in no way dependent on this specialization and can be used to optimize a general quadratic preference function. The investor's preference for a

1. This condition relies on a generalization of the central limit theorem to random variables which are not identically or independently distributed. Because the conditions are complex, the question of portfolio normality is probably best investigated via simulation.

particular sequence of portfolio vectors to be held during the multi-period planning horizon is assumed to be based on the expected value of a linear average of future portfolio market values, $\widetilde{M}(t/\tau)$, $t = 1, \ldots, T$, and the variance of the linear combination.

The objective function is given by

$$Z(\tau) = \lambda E\left(w_1\widetilde{M}(1/\tau) + \ldots + w_T\widetilde{M}(T/\tau)\right) - \mathrm{Var}\left(w_1\widetilde{M}(1/\tau) + \ldots + w_T\widetilde{M}(T/\tau)\right)$$

where $w_1, \ldots, w_T$ are weights indicating the relative importance of portfolio market values during the planning horizon.[1] λ is a parameter specifying the investor's desired tradeoff between increased expected market values and the risk associated with the overall portfolio policy.

Without loss of generality we can assume that

$$\sum_{t=1}^{T} w_t = 1$$

The weights can be considered in a number of ways. First, the decision maker may be faced with the incompatible multiple goals of providing superior performance in the short run at the cost of longer-run performance as opposed to the complementary goal of maximizing long-run gain independent of short-run considerations. The weights provide a means of ordering these conflicting goals in terms of their importance to the decision maker. Secondly, if terminal market value is of primary

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1. The weights w_t could be specified as functions of τ (i.e., $w_t(\tau)$). The reason for doing so would be that the relative tradeoffs between short-run and long-run considerations could change over time for an investment manager.

importance, the weights can be selected such that the model will produce a set of efficient tradeoffs between expected terminal portfolio market value and the risk measure associated with the multiple-period portfolio strategy.[1]

(2) The Model

At each portfolio review period the investor is assumed to have a joint prior distribution for security market values (market price plus value of dividends paid during the period) at the end of each of the T periods during the planning horizon. The mean of the investor's prior distribution represents his estimate of future security values (price plus dividends) and the covariance matrix of the prior distribution represents his estimate of the variances and covariances of the errors involved in forecasting future security values.[2]

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1. Two cases are possible

(i) $w_t = 0 \qquad t = 1, \ldots, T-1$

$w_T = 1.0$

In this case, the investor is interested only in the risk associated with his expected terminal portfolio.

(ii) $w_t^0 = 0 \qquad t = 1, \ldots, T-1$ (See original notation)

$w_T^0 = 1.0$

$w_{tt'} \neq 0 \qquad t, t' = 1, \ldots, T$

In this case, the investor would be concerned with the risk at the intermediate stages involved in reaching his terminal portfolio.

2. The Appendix contains a description of an estimation technique for adaptively determining the elements of the forecasting error covariance matrix.

Definition of Notation

$X_i(t/\tau)$ = The number of shares of security i the investor plans to hold (as of time τ) during period t of the planning horizon

$$i = 1, \ldots, N; \qquad t = 0, \ldots, T$$

(t = 0 represents actual holding at beginning of planning horizon.)

$\hat{V}_i(t/\tau)$ = The mean of the investor's prior distribution for $\tilde{V}_i(t/\tau)$ -- the total value of the security (market price plus any dividends paid during the period) at the end of period t of the planning horizon.

= $\hat{P}_i(t/\tau) + \hat{D}_i(t/\tau)$

$\hat{\sigma}_{iitt}(\tau)$ = The investor's estimate of the variance of the forecast error in his estimate of the value of security i at the end of period t.

$\hat{\sigma}_{ii'tt'}(\tau)$ = The estimated covariance between t period value forecast error on security i and the t' period error on security i'.

For compactness of notation, the following vectors are defined:

$$\underline{X}(t/\tau) = \left(X_i(t/\tau), \quad i = 1, \ \ldots, \ N\right)$$

$$\underline{X}(\tau) = \left(\underline{X}(t/\tau), \quad t = 1, \ \ldots, \ T\right)$$

$$\underline{\hat{V}}(t/\tau) = \left(\hat{V}_i(t/\tau), \quad i = 1, \ \ldots, \ N\right)$$

$$= \left(\hat{P}_i(t/\tau) + \hat{D}_i(t/\tau), \quad i = 1, \ \ldots, \ N\right)$$

$$\underline{\hat{V}}(\tau) = \left(\underline{\hat{V}}(t/\tau), \quad t = 1, \ \ldots, \ T\right)$$

$$= \left(\underline{\hat{P}}(t/\tau) + \underline{\hat{D}}(t/\tau), \quad t = 1, \ \ldots, \ T\right)$$

$\hat{\underline{\Sigma}}(\tau)$ = the multi-period value forecasting error covariance matrix

$= \left|\left| \hat{\sigma}_{ii'tt'}(\tau) \right|\right|$

$i, i' = 1, \ldots, N$

$t, t' = 1, \ldots, T$

As of the beginning of the planning horizon, the investor's ex ante knowledge about security performance during the planning horizon is contained in three matrices, $\hat{\underline{P}}(\tau)$, $\hat{\underline{D}}(\tau)$, and $\hat{\underline{\Sigma}}(\tau)$.

The estimated portfolio market value at the end of period t of the planning period is given by

$$\hat{M}(t/\tau) = \sum_{i=1}^{N} X_i(t/\tau) \cdot \hat{V}_i(t/\tau)$$

$$= \underline{X}'(t/\tau) \cdot \hat{\underline{V}}(t/\tau)$$

For a specified multi-period portfolio strategy, as represented by $\underline{X}(\tau)$, the weighted end of period portfolio market value is given by

$$\tilde{M}(\tau) = \sum_{t=1}^{T} w_t \tilde{M}(t/\tau)$$

The <u>expected</u> weighted portfolio market value, $\hat{M}(\tau)$, is given by

$$\hat{M}(\tau) = \sum_{t=1}^{T} w_t \hat{M}(t/\tau)$$

$$= \underline{W}' \underline{X}'(\tau) \hat{\underline{V}}(\tau)$$

where $\underline{W}$ is a diagonal matrix

$$\underline{W} = \begin{bmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_t \end{bmatrix}$$

The risk measure associated with this portfolio strategy is given by the forecast error variance associated with $\widetilde{M}(\tau)$.

$$\text{Var}\left(\widetilde{M}(\tau)\right) = \sum_{t=1}^{T} \sum_{t'=1}^{T} \sum_{i=1}^{N} \sum_{i'=1}^{N} w_t \, X_i(t/\tau) \, \hat{\sigma}_{ii'tt'}(\tau) \, X_{i'}(t'/\tau) \, w_{t'}$$

$$= \underline{W}' \, \underline{X}'(\tau) \, \underline{\hat{\Sigma}}(\tau) \, \underline{X}(\tau) \, \underline{W}$$

The problem is now to maximize the investor's objective function

$$Z(\tau) = \lambda \, E\left(\widetilde{M}(\tau)\right) - \text{Var}\left(\widetilde{M}(\tau)\right) \quad (\lambda \geq 0)$$

given a starting portfolio $\underline{X}(0/\tau)$ and the various investment policy constraints discussed below.

