LOT SIZE DETERMINATION IN MULTI-STAGE ASSEMBLY SYSTEMS

WALLACE B. CROWSTON

MICHAEL WAGNER

JANUARY, 1971 508-71

LOT SIZE DETERMINATION IN MULTI-STAGE ASSEMBLY SYSTEMS

In a multi-stage assembly system each stage requires inputs from a number of immediate predecessor stages and it supplies, in turn, one immediate successor stage. An efficient dynamic programming algorithm for lot size determination at all stages is derived for the infinite horizon case under the assumption of constant demand. For the finite horizon case with deterministic demand, an application of Benders' mixed integer programming algorithm is presented. For the special case of one predecessor for each stage, a dynamic programming algorithm is developed.

LOT SIZE DETERMINATION IN MULTI-STAGE ASSEMBLY SYSTEMS

Wallace B. Crowston and Michael Wagner

Sloan School of Management, M. I. T.

INTRODUCTION

The classical economic lot size model is used to determine the lot size that minimizes the sum of production and inventory carrying costs for a single stage system. Demand is assumed to be continuous and constant with no stockout permitted. A number of extensions to the basic single stage model have been devised [7] including provisions for non-instananeous production, and for discrete, but constant, demands. A further large class of extensions considers stochastic demands. A distinguishing characteristic of all of these models is that the objective is minimization of costs over an infinite horizon.

A different fundamental approach to determination of lot sizes is based on the assumption of discrete known demands occurring through a finite horizon. Such an approach allows consideration of non-constant demands and a time varying objective function. Manne [10], Dzielinski and Gomory [4], H. Wagner and Whitin [14], and H. Wagner [15] develop results for a single facilitv. Dantzig [3] introduces the concept of multi-facility systems in which production of items at one facility requires inputs from other facilities, and obtains solutions for a linear

cost structure. Veinott [13] and Zangwill [16-19] consider extensions to concave cost objectives including the important case of a production set up charge with linear production and holding cost.

A multi-stage assembly system is a special case of Veinott's general multi-facility system in that each facility or stage may have any number of predecessor stages but is restricted to at most a single successor. Gorenstein [5,6] considers systems of this form in the context of the finite horizon planning models of Manne [10], and Dzielinski and Gomory [4]. In this paper we develop a finite horizon model and present solution techniques for two cases: the multi-echelon system with each stage having a single predecessor, and the more general multi-stage assembly system. The former case has been treated by Zangwill [19] for concave objective functions. We modify his dynamic programming algorithm to take substantial advantage of the particular objective function under consideration. In the latter case we investigate the application of Benders' partitioning procedure [1] for mixed integer problems, and discuss how the assembly structure can be exploited to computational advantage.

For the case of an infinite horizon we show in this paper that the optimal lot size at any stage is an integer multiple of the lot size at the succeeding stage. Using this result a total cost model is formulated and a bounded dynamic programming algorithm is presented for the optimal solution of the problem. In a companion paper, the authors with Henshaw [2] discuss heuristic solution methods for this problem and give comparisons of computational times and solution values for the heuristic routines and a version of the dynamic programming algorithm to be presented below.

Schussel [11] discusses the problem and develops a heuristic for a more general criterion function than that described in this paper.

Problem Description

In a multi-stage system, the manufacture of final product requires completion of a number of operations or stages. A stage might consist of an operation such as procurement of raw materials, fabrication of parts, or assembly. A fixed sequence of operations is assumed, so that output from one stage serves as input to an immediate successor stage. The final stage is an exception in that its output is a finished product used to service customer demand. Output from any stage may be stored until needed in that stage's inventory.

A multi-stage assembly system is characterized by the restriction that each stage has at most one immediate successor. We emphasize that, in general, a stage may have any number of immediate predecessors. Examples of multi-stage assembly systems are depicted in Figures 1 and 2.

We shall denote a stage F_n, where n is an index ranging from 1 to N, and F_N is the final stage. Let a(n) be the index of the <u>immediate</u> successor of F_n, A(n) the set of indices of all successors, b(n) the set of indices for all <u>immediate</u> predecessors and B(n) for all predecessors. In Figure 2 for example

$$a(6) = [7],\ b(6) = [4,5],\ A(6) = [7,17],$$

$$B(7) = [1,2,3,4,5,6].$$

3 level, 1 stage per level system

FIGURE 1

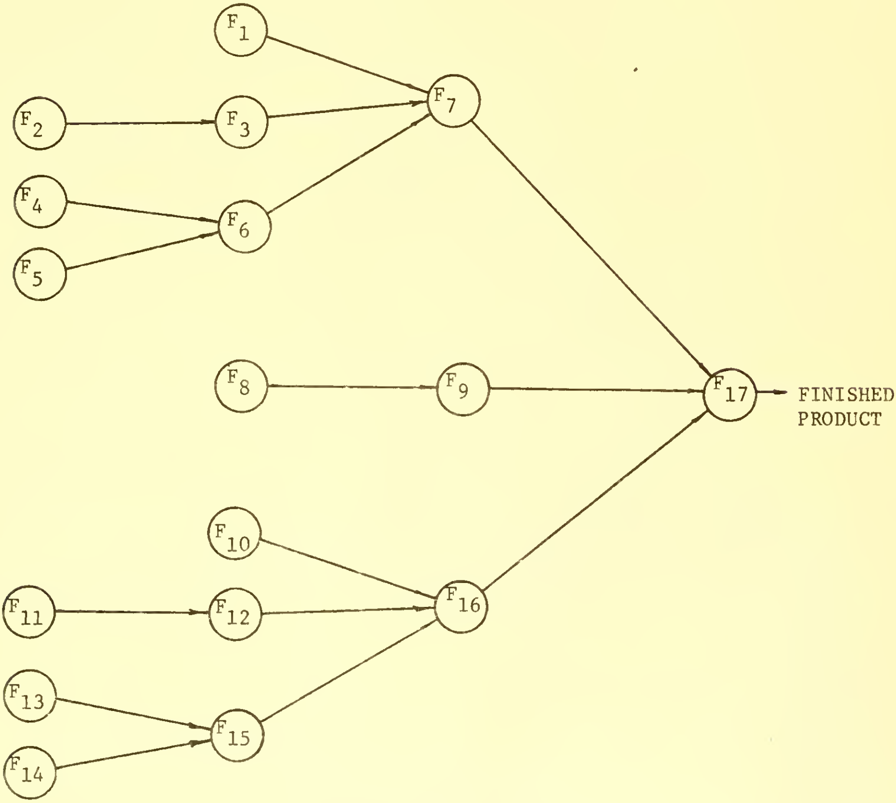


FIGURE 2

L_1 L_2 L_3 L_4

4 levels, multi-stage per level system

For expositional convenience, we introduce the notion of a level, where stages are assigned to levels according to: the final stage F_N is in level L_M, and F_n is in level L_M if its successor $F_{a(n)}$ is in L_{M+1}.

