DOCUMENT ROOM DOCUMENT ROOM 36-412
RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

#2

RESPONSE OF NONLINEAR DEVICES TO A PULSED SIGNAL AND GATED NOISE

D. B. ARMSTRONG

LOAN COPY

TECHNICAL REPORT NO. 214

OCTOBER 4, 1951

RESEARCH LABORATORY OF ELECTRONICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research) and the Air Force (Air Materiel Command), under Signal Corps Contract No. DA36-039 sc-100, Project No. 8-102B-0; Department of the Army Project No. 3-99-10-022; and under Bureau of Ordnance Contract NOrd 9661.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

RESEARCH LABORATORY OF ELECTRONICS

Technical Report No. 214 October 4, 1951

RESPONSE OF NONLINEAR DEVICES TO A PULSED SIGNAL AND GATED NOISE

D. B. Armstrong

Abstract

The analysis of the performance of nonlinear devices in the presence of signals and noise is extended to the case where the excitation consists of a pulsed signal and gated noise. The idealizing assumption is made that the pulses are rectangular. The method of analysis is an extension of that used by D. Middleton. The general formula is applied to a balanced diode phase detector circuit.

RESPONSE OF NONLINEAR DEVICES TO A PULSED SIGNAL AND GATED NOISE

I. Introduction

The theory of the behavior of signals and noise on passage through nonlinear devices has been treated by W. R. Bennett (1), S. O. Rice (2), D. Middleton (3, 4, 5) and others. Middleton in particular has developed general expressions from which one may determine the output signal spectrum and mean noise spectrum from a nonlinear device when the input excitation is an a-m signal and superimposed Gaussian noise, and the signal modulation is arbitrary in character. When applying Middleton's formulas to a specific nonlinear device and for a specified form of modulation envelope, one may encounter integrals of considerable complexity which are not always susceptible to analytic evaluation, even in series form. For example, in the relatively simple case where the modulation envelope is sinusoidal, Middleton has found it necessary to resort to a number of approximations in order to obtain a solution (4). It might therefore appear that in the case of a pulse-modulated signal, analytical methods would become prohibitively difficult.

The problem of pulse-modulated signals with a continuous noise background has been treated in approximate fashion by Van Vleck and Middleton (5). The primary approximation was the assumption that the output noise consisted solely of products formed by the intermodulation of the input noise components alone, while the output noise components due to the intermodulation of input signal and noise terms were ignored. This approximation is legitimate when the pulse duty cycle is much less than unity, since the mean input noise power then greatly exceeds the mean input signal power. However, if the noise is gated, and the gate duration is comparable to the pulse duration, the preceding approximation is invalid and a different approach to the problem must be sought.

One approach has been given by R. H. DeLano (6), who has obtained expressions for the mean signal and noise outputs from both linear and square-law detectors when the input consists of rectangular signal pulses and gated noise. His method is markedly different from the one followed in this report; a brief comparison of the two is of interest. The general procedure underlying DeLano's approach is that introduced by Bennett (1), which has been called the "direct method." It involves the counting of cross-modulation products at the output, to obtain the output noise and signal spectra. The approach used here follows a procedure introduced by Rice (2) and extended by Middleton (3), namely the evaluation of the autocorrelation function of the output by the "characteristic function method." In one respect, DeLano's method enjoys a wider applicability than that used here, in that it is capable of dealing with pulses of arbitrary shape, whereas the solution presented here applies only to rectangular pulses. On the other hand the method used here is valid for inputs having arbitrary spectral distributions whereas DeLano's is restricted to narrow-band inputs. Also our method is more rigorous, and it avoids the necessity for graphical analysis which is required by DeLano's method for the evaluation of the output noise spectrum. Both methods have one feature in common, namely that their development depends explicitly on the corresponding solution for continuous signal and noise inputs, that is, for a c-w carrier and ungated noise. Fortunately the solutions for many of the common nonlinear devices for this simple form of input are well known.

Section II of this report analyzes the problem in general terms, while Section III presents an application of the general results to a balanced diode phase detector circuit. Before proceeding with the general analysis, it is well to recall a fundamental limitation which applies to all the existing theoretical treatments in this particular field of nonlinear analysis. The limitation imposed is that the nonlinear circuit must not contain energy storage elements. In other words, the circuit is at most a combination of nonlinear and linear resistive elements, for which an over-all transfer characteristic can be specified. The reason for this restriction is that the presence of energy storage elements gives rise to nonlinear differential circuit equations, in general not solvable, whereas for circuits containing only resistive elements, the basic circuit equations, though nonlinear, are not differential in character.

Due to the above limitation, many practical circuits are not embraced by the existing theory, at least with any degree of rigor. An important example is the peak detector, which incorporates a parallel resistor-capacitor combination in series with a diode. It is true that existing methods have found approximate application to the peak detector problem, by assuming that the capacitors may be effectively replaced by biasing batteries. Since the balanced diode phase detector which is treated in Section III usually incorporates peak detectors, the assumption of biasing batteries might have been employed in the analysis but was not. Instead a somewhat idealized circuit is treated, in which purely resistive diode loads are postulated; the diodes are therefore assumed to act as half-wave linear detectors.