The solution will produce an updated multi-period portfolio vector, $\underline{X}(\tau)$, of which only the first period portfolio, $\underline{X}(1/\tau)$, is actually implemented. The revision procedure will be repeated one period later $(\tau + 1)$ when revised expectations about security returns over a new T period planning horizon have been formed.

(3) Transactions' Costs[1]

For the purposes of our research, common stock transactions' costs are assumed to be made up of two parts: brokerage fees and the costs associated with marketability of the shares, that is, the price spread resulting from large volume transactions.

On the basis of New York Stock Exchange data examined, and discussions with practicing investment managers, a transactions' costs curve was defined to give an increasing marginal cost of transactions. The total dollar cost per share for a transaction of a specific security is assumed to be an increasing function of the percentage of the average daily trading volume (for that security) accounted for by the purchase or sale in question.

Figure 2a relates the total dollar transactions' costs for security i to the number of shares purchased or sold. In Figure 2b we have approximated the previous curve by a piecewise linear representation. The change points for the marginal transactions' costs rates (i.e., the slopes of the linear segments) occur when the share purchases or sales amount to specified percentages of the total trading volume for security i.

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1. Since procedures for dealing with transactions' costs have been discussed elsewhere in detail for the single-period model, the discussion here will be brief. See G. A. Pogue, "An Extension of the Markowitz Model ...", op. cit.

Figure 2a

TOTAL TRANSACTIONS' COSTS CURVE

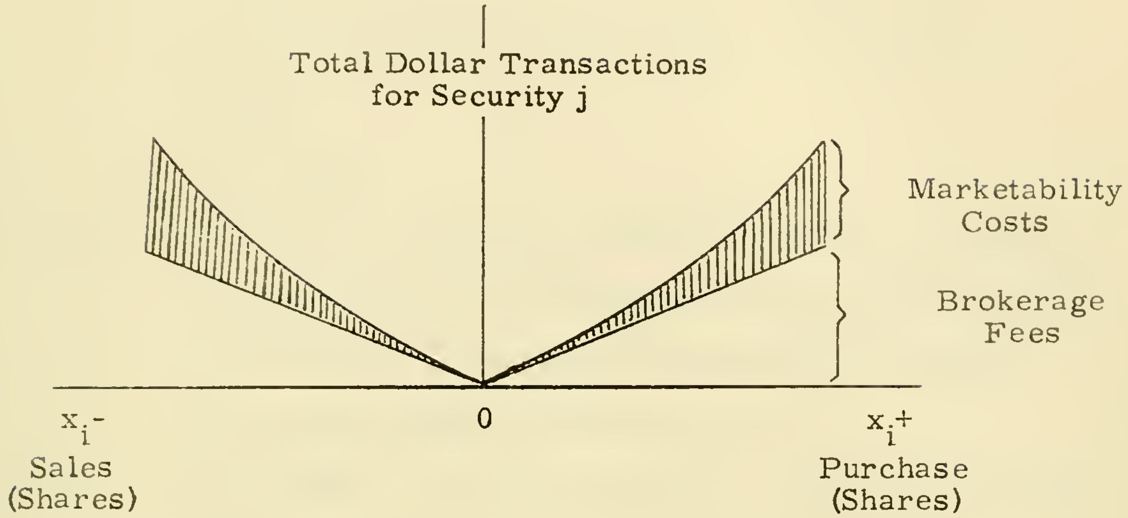
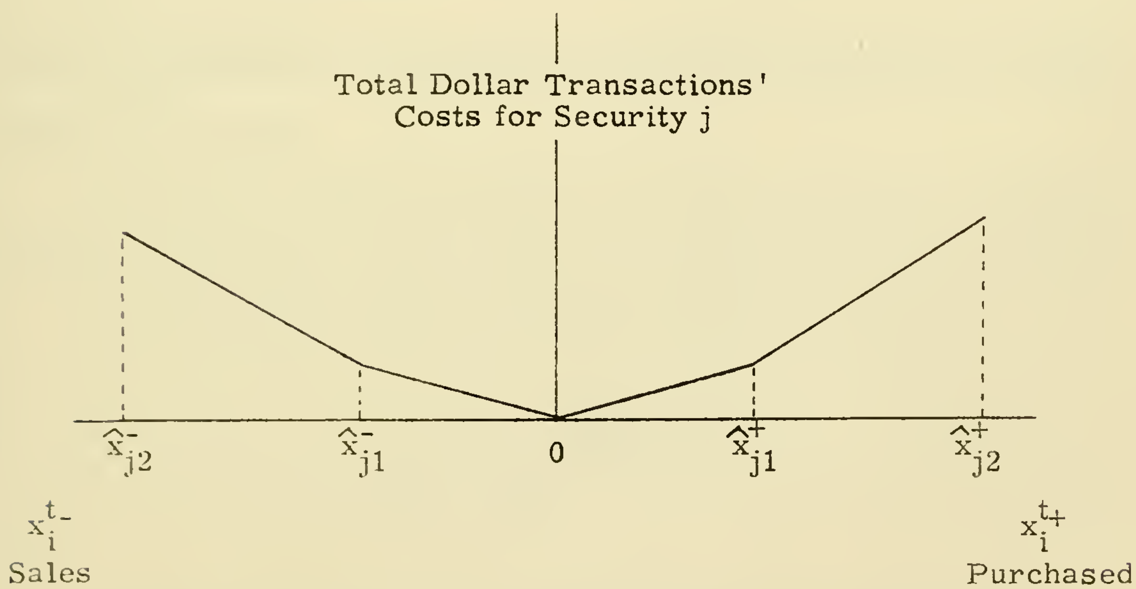


Figure 2b

PIECEWISE LINEAR APPROXIMATION
TO THE TRANSACTIONS' COSTS CURVE



Let c^{+}_{ji} = the percentage of the auction market price which must be paid for transactions in the j^{th} linear segment

r^{t+}_{ji} = the expected dollar transactions' costs per share for purchases in the j^{th} linear segment for security i at time t

$= c^{+}_{ji} \hat{P}_i(t/\tau)$

$\hat{x}^{+}_{ji}$ = the number of shares of security i which corresponds to a specified fraction δ_{ji} of the normal trading volume of security i. $\hat{x}^{+}_{ji}$ defines the upper limit of the j^{th} purchase segment of the cost curve.(1)

x^{t+}_{ji} = the number of shares of security i purchased in the j^{th} linear segment of the cost curve during period t.

x^{t+}_{i} = the total number of shares of security i purchased during period t

$$= \sum_{j=1}^{m+} x^{t+}_{ji}$$

Similar quantitites can be defined for the sales segments of the transactions' cost curve.

We can now define the number of shares of security i traded in terms of purchases or sales in the linear segments of the cost curve.

The number of shares of security i traded at period t

$$= X_i(t/\tau) - X_i(t - 1/\tau)$$

$$= x^{t+}_i - x^{t-}_i$$

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1. It is assumed that the cost per dollar of purchases, for transactions that occur between a fraction δ^{+}_{j-1i} and a fraction δ^{+}_{ji} of the period t trading volume, are incurred at a constant rate, c^{+}_{ji}.

