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Abstract

This report gives an expository account of a method due to Wiener (10). His method is to solve for the voltage across a nonlinear device in terms of the entire random voltage and then to get statistical averages, on the assumption that the current-voltage function of the nonlinear element and the system transfer function are given. We treat a case in which random voltage passes through a filter before entering the circuit in question. Explicit formulas depending only on the assumptions above can be given for the moments of all orders of the voltage across part of the circuit, and similarly, for its frequency spectrum. The method of computation explicitly requires the use of the Wiener theory of Brownian motion (11, 12), also associated with the names of Einstein, Smoluchowski, Perrin, and many others (7). Sections I and III form the main part of the paper; section II is an heuristic exposition of the Wiener theory of Brownian motion preparatory to section III.

A METHOD OF WIENER IN A NONLINEAR CIRCUIT

Introduction

The problem of noise is encountered in almost all electronic processes, and the properties of noise have been studied by many researchers (1 to 5); among them, Rice (4) and Middleton (5). The term "properties of noise" is used to indicate such measurable quantities as the average, the mean-square voltage (or current), the correlation function, and the power or energy associated with the noise. It also refers to the power and correlation function of all or part of the disturbance, when noise, or a signal and noise, is modified by passage through some nonlinear apparatus. The analytical descriptions of noise systems have been limited mainly to those cases where the random noise belongs to a normal random process whose statistical properties are well known.

In such cases, one avenue of approach is to study the actual random variation in time of the displacement or voltage appropriate to the problem. This variable is usually developed in a Fourier series in time the coefficients of which are allowed to vary in a random fashion. This Fourier method has been applied systematically to a whole series of problems by Rice and by Middleton. The other approach is the method of Fokker-Plank or the diffusion equation method (6), in which the distribution function of the random variables of the system fulfills a partial differential equation of the diffusion type. These two methods may be shown to yield identical results (7).

It should be noted that Kac and Siegert have developed a method for dealing with the exceptional case of rectified and filtered, originally normal, random noise. Because of rectification and filtering, the distribution law of noise is no longer gaussian; and therein lies the difficulty of the nonlinear problem. Their method yields the probability density of the output of a receiver in terms of eigenvalues and eigenfunctions of a certain integral equation (8). Recently Weinberg and Kraft have performed experimental study of nonlinear devices (9).

This report gives an expository account of a method due to Wiener (10). His method is to solve for the voltage across a nonlinear device in terms of the entire random voltage and then to get statistical averages, on the assumption that the current-voltage function of the nonlinear element and the system transfer function are given. We treat a case in which random voltage passes through a filter before entering the circuit in question. Explicit formulas depending only on the assumptions above can be given for the moments of all orders of the voltage across part of the circuit, and similarly, for its frequency spectrum. The method of computation explicitly requires the use of the Wiener theory of Brownian motion (11, 12), also associated with the names of Einstein, Smoluchowski, Perrin, and many others (7). Sections I and III form the main part of the paper; section II is an heuristic exposition of the Wiener theory of Brownian motion preparatory to section III.

The circuit with which we deal is indicated in Fig. 1, in which a nonlinear device D is connected in series with an admittance $H(\omega)$, the system transfer function, and a filter precedes the combination. A random noise voltage $v(t)$ of the Brownian distribution is impressed across the whole circuit. The voltage across the filter $v_f(t)$ is given (13) by

$$v_f(t) = \int_{-\infty}^{t} v(\tau)W(t - \tau)d\tau \tag{1}$$

where $W(t)$ is the weighting function of the filter, and $W(t) = 0$ for $t < 0$, which is assumed throughout this paper.
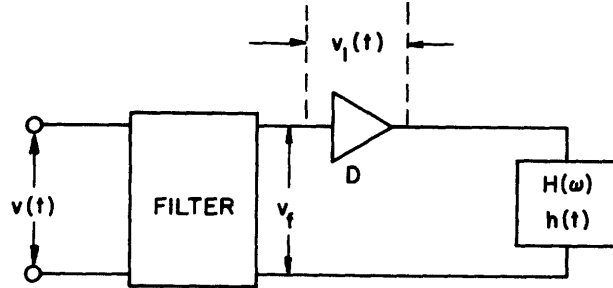

Fig. 1

Let us assume that the admittance function $2\pi H(\omega)$ is known, and let $h(t)$ be the current response of the system to a unit voltage impulse. If $v_1(t)$ denotes the voltage across the nonlinear device D, it is well known that the current through $H(\omega)$ is given by

$$\int_{-\infty}^{\infty} h(t - \tau) (v_f(\tau) - v_1(\tau))d\tau \tag{2}$$

In order to solve this equation we assume that the current-voltage function of the nonlinear element D is known; we take the current equal to $v_1(t) + \epsilon (v_1(t))^2$, where ϵ is a constant. Since the same current flows through D and $H(\omega)$, we have

$$v_1(t) + \epsilon (v_1(t))^2 = \int_{-\infty}^{\infty} h(t - \tau) (v_f(\tau) - v_1(\tau))d\tau \tag{3}$$

After substituting Eq. 1 in Eq. 3 we obtain

$$v_1(t) + \epsilon (v_1(t))^2 = \int_{-\infty}^{\infty} h(t - \tau) \left(\int_{-\infty}^{\infty} v(\sigma)W(\tau - \sigma)d\sigma \right) d\tau$$

$$- \int_{-\infty}^{\infty} h(t - \tau) v_1(\tau)d\tau \tag{4}$$

We wish to solve Eq. 4 for $v_1(t)$ in terms of our random voltage $v(t)$. For this step we assume that

$$v_1(t) = G_0 + \int_{-\infty}^{\infty} G_1(t - \tau)v(\tau)d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_2(t - \tau_1, t - \tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_3(t - \tau_1, t - \tau_2, t - \tau_3)v(\tau_1)v(\tau_2)v(\tau_3)d\tau_1 d\tau_2 d\tau_3 + \ldots \tag{5}$$

where G_0 is a constant. In the sequel, we shall take $G_0 = 0$ for the sake of simplicity.