II. Theory

The notation which is used in this and the following section is substantially the same as that employed by Middleton (3). The general method of analysis is to determine the autocorrelation function of the response of the nonlinear device in question, in terms of the parameters specifying the applied excitation voltage. Once the output autocorrelation is obtained, one may immediately derive from it the magnitude of each line component of the output signal spectrum, and also the mean power spectrum of the output noise. The block diagram of Fig. 1 illustrates the basic circuit under consideration. The excitation voltage $V(t)$ is the sum of a signal and a noise voltage. The source resistance is lumped together with all the remaining circuit elements inside the box indicated by broken lines. The current $I(t)$ may be taken as the response function of the device, whose transfer characteristic is therefore the functional relation between I and V. A typical form of characteristic, denoted by $I = g(V)$, is shown in Fig. 2.

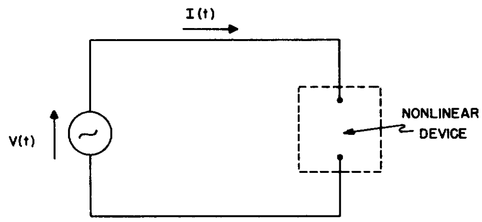

Fig. 1
Block schematic of basic nonlinear circuit.

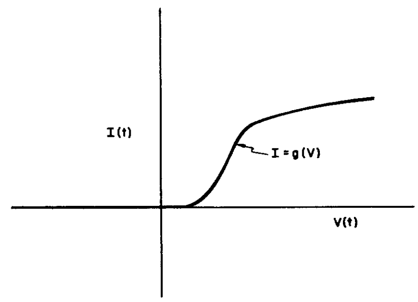

Fig. 2
A typical transfer characteristic of a nonlinear device.

The autocorrelation function of the response is defined by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} I(t)I(t+\tau)\, dt.$$

The particular method employed in this report to evaluate $R(\tau)$ is the "characteristic function method," originated by S. O. Rice (2). It is particularly useful in dealing with transfer characteristics which cannot be specified by a single analytic relation valid over the complete range of V. The procedure involves the replacement of the transfer relation by the inverse of the Laplace transform of this relation. This artifice leads to an expression for the autocorrelation of the response in terms of the (two-dimensional) characteristic functions of the applied signal and noise voltages respectively. Specifically the response autocorrelation is

$$R(\tau) = \frac{1}{4\pi^2} \int_{c} f(iz)\, dz \int_{c'} f(i\xi) F_S(z,\xi\,;\tau) F_N(z,\xi\,;\tau)\, d\xi \tag{1}$$

where

(a) f is the Laplace transform of the transfer characteristic

(b) F_S and F_N are the characteristic functions of the input signal and input noise voltages respectively

(c) z and ξ are complex variables

(d) c and c' are identical paths in the complex z and ξ planes. Each path consists of the real axis, except for a downward indentation at the origin to avoid a singularity.

The most convenient analytic representation for the excitation signal and noise voltages which compose V(t) are

$$\text{Signal} = E_S(t')\, A \cos \omega t \tag{2}$$

$$\text{Noise} = E_N(t')\, X(t). \tag{3}$$

In the former expression, $A \cos \omega t$ is a c-w carrier of amplitude A, and $E_S(t')$ is the signal envelope function. In the latter expression, X(t) represents the ungated, or continuous, noise voltage, while $E_N(t')$ is the gating function. For the problem at hand, if we let the pulse duration be γ and the gate duration be δ, while the interval between pulses is Δ, then $E_S(t')$ and $E_N(t')$ are as shown in Figs. 3a and 3b. Expressed analytically, the envelope and gating functions are

$$E_S(t') = \begin{cases} 1, 0 \leq t' \leq \gamma \\ 0, \gamma < t' < \Delta \end{cases} \tag{4}$$

$$E_N(t') = \begin{cases} 1, 0 \leq t' \leq \delta \\ 0, \delta < t' < \Delta \end{cases}. \tag{5}$$

Two remarks should be made concerning Eqs. 2-5. First, two separate time variables, t' and t, are employed, to emphasize that the E functions are generated by a source which is physically independent from the c-w carrier and noise sources. Hence t' and t are to be treated as independent variables in a mathematical sense. Actually the c-w carrier and noise sources are also mutually independent, so that their time variables should not both be represented by the same symbol t. However a distinction is unnecessary in this case because no use will be made of the time dependence of the ungated noise voltage. Second, Eqs. 4 and 5 specify that the leading edges of the signal pulse and noise gate coincide. This arbitrary arrangement is one of convenience and does not impose a limitation on the analysis, because the output autocorrelation is in theory independent of the position of the pulse in the gate provided the pulse is contained entirely within the gate.