The transactions' costs incurred

$$= \sum_{j=1}^{m+} r_{ji}^{t+} x_{ji}^{t+} + \sum_{j=1}^{m-} r_{ji}^{t-} x_{ji}^{t-}$$

The transactions costs will be included in the budget equation (described below), reducing the amount of resources available for reinvestment in a revised portfolio.(1)

Additionally, we require that each of the transaction's variables x_{ji}^{t+} and x_{ji}^{t-} be upper bounded

$$x_{ji}^{t+} \leq \hat{x}_{ji}^{+} \qquad j = 1, \ldots, m+$$

$$x_{ji}^{t-} \leq \hat{x}_{ji}^{-} \qquad j = 1, \ldots, m-$$

Because of the convexity of the transactions' cost curve, we need not be concerned about the possibility that $x_{j+1,i}^{t+} \geq 0$ while $x_{ji}^{t+} \leq \hat{x}_{ji}^{+}$. This condition will not arise because higher segments of the curve are more costly in terms of transactions' costs.

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1. A detailed study of transactions' costs data is required to estimate the marginal costs per dollar of transactions (c_{ji}^{+} or c_{ji}^{-}) which are associated with the various linear segments of the transactions' costs curve. The study would require the analysis of many large block transactions relating the total transactions' costs, as a fraction of the normal auction market price, to the percentage of the average daily trading volume of the security accounted for by the transaction. The price spread component of the transactions' costs is measured by comparing the normal auction market price (the price prior to the block purchase or sale) with the price at which the block was actually purchased or sold.

   Transactions data could most likely be pooled for similar types of securities to obtain a set of coefficients c_j^{+}, c_j^{-} which pertained to that set of securities (for example, common stocks traded on the New York Stock Exchange).

(4) Investor Restrictions and Policy Constraints

The maximization of the investor's objective function is carried out subject to a number of constraints representing policy and legal restrictions, as well as accounting identities.

(a) Liquidity Constraints

The investor may require that whenever the portfolio is revised a fraction, K, of the current market value of the firm be allocated to cash (or "near" cash items such as treasury bills). If we define the N^{th} security in the portfolio to be the cash position, then the liquidity requirement can be expressed as a lower bound on $X_N(t/\tau)$, the expected cash position during period t.

$$X_N(t/\tau) \geq K \left(\sum_{i=1}^{N} X_i(t/\tau) \; \hat{P}_i(t - 1/\tau) \right)$$

$$\text{for } t = 1, \ldots, T$$

where

$\hat{P}_i(t - 1/\tau)$ = the expected price of security i at the end of period t - 1 (and at the beginning of period t when the portfolio is revised).[1]

(b) Dividend Income Constraint

The investor may require that acceptable portfolios have a specified expected dividend income during each of the periods of the

1. Since the N^{th} security is defined to be cash, $\hat{P}_N(t/\tau) = 1.0$ for all t and τ.

time horizon. This requirement can be specified as a series of linear constraints of the following form:

$$\sum_{i=1}^{N} X_i(t/\tau)\ \hat{D}_i(t/\tau) \geq D^*(t/\tau)$$

where

$D^*(t/\tau)$ = the required level of expected dividend income

$\hat{D}_i(t/\tau)$ = the expected dividend paid on a share of security i during the t^{th} period of the planning horizon(1)

(c) Upper Bounds on Portfolio Holdings

In order that the portfolios generated be relevant for institutional investors, they must conform to legal and policy restrictions placed upon the portfolio manager by federal agencies and the management of his financial institution. In practice, many institutional investors (e.g., common trust funds) have legal restrictions on the percentage of their portfolio which can be invested in a single security, and the percentage of the stock of any one company which can be held. Other institutional investors (e.g., pension funds) adhere heuristically to such restrictions to avoid becoming formally involved as major shareholders in companies in which they invest. Thus, decision rules which recommend that the portfolio manager, for example, invest 24% of his portfolio in National Biscuit and 13% in American Airlines would probably be quite unacceptable. The upper bound constraints also serve to alleviate possible marketing problems by keeping the holdings of particular companies to levels which allow some marketing flexibility. A further use of upper bound constraints is to serve as a hedge against biases in

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1. $\hat{D}_N(t/\tau) = 0$ for all t and τ.

the input data by preventing portfolios from attaining radically undiversified postures at any point in time.

The above comments are included to emphasize the requirement that a useful portfolio selection model must incorporate upper bound constraints to handle legal and policy restrictions. Upper bound constraints are incorporated into the portfolio management model to apply to additional purchases only. It is not realistic, nor institutional practice, to immediately sell sufficient shares of any security which has become greater than the upper bound because of a shift in relative prices in order to bring the percentage allocations back into line.

Let

u_i = the percentage of the portfolio market value beyond which no additional shares of security i are purchased

q_i = the percentage of the capitalization of a firm beyond which no additional shares of security i will be purchased

$X_i^O(\tau)$ = the number of shares of security i outstanding at revision time τ

Now, it is required that

$$X_i(t/\tau) \leq \min \begin{cases} q_i X_i^O(\tau) \\ \max \begin{cases} X_i(t - 1/\tau) \\ \dfrac{u_i \hat{M}(t - 1/\tau)}{\hat{P}_i(t - 1/\tau)} \end{cases} \end{cases}$$

$$i = 1, \ldots, N \qquad t = 1, \ldots, T$$

This condition is achieved by defining additional non-negative variables $R_1(t)$, $R_2(t)$, $U_1(t)$, $U_2(t)$, $\phi_1(t)$ and $\phi_2(t)$ and requiring that

$$X_i(t/\tau) \leq q_i X_i^O(\tau) \qquad (1)$$

$$X_i(t/\tau) + R_1(t) - R_2(t) = X_i(t - 1/\tau) \qquad (2)$$

$$X_i(t/\tau) + U_1(t) - U_2(t) = \frac{u_i}{\hat{P}(t - 1/\tau)} \hat{M}(t - 1/\tau) \qquad (3)$$

$$X_i(t - 1/\tau) - \frac{u_i \hat{M}(t - 1/\tau)}{\hat{P}(t - 1/\tau)} = \phi_1(t) - \phi_2(t) \qquad (4)$$

$$\left(R_2(t)\right) \left(\phi_1(t)\right) = 0$$

$$\left(U_2(t)\right) \left(\phi_2(t)\right) = 0$$

$$\left(\phi_1(t)\right) \left(\phi_2(t)\right) = 0 \qquad (5)$$

$$t = 1, \ldots, T$$

The above condition can be obtained in a quadratic programming framework. The linear equalities (2) through (4) can be directly

incorporated for each t. The nonlinear equalities (5) can be incorporated via the addition of large penalty terms as the coefficients of these cross-product terms in the objective function.

(d) Portfolio Budget Constraints

These relationships insure that expected sources of funds (from security sales, dividend payments, and exogenous inflows) are balanced against expected uses of funds (arising from security purchases, transactions' costs, and exogenous outflows) during each period of the planning horizon.

The sources and uses relationships are given by

$$F(t/\tau) + \sum_{i=1}^{N} X_i(t - 1/\tau)\ \hat{D}_i(t - 1/\tau)$$

$$- \sum_{i=1}^{N} \left[X_i(t/\tau) - X_i(t - 1/\tau)\right] \hat{P}_i(t - 1/\tau)$$

$$- \sum_{i=1}^{N-1} \left[\sum_{j=1}^{m+} r_{ji}^{t+}\ x_{ji}^{t+} + \sum_{j=1}^{m-} r_{ji}^{t-}\ x_{ji}^{t-} \right] = 0$$

for t = 1, . . . , T

where

$F(t/\tau)$ = the estimated exogenous flow which is to be optimally invested or disbursed at the beginning of period t

The second term represents the total dividends accumulated during the previous period which is to be optimally invested at the beginning of period t. The third term represents the cash flow associated with the purchase or sale of securities. The final term represents the expected

transactions' costs incurred in revising the portfolio. The cash balance after the portfolio revision, $X_N(t/\tau)$, represents the net result of the transactions specified by the difference in the portfolio vectors $\underline{X}(t/\tau)$ and $\underline{X}(t - 1/\tau)$.

