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LIMIT LAWS FOR EXTREME ORDER STATISTICS FROM STRONG-MIXING PROCESSES[1]

by

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ABSTRACT

This paper considers the possible limit laws for a sequence of normalized extreme order statistics (maximum, second maximum, etc.) from a stationary strong-mixing sequence of random variables. It extends the work of Loynes who treated only the maximum process.

The maximum process leads to limit laws that are the same three types that occur when the underlying process is a sequence of independent random variables. The results presented here show that the possible limit laws for the k-th maximum process (k>1) from a strong-mixing sequence form a larger class than can occur in the independent case.

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1. Introduction. The limiting distributions of the extreme order statistics from a sequence of independent, identically distributed random variables have been exhaustively analyzed by Gnedenko [2] and Smirnov [8]. Many authors have generalized these results for the maximum term by relaxing the independence assumption in various ways, e.g. Loynes [5] showed that the only possible limit laws for the maximum term in a stationary strong-mixing seouence of random variables are the same three types that occur in the indeoendent case.

This paper extends the work of Loynes by considering the possible limit laws of order statistics of fixed rank other than the maximum. It is shown that these limit laws form a larger class than can occur in the independent case.

These results were motivated in part by a specific model from reliability theory. Consider a system of n identical components in parallel such that the lifetime of a component is dependent in a certain way (e.g. a mixing condition) on the lifetimes of its nearest neighbors. In effect we expect that if a particular component fails (say because of excess heat) its nearest neighbors are highly likely to be the next components to fail. We also assumed the system would continue to operate if only one component failed but the sustem itself would fail if two or more component failures occurred. The lifetime of the system is then represented by the (n-1)st order statistic of the sequence $X_1, X_2, \ldots, X_n$ of component lifetimes where the X_i are identically

distributed and satisfy a specified dependence relation. In our notation the n-th order statistic is the minimum. Since most of the literature discusses maximia rather than minima we will deal with the maximum and second maximum. A simple transformation converts our results to ones for minima.

2. Notation and Preliminary Results. If $\langle X_n : n \geq 1\rangle$ is a strictly stationary sequence of random variables with common distribution function $F(x) = P\{X_n \leq x\}$, the associated independent process of the process $\langle X_n : n \geq 1\rangle$ will be any sequence of mutually independent identically distributed random variables $\langle \hat{X}_n : n \geq 1\rangle$ with $P\{\hat{X}_n \leq x\} = F(x)$ for all x. Define the order statistics $Y_{i,n}$ by

$$Y_{i,n} = \begin{cases} i^{th} \text{ largest among } (X_1, X_2, \ldots, X_n) & i \leq n \\ Y_{n,n} & i > n \end{cases}$$

and let $\hat{Y}_{i,n}$ denote the order statistics of the associated independent process. We shall limit our discussion to $i=1,2$ and set $M_n = Y_{1,n}$ and $S_n = Y_{2,n}$. It will be technically convenient to consider the joint law of M_n and S_n.

Let $\mathcal{M}_a^b$ denote the σ-field generated by events of the form $\{(X_{i_1}, \ldots, X_{i_m}) \in E\}$, where $1 \leq a \leq i_1 < i_2 < \ldots < i_m \leq b$ and E is an m-dimensional Borel set. Then $\langle X_n, n \geq 1\rangle$ will be called strong-mixing (cf. [4]) if

$$\sup\{|P(AB) - P(A)P(B)| : A \in \mathcal{M}_1^m,\ B \in \mathcal{M}_{m+k}^\infty\} \leq \alpha(k) \downarrow 0 \qquad (k \to \infty). \tag{2.1}$$

Loynes [5] referred to (2.1) as uniform mixing.

The following lemma is a direct consequence of the work of Gnedenko [2].