The expression for $v_1(t)$ is now substituted in Eq. 4 and we shall equate linear part with linear part, quadratic part with quadratic part, and so on, involving the random function $v(t)$. Then we get these first-degree terms:

$$\int_{-\infty}^{\infty} G_1(t - \tau)v(\tau)d\tau + \int_{-\infty}^{\infty} h(t - \sigma)d\sigma \int_{-\infty}^{\infty} G_1(\sigma - \tau)v(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(t - \sigma)\left(\int_{-\infty}^{\infty} W(\sigma - \tau)v(\tau)d\tau\right)d\sigma \tag{6}$$

This expression can be true if we have

$$G_1(t - \tau) + \int_{-\infty}^{\infty} h(t - \sigma)d\sigma\, G_1(\sigma - \tau) = \int_{-\infty}^{\infty} h(t - \sigma)W(\sigma - \tau)d\sigma \tag{7}$$

The solution of this equation can be obtained by means of the Fourier transform method Let G be the Fourier transform of g. Then we have (14)

$$G(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega \tag{8}$$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t)e^{-i\omega t}dt \tag{9}$$

and since $H(\omega)$ is the Fourier transform of $h(t)$, we also have the following relation

$$h(t) = \int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega \tag{10}$$

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \tag{11}$$

Moreover, we let $W(t)$ be the Fourier transform of $w(\omega)$. Then

$$W(t) = \int_{-\infty}^{\infty} w(\omega)e^{i\omega t}d\omega \tag{12}$$

$$w(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(t)e^{-i\omega t}dt \tag{13}$$

If we take the Fourier transform of both sides of Eq. 7, we get

$$g_1(\omega) + 2\pi H(\omega)g_1(\omega) = 2\pi H(\omega)w(\omega)$$

or

$$g_1(\omega) = \frac{2\pi H(\omega)w(\omega)}{1 + 2\pi H(\omega)} \tag{14}$$

If there were no filter in our circuit, then we would get Eq. 14 without $2\pi w(\omega)$ in the numerator. By use of Eq. 8 we can find $G_1(t)$ from Eq. 14. From Eq. 4 we shall similarly get the second-degree terms:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(t - \tau_1, t - \tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 + \epsilon\left(\int_{-\infty}^{\infty} \cdot G_1(t - \tau)v(\tau)d\tau\right)^2$$

$$+ \int_{-\infty}^{\infty} h(t - \tau)d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(\tau - \tau_1, \tau - \tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 = 0 \tag{15}$$

Equation 15 will be true if the following expression holds:

$$G_2(t - \tau_1, t - \tau_2) + \epsilon G_1(t - \tau_1)G_1(t - \tau_2)$$

$$+ \int_{-\infty}^{\infty} h(t - \tau)G_2(\tau - \tau_1, \tau - \tau_2)d\tau = 0 \tag{16}$$

Let us assume that $G_2(\tau_1, \tau_2)$ is the double Fourier transform of $g_2(\omega_1, \omega_2)$, that is

$$G_2(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(\omega_1, \omega_2)e^{i(\omega_1 t_1 + \omega_2 t_2)}d\omega_1 d\omega_2 \tag{17}$$

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$$g_2(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(t_1, t_2) e^{-i(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2 \qquad (18)$$

Taking the double Fourier transform of both sides of Eq 16, we find, by using Eqs. 9, 11 and 18, that

$$g_2(\omega_1, \omega_2) + \epsilon g_1(\omega_1) g_1(\omega_2) + 2\pi H(\omega_1 + \omega_2) g_2(\omega_1, \omega_2) = 0$$

or

$$g_2(\omega_1, \omega_2) = - \frac{\epsilon g_1(\omega_1) g_1(\omega_2)}{1 + 2\pi H(\omega_1 + \omega_2)}$$

$$= - \frac{\epsilon (2\pi)^2 H(\omega_1) H(\omega_2) w(\omega_1) w(\omega_2)}{[1 + 2\pi H(\omega_1)][1 + 2\pi H(\omega_2)][1 + 2\pi H(\omega_1 + \omega_2)]} \qquad (19)$$

If there were no filter in the circuit, we would get Eq. 19 without the factor $(2\pi)^2 w(\omega_1) w(\omega_2)$. Similar remarks apply to all g_n's.

Multiple Fourier transforms of the G_n's are defined as in Eqs. 17 and 18:

$$G_n(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_n(\omega_1, \ldots, \omega_n) e^{i \sum_1^n \omega_k t_k} d\omega_1 \ldots d\omega_n \qquad (20)$$

$$g_n(\omega_1, \ldots, \omega_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} G_n(t_1, \ldots, t_n) e^{-i \sum_1^n \omega_k t_k} dt_1 \ldots dt_n \qquad (21)$$

Working in the same manner as for $g_1(\omega)$ or $g_2(\omega_1, \omega_2)$ we get from the third-degree terms of Eq. 4

$$g_3(\omega_1, \omega_2, \omega_3) = - \frac{2\epsilon\, g_1(\omega_1) g_2(\omega_2, \omega_3)}{1 + 2\pi H(\omega_1 + \omega_2 + \omega_3)}$$

$$= \frac{2\epsilon^2 (2\pi) H(\omega_1) w(\omega_1)}{[1 + 2\pi H(\omega_1)]} \cdot \frac{(2\pi) H(\omega_2) w(\omega_2)}{[1 + 2\pi H(\omega_2)]} \cdot \frac{(2\pi) H(\omega_3) w(\omega_3)}{[1 + 2\pi H(\omega_3)]}$$

$$\cdot \frac{1}{1 + 2\pi H(\omega_2 + \omega_3)} \cdot \frac{1}{1 + 2\pi H(\omega_1 + \omega_2 + \omega_3)} \qquad (22)$$