(5) Summary of Inter-Temporal Model

When new data become available at time τ, the updated price and dividend estimate vectors, $\hat{\underline{P}}(\tau)$ and $\hat{\underline{D}}(\tau)$, and the estimated value forecasting error covariance matrix, $\hat{\underline{\Sigma}}(\tau)$, provide a data base for updating the existing portfolio vector $\underline{X}(0/\tau)$ to obtain another which is more efficient when viewed in terms of the investor's revised expectations about security performance. The portfolio for the first period of the planning horizon is implemented, and the procedure is repeated one review period later when revised expectations about security returns over a new T period planning horizon have been formed.

The updated portfolio vector $\underline{X}(\tau)$ is selected to maximize

$$Z = \lambda \underline{W} \underline{X}'(\tau) \hat{\underline{V}}(\tau) - \underline{W} \underline{X}'(\tau) \hat{\underline{\Sigma}}(\tau) \underline{X}(\tau) \underline{W}$$

$$- Y \sum_{t=1}^{T} \left(R_2(t) \phi_1(t) + U_2(t) \phi_2(t) + \phi_1(t) \phi_2(t) \right)$$

where $\lambda \geq 0$ and Y is a very large positive number (e.g., 10^{10}), subject to the following constraints for each period t during the planning horizon, $t = 1, \ldots, T$.

(a) Transactions' Cost Curve Constraints

$$X_i(t/\tau) - X_i(t - 1/\tau) = \sum_{j=1}^{m+} x_{ji}^{t+} - \sum_{j=1}^{m-} x_{ji}^{t-}$$

$$x_{ji}^{t+} \leq \hat{x}_{ji}^{+} \qquad j = 1, \ldots, m+ \quad i = 1, \ldots, N$$

$$x_{ji}^{t-} \leq \hat{x}_{ji}^{-} \qquad j = 1, \ldots, m- \quad i = 1, \ldots, N$$

(b) Liquidity Constraints

$$X_N(t/\tau) \geq K \left[\sum_{i=1}^{N} X_i(t/\tau)\ \hat{P}_i(t - 1/\tau) \right]$$

(c) Dividend Income Constraints

$$\sum_{i=1}^{N} X_i(t/\tau)\ \hat{D}_i(t - 1/\tau) \geq D^{*}(t/\tau)$$

(d) Upper Bounds on Portfolio Holdings

$$X_i(t/\tau) \leq q_i\ X_i^{O}(\tau)$$

$$X_i(t/\tau) + R_1(t) - R_2(t) = X_i(t - 1/\tau)$$

$$X_i(t/\tau) + U_1(t) - U_2(t) = \frac{u_i\ \hat{M}(t - 1/\tau)}{\hat{P}(t - 1/\tau)}$$

$$X_i(t - 1/\tau) - \frac{u_i\ \hat{M}_i(t - 1/\tau)}{\hat{P}(t - 1/\tau)} = \phi_1(t) - \phi_2(t)$$

(e) Budget Constraints

$$F(t/\tau) + \sum_{k=1}^{N} X_i(t - 1/\tau)\ \hat{D}_i(t - 1/\tau)$$

$$- \sum_{i=1}^{N} \left[X_i(t/\tau - 1) - X_i(t - 1/\tau)\right] \hat{P}_i(t - 1/\tau)$$

$$- \sum_{i=1}^{N} \left[\sum_{j=1}^{m+} r_{ji}^{t+}\ x_{ji}^{t+} + \sum_{j=1}^{m-} r_{ji}^{t-}\ x_{ji}^{t-} \right] = 0$$

By allowing the parameter λ to vary in the range between 0 and ∞, a quadratic programming code can be used to generate the efficient frontier after the transactions' costs involved in revising the initial portfolio, $\underline{X}(0/\tau)$, and in planned revisions at future review points prior to the investment horizon. The set of portfolios obtained provide the optimal tradeoffs between the selected measure of portfolio return over the multi-period horizon and the risk associated with the portfolio policy.

The efficient frontier is illustrated in Fig. 3. It is convenient to normalize the return and risk measures by the market value of the initial portfolio, $M(0/\tau)$, so that they are expressed in relative terms[1]

---

1. Weighted Expected Return $= E\left[\dfrac{\underline{W}\ \underline{\tilde{M}}(\tau)}{M(0/\tau)}\right]$

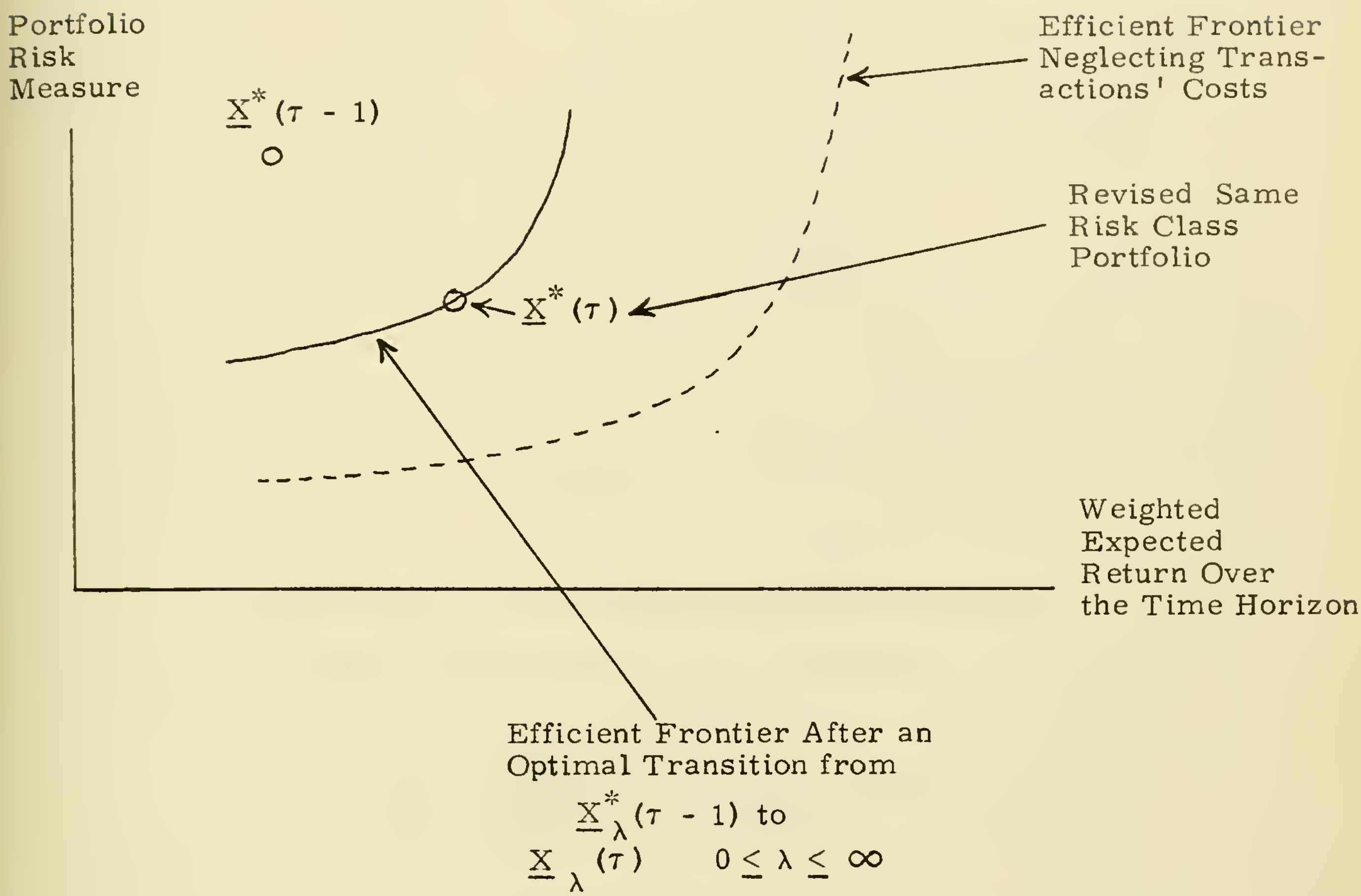
$$= \sum_{t=1}^{T} w_t \frac{\hat{M}(t/\tau)}{M(0/\tau)}$$

