WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

A MARKET SHARE THEOREM

David E. Bell*
Ralph L. Keeney‡
John D. C. Little‡

704-74
March, 1974

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

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*International Institute for Applied System Analysis, Laxenburg, Austria

‡M.I.T., Sloan School of Management & Operations Research Center

This work was supported in part by a grant from NABISCO, Inc. for research in marketing science.

Abstract

Many marketing models use variants of the relationship: Market share equals marketing effort divided by total marketing effort. Although the relation can be assumed directly, certain insight is gained by deriving it from more fundamental assumptions as follows. For a given customer group, each competitive seller has a real valued "attraction" with the properties: (1) attraction is non-negative, (2) two sellers with equal attraction have equal market share, (3) the market share for a given seller will be affected in the same manner if the attraction of any other seller is increased by a fixed amount.

A theorem proven states that if the relation between share and attraction satisfies the above assumptions, then share equals attraction divided by total attraction. Insofar as marketing factors can be assembled into an attraction function that satisfies the assumptions, the theorem provides a method for modeling market share.

1. Introduction

Marketing model builders frequently use relationships of the form (us)/(us + them) to express the effects of "us" variables on purchase probability and market share. For example, Hlavac and Little [1] hypothesize that the probability a car buyer will purchase his car at a given dealer is the ratio of the dealer's attractiveness (which depends on various dealer characteristics) to the sum of the same quantities over all dealers. Urban [2], in his new product model SPRINTER, makes the sales rate of a brand in a store depend on the ratio of a function of certain brand variables to the sum of such functions across brands. Kuehn and Weiss [3] make use of (us)/(us + them) formulations in a marketing game model, as does Kotler [4] in a market simulation. Mills [5] and Friedman [6] employ models of this form in game-theoretic analyses of competition. Urban [7] and Lambin [8] fit similar models to empirical data, Urban to a product sold in supermarkets and Lambin to a gasoline market.

In all these cases the result of the formulation is to bring a competitive effect into the model by simple normalization. That is, a quantity, let us call it <u>attraction</u>, is defined that relates only to marketing actions and uncontrolled variables of a specific selling entity. Then, by <u>adding</u> attractions over sellers and using the sum as a denominator, a market share is obtained for each seller. The result is a competitive model, since any seller's market share depends on the actions of every other seller. Time lags, market segmentation or other phenomena may subsequently be added so as better to represent other market features.

This approach to competition solves a dilemma for the model builder. Suppose he believes, for example, that salesmen affect sales. He can draw up a relation between sales and sales effort and try to calibrate it with field data. However, competitive actions clearly affect what happens and the model builder seems to need a new relationship for each possible level of activity of each competitor. The problem has suddenly become very complicated. Yet, it seems plausible that the salesmen's efforts can be viewed as enhancing the seller's position with the customers on some absolute scale. This can then interact with the effects created by other sellers measures on comparable absolute scales. The linear normalization offers a way to represent the interaction.

Normalized attraction models of this type can be postulated directly, but it is of interest to examine them more closely and ask what basic assumptions can be used to derive them. We shall demonstrate that under certain conditions such a normalization is mathematically required.

The present paper deals with share, whereas sales are also a needed output in most marketing models. A common approach is to relate total market sales to total marketing effort, thereby breaking the model building task into the two parts. However, only the first part will be studied here. It should also be pointed out that there are other approaches to modeling competitive interaction. For one such see Little [9].

2. Problem Definition

Given a finite set $S = \{s_1,\ldots,s_n\}$ of sellers which includes all sellers from whom a given customer group makes its purchases, suppose that for each seller $s_i \varepsilon S$ an "attraction" value $a(s_i)$ is calculated. We suppose the competitive situation can be completely determined by the vector of attractions

$$\underline{a} = (a(s_1),\ a(s_2),\ldots,a(s_n)) = (a_1,a_2,\ldots,a_n).$$

That is, the market share $m(s_i)$ of a seller is fully determined by $\underline{a}$.