And we obtain from the fourth-degree terms

$$g_4(\omega_1, \omega_2, \omega_3, \omega_4) = - \frac{\epsilon\, g_1(\omega_1) g_3(\omega_2, \omega_3, \omega_4) + 2\epsilon\, g_2(\omega_1, \omega_2) g_2(\omega_3, \omega_4)}{1 + 2\pi H(\omega_1 + \omega_2 + \omega_3 + \omega_4)} \qquad (23)$$

From this procedure it is easy to compute any g_n's. It will be noticed that

$g_2(\omega_1, \omega_2)$ has ϵ; $g_3(\omega_1, \omega_2, \omega_3)$ has ϵ^2; and in general, $g_n(\omega_1, ..., \omega_n)$ will have ϵ^{n-1} as its perturbation factor. By means of the Fourier transformation Eq. 20 we shall find G_n's, which will be substituted in Eq. 5 to find $v_1(t)$, the voltage across the nonlinear element D. Our next step is the computation of Eq. 5, involving the random voltage $v(t)$. For this purpose the next section of the report is devoted to an heuristic exposition of the Wiener theory of Brownian motion.

II

2.0. Before we give the Wiener theory of Brownian motion (11, 12), it is interesting to recall that in 1828 Robert Brown first observed tiny irregular motions of small particles suspended in water when viewed under a microscope. Each particle may be considered as bombarded on all sides by the molecules of the water moving under their velocities of thermal agitation. On an average the number striking the particle will be the same in all directions and the average momentum given to the particle in any direction will be zero. If we consider the interval of time to be sufficiently small for the individual impacts of the particles on one another to be discernible, there will be an excess of momentum in one direction or the other, resulting in a random kind of motion that is observable under a microscope. This motion is known as the Brownian motion.

According to the theory of Einstein (15) and Smoluchowski (16) the initial velocity over any ordinary interval of time is of negligible importance in comparison with the impulses received during the same interval of time. The motion observable with this time scale is so curious that it suggested to Perrin (17) the nondifferentiable continuous functions of Weierstrass (18). It is thus a matter of interest to the mathematician to discover the defining conditions and properties of these particle paths.

If we consider the x-coordinate of a Brownian particle, the probability that this should alter a given amount in a given time is, in the Wiener theory (11), assumed independent (a) of the entire past history of the particle, (b) of the instant from which the given interval is measured, and (c) of the direction in which the changes take place. Since displacements of a particle are independent and random by these assumptions, they will form a normal or Gaussian distribution by the central limit theorem (19), provided the time interval becomes sufficiently long. Because of minute motions of the Brownian particle it is not unreasonable to assume that the normal distribution law holds very nearly even in a sufficiently short interval of time, though it is still large when compared with intervals between molecular collisions. From these assumptions it is to be noted that displacements of each component of the Brownian particle form a stationary time series (20); and the mean-square motion or displacement in a given direction over a given time is proportional to the length of that time (21).

2.1. We shall consider the time equation of the path of a particle subject to the Brownian movement to be of the form

$$x = x(t)$$
$$y = y(t)$$
$$z = z(t) \qquad (24)$$

where t is time, and x, y, z are the coordinates of the particle. We shall limit our attention to the function x(t). Then the difference between $x(t_1)$ and $x(t)$ $(t_1 > t)$ may be regarded as the sum of the displacements incurred by the particle over a set of intervals consisting of the interval from t to t_1. If the constituent intervals are of equal size, then the probability distribution of the displacements accrued in the different intervals will be the same; and the probability that $x(t_1) - x(t)$ lies between a and b is very nearly of the form

$$\frac{1}{[2\pi\phi(t_1 - t)]^{1/2}} \int_a^b e^{-\frac{x^2}{2\phi(t_1 - t)}} dx \qquad (t_1 > t) \qquad (25)$$

It is the basic characteristic of the Brownian motion to accrue errors over the interval from t to t_1 $(t < t_1)$ such that the error is the sum of the independent errors incurred over the times from t to t_1 and from t_i to t_1 $(t < t_i < t_1)$. Thus we shall have

$$\frac{1}{[2\pi\phi(t_1 - t)]^{1/2}} e^{-\frac{x_1^2}{2\phi(t_1 - t)}} = \frac{1}{2\pi[\phi(t_1 - t_i)\phi(t_i - t)]^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\phi(t_i - t)} - \frac{(x_1 - x)^2}{2\phi(t_1 - t_i)}} dx$$

$$= \exp\left[-\frac{x_1^2}{2\phi(t_i - t) + 2\phi(t_1 - t_i)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{2}\left(\frac{\phi(t_1 - t_i) + \phi(t_i - t)}{\phi(t_1 - t_i)\phi(t_i - t)}\right)\right] dy$$

$$= \left(2\pi\phi(t_1 - t_i)\phi(t_i - t)\right)^{-\frac{1}{2}} \exp\left(-\frac{x_1^2}{2\phi(t_1 - t_i) + 2\phi(t_i - t)}\right) \qquad (26)$$

This cannot be true unless we have

$$\phi(t_1 - t) = \phi(t_1 - t_i) + \phi(t_i - t) \qquad (27)$$

Hence we must have

$$\phi(\mu) = A\mu \qquad (A > 0) \qquad (28)$$

where A is a constant. The probability, then, that $x(t_1) - x(t)$ lies between a and b is approximately of the form

$$\frac{1}{[2\pi A(t_1 - t)]^{1/2}} \int_a^b e^{-\frac{x^2}{2A(t_1 - t)}} dx \qquad (t_1 > t) \qquad (29)$$

Here A is a constant which we shall reduce to 1 by a proper choice of units. If we insert $t_i - t = t_1$, and $t_1 - t_i = t_2$ in Eq. 26, the fundamental identity reads

$$\frac{1}{[2\pi(t_1 + t_2)]^{1/2}} e^{-\frac{x_1^2}{2(t_1 + t_2)}} = \frac{1}{2\pi(t_1 t_2)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2t_1} - \frac{(x_1 - x)^2}{2t_2}} dx \qquad (30)$$

This equation states that the probability that x_1 should have changed by an amount lying between x_1 and $x_1 + dx_1$ after a time $t_1 + t_2$ is the probability that the change of x over time t_1 should be anything at all, and that at the end in an interval t_2 the particle should then find itself at a position between x_1 and $x_1 + dx_1$.