2. Portfolio Risk Measure $=$ Standard Deviation $\left[\underline{W} \cdot \dfrac{\underline{\tilde{M}}(\tau)}{M(0/\tau)}\right]$

$$\frac{\left[\underline{W}'\ \underline{X}'(\tau) \hat{\underline{\sum}}(\tau)\ \underline{X}(\tau)\ \underline{W}\right]^{1/2}}{M(0/\tau)}$$

Figure 3

THE EFFICIENT FRONTIER AFTER TRANSACTIONS' COSTS



(6) <u>Ex Post Risk Measures</u>

The ex ante risk measures used in the model are measures of the investor's inability to predict future market prices. In the ex ante case, where the risk estimates are used as decision parameters, the risk measures for securities were based on weighted averages of past squared forecasting errors. This was the result of the assumption that the forecasting error variances were locally stationary. The decision-

making requirements demanded that the estimates of forecasting variances used be as representative as possible of the current parameter values, without being overly sensitive to random fluctuations. As a result, exponentially weighted estimation procedures were used to discount past observations, which were becoming progressively less relevant in the formation of expectations about the current values of the forecasting error variances.

In the ex post case, the risk measures are used for evaluation of past performance rather than as a basis for future predictions. In this case an unweighted measure of the observed forecasting errors provides a measure of the total risk inherent in the projections of future portfolio market values.

An estimate of the coefficient of variation associated with t period forecasts of portfolio market value is given by

$$S(t/\tau) = \left[\frac{1}{\tau_0} \sum_{\nu=0}^{\tau_0} \left(\frac{\hat{M}(\tau-\nu/\tau-\nu-t) - M(\tau-\nu)}{M(\tau-\nu)} \right)^2 \right]^{1/2}$$

where

$M(\tau-\nu)$ = the realized market value at time $t-\nu$

$\hat{M}(\tau-\nu/\tau-\nu-t)$ = an estimate of $M(\tau-\nu)$ made at time $\tau-\nu-t$

τ_0 = the number of observations less one

The ex post risk measure must, however, take into consideration the multiple-period nature of the ex post evaluation process. The objective function for the ex ante selection process contained weighted

estimated future portfolio market values for several (i.e., T) future periods. A weighted ex post risk measure can be constructed in a manner analogous to the ex ante risk measure. Thus, a weighted coefficient of variation of multiple-period forecasting errors is given by

$$S(\tau) = \left[\sum_{t=1}^{T} \sum_{t'=1}^{T} w_t \, w_{t'} \, S(t,t'/\tau) \right]^{1/2}$$

where

$$S(t,t/\tau) = S(t/\tau)^2$$

$$S(t,t'/\tau) = \frac{1}{\tau_0} \sum_{\nu=0}^{\tau_0} \left[\frac{\hat{M}(\tau-\nu/\tau-\nu-t) - M(\tau-\nu)}{M(\tau-\nu)} \right] \cdot \left[\frac{\hat{M}(\tau-\nu/\tau-\nu-t') - M(\tau-\nu)}{M(\tau-\nu)} \right]$$

V. NUMERICAL EXAMPLES

In order to illustrate how the model might be utilized in a practical situation, a numerical example will be presented. Two situations are considered. First, the initial selection and the later revisions of a portfolio in a particular risk class, $\lambda = 3000$, are shown (see Table 3 and Exhibits 1 and 2). Secondly, the ex post evaluation of the 10-year performance of this managed portfolio is compared with the performance of other managed portfolios (i.e., other λ values) and unmanaged portfolios over the same time period (year end 1956 through year end 1965; see Table 4 and Fig. 4).

(a) Portfolio Selection and Revision

A portfolio in a specified risk class was selected at the end of 1956 and subsequently revised annually until the end of 1965 (see Table 2 for parameter values used in run). At each revision the portfolio was revised to the corresponding portfolio in the same risk class, $\lambda = 3000$.[1] Projections of future security returns and risk were based only on data that would have been available as of the simulated portfolio revision date.

The historical security data used in the security evaluation phase was obtained from the Standard and Poor's Annual Industrial Compustat tape. The security evaluation model used to predict stock prices was of the form

$$\hat{P}_j(t/\tau) = a_\tau + b_\tau\left(\hat{E}_j(t/\tau)\right) + c_\tau\left(\widehat{Cov}(\tilde{R}_j \tilde{R}_M/\tau)\right) + d_\tau\left(\hat{g}_j(t/\tau)\right)$$

$$t = 1, \ldots, T$$

where

$\hat{E}_j(t/\tau)$ = projected earnings per share

$\widehat{Cov}(\tilde{R}_j\tilde{R}_M/\tau)$ = the covariance of the return on security with the return on a market index (S and P 500)

$\hat{g}_j(t/\tau)$ = projected growth rate in earnings

a_τ, b_τ, c_τ, and d_τ = smoothed cross-sectional regression parameters

---

1. Transactions' costs were ignored at the beginning of 1957 to obtain a starting portfolio. The transactions' cost parameters used in subsequent portfolio revisions (see Table 3) were selected for illustration purposes only and do not represent the results of empirical estimation.

Table 3

Parameter Values for Numerical Example

Parameter	Symbol	Value
Number of periods in investment horizon	T	3
Duration of Period 1 Duration of Period 2 Duration of Period 3		1 year 1 year 3 years
Duration of portfolio management runs Year of initial selection Year of last revision		 1956 1965
Number of securities (including cash)	N	7
Objective function weights Period 1 Period 2 Period 3	 w_1 w_2 w_3	 0.60 0.30 0.10
Objective function risk coefficient	λ	3000
Purchase upper bound parameters (all securities except cash) Limit on company ownership Limit on portfolio investment	 q_i u_i	 0.10 0.40

Table 3 (cont.)