Attraction may be a function of the seller's advertising expenditure and effectiveness, the price of his product, the reputation of the company, the service given during and after purchase, location of retail stores and much more. Indeed, the attraction of an individual seller can, if we wish, be a function of these qualities for all the other sellers, or

$$a(s_i) = \emptyset_i(q_1,\ldots q_n;p_1,\ldots p_n;\ldots\ldots) \ ,$$

where q_j may be quality of service of seller j, p_j might indicate seller j's price, and so on. However, one would hope that most of a seller's attraction would be the result of his own actions and most model builders have treated it this way.

Since, by definition, attraction completely determines market share, it can be said that

$$m(s_i) = f_i(\underline{a}) \ , \quad i = 1,\dots,n \ ,$$

for some function f_i where $m(s_i)$ is the market share of seller i. Clearly,

$$\sum_{i=1}^{n} m(s_i) = 1$$

and

$$0 \leq m(s_i) \leq 1 \ , \quad i = 1,\dots,n \ ,$$

but otherwise the functions f_i are as yet arbitrary.

The aim here is to give conditions on the relationship between attraction and market share which force the simple linear normalization model

$$f_i(\underline{a}) = \frac{a_i}{\sum_{j=1}^{n} a_j} \ .$$

3. <u>Formal Development</u>

The assumptions are:

A1) The attraction vector is non-negative and non-zero,

$$\underline{a} \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} a_i > 0 \ .$$

A2) A seller with zero attraction has no market share,

$$a_i = 0 \longrightarrow m(s_i) = 0 \ .$$

A3) Two sellers with equal attraction have equal market share,

$$a_i = a_j \longrightarrow m(s_i) = m(s_j) .$$

A4) The market share of a given seller will be affected in the same manner if the attraction of any other seller is increased by a fixed amount Δ. Mathematically,

$$f_i(\underline{a} + \Delta e_j) - f_i(\underline{a}) , \quad \text{for } j \neq i ,$$

is independent of j, where e_j is the j^{th} unit vector.

Theorem. If a market share is assigned to each seller based only on the attraction vector and in such a way that assumptions A1 - A4 are satisfied, then market share is given by

$$m(s_i) = \frac{a(s_i)}{\sum_{j=1}^{n} a(s_j)} , \quad \text{for } i = 1,2,\ldots,n .$$

Proof Since the vector $\underline{a}$ completely defines the vector $(m(s_1),\ldots,m(s_n))$ then functions $f_1,\ldots,f_n$ exist such that

$$m(s_i) = f_i(\underline{a}) , \quad \text{for all } i = 1,\ldots,n ,$$

with

$$\sum_{i=1}^{n} f_i(\underline{a}) = 1 \tag{1}$$

and

$$f_i(\underline{a}) \geq 0 , \quad \text{for all } i = 1,\ldots,n . \tag{2}$$

Consider the set

$$\alpha = \{\underline{a} : a_1 \text{ is constant and } \sum_{i=1}^{n} a_i = A \text{ for some } A > 0\}$$

Let $\underline{\bar{a}}$, $\underline{\bar{\bar{a}}} \varepsilon \alpha$, $\bar{a} \neq \bar{\bar{a}}$, then it will be shown that

$$f_1(\underline{\bar{a}}) = f_1(\underline{\bar{\bar{a}}}) \ ,$$

from which it may be concluded that $f_i(\underline{a})$ is a function only of a_i and $\sum_{i=1}^{n} a_i$.