It is assumed throughout that demand is known with certainty. The objective is minimization of the cost of satisfying all demand with no backorders. Costs are assumed to depend upon the stage, F_n, there being a fixed charge for production setup, s_n (\$/setup), and a linear inventory holding cost, H_n (\$/unit/time). One unit at a stage is the quantity required in one unit of final product.

We will find it convenient to refer to an incremental inventory holding cost, h_n, defined by: $h_n = H_n - \sum_{m \varepsilon b(n)} H_m$. The concept of an incremental holding cost is closely related to that of "value added" at a production stage. In fact, the holding cost in many situations might be: $H_n = C_n I$, where C_n = total value of a completed stage n unit and I is a cost of carrying inventory and $h_n = c_n I$, where c_n = value per unit added by the stage n process. We note that a direct per unit production cost, p_n, can easily be added to the models discussed herein, but such a term has no effect upon the lot size decision and simply adds a constant to the total costs.

THE FINITE HORIZON MODEL

Introduction

A finite horizon model makes it possible to express non-constant demand for final product and time varying objective functions. The special case of a system with a single stage for each level, such as that depicted in Figure 1, has been treated by Zangwill [19]. He develops a dynamic programming algorithm under the assumption of a general concave cost function. Under the more restrictive assumption of a time invariant cost function, we present a characterization of the form of an optimal solution to the multi-stage per level assembly system. Application of Zangwill's algorithm is then discussed for the single stage per level case with simplifications arising from our restricted objective function noted. It is found that Zangwill's algorithm cannot be simply extended to the multiple stage per level case. We thus turn to a mixed integer linear programming approach and describe application of Benders' partitioning procedure.

Model Formulation

We assume that demand occurs at discrete points in time, production is instantaneous and that we wish to minimize costs over a finite number of time periods T. Then the problem of economic lot size determination can be given a mathematical programming formulation which shall be

referred to as Problem I:

Let Q_{nt} = Production quantity at stage n at time t,

Y_{nt} = Ending inventory at stage n at time t

$$d_{nt} = \begin{cases} 1 & \text{if } Q_{nt} > 0, \\ 0 & \text{if } Q_{nt} = 0; \end{cases}$$

given R_t = Demand for final product at time t

H_n = One period inventory holding cost

S_n = Production set up cost

then minimize $Z = \Sigma_n \Sigma_t (H_n \cdot Y_{nt} + S_n d_{nt})$ I.A

subject to $Q_{nt} + Y_{nt-1} - Y_{nt} - Q_{a(n)t} = 0$ n=1,...,N, t=1,...,T, I.B

$Q_{nt} \geq 0,\ Y_{nt} \geq 0$ for n,t

where $Y_{n0} = Y_{nT} = 0$ for all n

$Q_{a(N)t} = R_t$ for all t

$M_t d_{nt} - Q_{nt} \geq 0,\ d_{nt}$ 0-1 integer for all n,t I.C

where M_t is a suitably large constant, namely, $M_t = \sum_{m=t}^{T} R_m$.

Form of an Optimal Solution

This model is an example of the multi-facility economic lot-size model discussed by Veinott [11]. In this connection, we remark that the objective function I.A is concave. Furthermore the constraint set I.B is of the form Ax = b, where A is a Leontief matrix, that is, each column of A has exactly one positive element. Following Veinott, we obtain the following characterization of an optimal solution: a) production at a stage does not occur if entering inventory already exists; and b) production at a stage does not take place unless production also occurs simultaneously at the immediate successor stage. These results are summarized in Theorem 1.

Theorem 1: Form of the Optimal Solution. There exists an optimal solution to Problem I with the properties that

a) $Q_{nt} \cdot Y_{nt-1} = 0$ for all n,t; and

b) $Q_{nt} \cdot (1 - d_{a(n)t}) = 0$ for all n,t.

A detailed proof is given in the Appendix. Approximately stated, property a) is direct consequence of Veinott's Corollary 2 [13] which characterizes extreme point solutions of Leontief substitution systems. Property b) depends upon the time invariance of the cost functions H_{nt} and P_{nt}. We start with a presumed optimal solution and show that it can be modified so as to satisfy the conditions of Theorem 1 with

identical setup costs and at no increase in inventory holdings costs. Theorem 1 provides the basis for a dynamic programming algorithm for solution of the single predecessor case.

Dynamic Programming: One Stage per Level

We now consider the case in which each stage has no more than one predecessor. This model has been analyzed by Zangwill [19] for concave cost functions. He develops the dynamic programming recursion:

$$F_{nt}(\alpha,B) = \underset{\alpha-1\leq\gamma\leq B}{\text{Min}} \{P_{a(n)t}\sum_{m=\alpha}^{B} R_m + F_{a(n)t}(\alpha,n) + C_{nt}\sum_{m=\gamma+1}^{B} R_m + F_{nt+1}(\gamma+1,B)\} \tag{1}$$