Parameter	Symbol	Value
Marginal cost per transaction dollar (purchases and sales, all securities except cash, all periods)		
Cost curve segment 1	c_1	0.015
Cost curve segment 2	c_2	0.025
Cost curve segment 3	c_3	0.100
Trading volume break points as a percentage of average monthly trading volumes (purchases, sales, all securities except cash, all periods)		
Cost curve segment 1	δ_1	0.20
Cost curve segment 2	δ_2	0.50
Cost curve segment 3	δ_3	∞
Market value of initial portfolio (held in cash)	M(1956)	25 million
Liquidity requirement (fraction of portfolio market value to be held in cash)	K	0.01
Dividend income requirement	D^*	0

PORTFOLIO TRANSACTIONS AND EXPECTATIONS

7 Security Universe Lambda = 3000

BEGINNING YEAR 1957 DECISION REPORT

	1956 SHARES	TRANS SHARES	TRANS COSTS	1957 SHARES	MKT. V. (000)	P.C. M.V.	1957 SHARES	EXP. PRICE	P.C. M.V.	1958 SHARES	EXP. PRICE	P.C. M.V.	1961 SHARES	EXP. PRICE	P.C. M.V.
GENERAL MILLS, INC.	0.0	210.9	0.0	210.9	4749.7	19.0	210.9	24.8	17.3	240.7	26.4	18.2	240.7	31.4	15.2
EASTMAN KODAK CO.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	21.3	0.0	0.0	24.1	0.0	32.9	32.4	2.1
INTERNATIONAL BUSINESS MACHINE	0.0	208.2	0.0	208.2	9999.8	40.0	208.2	59.0	40.7	208.2	71.0	42.3	208.2	107.1	44.8
AMERICAN TELEPHONE + TELEGRAPH	0.0	0.0	0.0	0.0	0.0	0.0	0.0	30.2	0.0	0.0	31.4	0.0	0.0	34.9	0.0
SEARS, ROEBUCK + CO.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	17.8	0.0	0.0	20.3	0.0	0.0	27.3	0.0
NATIONAL DAIRY CORP.	0.0	516.3	0.0	516.3	10000.0	40.0	516.3	22.6	38.7	516.3	24.7	36.5	516.3	30.9	32.0
CASH AND DIVIDENDS	25000.0		0.0	250.0	250.0	1.0	1010.3	1.0	3.3	1065.5	1.0	3.0	2931.5	1.0	5.9
PORTFOLIO VALUE	25000.0				24999.5		30203.3			34974.2			49837.8		
EXPECTED DIVIDENDS	0.0						760.3			815.5			2681.5		
TRANSACTIONS COSTS			0.0							0.0			0.0		

BEGINNING YEAR 1958 DECISION REPORT

	1957 SHARES	TRANS SHARES	TRANS COSTS	1958 SHARES	MKT. V. (000)	P.C. M.V.	1958 SHARES	EXP. PRICE	P.C. M.V.	1959 SHARES	EXP. PRICE	P.C. M.V.	1962 SHARES	EXP. PRICE	P.C. M.V.
GENERAL MILLS, INC.	210.9	-13.9	5.9	197.0	4174.6	14.4	197.0	22.4	13.1	197.0	23.4	11.9	0.0	26.2	0.0
EASTMAN KODAK CO.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	23.0	0.0	2.7	25.9	0.2	203.2	35.2	13.1
INTERNATIONAL BUSINESS MACHINE	208.2	-9.0	6.0	199.3	13319.9	46.0	199.3	77.5	45.6	208.2	90.8	49.0	208.2	130.6	49.9
AMERICAN TELEPHONE + TELEGRAPH	0.0	0.0	0.0	0.0	0.0	0.0	0.0	30.5	0.0	0.0	31.4	0.0	0.0	34.1	0.0
SEARS, ROEBUCK + CO.	0.0	119.0	32.0	119.0	1600.8	5.5	119.0	16.9	6.0	0.0	18.0	0.0	0.0	20.7	0.0
NATIONAL DAIRY CORP.	516.3	0.0	0.0	516.3	9561.1	33.0	516.3	21.0	32.1	609.0	22.8	35.9	609.0	27.9	31.1
CASH AND DIVIDENDS	1033.8	-751.8	0.0	282.0	282.0	1.0	1113.3	1.0	3.3	1172.9	1.0	3.0	3190.0	1.0	5.9
PORTFOLIO VALUE	28982.3				28938.4		33852.6			38603.9			54517.3		
EXPECTED DIVIDENDS	783.8						831.3			890.9			2908.0		
TRANSACTIONS COSTS			43.9							141.7			294.3		

EXHIBIT 2*

SUMMARY DATA FROM A TEN YEAR INTER-TEMPORAL PORTFOLIO MANAGEMENT TRIAL

Number of Securities = 7 Purchase Upper Bound = 40%

Lambda = 3000 Weight Vector W = (.6, .3, .1)

(a) Performance Data and Expectations Held at the End of Each Year during the Trial

YR	YR END MV.(000)	DVNDS (000)	TRNS CSTS	1 YR YLD	5 YR YLD	ONE YR EXPECTATNS.. MV.(000)	YLD	CVAR	FIVE YEAR EXPECTATNS.. MV.(000)	YLD	CVAR	WTD EXPECTATNS (OF) MV.(000)	YLD	CVAR
56	25000.0	0.0	0.0	0.0		30203.3	20.8	5.4	49837.8	99.4	8.6	33598.0	34.4	6.2
57	28982.3	783.8	43.9	15.9	0.0	33852.6	16.8	4.7	54517.3	88.1	14.4	37344.5	28.9	5.1
58	38435.6	852.6	21.3	32.6	0.0	44226.6	15.1	5.8	72041.1	87.4	13.7	49041.2	27.6	5.7
59	52531.2	985.4	1262.5	36.7	0.0	57093.9	8.7	6.5	75322.9	43.4	8.2	59211.7	12.7	5.0
60	61853.6	1431.2	37.0	17.7	0.0	70087.0	13.3	6.8	86000.4	39.0	5.1	71800.3	16.1	4.9
61	83984.0	1501.2	1884.0	35.8	235.9	92337.3	9.9	6.5	100467.6	19.6	2.1	92633.2	10.3	4.3
62	80996.4	987.4	1346.2	-3.6	179.5	92739.6	14.5	7.3	110704.3	36.7	10.4	94111.2	16.2	4.6
63	82362.8	1164.0	185.3	1.7	114.3	95654.4	16.1	8.2	115127.4	39.8	16.0	97291.1	18.1	4.2
64	93420.7	1282.2	1619.6	13.4	77.8	104354.2	11.7	5.0	137454.4	47.1	15.4	107909.9	15.5	2.9
65	108276.0	2063.7	214.3	15.9	75.1	130603.7	20.6	3.8	176476.2	63.0	14.0	136655.1	26.2	2.9

* See Notes following Exhibit 2.