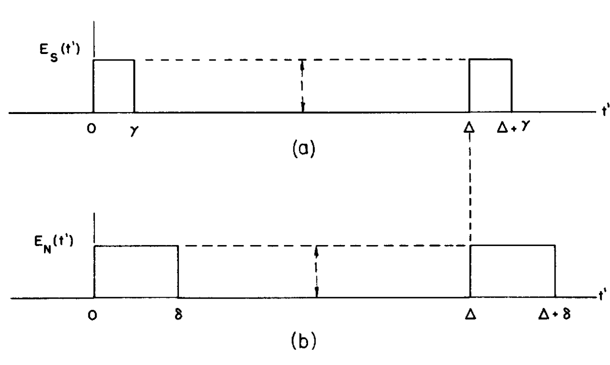

Fig. 3

Signal envelope and noise gate functions.

The characteristic functions F_S and F_N are expressible in terms of the applied signal and noise voltages as follows

$$F_S(z,\xi;\tau) = \left\langle\left\langle \exp\left[iz\, E_S(t')\, A\cos\omega t + i\xi\, E_S(t'+\tau)\, A\cos\omega(t+\tau)\right]\right\rangle_{AVt'}\right\rangle_{AVt} \tag{6}$$

$$F_N(z,\xi;\tau) = \left\langle \overline{\exp\left[iz\, E_N(t')\, X(t) + i\xi\, E_N(t'+\tau)\, X(t+\tau)\right]}\right\rangle_{AVt'} . \tag{7}$$

In Eqs. 6 and 7 the angle-brackets indicate time averages while the bar indicates a statistical average. Equation 7 may be rewritten in a form which indicates explicitly the manner of performing the statistical average over the random Gaussian variable X.

$$F_N(z,\xi;\tau) = \left\langle\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[iz\, E_S(t')\, X_1 + i\xi\, E_N(t'+\tau)\, X_2\right] W_2(X_1,X_2;\tau)\, dX_1\, dX_2\right\}\right\rangle_{AVt'} \tag{8}$$

where X_1 and X_2 represent X(t) and X(t+τ) respectively, and W_2 is the two-dimensional Gaussian probability distribution of X. Evaluation of the statistical average results in

$$F_N(z,\xi;\tau) = \left\langle\left\{\exp\left[-\frac{\psi_o}{2} z^2\, E_N(t')^2 - \frac{\psi_o}{2}\xi^2\, E_N(t'+\tau)^2 - \psi(\tau)\, z\xi\, E_N(t')\, E_N(t'+\tau)\right]\right\}\right\rangle_{AVt'} . \tag{9}$$

Here, $\psi(\tau)$ is the autocorrelation function of the ungated noise voltage excitation, while ψ_o denotes $\psi(o)$.

If Eqs. 6 and 9 are now substituted into Eq. 1, it will be seen that the evaluation of $R(\tau)$ requires three separate integrations to be performed:

(a) an average with respect to t' over the pulse interval Δ

(b) an average with respect to t over one cycle of the c-w carrier

(c) the contour integrations over c and c'.

In the present problem it is best to carry out the integrations in the order specified above. The first integration, which involves both the characteristic functions, may be written

$$\left\langle F_S \, F_N \right\rangle_{AVt'} = \frac{1}{\Delta} \int_0^{\Delta} \exp \Big[\alpha_1 E_S(t') + \alpha_2 E_S(t' + \tau) - \alpha_3 E_N(t')^2$$

$$- \alpha_4 E_N(t' + \tau)^2 - \alpha_5 E_N(t') E_N(t' + \tau) \Big] dt' \qquad (10)$$

where

$$\left\{ \begin{array}{l} \alpha_1 = iz \, A \cos \omega t ; \quad \alpha_2 = i\xi \, A \cos \omega (t+\tau) \\ \alpha_3 = \frac{\psi_o}{2} z^2 ; \quad \alpha_4 = \frac{\psi_o}{2} \xi^2 ; \quad \alpha_5 = \psi(\tau) \, z\xi . \end{array} \right\}$$

The integration is simple because of the particular form of the envelope functions E_S and E_N. The result is

$$\left\langle F_N \, F_S \right\rangle_{AVt'} = \Big[g_1(\tau) \exp(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5)$$
$$+ g_2(\tau) \exp(\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5)$$
$$+ g_3(\tau) \exp(- \alpha_3 - \alpha_4 - \alpha_5) + g_4(\tau) \exp(\alpha_1 - \alpha_3)$$
$$+ g_5(\tau) \exp(\alpha_2 - \alpha_4) + g_6(\tau) \exp(- \alpha_3)$$
$$+ g_7(\tau) \exp(- \alpha_4) + g_8(\tau). \Big] \qquad (11)$$

The functions $g_1(\tau), \ldots\ldots, g_8(\tau)$ are specified in Fig. 4, to which the conditions $\Delta \geqslant 2\delta, \ldots\ldots, \delta \geqslant 2\gamma$ pertain. For the case where $\gamma \leqslant \delta \leqslant 2\gamma$, all the g functions except g_2 remain as defined in Fig. 4, while g_2 is only slightly modified, as indicated in Fig. 5.