Let $\underline{a}^o = \min(\underline{\bar{a}} \ , \ \underline{\bar{\bar{a}}})$ taken componentwise and e_j be the j^{th} unit vector. Then if $\overset{o}{b}$ is defined as the smallest non-zero component of two vectors $(\underline{\bar{a}} - \underline{a}^o, \underline{\bar{\bar{a}}} - \underline{a}^o)$, some i and j exist such that we can define

$$\underline{\bar{a}}^1 = \underline{a}^o + \overset{o}{b}e_i \ , \quad \underline{\bar{a}}^1 \leq \underline{\bar{a}} \ ,$$

and

$$\underline{\bar{\bar{a}}}^1 = \underline{a}^o + \overset{o}{b}e_j \ , \quad \underline{\bar{\bar{a}}}^1 \leq \underline{\bar{\bar{a}}} \ ,$$

where either

$$\bar{a}_i^1 = \bar{a}_i \quad \text{or} \quad \bar{\bar{a}}_j^1 = \bar{\bar{a}}_j \ .$$

By assumption A4

$$f_1(\underline{\bar{a}}^1) = f_1(\underline{\bar{\bar{a}}}^1) \ .$$

Now define b^1 as the minimum non-zero element of $(\underline{\bar{a}} - \underline{\bar{a}}^1, \ \bar{\bar{a}} - \bar{\bar{a}}^1)$ and form

$$\underline{\bar{a}}^2 = \underline{\bar{a}}^1 + b^1e_i \ , \quad \underline{\bar{a}}^2 \leq \underline{\bar{a}} \ ,$$

and

$$\underline{\bar{\bar{a}}}^2 = \underline{\bar{\bar{a}}}^1 + b^1e_j \ , \quad \underline{\bar{\bar{a}}}^2 \leq \underline{\bar{\bar{a}}} \ ,$$

where either

$$\bar{a}_i^2 = \bar{a}_i \quad \text{or} \quad \bar{\bar{a}}_j^2 = \bar{\bar{a}}_j \ .$$

Again by A4

$$f_1(\underline{\bar{a}}^2) = f_1(\underline{\bar{\bar{a}}}^2) \ .$$

Since the number of zero elements of $(\underline{\bar{a}} - \underline{\bar{a}}^k, \underline{\bar{\bar{a}}} - \underline{\bar{\bar{a}}}^k)$ increases by at least one at each iteration of this procedure, and

$$f_1(\underline{\bar{a}}^k) = f_1(\underline{\bar{\bar{a}}}^k) \ , \quad \text{for all } k \ ,$$

we have

$$\underline{\bar{a}}^k = \underline{\bar{a}} \ , \quad \underline{\bar{\bar{a}}}^k = \underline{\bar{\bar{a}}} \ , \quad \text{for } k \geq n\text{-}1 \ .$$

Thus, $f_1(\underline{\bar{a}}) = f_1(\underline{\bar{\bar{a}}})$ as required, establishing the claim that the market share $m(s_1)$ is constant over the set $\mathcal{A}$ and hence depends only upon the quantities a_1 and A. So, in general, we will express $f_i(\underline{a})$ in the form $f_i(a_i, A)$. By A3

$$f_i(a,A) = f_j(a,A)$$

so that

$$f_i = f_j \ , \quad \text{for all } i,j = 1,\ldots,n. \tag{3}$$

By A2

$$f_i(0,A) = 0 \ . \tag{4}$$

Now suppose by contradiction that, for any fixed a and A,

$$f_i(a,A) = \lambda \neq a/A \ .$$

Assume $\lambda > a/A$; the case $\lambda < a/A$ being similar.

Consider two vectors $\underline{\bar{a}}$, $\underline{\bar{\bar{a}}}$ where

$$\bar{a}_i = 0 \ , \qquad i = 1,\dots,k-1,$$

$$\bar{a}_i = a \ , \qquad i = k,$$

$$\bar{a}_i = \bar{\bar{a}}_i \ , \qquad i = k+1,\dots,n,$$

$$\bar{\bar{a}}_i = a/k \ , \qquad i = 1,\dots,k \ .$$

By (3)

$$\sum_{i=k+1}^{n} f_i(\underline{\bar{a}}) = \sum_{i=k+1}^{n} f_i(\underline{\bar{\bar{a}}}) \ ,$$