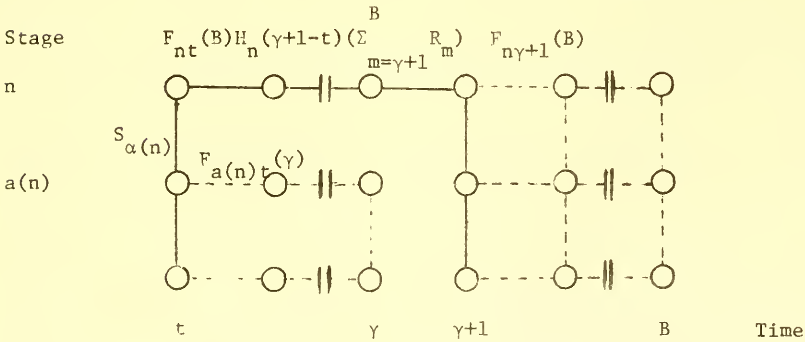
where $F_{nt}(\alpha,B)$ is the optimal cost of sending $\sum_{k=\alpha}^{B} R_k$ units from node (n,t) to final destinations (N,α), $(N,\alpha+1),\ldots,(N,B)$; and $C_{nt}(\cdot)$ and $P_{nt}(\cdot)$ are general concave holding and production cost functions, respectively. Theorem 1, property b) allows simplification of this recursion for the particular cost functions under consideration. Let $F_{nt}(B)$ be the optimal cost of sending $\sum_{k=t}^{B} R_k$ units through node (n,t) to final destinations (N,t), (N,t+1),...,(N,T) given that production does take place at (n,t). Then

$$F_{nt}(B) = \underset{t\leq\gamma\leq B}{\text{Min}} \{S_{a(n)} + F_{\alpha(n)t}(\gamma) + H_n(\gamma-t+1)(\sum_{m=\gamma+1}^{B} R_m) + F_{n\gamma+1}(B)\}. \tag{2}$$

The simplification arises from the consequence of property b) that if production occurs at any stage at time t, it must be partly to satisfy final demand at time t. Thus $\alpha = t$, given that production occurs, and the number of dynamic programming stages can be significantly reduced.

The recursions are computed in the usual dynamic programming fashion, starting with $F_{NT}(T,T)$, then $F_{NT-1}(T-1)$, $F_{NT-1}(T)$,..., $F_{b(N)T}(T)$,...,$F_{01}(T)$. H_0, $F_{a(N)t}(B)$, and $S_{a(N)}$ are taken to be zero for all t and B, and a(0) is the first stage. Production must take place at all stages for t = 1; hence $F_{01}(T)$ is the value of the optimal solution. The number of dynamic programming states required is $(N + 1)T \cdot (T + 1)/2$.

FIGURE 3



$$F_{nt}(B) = \min_{t \le \gamma \le B} \{S_{a(n)} + F_{a(n)t}(\gamma) + H_n(\gamma+1-t)(\sum_{m=\gamma+1}^{B} R_m) + F_{n\gamma+1}(B)\}$$

Mixed Integer Programming

The dynamic programming recursions (1), (2), cannot be extended in a straightforward fashion to cases in which stages have more than a single predecessor. Attempts to accomplish this extension lead to a computationally unreasonable number of dynamic programming states. Since Problem I is a mixed integer linear programming problem it can be solved by general methods. We discuss the application of the partitioning procedure of Benders.* We will show that the problem structure can be exploited to computational advantage in the single stage per level case, and that the multiple stages per level case decomposes usefully.

Benders' procedure requires repeated solution of two problems: a pure integer linear problem formed from the integer variables of the original problem, and a continuous linear problem which is the dual of the linear programming problem defined by treating all integer variables as constants in the original. Recasting Problem I with the integer variables d_{nt} treated as constants, we obtain Problem II

$$Z = \Sigma_n \Sigma_t S_n d_{nt} + \min_Q \Sigma_n \Sigma_t H_n \cdot Y_{nt} \qquad \text{II.A}$$

subject to $$Q_{nt} + Y_{nt-1} - Y_{nt} - Q_{a(n)t} = 0 \qquad \text{for all } n,t \qquad \text{II.B}$$

$$Q,Y \geq 0$$

where $$Q_{a(N)t} = R_t \qquad \text{for all } t \qquad \text{II.C}$$

*For a description of Benders' procedure, see Hu [8].

Associating the dual variables w_{nt} with the constraint set II.B and v_{nt} with the constraint set II.C, the dual of Problem II is Problem III:

$$Z = \Sigma_n \Sigma_t s_n d_{nt} + \max \Sigma_t R_t w_{nt} + \Sigma_n \Sigma_t m_t d_{nt} v_{nt} \qquad \text{III.A}$$

subject to

$$w_{nt} - \Sigma_{j\varepsilon b(n)} w_{jt} - v_{nt} \leq 0 \qquad \text{III.B}$$

$$w_{nt} - w_{nt-1} \leq 0 \qquad \text{III.C}$$

$$w_{nt}, v_{nt} \geq 0 \qquad \text{for all } n,t.$$

Benders' procedure uses Problem IV to generate constraints for the pure integer problem, Problem IV:

maximize z

subject to

$$z \geq \Sigma_t R_t w^m_{Nt} + \Sigma_n \Sigma_t (S_n - M_t v^m_{nt}) d^m_{nt} \qquad m=1,\ldots,M \qquad \text{IV}$$

where there have been M iterations and w^m, v^m are obtained from the m^{th} solution of Problem IV, $0 \leq d_{nt} \leq 1$ integer for all n.

We also insist $d_{n1} = 1$ if $R_1 > 0$ for all n. Problem IV returns values of the integer variables d_{nt} for the next solution of Problem III. When $Z = z$, the procedure terminates.

The point we wish to make is that the continuous Problem III can be solved with very little effort. In the case of single stages per level, Problem III is a simple shortest path problem and can be solved by first applying the recursion

$$w_{nt} = \min\,[w_{nt-1} + H_n;\ w_{b(n)t}] \quad \text{if } d_{nt} = 1 \qquad (3)$$

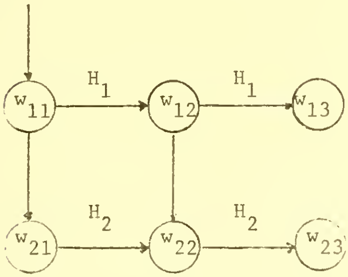
$$\quad = w_{nt-1} + H_n \quad \text{if } d_{nt} = 0$$

where $w_{0t} = 0$ for all t, $w_{n0} = 0$ for all n,

then, $v_{nt} = w_{nt} - w_{b(n)t}$ if $d_{nt} = 0$

$v_{nt} = 0$ if $d_{nt} = 1$.