Lemma 1. If there exists a sequence of constants $\langle \hat{a}_n > 0, \hat{b}_n : n \geq 1 \rangle$ so that $P\{\hat{M}_n \leq \hat{a}_n x + \hat{b}_n, \hat{S}_n \leq \hat{a}_n y + \hat{b}_n\}$ has a limiting distribution $\hat{H}(x,y)$, with $\hat{G}(x)$, the limiting distribution of $P\{\hat{M}_n \leq \hat{a}_n x + \hat{b}_n\}$ non-degenerate, then

$$\hat{H}(x,y) = \begin{cases} \hat{G}(y)\{1 + \log [\hat{G}(x)/\hat{G}(y)]\} & y < x \\ \hat{G}(x) & y \geq x \end{cases} .$$

Proof. Since $P\{\hat{M}_n \leq \hat{a}_n x + \hat{b}_n\} = F^n(\hat{a}_n x + \hat{b}_n) \to \hat{G}(x)$ we have by Lemma 4 of [2] that $n(1 - F(\hat{a}_n x + \hat{b}_n)) \to - \log \hat{G}(x)$ when $\hat{G}(x) \neq 0$.

For $x > y$

$$P\{\hat{M}_n \leq \hat{a}_n x + \hat{b}_n, \hat{S}_n \leq \hat{a}_n y + \hat{b}_n\} = F^n(\hat{a}_n y + \hat{b}_n)$$

$$+ nF^{n-1}(\hat{a}_n y + \hat{b}_n)[F(\hat{a}_n x + \hat{b}_n) - F(\hat{a}_n y + \hat{b}_n)]$$

and the result follows. Gnedenko also proved that $\hat{G}(x)$ has only three possible forms (except for scale and location parameters),

$$G_1(x) = \begin{cases} 0 & x \leq 0 \\ \exp\,[-(x^{-\alpha})] & x > 0,\ \alpha > 0 \end{cases}$$

(2.2) $$G_2(x) = \begin{cases} \exp[-(-x)^{\alpha}] & x < 0,\ \alpha > 0 \\ 1 & x \geq 0 \end{cases}$$

$$G_3(x) = \exp\,(-e^{-x}) \qquad -\infty < x < \infty.$$

The symbol $G(x)$ will be used to denote one of these types.

Lemma 1 shows that for an independent process there are only three possible types for the joint law $\hat{H}(x,y)$. It seems reasonable to conjecture that in view of Loynes' result for M_n from a strong-mixing process (only three possible limit laws) there would only be three possible types for the joint limit law of M_n and S_n when the underlying process is strong-mixing. This is not true as the following simple example demonstrates. Let $\langle Z_n : n \geq 1 \rangle$ be a sequence of independent identically distributed random variables with distribution function $T(\cdot)$ and assume that $T(\cdot)$ is in the domain of attraction of one of the three limit laws in (2.2), i.e. there exist constants a_n, b_n such that $T^n(a_n x + b_n) \to G(x)$.

<u>Example 1</u>. Let $X_n = \max\,(Z_n, Z_{n+1})$, $n=1,2,\ldots$. Then $\langle X_n : n \geq 1 \rangle$ is a stationary strong-mixing sequence, $P\{M_n \leq a_n x + b_n\} \to G(x)$ and

$P\{M_n \le a_n x + b_n,\ S_n \le a_n y + b_n\} \to H(x,y)$ where

$$(2.3) \qquad H(x,y) = \begin{cases} G(y) & y < x \\ G(x) & y \ge x \end{cases} .$$

Proof. Clearly $\langle X_n : n \ge 1 \rangle$ is a strong-mixing sequence. Now in this case $M_n = \max (X_1, \ldots, X_n) \equiv \max (Z_1, \ldots, Z_{n+1})$. If $M_n = Z_i$ $i=2,\ldots,n$ then $S_n = \text{second max } (\max (Z_1, Z_2), \ldots, \max (Z_n, Z_{n+1})) = M_n$. Therefore $P\{M_n = S_n\} \ge (n-1)/n+1$ and the example follows immediately since

$$P\{M_n \le a_n x + b_n\} = T^{n+1}(a_n x + b_n) \to G(x).$$

This method of constructing a strong-mixing sequence is due to Newell [6].

The limit law (2.3) is not of the same form as $\hat{H}(x,y)$ and we conclude that by weakening the independence assumption a larger class of limit laws is possible. In the next section we prove a result which limits the size of this class.