A quantity x whose changes are distributed after the manner just discussed is said to have them normally distributed. What really is distributed is the function x(t) representing the successive values of x. Without essential restriction we may suppose x(0) = 0. Our next problem is to establish a theory of integration which is based on a method of mapping. This method of mapping consists in making certain sets of functions x(t), which Wiener calls "quasi-intervals" (11), correspond to certain intervals of the line $0 \leqslant a \leqslant 1$. The quasi-intervals will be a set of all functions x(t) defined for $0 \leqslant t \leqslant 1$ such that

$$x(0) = 0$$
$$a_1 \leqslant x(t_1) \leqslant b_1$$
$$a_2 \leqslant x(t_2) \leqslant b_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n \leqslant x(t_n) \leqslant b_n$$
$$0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant 1 \qquad (31)$$

By our definition of probability, the probability that x(t) should lie in these quasi-intervals is

$$m(I) = \frac{1}{[(2\pi)^n t_1(t_2 - t_1) \ldots (t_n - t_{n-1})]^{1/2}} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n}$$

$$\exp\left[-\frac{\xi_1^2}{2t_1} - \frac{(\xi_2 - \xi_1)^2}{2(t_2 - t_1)} - \ldots - \frac{(\xi_n - \xi_{n-1})^2}{2(t_n - t_{n-1})}\right] d\xi_1 d\xi_2 \ldots d\xi_n \qquad (32)$$

We may let $a_i \to -\infty$ and $b_i \to \infty$. If $t_i = t_j$, we may use the larger of the (a_i, a_j) and the smaller of the (b_i, b_j). If we add an extra ordinate line at t = τ, $(t_i < \tau < t_{i+1})$ with $a_\tau = -\infty$, $b_\tau = \infty$, the value of m(I) will remain the same, as shown below

$$\left[(2\pi)^{n+1} t_1(t_2 - t_1) \cdots (t_i - t_{i-1})(\tau - t_i)(t_{i+1} - \tau) \cdots (t_n - t_{n-1})\right]^{-\frac{1}{2}}$$

$$\int_{a_1}^{b_1} \cdots \int_{a_i}^{b_i} \int_{-\infty}^{\infty} \int_{a_{i+1}}^{b_{i+1}} \cdots \int_{a_n}^{b_n}$$

$$\exp\left[-\frac{\xi_1^2}{2t_1} - \frac{(\xi_2 - \xi_1)^2}{2(t_2 - t_1)} - \cdots - \frac{(\eta - \xi_i)^2}{2(\tau - t_i)} - \frac{(\xi_{i+1} - \eta)^2}{2(t_{i+1} - \tau)} - \cdots - \frac{(\xi_n - \xi_{n-1})^2}{2(t_n - t_{n-1})}\right]$$

$$d\xi_1 d\xi_2 \cdots d\xi_i d\eta d\xi_{i+1} \cdots d\xi_n = m(I) \tag{33}$$

since we have

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(\eta - \xi_i)^2}{2(\tau - t_i)} - \frac{(\xi_{i+1} - \eta)^2}{2(t_{i+1} - \tau)}\right) d\eta$$

$$= \left(\frac{2\pi(\tau - t_i)(t_{i+1} - \tau)}{t_{i+1} - t_i}\right)^{1/2} \cdot \exp\left(-\frac{(\xi_{i+1} - \xi_i)^2}{2(t_{i+1} - t_i)}\right) \tag{34}$$

Hence there is no effect from the addition of extra ordinates, which is, of course, expected from the definition of the total compound probability; and the total probability of m(I) is 1.

As the Brownian particle x(t) moves in the quasi-intervals, there is a corresponding set of values of α. If we write x(t, α) for x(t) corresponding to each α, we have as a measure of the probability of each coincidence, the product given by Eq 32 for the set of intervals

$$x(0) = 0$$

$$a_1 \leqslant x(t_1, \alpha) \leqslant b_1$$

$$a_2 \leqslant x(t_2, \alpha) \leqslant b_2$$

$$\cdots \cdots \cdots \cdots$$

$$a_n \leqslant x(t_n, \alpha) \leqslant b_n$$

$$0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant 1 \tag{35}$$

In this process we may determine the integral (or average) of other functions of α determined as functionals of x(t, α). The function x(t, α) is called a random function, in which t can trace all possible courses traversed by all possible particles, while α is the argument which singles out the specific Brownian motion from all possible Brownian motions. The variable α is introduced for the purpose of integration, so that a certain range of α measures, by its length, the probability of the set of Brownian motion that

it represents; and an integration with respect to this variable yields a probability average of the quantity integrated. Physically, it is at least reasonable to assume that the Brownian motion of a particle is continuous. It has been shown to be almost continuous (18).