In the case of multiple predecessors, Problem III decomposes into a series of problems which may be solved by the recursion (3). The procedure is specified by replacing $w_{b(n)}$ with $w'_{b(n)} = \sum_{j \varepsilon b(n)} w_j$ in (3). Figure 4 illustrates the solution of Problem III for a two stage, single stage per level assembly structure.



Given: $d_{11} = d_{21} = d_{22} = 1$

$d_{12} = d_{13} = d_{23} = 0$

$w_{11} = 0,\ w_{12} = w_{11} + H_1,\ w_{13} = w_{12} + H_1$

$w_{21} = w_{11} + 0,\ w_{22} = \min(w_{12},\ w_{21} + H_2),\ w_{23} = w_{22} + H_2$

$v_{11} = v_{21} = v_{22} = 0,\ v_{12} = w_{12},\ v_{13} = w_{13},\ v_{23} = w_{23} - w_{13}$

FIGURE 4

THE INFINITE HORIZON MODEL

Introduction

In this section we focus on the problem of determining the set of optimal lot sizes in an N-stage assembly system under assumptions of constant discrete demand and infinite production rates with no back-orders. We shall refer to this as the Basic Problem. In addition to the Basic Problem described above we briefly discuss the case of non-instantaneous production and the case of delivery delay between stages. We also examine the implications of these models for the development of heuristic solution techniques for more complicated multi-facility structures.

For the special case of a single predecessor for each stage, or serial production, there are two recent contributions. The model of Taha and Skeith [12] allows non-instantaneous production, delay between stages and back-orders for the product of the final stage. They assume that in an optimal solution the lot-size at a stage is an integer multiple of the lot-size at the succeeding stage and suggest the problem be solved by examining all combinations of such integer values. Jensen and Khan [9] also allow non-instantaneous production but do not use the assumption of positive integers. Instead they have constructed a simulation model which evaluates the average inventory at a stage, given the lot size at that stage and at the succeeding stage, along with the production rate at both stages. A dynamic programming algorithm is then formulated in which the simulation model is used in evaluation of each functional equation. They note that high average inventories result if the integer multiple

assumption is not followed and discuss a problem for which non-constant lot size is optimal.

For the multiple predecessor case Schussel [11] develops a simulation model and heuristic decision rule which again assumes that integer multiples are optimal. He adds a "learning curve" function so that unit production cost decreases with lot size and allows costs to be discounted over time. Crowston, Wagner and Henshaw [2] tested four heuristic rules and compared them with a version of the dynamic programming algorithm developed in this paper.

In this section we prove that under certain assumptions the "integer multiple" assumption used by others is correct. A particularly simple model of the total cost structure is then formulated and a dynamic programming algorithm is developed to find optimal lot sizes for all stages in the system. It is shown that the cost structure may be used to develop upper and lower bounds on all lot sizes and thus increase the efficiency of the dynamic programming algorithm.

Form of the Optimal Solution

We consider only solutions which can be characterized by a single lot-size for each stage. Let $k_n = Q_n/Q_{a(n)}$ and $K_n = Q_n/Q_N$. A particular solution is given by $k^j = \{k_1^j, k_2^j, \ldots, k_{N-1}^j, 1\}$ and Q_N^j or by $K^j = \{K_1^j, K_2^j, \ldots, K_{N-1}^j, 1\}$ and Q_N^j. Then it can be shown that the ratio of lot sizes between successor and predecessor stages, k_n, must be a positive integer. This result is summarized in Theorem 2.

Theorem 2: Form of the Optimal Solution. If the set of all solutions to the Basic Problem which can be characterized by a set of rational lot size multiples k^j and final stage quantity Q_N^j, a minimum cost solution exists with Q_N^j and k^j all positive integers.

A detailed proof is given in the Appendix. An expression is derived for the costs associated with a lot size Q_n given $Q_{a(n)}$. This function is shown to be minimized with $k_n = Q_n/Q_{a(n)}$ a positive integer. Proof then follows by induction over the levels of the system.

We wish to emphasize that the assumption of a time-invariant lot size for each stage is quite strong. The possibility of cyclic lot sizes, for example, is thus eliminated. The restriction may be justified, in some cases, by the costs of administering changing lot sizes. In any event, Theorem 2 leads to computationally powerful algorithms for finding the optimum in a class of easily implemented solutions. Given the results of Theorem 2, we now derive expressions for the total costs of a particular solution k^j, Q_N^j.

Development of the Model

If all lot sizes within the system were equal, then, given instantaneous production and no transfer delay, inventory would be held only at the final stage. If $Q_n \neq Q_{a(n)}$ then in-process inventory occurs at F_n and the average level of such inventory is a complicated function of Q_n and $Q_{a(n)}$. The value of a unit of such inventory would be $C_n = \sum_{m \varepsilon B(n)} c_m + c_n$ and the carrying cost would be IC_n. In Figure 5 we show the inventory at each stage of a 3-stage production process with $K_1 = 6$, $K_2 = 2$ and $K_3 = 1$.