3. Possible Limit Laws.

Theorem 1. Let $\langle X_n : n \geq 1\rangle$ be a stationary strong-mixing sequence. If there exists a sequence of constants $\langle a_n > 0,\ b_n : n \geq 1\rangle$ so that $P\{M_n \leq a_n x + b_n,\ S_n \leq a_n y + b_n\}$ has a limiting distribution, $H(x,y)$, with $G(x)$, the limiting distribution of $P\{M_n \leq a_n x + b_n\}$ non-degenerate, then

$$H(x,y) = \begin{cases} G(y)\{1 - \rho[(\log G(x))/\log G(y)] \log G(y)\} & y < x \\ G(x) & y \geq x \end{cases}$$

where $\rho(s)$, $0 \leq s \leq 1$, is a concave, monotone non-increasing function which satisfies

$$\rho(0)(1-s) \leq \rho(s) \leq 1 - s.$$

$G(\cdot)$ is one of the three types (2.2) and we interpret $(\infty/\infty) = 1$, $(0/0) = 1$, and $(0/\infty) = 0$.

Proof. The essential idea (cf. Loynes [5]) is to break up sets of km random variables (k fixed, $m \geq 1$) into k blocks of length m-q separated by blocks of length q. The fact that the blocks of length m-q are "nearly independent" because of the strong-mixing property is then used to obtain a functional equation for $H(x,y)$. For $x \leq y$ the assumptions of the theorem are the same as for Loynes' result. Therefore $G(\cdot)$ is one of the three types (2.2).

Given an $\varepsilon > 0$ and a fixed positive integer $k \geq 2$, choose q so that $(k+1)(k-1)\alpha(q) < \varepsilon/2$ and define

$$M_m = \max_{1 \le j \le m} X_j \qquad m > q$$

$$M_{m,i} = \max_{q+1 \le j \le m} X_{(i-1)m+j} \qquad 1 \le i \le k$$

$$M'_{m,i} = \max_{1 \le j \le q} X_{(i-1)m+j}$$

$$\tilde{M}_n = \max \{X_{q+1}, X_{q+2}, \ldots, X_m, X_{q+m+1}, X_{q+m+2}, \ldots, X_{2m},$$

$$\ldots, X_{q+(k-1)m+1}, X_{q+(k-1)m+2}, \ldots, X_{km}\} \qquad n = km\,.$$

The quantities S_m, $S_{m,i}$, $S'_{m,i}$, $\tilde{S}_n$ are defined similarly for the second maximum. Denote by $E_{m,i}$ the event $\{S_{m,i} < M'_{m,i}\}$. It is shown in Lemma 2 at the end of this section that $\lim_{m\to\infty} P\{E_{m,1}\} = 0$. Furthermore $P\{E_{m,i}\} = P\{E_{m,1}\}$ for $2 \le i \le k$. Since q is now fixed we may choose an N so that for $m > N$, $kP\{E_{m,1}\} < \varepsilon/2(k+3)$.

The first step is to show that

$$(3.1) \qquad \lim_{\substack{m\to\infty \\ n=km}} \Big| P\{M_n \le a_n x + b_n,\ S_n \le a_n y + b_n\} - P^k\{M_m \le a_n y + b_n\}$$

$$- k[P\{M_m \le a_n x + b_n,\ S_m \le a_n y + b_n\} - P\{M_m \le a_n y + b_n\}]$$

$$\cdot P^{k-1}\{M_m \le a_n y + b_n\} \Big| = 0\,.$$

For $x \geq y$ and $m > N$

$$(3.2)\quad P\{\tilde{M}_n \leq a_n x + b_n,\ \tilde{S}_n \leq a_n y + b_n\}$$

$$- P\{M_n \leq a_n x + b_n,\ S_n \leq a_n y + b_n\} \leq \sum_{i=1}^{k} P\{E_{m,i}\} \leq \varepsilon/2(k+3).$$

Now

$$P\{\tilde{M}_n \leq a_n x + b_n,\ \tilde{S}_n \leq a_n y + b_n\}$$

$$= P\{\tilde{M}_n \leq a_n y + b_n\}$$

$$+ \sum_{i=1}^{k} P\{a_n y + b_n < M_{m,i} \leq a_n x + b_n,\ S_{m,i} \leq a_n y + b_n;$$

$$M_{m,j} \leq a_n y + b_n,\ 1 \leq j \leq k,\ j \neq i\}$$

and the strong-mixing property implies that

$$(3.3)\quad |P\{\tilde{M}_n \leq a_n x + b_n,\ \tilde{S}_n \leq a_n y + b_n\} - P^k\{M_{m,1} \leq a_n y + b_n\}$$

$$- kP\{a_n y + b_n < M_{m,1} \leq a_n x + b_n,\ S_{m,1} \leq a_n y + b_n\}$$

$$\cdot P^{k-1}\{M_{m,1} \leq a_n y + b_n\}| \leq (k-1)\alpha(q) + k(k-1)\alpha(q) \leq \varepsilon/2.$$