2.2. If we wish to determine the average (or expectation) with respect to a of $x(t_1, a)$ ... $x(t_n, a)$, we have, for $0 = t_0 \leqslant t_1 \leqslant ... \leqslant t_n$,

$$\int_0^1 x(t_1, a) x(t_2, a) \ldots x(t_n, a) da$$

$$= (2\pi)^{-\frac{n}{2}} \left[t_1(t_2 - t_1) \ldots (t_n - t_{n-1}) \right]^{-1/2}$$

$$\int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_n \cdot \xi_1 \xi_2 \ldots \xi_n \cdot \exp\left(-\frac{\xi_1^2}{2t_1} - \frac{(\xi_2 - \xi_1)^2}{2(t_2 - t_1)} - \ldots - \frac{(\xi_n - \xi_{n-1})^2}{2(t_n - t_{n-1})} \right) \tag{36}$$

Let us compute a simple case

$$\int_0^1 x(t_1, a) x(t_2, a) da$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_1(t_2 - t_1)^{-\frac{1}{2}} \xi_1 \xi_2 \, e^{-\frac{\xi_1^2}{2t_1} - \frac{(\xi_2 - \xi_1)^2}{2(t_2 - t_1)}} d\xi_1 d\xi_2 \tag{37}$$

for $t_1 < t_2$. Let

$$\eta_1 = \frac{\xi_1}{(t_1)^{1/2}} , \quad \eta_2 = \frac{\xi_2 - \xi_1}{(t_2 - t_1)^{1/2}} \tag{38}$$

Then we have the Jacobian with respect to ξ_1, ξ_2:

$$J(\eta_1, \eta_2) = \frac{\partial(\xi_1, \xi_2)}{\partial(\eta_1, \eta_2)} = \begin{vmatrix} (t_1)^{1/2} & 0 \\ (t_1)^{1/2} & (t_2 - t_1)^{1/2} \end{vmatrix} = [t_1(t_2 - t_1)]^{1/2}$$

and

$$\xi_1 = \eta_1 (t_1)^{1/2}, \quad \xi_2 = \eta_2 (t_2 - t_1)^{1/2} + \eta_1 (t_1)^{1/2}$$

Then we get

$$\int_0^1 x(t_1, \alpha)x(t_2, \alpha)d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_1(t_1)^{1/2} \left[\eta_2(t_2 - t_1)^{1/2} + \eta_1(t_1)^{1/2} \right] e^{-\frac{\eta_1^2 + \eta_2^2}{2}} d\eta_1 d\eta_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[t_1 \eta_1^2 e^{-\frac{\eta_1^2 + \eta_2^2}{2}} + \eta_1 \eta_2 \left[t_1(t_2 - t_1) \right]^{1/2} e^{-\frac{\eta_1^2 + \eta_2^2}{2}} \right] d\eta_1 d\eta_2$$

$$= \frac{t_1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_1^2 e^{-\frac{\eta_1^2 + \eta_2^2}{2}} d\eta_1 d\eta_2 = t_1 \qquad (t_1 < t_2) \tag{39}$$

The second term involving η_1, η_2 drops out, since this term is an odd function with respect to η_1 and η_2. The result given above may be written

$$\int_0^1 x(t_1, \alpha)x(t_2, \alpha)d\alpha = \min(t_1, t_2) \tag{40}$$

where $\min(t_1, t_2)$ denotes the minimum of t_1 and t_2. In general, we have

$$\int_0^1 x(t_1, \alpha)x(t_2, \alpha) \ldots x(t_{2n}, \alpha)d\alpha$$

$$= \sum \min(t_{\nu_1}, t_{\nu_2}) \min(t_{\nu_3}, t_{\nu_4}) \ldots \min(t_{2n-1}, t_{2n}) \tag{41}$$

where the summation is on all of the permutations $(\nu_1, \ldots, \nu_{2n})$ of 1, 2, ..., 2n. And

$$\int_0^1 x(t_1, \alpha)x(t_2, \alpha) \ldots x(t_{2n+1}, \alpha)d\alpha = 0 \tag{42}$$

Equation 36 can be also written

$$\int_0^1 x(t_1, \alpha)x(t_2, \alpha) \ldots x(t_n, \alpha)d\alpha \qquad \text{(n is even)}$$

$$\sum \prod \int_0^1 x(t_i, \alpha)x(t_j, \alpha)d\alpha \tag{43}$$

where the Σ is taken over all partitions of $t_1, \ldots, t_n$ into distinct pairs, and the $\prod$ over

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all the pairs in each partition. This means that the averages of the products of $x(t_k, a)$ by pairs will enable us to get the averages of all polynomials in these quantities, thereby giving their entire statistical distribution.

At this point we introduce a definition: If $\rho(t)$ is a function of limited total variation (22) over (0, 1), we shall write

$$\int_0^1 \rho(t)dx(t, a) = \rho(1)x(1, a) - \int_0^1 x(t, a)d\rho(t) \tag{44}$$

If, then, $\rho_1(t)$ and $\rho_2(t)$ are of limited total variation over (0, 1), and $\rho_1(1) = 0$, $\rho_2(1) = 0$, we get, by use of Eqs. 44 and 39,

$$\int_0^1 da \int_0^1 x(t_1, a)d\rho_1(t_1) \int_0^1 x(t_2, a)d\rho_2(t_2)$$

$$= \int_0^1 d\rho_1(t_1) \int_0^1 d\rho_2(t_2) \int_0^1 x(t_1, a)x(t_2, a)da$$

$$= \int_0^1 t_1 d\rho_1(t_1) \int_{t_1}^1 d\rho_2(t_2) + \int_0^1 t_2 d\rho_2(t_2) \int_{t_2}^1 d\rho_1(t_1)$$

$$= -\int_0^1 t\left[\rho_2(t)d\rho_1(t) + \rho_1(t)d\rho_2(t)\right]$$

$$= \int_0^1 td\left[\rho_1(t)\rho_2(t)\right] = \int_0^1 \rho_1(t)\rho_2(t)dt \tag{45}$$