EXHIBIT 2 (continued)

(b) PORTFOLIOS HELD DURING YEARS INDICATED = Number of Shares (000) and Percentage of Beginning Year Market Value

	1957 SHARES	1957 PC.	1958 SHARES	1958 PC.	1959 SHARES	1959 PC.	1960 SHARES	1960 PC.	1961 SHARES	1961 PC.
GENERAL MILLS, INC.	210.9	19.0	197.0	14.4	181.9	11.9	9.0	0.6	0.0	0.0
EASTMAN KODAK CO.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
I.B.M.	208.2	40.0	199.3	46.0	199.3	49.9	116.9	32.5	133.4	39.1
AM. T. & T.	0.0	0.0	0.0	0.0	0.0	0.0	439.2	34.1	439.2	33.3
SEARS, ROEBUCK + CO.	0.0	0.0	119.0	5.5	160.2	6.8	0.0	0.0	0.0	0.0
NATIONAL DAIRY CORP.	516.3	40.0	516.3	33.0	536.7	30.5	570.7	27.5	581.8	25.8
CASH	250.0	1.0	282.0	1.0	375.8	1.0	2748.2	5.4	1123.6	1.8

	1962 SHARES	1962 PC.	1963 SHARES	1963 PC.	1964 SHARES	1964 PC.	1965 SHARES	1965 PC.	1966 SHARES	1966 PC.
GENERAL MILLS, INC.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
EASTMAN KODAK CO.	0.0	0.0	303.6	19.1	395.5	26.3	573.3	40.1	573.3	49.5
I.B.M.	133.4	42.6	98.5	30.5	91.7	26.6	0.0	0.0	0.0	0.0
AM. T. & T.	430.0	31.8	299.2	22.9	214.6	16.7	529.8	40.1	654.5	39.3
SEARS, ROEBUCK + CO.	0.0	0.0	0.0	0.0	33.0	1.8	0.0	0.0	84.0	5.2
NATIONAL DAIRY CORP.	0.0	0.0	31.5	1.3	31.5	1.2	31.5	1.3	0.0	0.0
CASH	20962.2	25.5	20953.8	26.3	22497.3	27.4	16905.0	18.4	6390.6	5.9

Notes for Exhibit 2 -- Explanation of Column Headings

Column Heading	Description
YR	Year end at which portfolio revision takes place.
YR END MV	Year end market value, including dividends (before revision)
DVNDS	Portfolio dividends accumulated during the previous year
TRNS CSTS	Transactions' costs incurred in revising portfolio
1 YR YLD	Percentage growth in portfolio market values during the previous year -- $\left[MV(t) - MV(t-1) \;/\; MV(t-1)\right] \cdot 100$
5 YR YLD	Percentage increase in portfolio market value during the previous five years -- $\left[\left(MV(t) - MV(t-5)\right) / MV(t-5)\right] \cdot 100$

One-Year Expectation

MV	Expected market value at end of next year
YLD	Estimated percentage increase in portfolio value during the next year $= \left[\frac{\hat{M}(1/\tau)}{M(0/\tau)} - 1.0\right] \cdot 100$
CVAR	The coefficient of variation associated with the expected one-year return $= \left[\frac{\left[\mathrm{Var}\,\tilde{M}(1/\tau)\right]^{1/2}}{M(0/\tau)}\right] \cdot 100$

Notes for Exhibit 2 (Cont.)

Five-Year Expectation

MV	Expected market value at the end of the five-year planning horizon (including dividends) of portfolio held during planning period 3 (i.e., years 3, 4, and 5).
YLD	Estimated percentage increase in portfolio value over five-year planning horizon
CVAR	The coefficient of variation associated with expected five-year return

Weighted Expectations (Objective Function)

MV = a weighted average of expected portfolio market values

$$= w_1 \hat{M}(1/\tau) + w_2 \hat{M}(2/\tau) + w_3 \hat{M}(3/\tau)$$

$$\text{YLD} = \left[\frac{w_1 \hat{M}(1/\tau) + w_2 \hat{M}(2/\tau) + w_3 \hat{M}(3/\tau)}{M(0/\tau)} - 1.0 \right] 100$$

$$\text{CVAR} = \left[\frac{\left[\text{Var}\left(w_1 \tilde{M}(1/\tau) + w_2 \tilde{M}(2/\tau) + w_3 \tilde{M}(3/\tau) \right) \right]^{1/2}}{M(0/\tau)} \right] 100$$

Table 4

Ex Post Portfolio Return* and Risk**

Number of Securities = 7

Purchase Upper Bound = 40%

Weight Vector (w_1, w_2, w_3) = (0.6, 0.3, 0.1)

Portfolio λ Value (Managed Portfolios)	1 Year		5 Year		WTD***	
	Ret	Risk	Ret	Risk	Ret	Risk
700	12.7	8.0	81.8	19.2	22.5	14.1
1000	14.0	8.8	92.9	23.3	25.0	14.9
2000	16.7	10.8	116.3	30.7	30.0	17.9
3000	17.7	11.8	125.8	30.3	31.9	18.2
7500	18.7	12.7	136.0	28.2	33.9	16.7
Equal Dollar Portfolio	18.0	11.5	128.8	31.8	32.5	17.2
Market Weighted Portfolio	16.6	11.7	115.5	36.3	29.8	18.8

* Geometric Growth Rate compounded on interval specified -- percentages.