Finally we have that

$$(3.4)\quad P\{M_{m,1} \leq a_n x + b_n,\ S_{m,1} \leq a_n y + b_n\}$$

$$- P\{M_m \leq a_n x + b_n,\ S_m \leq a_n y + b_n\} \leq P\{E_{m,1}\} \leq \varepsilon/2k(k+3).$$

Combining (3.2), (3.3), and (3.4) gives

$$\Big|P\{M_n \le a_n x + b_n,\ S_n \le a_n y + b_n\} - P^k\{M_m \le a_n y + b_n\}$$

$$- k[P\{M_m \le a_n x + b_n,\ S_m \le a_n y + b_n\} - P\{M_m \le a_n y + b_n\}]$$

$$\cdot P^{k-1}\{M_m \le a_n y + b_n\}\Big| < \varepsilon$$

from which (3.1) follows. In particular, when $x = y$

$$(3.5) \qquad \lim_{m\to\infty} \big|P\{M_n \le a_n x + b_n\} - P^k\{M_m \le a_n x + b_n\}\big| = 0.$$

Set $\hat{a}_m = a_n/a_m$, $\hat{b}_m = (b_n - b_m)/a_m$, and $F_m(t) = P\{M_m \le a_m t + b_m\}$.

We have remarked that $F_m(x) \to G(x)$ and from (3.5) it follows that $F_m(\hat{a}_m x + \hat{b}_m) \to G^{1/k}(x)$. A theorem of Khintchine ([3], p. 40) states that $G^{1/k}(y)$ and $G(y)$ must be of the same type, i.e., there exist real-valued constants $\alpha_k > 0$ and β_k such that $G^k(\alpha_k y + \beta_k) = G(y)$ and

$$\lim_{m\to\infty} \hat{a}_m = \alpha_k\ , \quad \lim_{m\to\infty} \hat{b}_m = \beta_k .$$

Let

$$Q_m(x,y) = kP^{k-1}\{M_m \le a_m y + b_m\}P\{M_m \le a_m x + b_m,\ S_m \le a_m y + b_m\},$$

$$Q(x,y) = kG^{k-1}(y)H(x,y),$$

and assume that (x,y) and $(\alpha_k x + \beta_k,\ \alpha_k y + \beta_k)$ are points of continuity for H. Equation (3.1) is equivalent to

$$\lim_{m\to\infty} Q_m(\hat{a}_m x + \hat{b}_m,\ \hat{a}_m y + \hat{b}_m) = H(x,y) + (k-1)G(y).$$

But a standard argument ([3], p. 41) shows that

$$\lim_{m\to\infty} Q_m(\hat{a}_m x + \hat{b}_m,\ \hat{a}_m y + \hat{b}_m) = Q(\alpha_k x + \beta_k,\ \alpha_k y + \beta_k)$$

and therefore

$$\text{(3.6)} \quad H(x,y) = G(y) + k[H(\alpha_k x + \beta_k,\ \alpha_k y + \beta_k) - G(\alpha_k y + \beta_k)] \cdot G^{(k-1)/k}(y)$$

when $x \geq y$.

It is apparent from (2.1) that $G(t)$ is continuous and strictly increasing for t such that $0 < G(t) < 1$. Furthermore

$$\text{(3.7)} \quad G(y) \leq H(x,y) \text{ and } \qquad y \leq x$$

$$\text{(3.8)} \quad H(x_2,y) - H(x_1,y) \leq G(x_2) - G(x_1) \qquad y \leq x_1 \leq x_2.$$

From (3.7) we have that $H(x,y) = 1$ if $G(y) = 1$ and from (3.6) $H(x,y) = 0$ if $G(y) = 0$. Finally (3.8) implies that $H(x,y) = \lim_{t\to\infty} H(t,y)$ if $G(x) = 1$, and $H(x,y) = 0$ if $G(x) = 0$. Therefore if $x \geq y$ it is possible to express $H(x,y)$ in the form

$$H(x,y) = R(-\log G(x),\ -\log G(y)) \equiv R(u,v)$$

Moreover $H(x,y)$ has the same form if $x < y$, so that we may take $R(u,v) = e^{-u}$ for $u \geq v$.

