



WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

LINEAR PROGRAMMING AND CAPITAL BUDGETING:

COMMENT

2/10/70
Stewart C. Myers

March 1970

445-70

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139



LINEAR PROGRAMMING AND CAPITAL BUDGETING:

COMMENT

Stewart C. Myers
March 1970

445-70

LINEAR PROGRAMMING AND CAPITAL BUDGETING: COMMENT

Stewart C. Myers

This paper is the latest, and I hope the last chapter, in the controversy begun by William H. Baumol and Richard E. Quandt in their criticism [1] of Weingartner's work on capital budgeting under capital rationing [5]. A thorough review and partial resolution of the matter has been offered by Willard T. Carleton in a recent article [2]. My purpose here is to try to complete the job.

Background

The Weingartner model, reduced to essentials, is as follows:

$$\text{Maximize} \quad \sum_{j=1}^J \sum_{t=0}^T [a_{jt}/(1+k)^t]x_j \quad (1)$$

$$\text{Subject to} \quad - \left[\sum_{j=1}^J a_{jt}x_j \right] \leq M_t; \quad t = 0, 1, \dots, T \quad (2)$$

$$x_j \geq 0 \quad j = 1, \dots, J.$$

- where: k = a fixed discount rate, the "cost of capital";
 a_{jt} = the net cash flow, possibly negative, obtained from project j in period t ;¹
 x_j = the number of units of j constructed;
 M_t = the fixed amount of cash available at t .

¹Weingartner actually makes a distinction between cash outlays and cash returns. However, the distinction is not necessary within the restricted scope of this paper.

This follows Carleton's presentation [2], to which the reader is referred for a full summary of the controversy and for discussion of a number of side issues not treated here.

The slipperiest issue raised by Baumol and Quandt is that Weingartner's model

appears to run afoul of the Hirshleifer difficulty²: the discount factors $(1+k)^{-t}$ in solution must equal ratios of the internal discount factors. Since we acknowledge the existence of capital rationing, k must itself be internally determined and hence be independent of monetary phenomena. However, if we assume that k is a true marginal opportunity rate, then it turns out that we cannot use present value discounting in the primal objective function until we know the values of the dual variables.³

If the objective function cannot be specified until a solution is found, then the problem itself does not make sense, as Baumol and Quandt amply demonstrate.

One possible response to this issue is to argue that capital rationing is not "hard", in the sense of a absolute limit on external finance, but a tentative restriction posited as an aid to planning and control. This is the tack taken by Weingartner [6] and, in a more elaborate way, by Carleton [2]. Both have offered quite plausible interpretations of the basic model under these circumstances.

But the matter is not yet satisfactorily resolved for the case of

²Hirshleifer [3].

³Carleton [2], pp. 826-827.

"hard" rationing -- the case Baumol and Quandt were concerned with.

The Baumol-Quandt Solution

Baumol and Quandt recast the basic model as follows:⁴

$$\text{Maximize} \quad \sum_{t=0}^T U_t W_t \quad (3)$$

$$\text{Subject to} \quad - \left[\sum_{j=1}^J a_{jt} x_j \right] + W_t \leq M_t; \quad t = 1, 2, \dots, T \quad (4)$$

$$W_t, x_j \geq 0.$$

where; W_t = cash withdrawn for owners' consumption in period t ;

U_t = marginal utility of consumption in t , assumed constant.

This solves the problem, but not in a wholly useful way. First, the objective is a corporate one, and it is not clear that the usual idea of a project's present value retains meaning.⁵ Second, and more important, the idea of maximizing "utility," while perhaps useful to the individual investor, seems both vague and arbitrary for corporate capital budgeting decisions. As Carleton notes,⁶ "Precluded from using the company's cost of capital, [Baumol and Quandt] invoke a subjective discounting procedure whose welfare implications for resource allocation are quite suspect."

⁴[1], p. 326.

⁵See Carleton's discussion of this point. [2], p. 829.

⁶Ibid.

Resolution

For corporate decisions, the resolution of this matter of "hard" capital rationing is so simple that one can only apologize for having taken so much space to describe the problem.

We assume a world of certainty, and the absence of market imperfections except for capital rationing.

First, consider Eq. (4), the constraint in the Baumol-Quandt formulation. This will always be a strict equality in the optimal solution, since increasing W_t always increases the objective function.⁷ Solving for W_t and substituting in the objective function, Eq. (3), the problem can be rewritten

$$\text{Maximize} \quad \sum_{t=0}^T U_t [M_t + \sum_{j=1}^J a_{jt} x_j] \quad (3a)$$

$$\text{Subject to} \quad - \left[\sum_{j=1}^J a_{jt} x_j \right] + W_t = M_t; \quad t = 0, 1, \dots, T \quad (4a)$$

$$x_j, W_t \geq 0.$$

The constant term $\sum U_t M_t$ can be dropped from the objective function without harm.

The differences of Eqs. (3a) and (4a) from (1) and (2) are the inclusion of "slack" variables W and the use of marginal utilities, U_t , rather than the discount factors $1/(1+k)^t$. The extra slack variables are

⁷It is also an equality because sources and uses of funds have to be equal. Any "slack" in such an equation can, and should, be interpreted as another investment project -- i.e., investment in cash or liquid assets.

not important, but the latter difference appears to be.

In a certain world, however, investors facing a prevailing interest rate k will all adjust their portfolios so that the following conditions hold⁸

$$\frac{U_t}{U_{t-1}} = \frac{1}{1+k} \quad (5)$$

$$\frac{U_t}{U_0} = \frac{1}{(1+k)^t} \quad (6)$$

The interest rate k is, of course, the firm's "cost of capital."