2.3. Thus far we have considered the Brownian motion $x(t, a)$, where t is positive in the finite range $0 < t < 1$. We shall now give a general form valid for t running over the whole real infinite line. Let us write

$$\psi(t, a, \beta) = x(t, a) \qquad (t > 0)$$

$$\psi(t, a, \beta) = x(-t, \beta) \qquad (t < 0)$$

where a and β have independent uniform distributions over (0, 1). Then $\psi(t, a, \beta)$ gives a distribution over $(-\infty < t < +\infty)$. To map a square on a line segment is merely to write our coordinates in the square in the decimal form

$$a = .a_1 a_2 \ldots a_n \ldots$$

$$\beta = .\beta_1 \beta_2 \ldots \beta_n \ldots \tag{47}$$

and to put

$$\gamma = \cdot a_1 \beta_1 a_2 \beta_2 \cdots a_n \beta_n \cdots \tag{48}$$

This mapping is one-one for almost all points both in the line-segment and the square (12): whence we get a new definition for the random function

$$\psi(t, \gamma) = \psi(t, a, \beta) \qquad (-\infty < t < +\infty) \tag{49}$$

In order to integrate $\psi(t, \gamma)$ with respect to γ in a manner similar to that used in Eq. 44 we wish to define that

$$\int_{-\infty}^{\infty} P(t) d\psi(t, \gamma) = - \int_{-\infty}^{\infty} P'(t) \psi(t, \gamma) dt \tag{50}$$

holds if $P(t)$ is assumed to vanish sufficiently rapidly to zero at $\pm\infty$, and is a sufficiently smooth function. With this definition we have, formally,

$$\int_0^1 d\gamma \int_{-\infty}^{\infty} P_1(t) d\psi(t, \gamma) \int_{-\infty}^{\infty} P_2(t) d\psi(t, \gamma)$$

$$= \int_0^1 d\gamma \int_{-\infty}^{\infty} P_1'(t) \psi(t, \gamma) dt \int_{-\infty}^{\infty} P_2'(t) \psi(t, \gamma) dt$$

$$= \int_{-\infty}^{\infty} P_1'(s) ds \int_{-\infty}^{\infty} P_2'(t) dt \int_0^1 \psi(s, \gamma) \psi(t, \gamma) d\gamma \tag{51}$$

If t_1 and t_2 are of the same sign, and $|t_1| < |t_2|$, we obtain by the same process (23) as that used in Eq. 39

$$\int_0^1 \psi(t_1, \gamma) \psi(t_2, \gamma) d\gamma = \int_0^1 x(|t_1|, \gamma) x(|t_2|, \gamma) d\gamma = |t_1| \tag{52}$$

And if t_1 and t_2 are of opposite signs, we get

$$\int_0^1 \psi(t_1, \gamma) \psi(t_2, \gamma) d\gamma = 0 \tag{53}$$

Using Eq. 52 in Eq. 51, we get, as in Eq. 45,

$$\int_0^1 d\gamma \int_{-\infty}^{\infty} P_1(t) d\psi(t, \gamma) \int_{-\infty}^{\infty} P_2(t) d\psi(t, \gamma)$$

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$$= \int_{-\infty}^{\infty} P_1(t)P_2(t)dt \qquad (54)$$

It is to be noted that

$$\int_0^1 d\gamma \int_{-\infty}^{\infty} P(t + \tau_1)d\psi(t, \gamma) \int_{-\infty}^{\infty} P(t + \tau_2)d\psi(t_2, \gamma)$$

$$= \int_{-\infty}^{\infty} P(s)P(s + \tau_1 - \tau_2)ds \qquad (55)$$

and when n is even, we have

$$\int_0^1 d\gamma \prod_{k=1}^{n} \int_{-\infty}^{\infty} P(t + \tau_k)d\psi(t, \gamma) = \sum \prod \int_{-\infty}^{\infty} P(s)P(s + \tau_i - \tau_j)ds \qquad (56)$$

where the sum is over all partitions of $\tau_1, \ldots, \tau_n$ into pairs, and the product is over the pairs in each partition.

If we put

$$P(s)P(t) = P(s, t) = P(t, s) \qquad (57)$$

and define $P(s, t)$ as possessing the similar properties with respect to each variable as $P(t)$ itself, then we may put

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(t_1, t_2)d\psi(t_1, \gamma)d\psi(t_2, \gamma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t_1, \gamma)\psi(t_2, \gamma)d \; P(t_1, t_2) \qquad (58)$$

In general, we define a symmetrical function of n variables $P(t_1, t_2, \ldots, t_n)$ such that it vanishes sufficiently rapidly to zero at $\pm \infty$ for each variable and is a sufficiently smooth function. We may thus define

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} P(t_1, t_2, \ldots, t_n)d\psi(t_1, \gamma)d\psi(t_2, \gamma) \ldots d\psi(t_n, \gamma)$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \psi(t_1, \gamma)\psi(t_2, \gamma) \ldots \psi(t_n, \gamma)d \; P(t_1, t_2, \ldots, t_n) \qquad (59)$$

Coming back to Eq. 56, we put $n = 4$, and $\tau_1 = \tau_2 = \tau_3 = \tau_4$, and

$$P(s_1)P(s_2)P(s_3)P(s_4) = P(s_1, s_2)P(s_3, s_4) = P(s_2, s_1)P(s_4, s_3)$$

$$= P(s_1, s_2, s_3, s_4) \qquad (60)$$

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Then we get, formally,

$$\int_0^1 d\gamma \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty P(t_1, t_2, t_3, t_4) d\psi(t_1, \gamma) d\psi(t_2, \gamma) d\psi(t_3, \gamma) d\psi(t_4, \gamma)$$

$$= (4 - 1) \int_{-\infty}^\infty \int_{-\infty}^\infty P(s_1, s_1, s_2, s_2) ds_1 ds_2 \tag{61}$$

where $(4 - 1)$ is the number of ways of dividing four terms into pairs; and in general we have, from Eq. 59,

$$\int_0^1 d\gamma \underbrace{\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty}_{n \text{ times}} P(t_1, t_2, \ldots, t_n) d\psi(t_1, \gamma) d\psi(t_2, \gamma) \ldots d\psi(t_n, \gamma)$$

$$= (n - 1)(n - 3) \ldots 5 \cdot 3 \cdot 1 \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty P(t_1, t_1, t_2, t_2, \ldots, t_{\frac{n}{2}}, t_{\frac{n}{2}}) dt_1 \ldots dt_{\frac{n}{2}} \tag{62}$$

if n is even; and 0 if n is odd.