With $0 < u < v < \infty$ equation (3.6) takes the simple form

$$f(u,v) = kf(u/k,\ v/k) \qquad k \geq 1 \tag{3.9}$$

where $f(u,v) \equiv (R(u,v) - e^{-v})e^{v}$. Now $H(x,y)$ is a bivariate distribution function so that

$$\lim_{x \downarrow x',\, y \downarrow y'} H(x,y) = H(x',y')$$

and since $G(\cdot)$ is monotone increasing except when $G(\cdot) = 0$ or 1 we have

$$\lim_{r \downarrow r' > 0} R(u/r,\ v/r) = R(u/r',\ v/r') \qquad 0 < u < v < \infty\ . \tag{3.10}$$

Using (3.9) it is easy to show that $f(u,v) = rf(u/r,\ v/r)$ for all rational $r > 0$. Then (3.10) implies that $f(u,v) = zf(u/z,\ v/z)$ for all real $z > 0$. Now let $z = v > 0$ so that

$$f(u,v) = vf(u/v,\ 1) \equiv v\rho(u/v)$$

and

$$R(u,v) = e^{-v}[1 + v\rho(u/v)] \qquad u < v,\ \theta < v < \infty. \tag{3.11}$$

It is clear that $\rho(1) = 0$ and from (3.8) we obtain

$$0 \leq \rho(s_2) - \rho(s_1) \leq e^{-s_2+1} - e^{-s_1+1} \tag{3.12}$$

where $s_1 = -\log G(x_1)$, $s_2 = -\log G(x_2)$ and $\log G(y) = -1$. Therefore $\rho(\cdot)$ is continuous and monotone decreasing on $[0,1]$ and $\rho(\cdot) \geq 0$.

In order to show that $\rho(\cdot)$ is concave we will use the following lemma whose proof may be found in [7].

Lemma. Let r, s, Δr, Δs be any such numbers as $0 < r < s$, $\Delta r = \epsilon r$ and $\Delta s = \epsilon s$ $(\epsilon > 0)$, or $r = s = 0$ and $0 < \Delta r < \Delta s$. Then

$$(3.13) \qquad \frac{\psi(s+\Delta s) - \psi(s)}{\Delta s} \leq \frac{\psi(r+\Delta r) - \psi(r)}{\Delta r}$$

is necessary and sufficient for the continuous bounded function $\psi(\cdot)$ to be concave in $[0,\infty)$.

For convenience in applying this lemma we extend the domain of definition of $\rho(\cdot)$ by letting $\rho(s) = 0$ for $1 \leq s < \infty$.

Since $H(x,y)$ is a distribution function

$$H(x_2,y_2) - H(x_1,y_2) \geq H(x_2,y_1) - H(x_1,y_1) \tag{3.14}$$

with $x_2 \geq x_1$, $y_2 \geq y_1$. When $x_2 > x_1$, $y_2 \geq y_1$, $0 < G(x_1) < 1$, $G(y_1) > 0$ and $G(y_2) < 1$ this is equivalent to

$$\frac{G(y_2)}{G(y_1)}\left[\frac{\rho(s_1) - \rho(s_2)}{s_1 - s_2}\right] \leq \frac{\rho(r_1) - \rho(r_2)}{r_1 - r_2} \tag{3.15}$$

with $r_i = (\log G(x_i))/\log G(y_1)$ and $s_i = (\log G(x_i))/\log G(y_2)$, $i = 1,2$.