As we have implied above, existing models have been based on a determination of average inventory at a stage. A simpler but mathematically equivalent formulation results from an expression of the inventory in the total system that has undergone the activity of a particular stage. Figure 6 illustrates such system-wide inventory for the 3-stage problem. Since the demand on the system is assumed constant, the total system inventory of units that have undergone the activity of F_n will decline at rate R between successive production of Q_n. Given the optimality condition of Theorem 2, Q_n and $Q_{a(n)}$ will be produced simultaneously. Since this is true for all stages F_m, $m \varepsilon A(n)$, at the instant before Q_n is produced the complete system inventory of units that have undergone the activity of F_n will be zero. At that point in time a lot is produced and the system inventory becomes Q_n with units possibly located at F_n and F_m, $m \varepsilon A(n)$. We observe that the system inventory of the product of any stage has the familiar saw-tooth pattern of the basic E.O.Q. model and thus the average inventory for the product of F_n would be

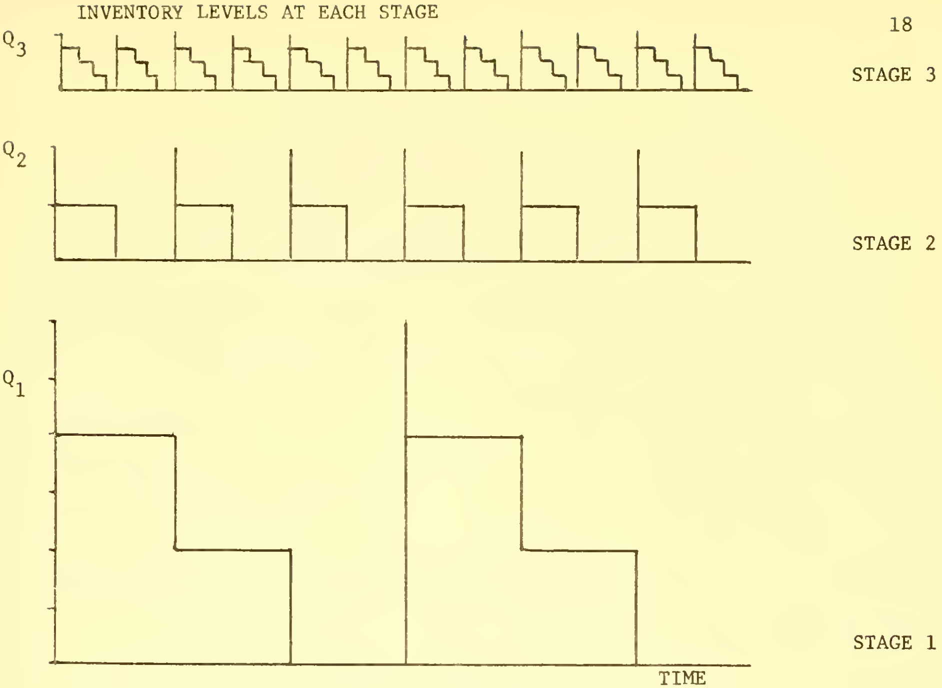


FIGURE 5

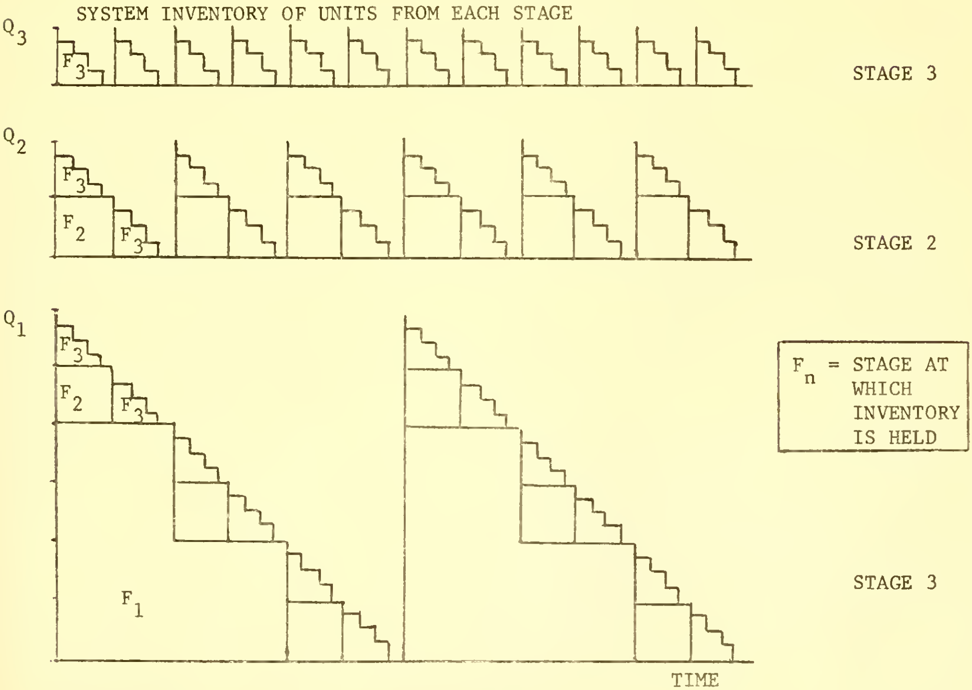


FIGURE 6

$\frac{Q_n - 1}{2}$ assuming discrete demand. The value of this inventory will be c_n per unit and therefore the holding cost will be

$(\frac{Q_n - 1}{2})h_n$ or $(\frac{Q_n - 1}{2})c_n I$.

The total cost for the product of F_m, including set-up and inventory carrying cost will be

$$TC_n = \frac{RS_n}{Q_n} + (\frac{Q_n - 1}{2})\ h_n \tag{4}$$

and the total cost for the system, s, will be

$$TC_s = \sum_{n=1}^{N} \frac{RS_n}{Q_n} + (\frac{Q_n - 1}{2})\ h_n$$

or

$$= \sum_{n=1}^{N} \frac{RS_n}{K_n Q_N} + (\frac{K_n Q_N - 1}{2})\ h_n\ . \tag{5}$$

Note that for a particular vector K^j the optimal value of Q_N will be

$$Q_N^j = \sqrt{\frac{2R\ \Sigma\ \frac{S_n}{K_n}}{\Sigma h_n K_n}} \tag{6}$$

Simple Extensions of the Model

In this section we briefly consider a special case of non-instantaneous production and the case of transfer delay between stages. If we assume production rate p_n at F_n and given $p_n \geq p_{a(n)}$ then the result of Theorem 2 applies. The cost function for the product of F_n will be

$$TC_n = RS_n/K_nQ_N + [(K_nQ_N - 1)/2][1 - R/p_n]h_n . \qquad (7)$$

Finally we observe that a transfer delay between stages simply adds a constant inventory term to either equation (4) or (7) and therefore does not affect the optimal solution.