III

3.1. Using the results of section II, we first compute the average of $v(t)$ across the nonlinear element D. Since $v(t)$ is a random noise voltage, we now write $v(t, \gamma)$ for $v(t)$, where γ is a usually suppressed parameter of distribution. Writing in full, we have

$$\text{Average of } v_1(t) = \int_0^1 v_1(t, \gamma) d\gamma$$

$$= \int_0^1 d\gamma \left[\int_{-\infty}^\infty G_1(t - \tau) dv(\tau, \gamma) + \int_{-\infty}^\infty \int_{-\infty}^\infty G_2(t - \tau_1, t - \tau_2) dv(\tau_1, \gamma) dv(\tau_2, \gamma) \right.$$

$$\left. + \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty G_3(t - \tau_1, t - \tau_2, t - \tau_3) dv(\tau_1, \gamma) dv(\tau_2, \gamma) dv(\tau_3, \gamma) + \ldots \right] \tag{63}$$

In view of Eq. 42, the first term is zero; but the second term is, by Eqs. 54 and 58,

$$\int_{-\infty}^\infty G_2(\tau, \tau) d\tau \tag{64}$$

. It is now necessary to express an average of G_n (n even) in terms of an average of g_n. For that purpose, let us start with

$$K(s) = \int_{-\infty}^{\infty} k(\omega)e^{i\omega s}dw \tag{65}$$

Then by the Fourier transformation, we get

$$k(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(s)e^{-i\omega s}ds \tag{66}$$

And the theory of convolution gives us

$$\int_{-\infty}^{\infty} K(s)K(s + \tau)ds = 2\pi \int_{-\infty}^{\infty} k(\omega)k(-\omega)e^{i\omega\tau}d\omega \tag{67}$$

If we place $\tau = 0$ in Eq. 67, we get

$$\int_{-\infty}^{\infty} K(s)K(s)ds = 2\pi \int_{-\infty}^{\infty} k(\omega)k(-\omega)d\omega \tag{68}$$

Putting

$$K(s)K(s) = R_2(s, s)$$

$$k(\omega)k(-\omega) = r_2(\omega, -\omega) \tag{69}$$

Eq. 68 reads

$$\int_{-\infty}^{\infty} R_2(s, s)ds = 2\pi \int_{-\infty}^{\infty} r_2(\omega, -\omega)d\omega \tag{70}$$

In general, we have

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R_n(s_1, s_1, s_2, s_2, \ldots, s_{\frac{n}{2}}, s_{\frac{n}{2}})ds_1 ds_2 \ldots ds_{\frac{n}{2}}$$

$$= (2\pi)^{\frac{n}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_n(\omega_1, -\omega_1, \omega_2, -\omega_2, \ldots, \omega_{\frac{n}{2}}, -\omega_{\frac{n}{2}})d\omega_1 d\omega_2 \ldots d\omega_{\frac{n}{2}} \tag{71}$$

By using Eq. 70, we can write Eq. 64 as

$$\int_{-\infty}^{\infty} G_2(\tau, \tau)d\tau = 2\pi \int_{-\infty}^{\infty} g_2(\omega, -\omega)d\omega \tag{72}$$

which is, in turn, evaluated by Eq. 19: that is,

$$- 2\pi\epsilon \int_{-\infty}^{\infty} \frac{H(\omega)H(-\omega)w(\omega)w(-\omega)}{(1 + 2\pi H(\omega))(1 + 2\pi H(-\omega))(1 + 2\pi H(0))} \, d\omega \tag{73}$$

The next nonvanishing term is

$$3(2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_4(\omega_1, -\omega_1, \omega_2, -\omega_2) d\omega_1 \, d\omega_2 \tag{74}$$

which can be computed by Eq. 23 together with Eqs. 14, 19, and 22. This term has ϵ^3 as its factor.

The average of $(v_1(t))^2$ can be computed in the same manner, which reads

$$2\pi \int_{-\infty}^{\infty} g_1(\omega)g_1(-\omega) d\omega$$

$$+ 4\pi^2 \left[2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(\omega_1, \omega_2)g_2(-\omega_1, -\omega_2) d\omega_1 d\omega_2 + \left(\int_{-\infty}^{\infty} g_2(\omega, -\omega) d\omega \right)^2 \right]$$

$$+ 2A\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\omega_1)g_3(-\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2$$

$$+ 8\pi^3 \left[6 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_3(\omega_1, \omega_2, \omega_3)g_3(-\omega_1, -\omega_2, -\omega_3) d\omega_1 d\omega_2 d\omega_3 \right.$$

$$\left. + 9 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_3(\omega_1, \omega_2, -\omega_2)g_3(-\omega_1, \omega_3, -\omega_3) d\omega_1 d\omega_2 d\omega_3 \right]$$

$$+ \ldots \tag{75}$$

Since we have
average $[v_1(t) + \epsilon(v_1(t))^2] =$ average $v_1(t) + \epsilon$ average $(v_1(t))^2$, Eqs. 73, 74, and 75 will give the average current across the nonlinear element, or $H(\omega)$.

The higher moments of $v_1(t)$ can be similarly computed, and also those of $(v_f(t) - v_1(t))$, the voltage across $H(\omega)$.

3.2. The most useful statistical characteristic of the random function in the Wiener theory is the correlation function. The autocorrelation function $\phi_{11}(\tau)$ of $f_1(t)$ is defined as

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$$\phi_{11}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} f_1(t)f_1(t + \tau)dt \qquad (76)$$

where

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} f_1^2(t)dt < +\infty \qquad (77)$$

Then the power density spectrum of $f_1(t)$ is given by

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{11}(\tau) \cos \omega\tau \, d\tau \qquad (78)$$

and its Fourier transform is

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega) \cos \omega\tau \, d\omega \qquad (79)$$

This relation between the autocorrelation function and the power density spectrum of a stationary random process was first proved by Wiener (11). Since then, correlation analysis has been applied to various communication problems by the school, in particular, of Lee and Wiesner (24-29).