If now we let $F_{Sc}(z,\xi;\tau)$ and $F_{Nc}(z,\xi;\tau)$ represent the characteristic functions of the c-w carrier A cos ωt and the ungated noise X(t) respectively, it may be shown that the exponential functions in Eq. 11 are related to F_{Sc} and F_{Nc} as follows

$$\left. \begin{array}{r} \exp(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) = F_{Nc}(z,\xi;\tau) \, F_{Sc}(z,\xi;\tau) \\ \exp(\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5) = F_{Nc}(z,\xi;\tau) \, F_{Sc}(z,o;\tau) \\ \exp(- \alpha_3 - \alpha_4 - \alpha_5) = F_{Nc}(z,\xi;\tau) \\ \exp(\alpha_1 - \alpha_3) = F_{Nc}(z,o;\tau) \, F_{Sc}(z,o;\tau) \\ \exp(\alpha_2 - \alpha_4) = F_{Nc}(o,\xi;\tau) \, F_{Sc}(o,\xi;\tau) \\ \exp(- \alpha_3) = F_{Nc}(z,o;\tau) \\ \exp(- \alpha_4) = F_{Nc}(o,\xi;\tau). \end{array} \right\} \qquad (12)$$

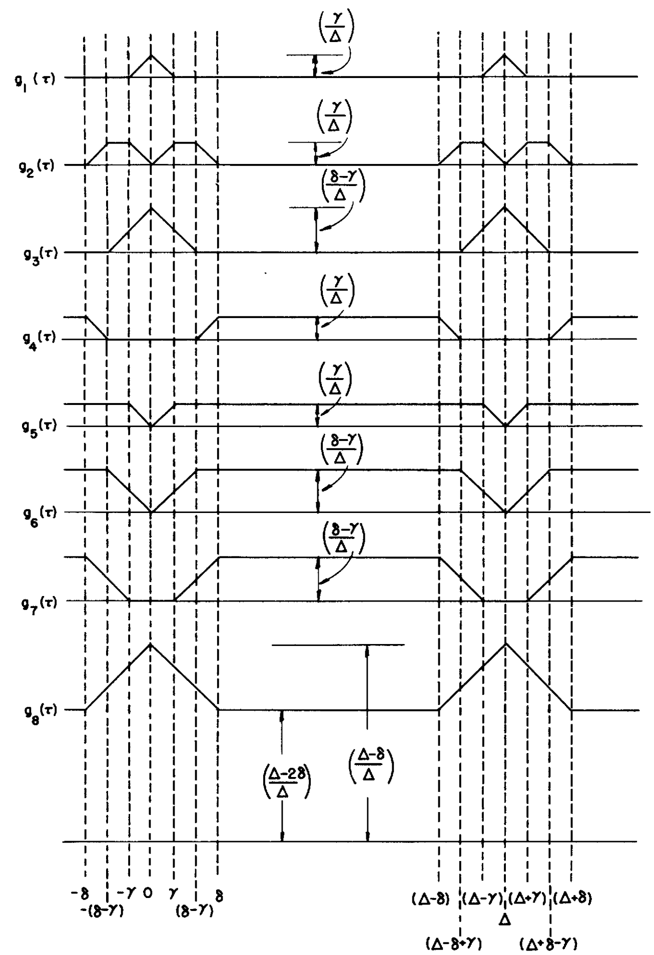

Fig. 4

g functions for $\Delta \geqslant 2\delta$; $\delta \geqslant 2\gamma$.

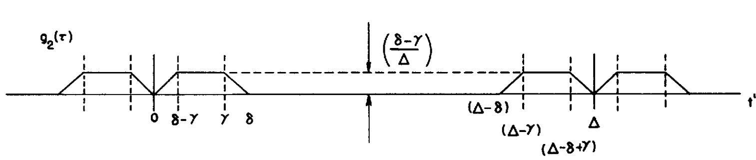

Fig. 5

$g_2(T)$ for the case where $\gamma \leqslant \delta \leqslant 2\delta$.