** Coefficient of Variation of Market Value Forecast Errors -- percentages.

*** Weighted Returns and Coefficients of Variation -- percentages.

Figure 4

Inter-Temporal Period Portfolio Management Model

Ex Post Annual Asset Growth and Risk Evaluation

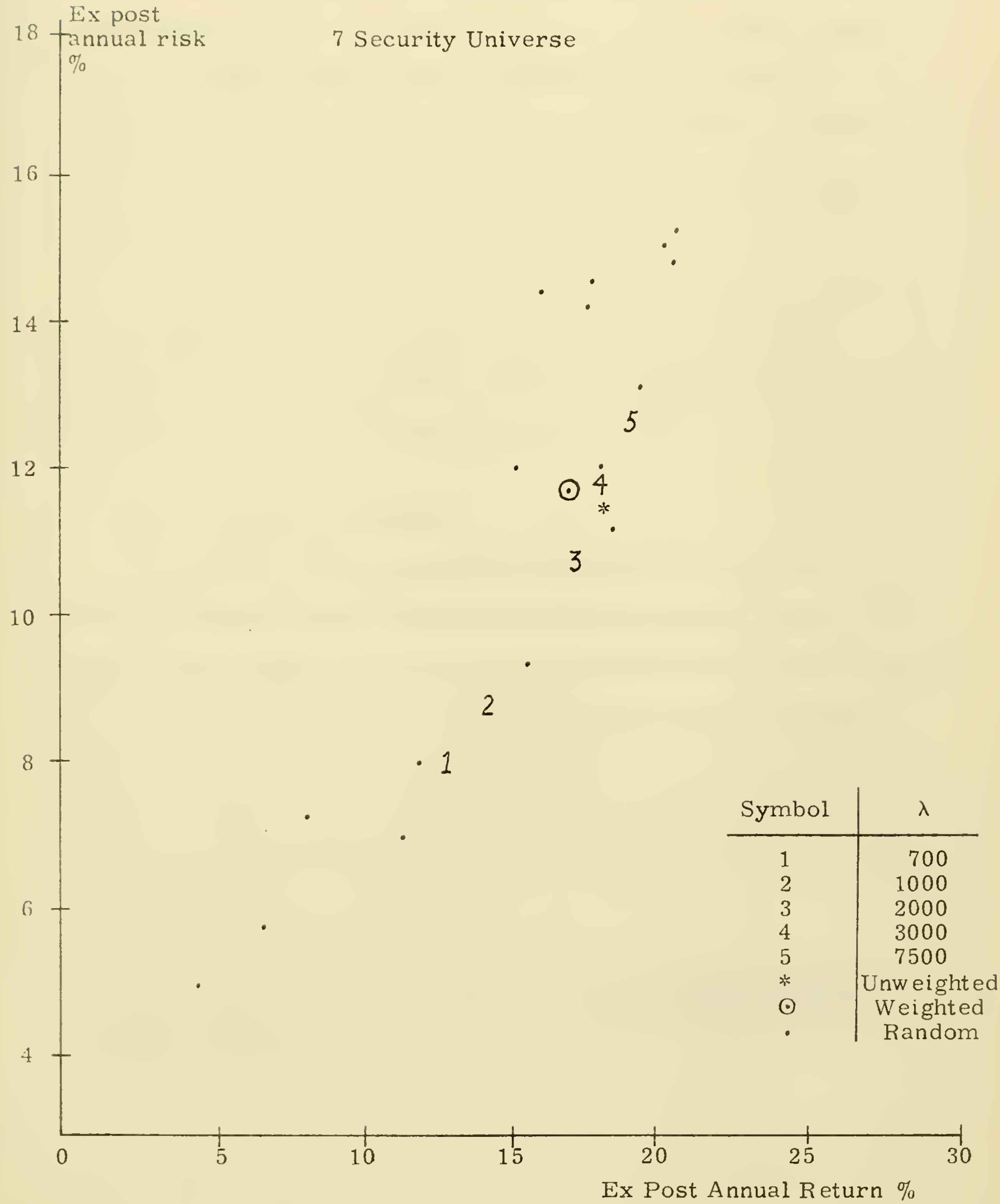


Exhibit 1 displays the optimal decision for the initial portfolio selection at the beginning of 1957 and the revised multiple-period portfolio vector obtained one year later. For each of the two years, the exhibit shows the initial portfolio, the transactions' costs involved in revising that portfolio as well as projections of the portfolios which, on the basis of current data, are expected to be held during Periods 2 and 3 of the planning horizon.

Exhibit 2 displays summary data for the ten-year portfolio management trial. Part (a) contains year-end performance and expectations data. Part (b) lists the portfolio that would have been held if the procedure had been applied in practice.[1]

(b) Ex Post Evaluation

The model was used (with different values) to obtain the same type of data for several other "managed" portfolios. Also, two market index portfolios were set up at the beginning of 1957 and managed over the ten-year period with dividends re-invested and transactions' costs deducted. These portfolios correspond to first, an equal dollar investment portfolio in the common stock universe considered, and second, a market value proportions portfolio in which the securities are

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1. The quadratic programming code used in obtaining the optimal portfolios was originally written at the Rand Corporation (QP4 — Wolfe Algorithm — Fortran 4) and converted at MIT for the IBM 360 computer.

weighted according to the total market value of their outstanding shares. In addition, sixteen random portfolios were chosen. The random portfolios were not revised during the ten-year test. The initial capitalization of each comparison portfolio was twenty-five million dollars.

The ex post results for the ten-year trials are summarized in Table 4, showing the achieved returns and volatility measures for the managed and market index portfolios. Data from Table 4 and the results from the random portfolio tests are plotted in Fig. 4. The evaluation period in this exhibit is one year. The ex post results for all review intervals (1-year, 5-year, and weighted) show a high correlation between ex ante risk and return expectations and ex post realizations.

VI. CONCLUDING REMARKS

In this paper the author has extended his previous work on single-period portfolio selection models to include consideration of inter-temporal effects. The model presented retains the important practical property of simpler models of computational feasibility for security universes of size relevant to institutional investors. The computational technology currently exists for handling security universes of practical size. The solution procedure is a modified version of the simplex method used for solving linear programming problems.(1) The number of constraints in the resulting modified linear programming problem is 23N-8, where N is the number of securities in the selection universe.

1. Modified to enforce restricted basis entry requirements.

For universes of practical size (e.g., 200 securities), the resulting modified linear programming problem would have approximately 4600 rows, which is well within the capabilities of existing large-scale linear programming codes. The author has treated up to 30 stock universes in tests of the model. In these trials, approximately ten minutes of IBM 360-65 computation time was required for a ten-year example, including the initial security evaluation and final ex post reporting phases.

APPENDIX

A HEURISTIC PROCEDURE FOR DEVELOPING EX ANTE RISK MEASURES

Given the investor's estimates of security prices and dividends for each period during the multiple-period planning horizon, smoothing techniques can be used to obtain estimates of the current value forecasting error variances from the errors observed in past value forecasts. Weighted estimation techniques are used to allow for the inherent non-stationary nature of the forecast error time series.

Let $V_i(t/\tau)$ = the investor's estimate of the future value of security i (i = 1, ..., N) at time periot t (t = 1, ..., T), during the planning horizon based on information up to the beginning of the planning horizon (time τ). We require estimates of the forecase error variances associated with the current value level projections. This will entail estimation of a covariance matrix $\underline{\Sigma}(\tau)$ whose elements are the forecasting error variances and covariances associated with the multi-period prediction errors for all securities and all forecast intervals.

Assuming the series of forecasting errors to be locally stationary, an exponentially weighted estimator of the t period forecasting error variance for security i, based on data up to period τ, is given by

$$\hat{\sigma}_{iitt}(\tau) = \sum_{\nu=0}^{\tau} \alpha(1-\alpha)^{\nu}\left(V_i(\tau-\nu) - \hat{V}_i(\tau-\nu/\tau-\nu-t)\right)^2 + (1-\alpha)^{\tau+1}\,\hat{\sigma}_{iitt}(0)$$

where

α = a smoothing parameter $(0 < \alpha < 1)$

$\hat{\sigma}_{iitt}(0)$ = the initialization value of the t period forecast error variance for security i

A relationship which is useful for updating the estimate of the forecasting error variance when a new observation $V_i(\tau)$ becomes available is easily obtained from the above expression:

$$\hat{\sigma}_{iitt}(\tau) = \alpha\Big(V_i(\tau) - \hat{V}_i(\tau/\tau - t)\Big)^2 + (1 - \alpha)\ \hat{\sigma}_{iitt}(\tau - 1)$$

The above expressions can be generalized to obtain estimated forecasting error variances for all periods $(t = 1, \ldots, T)$ of the planning horizon for all securities $(i = 1, \ldots, N)$.

Let $\sigma_{ii'tt'}(\tau)$ = the covariance between the t period forecasting error for security i and the t' period error for security i' at time τ and let $\hat{\sigma}_{ii'tt'}(\tau)$ be an exponentially weighted estimator of $\sigma_{ii'tt'}(\tau)$. The updating relationship for $\hat{\sigma}_{ii'tt'}(\tau)$ is given by

$$\hat{\sigma}_{ii'tt'}(\tau) = \alpha\Big(V_i(\tau) - \hat{V}_i(\tau/\tau - t)\Big)\Big(V_{i'}(\tau) - \hat{V}_{i'}(\tau/\tau - t')\Big) + (1 - \alpha)\ \hat{\sigma}_{ii'tt'}(\tau)$$

where

i, i' = $1, \ldots, N$

t, t' = $1, \ldots, T$

An ex ante "risk matrix" $\hat{\underline{\Sigma}}(\tau)$ can now be defined by

$$\hat{\underline{\Sigma}}(\tau) = \left\| \hat{\sigma}_{ii'tt'}(\tau) \right\|$$

The above expressions permit the elements of the forecasting error covariance matrix to be generated adaptively as new security price and divident data become available during the continuing portfolio management process.