MULTI-ITEM PRODUCTION PLANNING --

AN EXTENSION OF THE HMMS RULES

328-68

Barnard E. Smith*
and
Gary L. Bergstrom**

*Visiting Professor, Indian Institute of Management, Calcutta, India.

**Instructor, Sloan School of Management, M.I.T.

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The Linear Decision Rules (LDR) proposed by Holt, Modigliani, Muth and Simon for the production planning problem determine an optimum plan in terms of an aggregate production rate and work force level. The criteria of the LDR assume we wish to make decisions so as to minimize costs over a specified time horizon, given estimates of future aggregate demand. This paper extends the LDR to a multi-item formulation (MDR) which solves directly for the optimum sales, production, and inventory levels for individual items in future periods.

To remove the restriction of specified demand, revenue curves are estimated for each item in each time period. The MDR model then seeks a solution to maximize profit for the firm over the time horizon of interest. The practical feasibility of the MDR is demonstrated by an application in a firm producing a line of electric motors. The results of the MDR are compared to management's proposed plan and some important differences are detected.

INTRODUCTION

Within recent years a great amount of interest has been shown in the aggregate planning problem. A considerable portion of this can be traced to the pioneering work done by Holt, Modigliani, Muth, and Simon (HMMS) in developing their Linear Decision Rules (LDR) [4]. Since this time most of the research which has dealt with aggregate planning models of this type has

concentrated upon two areas: methods of improving the estimation of cost coefficients for the model or alternative computational techniques. Examples of the former include the work by Van De Panne and Bosje [9] and Kriebel [6] on cost coefficient estimation and errors. In the second category, Buffa and Taubert [2] and Taubert [8] have recently described the use of direct computer search techniques in solving aggregate planning problems. The objectives of this paper are considerably different in that it extends the capabilities of the LDR model in two new directions.

The LDR is designed to make decisions on aggregate production rate and employment level for the upcoming period. Because of the aggregate nature of this formulation, it is not possible to solve directly for the optimum production rates for individual products. Therefore in situations where no reasonable dimension for aggregation exists, the breakdown of an aggregate production plan into individual item plans may result in a schedule which is far from optimum. As an extreme example of this, consider the situation where a facility's two products are lawn mowers and snow blowers. In this case specification of an aggregate production plan neglects the most interesting question; namely, the correct production plan for each individual item. Another problem arises when we consider, for example, that one measure of aggregation may be quite adequate for representing work force related costs while at the same time it is a very poor measure of inventory associated costs. One of the goals of this paper is to extend the LDR model to enable determination of the optimum plan for each individual item to be produced in a facility.

To calculate aggregate production rates and employment levels for future periods the LDR requires estimates of expected future demands. Two decision rules are then employed, one to determine aggregate production rates and one to determine employment levels. The criteria employed is the minimization of expected costs due to (1) hiring and layoffs, (2) inventory and back orders, (3) payroll, and (4) overtime and undertime. From the viewpoint of the entire firm, of course, we are sub-optimizing since we have assumed that demands are fixed. With the demand curves facing many products this is not likely to be a good assumption, however. Our second goal here is to remove the LDR restriction that demands or forecasts of demands for future periods have been specified. This will be done by estimating a revenue curve for each product in each time period. Once revenue curves have been specified we can maximize profit across a time horizon of interest by calculating optimal sales quantities from the model. At the same time we also determine the optimal production rates and inventory levels for this sales program on a product-by-product basis. The multi-item decision rule we propose here (which we will refer to as the MDR), incorporates both market conditions and production related costs. Therefore, it is likely to be a much better approximation of the planning problems facing a firm than a formulation which looks at production and market planning problems independently.

The MDR Model

Because of the nature of cost and revenue functions, the estimation of overall optimal sales and production plans for a firm will be performed

by a quadratic model whose criteria is maximization of expected profits over a specified time horizon. This is done by expressing revenues and costs as functions of the important variables and then maximizing the total profit function. Within the structure of this model, four planning variables will be considered: sales, production, workforce, and inventory. In order to determine the interrelationships and effects of these variables, the overall revenue and cost structure of the firm must be considered.

Revenue and Cost Functions

Broadly speaking, the total costs incurred by a firm can be usefully subdivided into the following categories:

1. Inventory related costs (IC)
2. Production related costs
    a. Fixed costs (FC)
    b. Variable costs (VC)

The total revenue a firm receives from sales of all of its products we shall call R. Consequently, we may express the total profit this firm earns during a certain period t as:

(1) $$P_t = R_t - (FC)_t - (VC)_t - (IC)_t$$

$$t = 1, \ldots, T$$

with P_t being the profit earned in period t and T the total number of periods.