On substituting the results of Eqs. 11 and 12 into Eq. 1, we obtain

$$R(\tau) = \left[\frac{g_1(\tau)}{4\pi^2} \int_C f(iz)\, dz \int_{C'} f(i\xi)\, F_{Nc}(z,\xi;\tau)\, F_{Sc}(z,\xi;\tau)\, d\xi \right.$$
$$+ \frac{g_2(\tau)}{4\pi^2} \int_C f(iz)\, F_{Sc}(z,o;\tau)\, dz \int_{C'} f(i\xi)\, F_{Nc}(z,\xi;\tau)\, d\xi$$
$$+ \frac{g_3(\tau)}{4\pi^2} \int_C f(iz)\, dz \int_{C'} f(i\xi)\, F_{Nc}(z,\xi;\tau)\, d\xi$$
$$+ \frac{g_4(\tau)}{4\pi^2} \int_C f(iz)\, F_{Nc}(z,o;\tau)\, F_{Sc}(z,o;\tau)\, dz \int_{C'} f(iz)\, d\xi$$
$$+ \frac{g_5(\tau)}{4\pi^2} \int_C f(iz)\, dz \int_{C'} f(i\xi)\, F_{Nc}(o,\xi;\tau)\, F_{Sc}(o,\xi;\tau)\, d\xi$$
$$+ \frac{g_6(\tau)}{4\pi^2} \int_C f(iz)\, F_{Nc}(z,o;\tau)\, dz \int_{C'} f(i\xi)\, d\xi$$
$$+ \frac{g_7(\tau)}{4\pi^2} \int_C f(iz)\, dz \int_{C'} f(i\xi)\, F_{Nc}(o,\xi;\tau)\, d\xi$$
$$\left. + \frac{g_8(\tau)}{4\pi^2} \int_C f(iz)\, dz \int_{C'} f(i\xi)\, d\xi. \right] \qquad (13)$$

Although not indicated explicitly in the above equation, it should be remembered that a time average over t has yet to be applied to F_{Sc} wherever it appears.

A close examination of the individual terms on the right side of Eq. 13 will show that a significant simplification can be made in most cases. Specifically, it will be noted that each of the last five terms contains either the integral $\int_C f(iz)\, dz$ or the integral $\int_{C'} f(i\xi)\, d\xi$, or both. It will become apparent later (Eq. 14) that when the response current is zero for zero excitation voltage, that is, $I = 0$ when $V = 0$, the above two integrals are zero, so that the last five terms of Eq. 13 vanish. This situation always holds when the nonlinear elements in the circuit are passive and no biasing batteries are present. In the case where a biasing battery is inserted in series with the excitation voltage V, so that I assumes a constant, nonzero value when $V = 0$, then the last five terms of Eq. 13 are nonzero. It is of interest to determine their significance in this circumstance. First it will be noted that F_{Nc} and F_{Sc} appear in these terms as functions of only one of the complex variables z, ξ, the other being zero. As a result neither F_{Nc} or F_{Sc} is a function of τ, even though τ is specified as one of the variables. Consequently the integrals in the last five terms are independent of τ, so that this variable appears only in the $g(\tau)$ functions. Furthermore, it can be shown that the associated integrals have the following significance.

(a) $$\int_C f(iz)\, dz = \int_{C'} f(i\xi)\, d\xi = I_p, \text{ the constant value of } I(t) \text{ during the interval } \delta < t' < \Delta \text{ when } V(t) = 0 \quad (14)$$

(b) $$\int_C f(iz)\, F_{Nc}(z,o;\tau)\, F_{Sc}(z,o;\tau)\, dz = \int_{C'} f(i\xi)\, F_{Nc}(o,\xi;\tau)\, F_{Sc}(o,\xi;\tau)\, d\xi$$

$= I_S$, the average value of $I(t)$ during the interval $o \leqslant t' \leqslant \delta$ when both signal and noise excitation is applied (15)

(c) $$\int_C f(iz)\, F_{Nc}(z,o;\tau)\, dz = \int_{C'} f(i\xi)\, F_{Nc}(o,\xi;\tau)\, d\xi$$

$= I_{N-S}$, the average value of $I(t)$ during the interval $\gamma < t' \leqslant \delta$ when noise excitation alone is applied. (16)

It is therefore apparent that the last five terms of Eq. 13 reduce to

$$\left\{\left[g_4(\tau) + g_5(\tau)\right] (I_S\, I_p)\right\} + \left\{\left[g_6(\tau) + g_7(\tau)\right] (I_{N-S}\, I_p)\right\} + \left\{g_8(\tau)\, (I_p^2)\right\}. \quad (17)$$

Finally, it may be shown that the first brace in Eq. 17 represents the two-way crosscorrelation between a rectangle of height I_S occupying the interval (o,γ) and a rectangle of height I_p occupying (δ, Δ); the second brace represents a similar crosscorrelation between rectangles of heights I_{N-S} and I_p occupying intervals (γ,δ) and (δ,Δ) respectively; while the third brace represents the autocorrelation of the rectangle I_p occupying the interval (δ,Δ).

The conclusion to be drawn is that the frequency spectrum corresponding to the terms composing Eq. 17 is purely a video line spectrum, representing the periodic transitions of $I(t)$ to the nonzero value I_p during intervals $\left[n\Delta+\delta,\ (n+1)\Delta\right]$. As remarked previously, if $I_p = 0$ then Eq. 17 reduces to zero.