The Dynamic Programming Algorithm

The dynamic programming algorithm is written in terms of the simplest cost structure although it is clear that it could be modified to include the cost function for non-instantaneous production. Solution proceeds from the raw material stage to F_N with the recurrence relation defined as follows.

Let L denote the set of all positive integers; $u_n(Q_n)$, the optimal cost at F_n and all prior stages F_m, $m \in B(n)$ given Q_n. Then

$$u_n(Q_n) = \frac{(Q_n - 1)}{2} h_n + \frac{RS_n}{Q_n} + \sum_{m \in b(n)} \underset{\ell \in L}{\text{minimum}}\ u_m(\ell Q_n) \ . \qquad (8)$$

Optimal solutions for the system of Figure 2 have been obtained with this algorithm in approximately ten seconds of computation time on a time-shared GE 645 system [2].

We note that an inventory space constraint at F_n may be included directly as an upper bound on Q_n. Other bounds are possible given the form of cost function (4). We will now develop both upper and lower bounds on Q_n so as to improve the theoretical efficiency of the dynamic programming algorithm.

If we assume a problem with cost structure (4), then at F_n a lower bound, b_n, on the cost of system inventory of that stage will be

$$b_n = \frac{RS_n}{\sqrt{\frac{2RS_n}{h_n}}} + \left(\sqrt{\frac{2RS_n}{h_n}} - 1\right) \left(\frac{h_n}{2}\right) \ . \qquad (9)$$

This assumes no interdependency between successive stages. Then a lower bound for the complete system, B_s will be

$$B_s = \sum_{n=1}^{N} b_n \ .$$

An upper bound on total cost B_h for the system may be derived from a feasible heuristic solution [2] to the problem. Thus an upper bound on the cost of the product of F_n will be

$$b_n + (B_h - B_s) \ .$$

Now setting expression (4) equal to the upper bound on cost, that is

$$\frac{RS_n}{Q_n} + (Q_n - 1)\frac{h_n}{2} = b_n + (B_h - B_s)$$

we may solve directly for upper and lower bounds, Q_n^u and Q_n^ℓ on Q_n.

In addition, from Theorem 2, $Q_n \geq Q_{a(n)}$. Therefore

$$Q_n \max = \min \begin{cases} Q_n^u \\ Q_m^u, \quad m \in B(m) \end{cases}$$

and

$$Q_n^\ell \min = \max \begin{cases} Q_n^\ell \\ Q_m^\ell, \quad m \in A(n) \end{cases} \ .$$

Similar bounds may be calculated for the cost structure of equation (7).

Implications of the Model

A variety of heuristic rules have been suggested for problems having a structure similar to that of the multi-stage assembly system [2,9]. In addition in industrial applications heuristics such as "constant lot size" at all stages, where the lot size is taken to be

$Q_n = \sqrt{\frac{2RS_N}{H_N}}$, or "independent determination of lot size" at each stage

are used. For the cost structure of (4) the optimal "constant lot

size" would be $\sqrt{\frac{2R\Sigma S_n}{\Sigma h_n}}$ although experimentation shows [2] that this

is a poor decision rule. If "independent determination of lot size"

is used, a common model is $Q_n = \sqrt{\frac{2RS_n}{H_n}}$. This implies the carrying cost of a unit of in-process inventory of F_n is a function of the total value of its components. Our model indicates that this results in double-counting and that the use of the incremental carrying cost, that is

$Q_n = \sqrt{\frac{2RS_n}{h_n}}$ is appropriate.

Finally we would suggest that if heuristic decision rules are constructed for the more complicated case of multiple successors, incremental costs are again appropriate.

APPENDIX

Theorem 1: Form of the Optimal Solution (Finite Horizon). There exists an optimal solution to Problem I with the properties that

a) $Q_{nt} Y_{nt-1} = 0$, and

b) $Q_{nt} (1 - d_{a(n)t}) = 0$ for all n,t.

Proof: We will give a procedure for modifying a presumed optimal solution at no increase in cost to a solution satisfying the properties of Theorem 1. First we state Lemma 1 which is a restatement of Corollary 2 by Veinott [13].

Lemma 1. Let $d_{nt} = d^*_{nt}$ for all n,t. Then the resulting system of equations I.B with the column corresponding to Q_{nt} removed if $d_{nt} = 0$ is of the form Ax = b, where A is a Leontief matrix. Furthermore, I.A is concave. Thus, there exists an optimal solution to Problem I, given $d = d^*$, with the property $Q_{a(n)t}(1 - d_{a(n)t})$ for all n,t.

Proof of Theorem 1: Start with a presumed optimal solution (Q^*, Y^*, d^*). Let $d_{nt} = d^*_{nt}$ for n,t. Applying Lemma 1, obtain a new optimal solution, $(Q, Y | d_{nt} = d^*_{nt})$, with property a) satisfied. Now suppose $Q_{nt} (1 - d_{a(n)t}) \neq 0$ for some n and t. In addition, assume for the moment that $d_{jt} = 0$ for all $j \in b(n)$. Then the cost change resulting from transferring production one time unit in the future (i.e. $Q'_{nt} = 0$, $d'_{nt} = 0$, $Q'_{nt+1} = Q_{nt+1} + Q_{nt}$, $d'_{nt+1} = 1$) is given by $Z' - Z = (\sum_{j \in b(n)} H_j - H_n) Q_n$.

If $d'_{jt+1} = 0$ for all $j \in b(n)$ and $d_{a(n)t+1} = 0$, then the process can be repeated at the same change in cost. Thus, if $Z' - Z \leq 0$, production can be transferred to a future time period τ at a savings of $(\tau - t)(Z' - Z)$ until either $d'_{j\tau} = 1$ for some $j \in b(n)$, or $d_{a(n)\tau} = 1$. If $Z' - Z \geq 0$, then production can similarly be transferred forward in time. In both cases, the resulting solution maintains property a).