and by (1)

$$\sum_{i=1}^{k} f_i(\underline{\bar{a}}) = \sum_{i=1}^{k} f_i(\underline{\bar{\bar{a}}}) \ ,$$

so that by (3) and (4)

$$\lambda = kf_i(a/k,A) \ ,$$

or

$$f_i(a/k,A) = \lambda/k \ .$$

Now consider a vector $\underline{a}$ with

$$a_i = a/k \ , \qquad i = 1,\dots,n-1,$$

and

$$a_n = A - (n-1)a/k \ ,$$

where

$$A \geq (n-1)a/k \ .$$

Now

$$\sum_{i=1}^{n} f_i(\underline{a}) = \sum_{i=1}^{n} f_i(a_i, A)$$

$$= (n-1)\lambda/k + f_n(A - (n-1)a/k)$$

$$\geq (n-1)\lambda/k \quad \text{by } (2) \ .$$

Hence, there is a contradiction if k and n can be chosen such that

$$(n-1)\lambda/k > 1 \ ,$$

and

$$(n-1)a/k \leq A \ .$$

That is, if

$$(n-1)a/A \leq k < \lambda(n-1) \ ,$$

or

$$a/A \leq k/n-1 < \lambda \ . \tag{5}$$

Obviously, (5) can be satisfied for some values of n and k. Hence,

$$f_i(a,A) = a/A \ ,$$

and the theorem is proved.

4. Discussion

The key point of the mathematical analysis is that, subject to certain basic assumptions relating the vector quantity, attraction, to the scalar quantity, market share, mathematical consistency implies that market share is a simple linear normalization of attraction. Let us look at the implications of the assumptions used.

Assumptions A1 and A2 are rather inconsequential and made to simplify the analysis. A2 states that sellers with zero attraction will have no market share. A1 requires attraction to be non-negative and says the attraction of at least one firm must be positive. Otherwise there would be no active sellers in the market. Assumption A3 does have some substance. It says that if two competing sellers have equal attraction, then they will have an equal share of the market. If attraction were simply defined as advertising, for instance, then one could argue against A3 in many cases. Clearly, there are other factors which influence market share. Thus, A3 helps make clear to the model builder what he must include in his attraction function to obtain a sensible result from the model.

A crucial assumption is A4. It states that if the attraction of a competitor of s_i increases by some amount Δ, then the new market share of s_i will not depend on which competitor made the increase. A4 does not say the market share of s_i would remain fixed. Intuitively, we would expect, in fact, a drop in seller i's share if competitors increased their attraction. Is A4 reasonable?

We can think of two possible sources of deviations from A4: non-linearty and asymmetry. Nonlinearity would be evidenced if adding an increment to a small attraction produced a different effect (on others) from adding the same amount to a large attraction. To some extent, however, this is a matter of the scale along which attraction is measured. There is a clear advantage if attraction is additive in the sense of A4.

Asymmetry could arise if changes in attraction of one seller were differentially effective on the customers of another. Aspects of asymmetry can be formally considered in the linear normalization model by making attraction of seller i partially dependent on some of the qualities of seller j. However, in general, our assumptions do not accomodate asymmetry, and, an extension of the theory would be required. In some situations market segmentation would be sufficient to represent asymmetric effects. Thus a marketing action may increase attractiveness more in one group than another (for example, a sportier car may appeal more to younger people). The algebra of market segmentation is described below.

To understand the implications of the theorem further, we present two corollaries. However, either of them could be made as an assumption to replace A4. Then A4 would follow as a corollary.

C1: The market share of seller i depends only on his attraction a_i and the sum of all attractions.

C2: If the attraction of seller i increases by an amount λ and if the attraction of seller j decreases by the same amount λ, while the attraction of all other sellers s_k, $k \neq i,j$, remains the same, then the market share of sellers s_k, $k \neq i,j$ remains constant.