The first three terms of Eq. 13 all make a contribution to the noise spectrum of the response, because both the complex variables are present in F_{Nc}. A simple physical interpretation is possible for the first and third terms. The first term represents the response that would be obtained if the excitation which occurs during the interval (o,γ) alone were applied, I(t) being zero at other times. Similarly the third term represents the response that would be obtained if the excitation occurring in the interval (γ,δ) only were applied, I(t) again being zero at other times. The second term does not have a correspondingly simple interpretation; however, in some sense it represents additional terms in the response resulting from the simultaneous presence of excitation in the intervals (o,γ) and (γ,δ). This interpretation is prompted by the fact that $g_2(\tau)$ is the two-way crosscorrelation between pulses of unit height occupying intervals (o,γ) and (γ,δ) respectively.

It is not necessary to pursue the general analysis beyond this point, because the solutions of the integrals in the first three terms of Eq. 13 have been obtained previously (1, 2). Those solutions are employed in the following section, where the above analysis is applied to the balanced diode phase detector.

III. Diode Phase Detector

An idealized form of the balanced diode phase detector, together with its driving sources, is illustrated schematically in Fig. 6. The sources A and B generate voltages V_A and V_B, which are then added and subtracted to produce voltages $(V_A + V_B)$ and $(V_A - V_B)$ respectively. The latter are applied to separate diodes, which are assumed to act as ideal linear half-wave detectors. The output voltage V_o, representing the response of the device, is the difference of the voltages V_1 and V_2, appearing across the load resistances R. The capacitors C, shown in broken lines, are generally present in practice, but are omitted in the theoretical model (see Sec. I).

For the purposes of the analysis of this section, the frequency characteristics of sources A and B are specified to be identical, and to have the general shape shown in Fig. 7. Furthermore the center frequency of the spectrum is assumed to be much larger than the spectral width, which in turn is sufficiently wide to permit the generation of substantially rectangular signal pulses. The former assumption results in the spectrum having approximately arithmetic symmetry about its center frequency, a fact which is utilized to simplify the analysis. Specifically, it allows one to replace the autocorrelation function $\psi(\tau)$ of the ungated noise voltage from source A, by the expression $\left[\psi_o\, r(\tau) \cos\, \omega t\right]$, where $\omega/2\pi$ is the center frequency of the source spectrum and $r(\tau)$ is the normalized autocorrelation of the noise envelope. In practice the bandpass characteristic of Fig. 7 may result from an amplifier filter combination, as for example an i-f amplifier.

Additional assumptions upon which the analysis is predicated are

(a) The noise components of V_A and V_B are generated by physically separate sources, so that they are statistically independent.

(b) The pulsed carrier voltages composing the signal components of V_A and V_B have the same frequency, but have a phase difference of ϕ radians.

(c) The signal-to-noise ratio associated with V_A is equal to that associated with V_B.

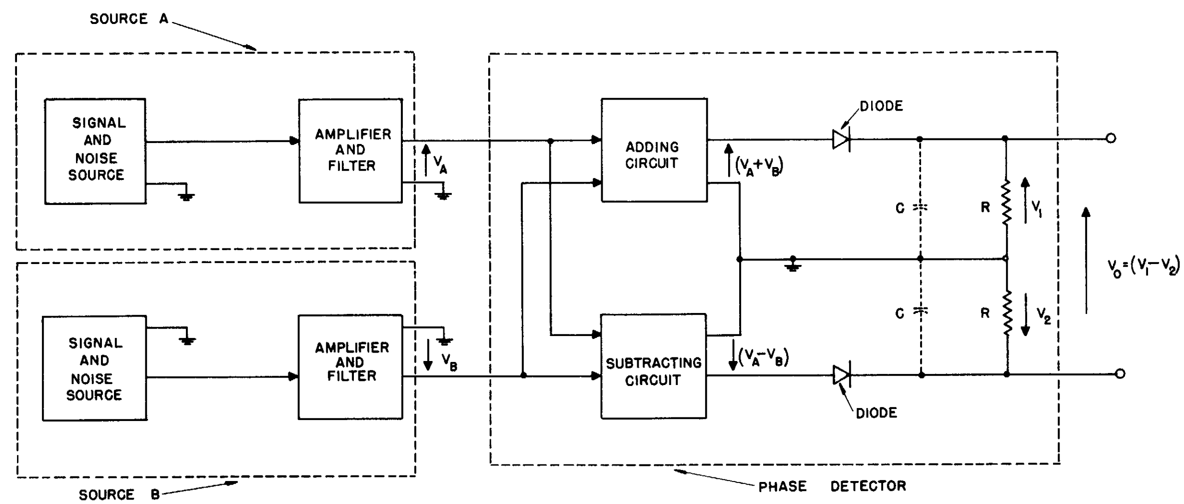

Fig. 6

Schematic of diode phase detector circuit and its driving sources.