Since time averages and ensemble averages of stationary random processes are equivalent by the ergodic hypothesis, we can compute the average of $v_1(t)v_1(t + \tau)$, the autocorrelation function of $v_1(t)$. Writing $v_1(t, \gamma)$ for $v_1(t)$, we have

$$\text{average } [v_1(t)v_1(t + \sigma)] = \int_0^1 v_1(t, \gamma)v_1(t + \sigma, \gamma)d\gamma$$

$$= \int_0^1 d\gamma \left[\int_{-\infty}^{\infty} G_1(t - \tau)dv(\tau, \gamma) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(t - \tau_1, t - \tau_2)dv(\tau_1, \gamma)dv(\tau_2, \gamma) \right.$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_3(t - \tau_1, t - \tau_2, t - \tau_3)dv(\tau_1, \gamma)dv(\tau_2, \gamma)dv(\tau_3, \gamma) + \dots \bigg]$$

$$\times \left[\int_{-\infty}^{\infty} G_1(t + \sigma - \tau)dv(\tau, \gamma) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_2(t + \sigma - \tau_1, t + \sigma - \tau_2)dv(\tau_1, \gamma)dv(\tau_2, \gamma) \right.$$

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$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_3(t + \sigma - \tau_1, t + \sigma - \tau_2, t + \sigma - \tau_3) dv(\tau_1, \gamma) dv(\tau_2, \gamma) dv(\tau_3, \gamma)$$

$$+ \dots \Big] \tag{80}$$

By using Eq. 62 and then by using Eq. 71, we obtain as before

$$2\pi \int_{-\infty}^{\infty} g_1(\omega) g_1(-\omega) e^{i\omega\sigma} d\omega + 4\pi^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 g_2(\omega_1, -\omega_1) g_2(\omega_2, -\omega_2)$$

$$+ 8\pi^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 g_2(\omega_1, \omega_2) g_2(-\omega_1, -\omega_2) e^{i\sigma(\omega_1 + \omega_2)}$$

$$+ 3(2\pi)^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 g_1(\omega_1) g_3(-\omega_1, \omega_2, -\omega_2) e^{i\sigma\omega_1}$$

$$+ 3(2\pi)^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 g_1(\omega_1) g_3(-\omega_1, \omega_2, -\omega_2) e^{i\sigma\omega_1} + \dots \tag{81}$$

In the expression given above the second term, independent of σ, is a constant representing a dc component, and the third term can be written as

$$8\pi^2 \int_{-\infty}^{\infty} e^{i\sigma\omega} d\omega \int_{-\infty}^{\infty} g_2(\omega_1, \omega - \omega_1) g_2(-\omega_1, \omega_1 - \omega) d\omega_1 \tag{82}$$

It means that to the frequency spectrum $2\pi g_1(\omega) g_1(-\omega)$ present with no rectification, there have been added

$$8\pi^2 \int_{-\infty}^{\infty} g_2(\omega_1, \omega - \omega_1) g_2(-\omega_1, \omega_1 - \omega) d\omega_1 \tag{83}$$

and similar terms from the fourth and fifth terms in expression 81.

3.3. Let us compute the dc component in expression 81, that is

$$4\pi^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 g_2(\omega_1, -\omega_1) g_2(\omega_2, -\omega_2) \tag{84}$$

We assume that

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$$W(t) = \frac{1}{T_o} e^{-t\,T_o} \qquad t \geq 0$$

$$= 0 \qquad\qquad t < 0 \tag{85}$$

where T_o is a constant. Then we get from Eq. 13

$$w(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(t) e^{-i\omega t} dt$$

$$= \frac{1}{2\pi T_o} \frac{1}{\frac{1}{T_o} + i\omega} \tag{86}$$

And if we assume that

$$h(t) = E e^{-at} \qquad t \geq 0$$

$$= 0 \qquad\qquad t < 0 \tag{87}$$

where E and a are positive constants. Then we have from Eq. 11

$$H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt$$

$$= \frac{1}{2\pi} \frac{E}{a + i\omega} \tag{88}$$

Into Eq. 19 we substitute Eqs. 85 and 88, and we have

$$g_2(\omega, -\omega) = -\frac{\epsilon g_1(\omega) g_1(-\omega)}{1 + 2\pi H(\omega - \omega)}$$

$$= -\frac{\epsilon (2\pi)^2 \frac{E}{2\pi} \frac{1}{(a + i\omega)} \frac{E}{2\pi} \frac{1}{(a - i\omega)} \frac{1}{(2\pi T_o)^2} \frac{1}{(\frac{1}{T_o} + i\omega)(\frac{1}{T_o} - i\omega)}}{(1 + 2\pi \frac{E}{2\pi} \cdot \frac{1}{a + i\omega})(1 + 2\pi \cdot \frac{E}{2\pi} \cdot \frac{1}{a - i\omega})(1 + 2\pi \cdot \frac{E}{2\pi a})} \tag{89}$$

It is now easy to evaluate

$$\int_{-\infty}^{\infty} g_2(\omega, -\omega) d\omega \tag{90}$$

by the calculus of residues, since the simple poles in expression 90 are, in view of Eq. 89, at $\pm i(a + E)$ and $\pm i(1/T_o)$. Hence the value of expression 90 is

$$-\epsilon \frac{2\pi}{2} \frac{E^2}{(2\pi)^2 (1 + \frac{E}{a})} \frac{T_o^2}{[1 - (a + E)^2 T_o^2]} \left(\frac{1}{a + E} - \frac{1}{T_o}\right) \tag{91}$$

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Expression 84 shows that it is a product of two $g_2(\omega, -\omega)$'s: thus we get a first approximation of the dc component of the autocorrelation function of $v_1(t)$

$$\frac{1}{4} \frac{\epsilon^2 E^4}{(1 + \frac{E}{a})^2} \frac{T_o^4}{[1 - (a + E)^2 T_o^2]^2} \left(\frac{1}{a + e} - \frac{1}{T_o}\right)^2 \tag{92}$$

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