Corollary C1 says that in considering the market share of seller i, one can aggregate the other sellers together, take their aggregated attraction to be the sum of their individual attractions, and then focus on seller i versus the rest. Corollary C2 is similar in spirit but less encompassing. C2 is local, whereas C1 is global. One point worth noting is that A4 is an assumption concerned with what happens when the total attraction, i.e., the sum, increases. The alternatives C1 and C2, on the other hand, concern the reaction of the market when total attraction remains constant.

Considerations for Model Builders. The main point for model builders is that a simple model which focuses on the attributes of a single seller, is sufficiently rich to model a fully competitive market.

It is instructive to point out an appealing method that cannot be used to deduce the normalization model. At first glance it appears that, since market share is, by definition, the ratio of sales to total sales, it would be sufficient to assume that sales are proportional to the seller's attraction function. Calculation of share immediately gives the normalization model. However, this will only be valid in a totally non-competitive market where the marketing activities of one seller do not influence the sales of another. If, for example, the market is of fixed size in total sales, individual sales cannot be linear with the attraction function. Furthermore, sales cannot be independent of competitive attraction.

Notice we have not deduced any specific results about market behavior, but rather some mathematical rules of the game. Thus, if someone asserts an attraction function depending on, say, advertising and price, and it is wrong, then the calculation of market shares will be wrong. Once attraction is specified, however, we can answer such questions as what is the impact on market share of incremental changes in price or advertising or any of the other factors composing attraction.

Another interesting aspect of this model is the quantity A, the total attraction of the sellers. One might construct a model of the size of the market as a function of A. Combining this with the market share, one could calculate for a given seller the total increase in his number of sales generated by increases in attraction. Part of these new sales would be due to an increased market size and part to an increased market share. In fact, one could consider A_1, $A_2 \ldots$, A_m to be the attractions of a number of different product classes which compete with each other for consumers. For instance, A_1 may represent the total attraction of radios, A_2 televison sets, A_3 stereo systems, and so on. One might postulate a different model for computing the share of the electronic media market held by each of these product classes. Combining this with our model for individual sellers within a segment provides a more sophisticated competitive model.

Assumptions A1 - A4 essentially make $a(\cdot)$ an unnormalized probability function on the set of sellers. For an alternative axiomation that closely parallels probability, see the Appendix. Market share, on the other hand, satisfies all the axioms of probability theory and so, mathematically speaking, is a probability function defined on the set of sellers. The statement of the assumptions and results is in terms of market share, but the term "probability of purchase" could clearly be substituted without affecting the mathematical

development. Notice that the results refer to probability of purchase from a seller given that a purchase will be made. In other words, the sum of the purchase probabilities is presumed to be one. Obviously, the probability of no purchase can be introduced as an extension of the model.

The fact that market share has the mathematical properties of a probability can be helpful in various ways. For example, if several customer groups or markets segments are identified, the concept of conditional market share becomes useful. Let

$C = \{c_1, \ldots, c_r\}$ = a set of r customer groups,

$a(s_i|c_j)$ = attraction of seller s_i within customer group j,

$p(c_j)$ = proportion of total sales coming from customer group c_j .

Then assuming that A1 - A4 hold for each customer group, the market share of s_i within customer group j is

$$m(s_i|c_j) = a(s_i|c_j)/\sum_{k=1}^{n} a(s_k|c_j),$$

and so the total market share is

$$m(s_i) = \sum_{j=1}^{r} m(s_i|c_j)p(c_j).$$

By partitioning the population into groups or segments a complex model can be built up from simple elements. Different marketing variables, say, price, promotion, advertising, and distribution, may impinge differently on different segments, which may, in turn, respond differently. The responses would define a relative attraction function which would then be assembled as shown above. Thus, the adoption of a basic normalized attraction model does not mean that all share expressions end up as simple ratios.