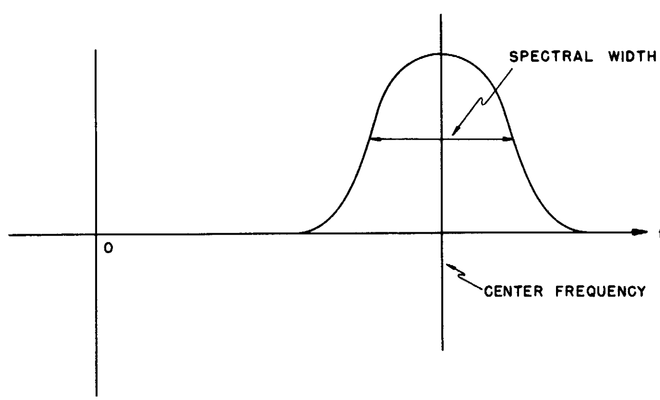

Fig. 7

General shape of frequency spectrum of sources A and B.

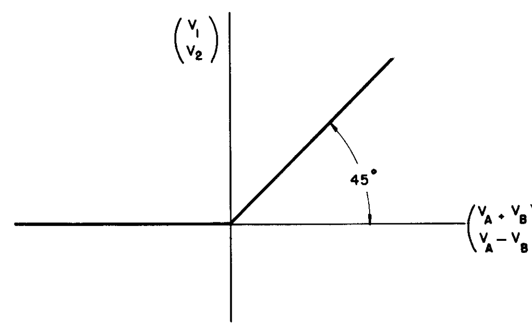

Fig. 8

Transfer characteristic associated with the diode phase detector.

(d) The ratio of rms voltages $\left[(V_B)\ \text{rms}\right]/\left[(V_A)\ \text{rms}\right]$ is denoted by a, where $0 < a \leqslant 1$. This means that if the mean power of the ungated noise output from source A is ψ_o, then that from source B is $a^2\psi_o$.

(e) The transfer characteristic relating V_1 and $(V_A + V_B)$, also relating V_2 and $(V_A - V_B)$, is as specified in Fig. 8. The Laplace transform of this characteristic is $f(iz) = (1/iz)^2 = -z^{-2}$.

It is evident from the transfer characteristic of Fig. 8 that the last five terms of Eq. 13 vanish. The evaluation of the first three terms of Eq. 13, using the formulas of reference 2, leads to the following expression for the autocorrelation of the rectified or low-frequency component of the response voltage V_o

$$R_o(\tau) = \left[\mu\psi_o\right] \times \Bigg[\frac{g_1(\tau)}{8} \sum_{m=0}^{\infty} e_m \left(\frac{A^2}{2\psi_o}\right)^m \sum_{q=0}^{\infty} \Bigg\{ \left[\frac{r(\tau)^{m+2q}}{q!\,(q+m)!\,(\Gamma(m+1))^2 \left(\Gamma\left(\frac{3}{2} - m - q\right)\right)^2} \right]$$

$$\times \Bigg[M_1^{2m} \left({}_1F_1(m + q - \tfrac{1}{2};\ m + 1;\ -\frac{A^2M_1^2}{2\psi_o}) \right)^2$$

$$+ M_2^{2m} \left({}_1F_1(m + q - \tfrac{1}{2};\ m + 1;\ -\frac{A^2M_2^2}{2\psi_o}) \right)^2 - 2\left(\frac{\nu}{\mu}\right)^{m+2q} M_1^m M_2^m$$

$$\times {}_1F_1(m + q - \tfrac{1}{2};\ m + 1;\ -\frac{A^2M_1^2}{2\psi_o})\ {}_1F_1(m + q - \tfrac{1}{2};\ m + 1;\ -\frac{A^2M_2^2}{2\psi_o}) \Bigg] \Bigg\}$$

$$+ \frac{g_2(\tau)}{8} \sum_{q=0}^{\infty} \Bigg\{ \left[\frac{r(\tau)^{2q}\left(1 - \left(\frac{\nu}{\mu}\right)^{2q}\right)}{(q!)^2 \left(\Gamma\left(\frac{3}{2} - q\right)\right)^2} \right]$$

$$\times \left[{}_1F_1(q - \tfrac{1}{2};\ 1;\ -\frac{A^2M_1^2}{2\psi_o}) + {}_1F_1(q - \tfrac{1}{2};\ 1;\ -\frac{A^2M_2^2}{2\psi_o}) \right] \Bigg\}$$

$$+ \frac{g_3(\tau)}{4} \sum_{q=0}^{\infty} \left\{ \left[\frac{r(\tau)^{2q}\left(1 - \left(\frac{\nu}{\mu}\right)^{2q}\right)}{(q!)^2 \left(\Gamma\left(\frac{3}{2} - q\right)\right)^2} \right] \right\} \Bigg] \tag{18}$$

where

$$\begin{cases} e_m = \begin{Bmatrix} 1, m = 0 \\ 2, m \neq 0 \end{Bmatrix} \\ \left(\frac{\nu}{\mu}\right) = \left(\frac{1 - a^2}{1 + a^2}\right) \\ M_1 = \sqrt{1 + \frac{2a}{1 + a^2} \cos \phi} \\ M_2 = \sqrt{1 - \frac{2a}{1 + a^2} \cos \phi}. \end{cases}$$