Define recursively a chain of predecessors, J_{nt}, as a set of ordered pairs (i,t) where $(n,t) \in J_{nt}$ and $(i,t) \in J_{nt}$ if $(a(i),t) \in J_{nt}$. Let $P(J_{nt})$ be the set of predecessors of J_{nt}, that is, $(i,t) \in P(J_{nt})$ if $(a(i),t) \in J_{nt}$ and $(a(i),t) \notin P(J_{nt})$. We can now treat the entire chain of predecessors J_{nt} as a single stage transferring production to the future if $H_n \geq \sum_{i \in P(J_{nt})} H_i$, otherwise transferring production to an earlier time period. The procedure must terminate since each iteration reduces by at least one the number of cases where $d_{nt} \neq d_{a(n)t}$ for some n and t, and there is a finite number of these.

Theorem 2: Form of the Optimal Solution (Infinite Horizon). Of the set of all solutions to the Basic Problem which can be characterized by a set of rational lot size multiples k^j and Q_N^j, a minimum cost solution exists with Q_N^j and k^j all positive integers.

Proof: We will use Proposition 1, Proposition 2, and Lemma 2.

Proposition 1: An optimal solution to the Basic Problem with rational lot size multipliers k^j is in phase, that is, for each stage n, there is some point in time at which production occurs simultaneously with production at the successor stage a(n).

Proof: Since Q_n and $Q_{a(n)}$ are rational, stage n inventory levels cycle with period D where $D = q_n Q_n = q_{a(n)} Q_{a(n)}$ and q_n, $q_{a(n)}$ relatively prime integers. Let Δt be the smallest interval of time between production at stage n and subsequent production at stage n + 1 during the cycle. If $\Delta t \neq 0$, then all production at stage n (and stage n's predecessors B(n)) can be transferred to the future by the amount Δt with no increase in setup costs and reduced inventory costs.

Proposition 2: In a single stage system with constant discrete demand, R, and with the system in phase in accordance with Proposition 1, the total cost/unit time associated with lot-size Q_1 is given by

$$Z(Q_1) = S_1^R/Q_1 + h_1(Q_1 - 1)/2 + RH_1(q_2 - 1)/q_2$$

where q_2 is defined by $q_1/q_2 = Q/R$ and q_1, q_2 are relatively prime integers.

Proof: There are three components of cost to consider:

1. The set-up cost-- S_1R/Q_1.

2. The familiar inventory cost which arises from periodic addition to the entire system of the amount Q_1, and the intermittent flow out of the system of R units-- $h_1(Q_1 - 1)/2$. Note that the holding cost is taken to be h_1 even though the physical product does not remain in Stage 1 inventory. As will be discussed later, this approach is correct so long as the holding cost h_n is taken to be the value added at F_n.

3. The permanent Stage 1 inventory that must be maintained to ensure that product is always available when required. Since Q_1 and R are assumed to be rational, we can find a cycle. The permanent component of inventory is the amount which must be on hand at the beginning of the cycle to insure that Stage 1 inventory remains non-negative. This amount can be found assuming that Stage 1 inventory is zero at the start of the cycle, and finding the minimum (most negative) level which is attained during the cycle.

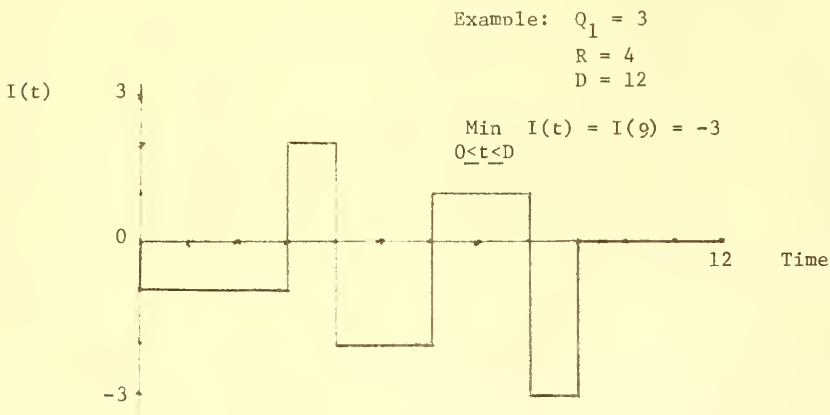


FIGURE 7

The level of inventory at any time $t \geq 0$ measured from the beginning of the cycle is

$$I(t) = [t/Q_1 + 1]Q_1 - [t/R + 1]R \text{ where [ ] denotes integer part.}$$

$$= ([t/Q_1] - R/Q_1 [t/R])Q_1 + Q_1 - R .$$

$I(t)$ is clearly minimized for some t such that t/R is integral, that is, immediately following a withdrawal to satisfy demand.

$$\min I(t) = \min_{\ell \text{ integer}} ([\ell R/Q_1] - R/Q_1[\ell R/R])Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} -(\ell R/Q_1 - [\ell R/Q_1])Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} -(\ell q_2/q_1 - [\ell q_2/q_1])Q_1 + Q_1 - R$$

$$= \min_{\ell \text{ integer}} -(\ell q_2 \bmod q_1)\ Q_1/q_1 + Q_1 - R.$$

Since q_2, q_1 relatively prime, $\ell q_2 \bmod q_1$ takes on all the values $1,2,\ldots,q_1-1$. In particular, for some ℓ, $\ell q_2 \bmod q_1 = q_1 - 1$.

$$\min I(t) = Q_1(1 - q_1)/q_1 + Q_1 - R = R(1 - q_1 + q_1 - q_2)/q_2$$

$$= - R(1 - q_2)/q_2 \ .^{*}$$

Thus the permanent inventory component costs $RH_1(1 - q_2)/q_2$.

*This result was suggested by William M. Hawkins, Sloan School of Management.

Lemma 2: A function

$$Z(Q) = C_1/Q + C_2(Q - 1)/2 + Q_2C_2(q_2 - 1)/q_2$$

where C_1, C_2,Q_2 are constants and q_2 defined as in Proposition 2 is minimized for $q_2 = 1$, that is, with Q/Q_2 an integer.