If we set $\varepsilon = (s_1-s_2)/s_2 = (r_1-r_2)/r_2$ when $r_2 > 0$, then (3.15) becomes

$$\frac{G(y_2)}{G(y_1)}\left[\frac{\rho(s_2+\Delta s_2) - \rho(s_2)}{\Delta s_2}\right] \leq \frac{\rho(r_2+\Delta r_2) - \rho(r_2)}{\Delta r_2} \tag{3.16}$$

where $\Delta s_2 = \varepsilon s_2$ and $\Delta r_2 = \varepsilon r_2$. Since (3.15) must hold for $G(y_2)/G(y_1)$ arbitrarily close to 1, (3.13) is satisfied. When $r_2 = s_2 = 0$, (3.15) again implies that (3.13) holds and therefore $\rho(\cdot)$ is concave in [0,1].

Finally from (3.12)

$$\frac{\rho(s)}{s - 1} \geq \frac{e^{-s+1} - 1}{s - 1} \qquad 0 \leq s < 1$$

so that $\liminf_{s\uparrow 1}[\rho(s)/(s-1)] \geq -1$. Since $\rho(s)$ in concave we conclude that $\rho(s) \leq 1 - s$. Substituting for (u,v) in (3.11) completes the proof.

Berman [1] and Sibuya [7] have used similar arguments to obtain limiting forms for bivariate extreme value distributions. The techniques used above generalize to the third maximum etc. but with increasing complexity.

Lemma 2. If $\langle X_n : n \geq 1\rangle$ is a strictly stationary ergodic sequence then $\lim_{m\to\infty} P\{E_{m,1}\} = 0$.

Proof. For ease of notation let $E_m = E_{m,1}$ and $M'_m = M_{m,1}$ and assume that $m \geq q + 1$. Clearly $E_{m+1} \subset E_m$ and therefore it is sufficient to show that $P\{\bigcap_{m=q+1}^{\infty} E_m\} = 0$. If $\mathcal{R}$ represents the rational numbers r such that $P\{X_1 \leq r\} < 1$ then

$$P\{\bigcap_m E_m\} \leq \sum_{r\in\mathcal{R}} P\{(\bigcap_m E_m) \cap (M'_m \leq r)\}$$

and

$$P\{(\bigcap_m E_m) \cap (M'_m \leq r)\}$$

$$\leq \sum_{i=q+1}^{\infty} P\{X_{q+1} \leq r, X_{q+2} \leq r, \ldots, X_i \leq r, X_{i+2} \leq r, \ldots\}$$

$$+ P\{X_j \leq r;\ j \geq q + 2\}.$$

By the strong law of large numbers for strictly stationary ergodic sequences

$$\lim_{n\to\infty} \left(\sum_{j=i+1}^{n} 1_{\{X_j \leq r\}} \right) / (n-1) = P\{X_1 \leq r\} < 1$$

almost surely. This implies that $P\{X_{i+1} \leq r, X_{i+2} \leq r, \ldots\} = 0$ for $i \geq 0$ and $P\{\bigcap_m E_m\} = 0$ follows immediately.

4. Conclusions. Theorem 1 clearly includes Lemma 1 and hence for an independent process $\rho(s) = 1 - s$. In Example 1, $\rho(s) \equiv 0$. Let $\langle Z_n, n \geq 1 \rangle$ be as in Example 1 and set

$$X_n = \max (Z_{(n-1)k+1}, Z_{(n-1)k+2}, \ldots, Z_{(n-1)k+\ell}) \quad n=1,2,\ldots$$

where k and ℓ are fixed positive integers. The sequence $\langle X_n : n \geq 1 \rangle$ is strong-mixing and it is possible to show that there exist constants a_n and b_n so that $P\{M_n \leq a_n x + b_n, S_n \leq a_n y + b_n\}$ converges and $H(x,y)$ is of the form given in Theorem 1 with $\rho(s) = c(1 - s)$ where c is a rational number, $0 \leq c \leq 1$, which is a function of k and ℓ. The proof is not difficult but rather tedious and the details will be omitted. Thus far we have only succeeded in constructing examples where $\rho(\cdot)$ is linear. The problem of finding a strong-mixing sequence leading to a strictly concave $\rho(\cdot)$ or sharpening Theorem 1 to exclude this case is still open.

In the reliability model mentioned earlier, we note that Theorem 1 implies that $P\{S_n \leq a_n x + b_n\} \to G(x)[1 - \rho(0) \log G(x)]$ and therefore the strong-mixing assumption can have a considerable effect on the asymptotic distribution of the second maximum ($\rho(0) = 1$ in the independent case). The consequences of this result are currently being explored.

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