The quantity $A^2/2\psi_o$ represents the ratio of the mean-square power of the c-w carrier A cos ωt, to the mean-square power of the ungated noise, at the output of either source. Therefore this quantity may be replaced by a single symbol Q, and defined as the signal-to-noise ratio at the input to the balanced phase detector. Frequently one is interested in evaluating the signal-to-noise ratio for the response, a quantity which we shall denote by S. Since it is the d-c component of the response which is usually selected to provide the signal information (i.e. the information concerning the value of the carrier phase difference, ϕ), one usually follows the phase detector with a low-pass filter whose upper half-power point is of the order of a few cycles. The output of the filter therefore includes low frequency noise in addition to direct current. If $W_{oN}(f)$ denotes the mean power spectrum of the rectified noise in the phase detector response, then the mean noise power passed by the low-pass filter is to a good approximation $\left[W_{oN}(o)\Delta f\right]$, where Δf is the effective filter bandwidth. The output signal-to-noise ratio may therefore be defined as

$$S = \left[\frac{\text{square of d-c response}}{W_{oN}(o)\Delta f}\right]. \qquad (19)$$

The square of the d-c response is just the constant term of Eq. 18, and is denoted by R_{d-c}. It is obtained by selecting the m = q = 0 term of the first series and the q = 0 terms of the second and third series, and performing the average over the $g(\tau)$ functions. Since the contribution of the last two series to the constant term is zero, one obtains

$$R_{d-c} = \left\langle g_1(\tau)\right\rangle_{AV_\tau} \left(\frac{\mu\psi_o}{2\pi}\right) \left\{ {}_1F_1\left(-\tfrac{1}{2};\ 1;\ -QM_1^2\right) - {}_1F_1\left(-\tfrac{1}{2};\ 1;\ -QM_2^2\right)\right\}^2. \qquad (20)$$

The part of $R_o(\tau)$ which remains after extracting the m = q = 0 and q = 0 terms from the respective series represents the autocorrelation of the noise response alone. If it is denoted by $R_{oN}(\tau)$, then the zero-frequency value of the mean noise power spectrum is given by

$$W_{oN}(o) = 4 \int_0^\infty R_{oN}(\tau)\, d\tau. \tag{21}$$

When Eq. 18 is substituted into Eq. 21, it is seen that the following integrals must be evaluated

$$\left.\begin{array}{ll} \text{(a)} & \int_0^\infty g_1(\tau)\, r(\tau)^{m+2q}\, d\tau \\ \text{(b)} & \int_0^\infty g_2(\tau)\, r(\tau)^{2q}\, d\tau \\ \text{(c)} & \int_0^\infty g_3(\tau)\, r(\tau)^{2q}\, d\tau. \end{array}\right\} \tag{22}$$

These integrations will be of varying difficulty, depending upon the form of $r(\tau)$, which is determined by the spectrum of the source. If the integration cannot be carried out analytically, recourse must be had to numerical or graphical integration. It is usually necessary to evaluate only a few of the above integrals, because the series in Eq. 18 converge quite rapidly for a wide range of Q values (i.e. $Q \ll 1$ to $Q \gg 1$).

It has been mentioned that the above analysis applies to the circuit of Fig. 6 only when capacitors C are absent. Empirical measurements show that, for a given Q, the value of S may be appreciably increased by the addition of these capacitors to the circuit. The improvement is most marked for values of $Q \gg$ unity. The reason for the improvement is that the insertion of the capacitors causes the diodes to act as peak detectors. Consequently when $Q \gg 1$, so that the pulse rises well above the noise background, only that noise which is riding on or near the top of the pulse is effective in contributing to the output noise; the noise in that part of the gate not occupied by the pulse does not pass through the diodes because of the bias applied to them by the capacitors. The effect just described has in fact been noticed for values of Q as low as 0.5.

Acknowledgment

The author wishes to express his appreciation to Professor H. J. Zimmermann for his continued interest and encouragement during the course of the above research.

References

1. W. R. Bennett: B.S.T.J. 23, 97, 1944
2. S. O. Rice: B.S.T.J. 24, 46, 1945
3. D. Middleton: Quart. App. Math. 5, 445, 1947
4. D. Middleton: Proc. I.R.E. 36, 1467, 1948
5. J. H. Van Vleck, D. Middleton: J. App. Phys. 17, 940, 1946
6. R. H. DeLano: Proc. I.R.E. 37, 1120, 1949