Proof: Suppose Q_1^* minimizes Z and Q_1^*/Q_2 not integer. Define Q_1 by $Q_1^* = Q_1 + \Delta Q_2$ with Q_1/Q_2 an integer and $0 < \Delta Q_2 \leq Q_2$. This can be done because Q_1^* is clearly not zero. Then

$$Z(Q_1^*) = C_1/(Q_1 + \Delta Q_2) + C_2(Q_1 + \Delta Q_2)/2 + Q_2C_2(q_2 - 1)/q_2$$

since Q_1^* not integer, $(q_2 - 1)/q_2 \geq 1/2$

$$Z(Q_1^*) \geq C_1/(Q_1 + \Delta Q_2) + C_2(Q_2 + \Delta Q_2)/2 + C_2Q_2/2$$

$$\geq C_1/(Q_1 + Q_2) + C_2(Q_1 + Q_2)/2 + C_2\Delta Q_2/2 \qquad \text{since } Q_2 \geq \Delta Q_2.$$

$$\geq C_1/(Q_1 + Q_2) + C_2(Q_1 + Q_2) = Z(Q_1 + Q_2).$$

Thus $Z(Q_1 + Q_2) \leq Z(Q_1^*)$, and $(Q_1 + Q_2)/Q_2$ is integer bv construction.

Proof of Theorem 2 follows bv induction over the levels of the multi-stage system. We assume we have an optimal solution Q^* and show that it must be integer. Consider the stages belonging to the first level, L_1. If $n \in L_1$, then $h_n = H_n$. Substituting $Q^*_{a(n)}$ for R in

Proposition 2, $Z(Q_n) = S_n/Q_n + h_n(Q_n - 1)/2 + H_n((q_{a(n)} - 1)/(q_a(n))Q_{a(n)}$.

Lemma 2 applies implying k_n is a positive integer.

Now suppose k_i is integer for all stages F_i, $i \in L_1 \cup L_2 \cup, \ldots, \cup L_{j-1}$.

Let $n \in L_j$. Then the total cost associated with the choice of lot size Q_n is evidently

$$Z(Q_n) = RS_n/Q_n + h_n(Q_n - 1)/2 + Q_{a(n)}H_n(1 - q_{a(n)}/q_{a(n)}$$

$$+ \sum_{i \varepsilon b(n)} Z(k_i Q_n).$$

Noting that $(1 - q_{a(i)})/q_{a(i)} = 0$ if k_i is integral,

$$Z(k_i Q_n) = S_i/k_i Q_n + h_i(k_i Q_n - 1)/2 + \sum_{\ell \varepsilon b(i)} Z(k_\ell k_i Q_n)$$

thus, $Z(Q_n) = R/Q_n \cdot \sum_{i \varepsilon B(n)} S_i + Q_n/2 \sum_{i \varepsilon B(n)} h_i + H_n Q_{a(n)} (1 - q_{a(n)})/q_{a(n)}$.

Since, by definition, $H_n = \sum_{i \varepsilon B(n)} h_i$, Lemma 2 applies directly. The induction argument proves the theorem for all stages including the final stage if $Q_{a(N)}$ is taken to be R.

REFERENCES

1. Benders, J. F.,"Partitioning Procedures for Solving Mixed-Variables Programming Problems", Numerische Mathematik, 4, (1962), 238-252.

2. Crowston, W. B., M. Wagner, and A. Henshaw, "A Comparison of Exact and Heuristic Routines for Lot Size Determination in Multi-Stage Assembly Systems", Sloan School Working Paper No. 475-70, M.I.T., (August, 1970).

3. Dantzig, G. B., "Optimal Solution to a Dvnamic Leontief Model with Substitution", Econometrica, 23, (1955), 295-302.

4. Dzielinski, B. and R. Gomory, "Optimal Programming of Lot Sizes, Inventories, and Labor Allocations", Management Science, 2, No. 9, (July, 1965), 874-890.

5. Gorenstein, S., "Planning Tire Production", Management Science, 17, No. 2, (October, 1970).

6. Gorenstein, S., "Programming for Economic Lot Sizes with Precedences between Items--An Assembly Model", I.B.M. Technical Report No. 320-2995, (October, 1970).

7. Hadley, G. and T. M. Whitin, Analysis of Inventorv Systems, Prentice-Hall, Inc., Englewood Cliffs, N. J., (1963).

8. Hu, T. C., Integer Programming and Network Flows, Addison-Wesley Publishing Company, Reading, Mass., (1969).

9. Jensen, P. A. and H. A. Khan, "An Algorithm to Find the Optimum Schedule for a Multi-Stage Production System", presented at the National Meeting of ORSA, (October, 1970).

10. Manne, A. S., "Programming of Economic Lot Sizes", Management Science, 4, No. 2, (January, 1958), 115-135.

11. Schussel G., "Job-Shop Lot Release Sizes", Management Science, 14, No. 8, (1968), B449-B472.

12. Taha, A. H. and R. W. Skeith, "The Economic Lot Size in Multistage Production Systems", AIIE Trans., (June, 1970).

13. Veinott, A. F., Jr., "Minimum Concave-Cost Solutions of Leontief Substitution Models of Multi-Facilitv Inventory Systems", Operations Research, 17, (March-April, 1969), 262-291.

14. Wagner, H. M. and T. M. Whitin, "Dynamic Version of the Economic Lot Size Model", Management Science, 5, No. 1, (October, 1959), 89-96.

15. Wagner, H. M., "A Postscript to 'Dynamic Problems of the Firm'", Naval Research Logistics Quarterly, 7, (1960), 7-12.

16. Zangwill, W. I., "A Deterministic Multi-Period Production Scheduling Model with Backlogging", Management Science, 13, (1966), 105-119.

17. Zangwill, W. I., "A Deterministic Multiproduct Multifacility Production and Inventory Model", Operations Research, 14, (1966), 486-507.

18. Zangwill, W. I., "Minimum Concave Cost Flows in Certain Networks", Management Science, 14, (1968), 429-450.

19. Zangwill, W. I., "A Backlogging Model and Multi-Echelon Model of a Dynamic Economic Lot Size Production System--A Network Approach", Management Science, 15, No. 9, (May, 1969).