We can scale any utility function so that $U_0 = 1$. At equilibrium, then, the following relationship will hold for all investors:

$$U_t = \frac{1}{(1+k)^t} \quad (6a)$$

In other words, the firm can use the observed interest rate k to infer the marginal utilities required by the Baumol-Quandt formulation.

Substituting Eq. (6a) in (4a), the Baumol-Quandt formulation turns out to be exactly equivalent to the original Weingartner model, Eqs. (1) and (2).

Conclusion

We may summarize as follows:

1. Weingartner was right in the first place; the "Hirshleifer Problem" does not apply.

⁸See the Appendix to this paper.

2. Since the problem does not exist, Baumol and Quandt's "solution" is no different than the original model, as they would have discovered had they pushed their analysis a bit further.
3. The existence of capital rationing should not change the firm's basic objective. Shareholders' utility is maximized when the firm's current market value (the present value of future dividends) is maximized; and the firm's market value is the sum of the present values of accepted projects.
4. It is not true that "if during period t capital is in short supply and is effectively limited to the amount M_t , then . . . the firm is thereby necessarily cut off from the capital market and, as a consequence, from any external discounting criteria."⁹ The firm always has the option of paying dividends, and the relative values of dividend payments at different points in time is determined by the interest rate k regardless of whether capital is rationed. The "cost of capital" k thus serves perfectly well as an external discounting criterion in the case of "hard" capital rationing.

⁹Baumol and Quandt [1], p. 322.

APPENDIX

Consider an investor in a world of certainty where security prices are determined in the usual way:

$$P_j = \sum_{t=0}^T \frac{R_j(t)}{(1+k)^t} \quad (\text{A.1})$$

where: $P_j(0)$ = price per unit of security j at the beginning of period $t = 0$;

$R_j(t)$ = cash return of j in t , except in the (arbitrary) horizon period T , for which $R_j(T)$ = any cash payment at T plus $P_j(T)$;

k = the rate of interest, here assumed independent of t .

The investor's problem is to maximize the utility of his consumption stream over the period $t = 0, 1, \dots, T$, subject to the constraint that current wealth, E , be divided between investment and current consumption, $C(0)$. The corresponding Lagrangian expression is

$$\begin{aligned} \Psi = & U(C(0), C(1), \dots, C(T)) \\ & + \sum_j x_j (P_j(0) - R_j(0)) + C(0) - E \end{aligned}$$

where: x_j = the number of units of security j purchased; for $t = 1, 2, \dots, T$

$C(t)$ = endowed income (assumed given) plus $\sum_j x_j R_j(t)$.

Note that x_j can be negative--this represents borrowing or selling short.

Define $U_t \equiv \partial U / \partial C(t)$, the marginal utility of consumption in t . Then the conditions for the optimum are

$$\frac{\partial \Psi}{\partial x_j} = \sum_t U_t R_j(t) + \lambda (P_j(0) - R_j(0)) = 0, \quad (\text{A.3})$$

$$\frac{\partial \Psi}{\partial C(0)} = U_0 + \lambda = 0.$$

Since $\lambda = -U_0$, we have the following result for any security j .

$$P_j(0) = \sum_{t=0}^T \frac{U_t}{U_0} R_j(t) \quad (\text{A.4})$$

Moreover, this holds for all investors, since each one faces the same prices $P_j(0)$ and has the same (certain) estimates of returns $R_j(t)$.

Comparing Equations (A.1) and (A.4) it is apparent that

$$\frac{U_t}{U_0} = \frac{1}{(1+k)^t} \quad (\text{A.5})$$

The firm can use the interest rate k as a "cost of capital" and be assured that its investment decisions maximize any shareholders' utility.

This demonstration is of course, simply a compressed statement of the classical theory of security valuation. See, for example, Hirshleifer [3] or Myers [4].

REFERENCES

1. William Baumol and Richard E. Quandt, "Investment and Discount Rates Under Capital Rationing--A Programming Approach", Economic Journal, LXXV (June 1965), pp. 317-329.
2. Willard T. Carleton, "Linear Programming and Capital Budgeting Models: A New Interpretation", Journal of Finance, XXIV (December 1969), pp. 825 - 833.
3. Jack Hirshleifer, "On the Theory of Optimal Investment Decision", Journal of Political Economy, LXVI (August 1958), pp. 329 - 352.
4. Stewart C. Myers, "A Time - State - Preference Theory of Security Valuation", Journal of Financial and Quantitative Analysis, III (March 1968), pp. 1 - 33.
5. H. Martin Weingartner, Mathematical Programming and The Analysis of Capital Budgeting Problems, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1963.
6. _____, "Criteria for Programming Investment Project Selection", Journal of Industrial Economics, XV (November 1966), pp. 65 - 76.

MAR 19 74

LIBRARY

Date Due

FEB 05 '78

JUL 6 '78

APR 02 '77

NOV 3

Lib-26-67

MIT LIBRARIES DUPL
3 9080 003 702 237

441-75

MIT LIBRARIES DUPL
3 9080 003 702 211

443-70

MIT LIBRARIES DUPL
3 9080 003 671 200

444-70A

M.T. .

MIT LIBRARIES DUPL
3 9080 003 671 242

445-70

MIT LIBRARIES DUPL
3 9080 003 671 218

446-70

MIT LIBRARIES DUPL
3 9080 003 671 358

447-70

MIT LIBRARIES DUPL
3 9080 003 702 344

448-70

MIT LIBRARIES DUPL
3 9080 003 702 310

449-70

MIT LIBRARIES DUPL
3 9080 003 671 309

450-70

MIT LIBRARIES DUPL
3 9080 003 671 325

451-70

MIT LIBRARIES DUPL
3 9080 003 671 333

452-70