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Appendix

Attraction As An Unnormalized Probability

An alternative axiomization of the linear normalized market share model brings out the close mathematical connection between attraction and probability theory.

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ = set of all sellers

$S \subset \mathcal{S}$ = a subset of sellers

$a(S)$ = attraction of a subset of sellers.

A sufficient set of axioms is:

B1. Attraction is non negative ,

$$a(s_i) \geq 0 , \qquad s_i \in \mathcal{S} .$$

B2. The attraction of a subset of sellers is the sum of the attractions of the sellers in the subset.

$$a(S) = \sum_{s_i \in S} a(s_i), \qquad S \subset \mathcal{S} .$$

B3. $a(s_i)$ is finite for all $s_i \in \mathcal{S}$ and $a(s_i) > 0$ for at least one s_i.

B4. If two subsets of sellers have equal attractions, their market shares are equal,

$$a(S_1) = a(S_2) \rightarrow m(S_1) = m(S_2) , \quad S_1, S_2 \subset \mathcal{S} .$$

The proof of the market share theorem is much the same as before. The intermediate result

$$f_1 (\bar{\underline{a}}) = f_1 (\bar{\bar{\underline{a}}})$$

can be obtained as follows. Define

$$S = \{s_2, \ldots, s_n\}$$

For $\underline{a} = \bar{\underline{a}}$,

$$a(S) = \sum_{i=2}^{n} \bar{a}_i = A - a_1 \quad \text{by B2.}$$

For $\underline{a} = \underline{\bar{\bar{a}}}$,

$$a(S) = \sum_{i=2}^{n} \bar{\bar{a}}_i = A - a_1 \qquad \text{by B2.}$$

Therefore, denoting the market share of S given $\underline{a} = \underline{a}'$ by $m(S|\underline{a} = \underline{a}')$,

$$m(S|\underline{a} = \underline{\bar{a}}) = m(S|\underline{a} = \underline{\bar{\bar{a}}}) \qquad \text{by B4,}$$

and so

$$f_1(\underline{\bar{a}}) = 1 - m(S|a = \underline{\bar{a}}) = 1 - m(S|a = \underline{\bar{\bar{a}}}) = f_1(\underline{\bar{\bar{a}}})$$

as desired. The argument that $f_i(\underline{a})$ can be written $f_i(a,A)$ and $f_i = f_j$ for all i,j is the same. Since by B2 and B4 there is an equivalence between a single seller and a set of sellers with the same total attraction, we can extend the notation to $f_S(a,A)$ and $f_S = f_i = f_j$ for all i,j,S.

By definition $a(\mathcal{S}) = A$ and $m(\mathcal{S}) = 1$ so that $f_i(A,A) = f_S(A,A) = 1$. Consider a seller, say s_1, with zero attraction. Let $S = \{s_2, \ldots, s_n\}$, then

$$f_1(0,A) + f_S(A,A) = 1,$$

and so $f_1(0,A) = 0$. This establishes (4) without assuming A2. The rest of the proof is the same.

Axioms B1 and B2 are two of the three axioms of finite sample space probability theory. (See, for example, Parzen [10].) The third probability axiom is that the probability of a certain event is 1. B3 states two properties implied by this, namely, finiteness and at least one positive value, but stops short of the unity normalization. Thus B1 - B3 create attraction as an unnormalized probability function. B4 makes the connection to share. Share itself satisfies all the axioms of probability and so is a probability function defined on the set of sellers.

The axiomization B1-B4 is very appealing but was not chosen as the

basic approach because it introduces the additivity assumption by means of the attraction of a set of sellers. The concept of the attraction of a set seems a little artificial. This is because attraction has been discussed as a property of an individual seller and, although our final result implies that the concept can be extended to sets it seems more natural to have this as a deduction than an assumption. The approach chosen is to use A4, which expresses additivity in terms of increments of an individual sellers' attraction so that no concept of collective attraction is required.