In this model, fixed costs are assumed to be constant within the time horizon under consideration and therefore beyond control of the planning process. As a result they will not be included in the total profit function.

Holt, Modigliani, Muth, and Simon (HMMS) have argued that both (VC) and (IC) can be described adequately in terms of three primary variables [4]:

Production rate in units/period

Number of production workers employed

Inventory level in units

R can be described very satisfactorily in terms of one additional variable:

Sales rate in units/period

Expressions for Revenue and Cost Components

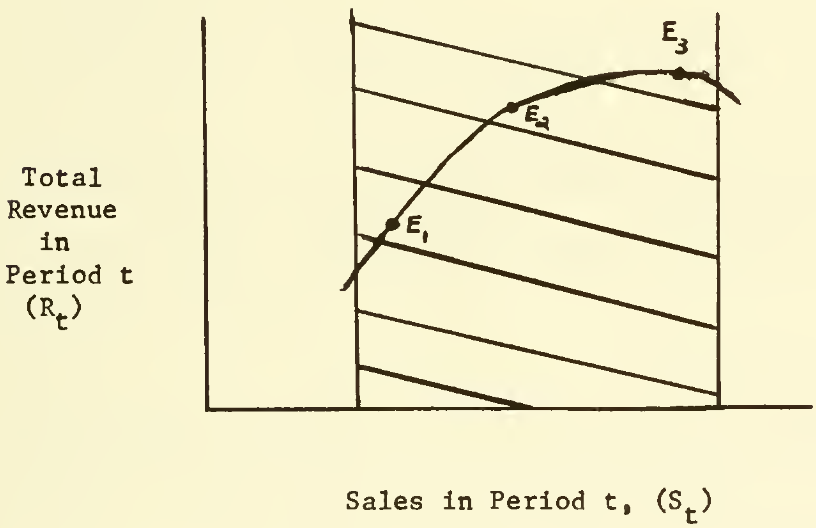
Having outlined the nature of the total profit function, the next concern is the expression of each of its components in terms of the relevant variables. In the HMMS model, expressions are developed for the various portions of (VC) and (IC). Certain of these expressions will be adopted without change while others will be modified to various degrees. Since HMMS offer no expression for R, a suitable one must be derived.

The Total Revenue Term

The classical concept of total revenue described extensively in the micro-economic literature provides a starting point for derivation of the desired revenue term. Basically, total revenue for any volume of output is

the product of that volume and the price at which it can be sold as determined from the demand curve. The general dependency of total revenue on sales rate is given in the following figure where E_1, E_2, and E_3 are estimates of points on the total revenue curve.

Figure 1



The form of this revenue curve suggests that within the shaded region the revenue function can be well represented by a quadratic expression in S_t.

(2) $$R_t = r_1 + r_2 S_t + r_3 S_t^2$$

where r_1, r_2, r_3 are constants.

This expression represents the revenue accruing to the firm from the sales of a single product. Now consider the case where multiple products are produced and sold. If the demand for each of n products in the firm's line can be considered to be independent of the demand for all others, then a

separate revenue expression of the same form as (2) can be specified for each product in each period.

$$R_{it} = r_{1it} + r_{2it}S_{it} + r_{3it}S_{it}^{2} \tag{3}$$

where R_{it} = total revenue from sales of product i in period t

S_{it} = sales of product i in period t

$i = 1, \ldots, n$

$r_{1it}, r_{2it}, r_{3it}$ are constants.

In situations where interdependencies of demand are shown to exist (3) could be easily generalized to reflect these conditions by the incorporation of additional variables.

The expression for total revenue, R_t, may now be rewritten in terms of the revenue from each of the individual products:

$$R_t = \sum_{i=1}^{n} (r_{1it} + r_{2it}S_{it} + r_{3it}S_{it}^{2}) \tag{4}$$

The Inventory Carrying Costs Term

Economic lot-size formulas indicate that the optimal reorder quantity and the optimal safety stock on an item increase roughly as the square root of its sales rate. This is also approximately true with respect to optimal net inventory (inventory minus back orders), since back orders are generally small in relation to inventory. Therefore, the square root relation between total inventory and sales rate still dominates the relationship between net inventory and order rate. Using this knowledge, HMMS show that the expected

costs of inventory, back orders, and setups can be reasonably described over a range by a quadratic expression in which cost rises as the square of the deviation of net inventory from the optimal level [4, p. 57]. More specifically:

(5) $$(IC)_t = C_7 [I_t - (C_8 + C_9 S_t)]^2 + C_{13}$$

in this expression:

$(IC)_t$ = expected inventory, back order, and setup costs during period t

I_t = net inventory at the end of period t (inventory less the backlog of unfilled orders)

S_t = aggregate sales rate during period t

C_7, C_8, C_9, C_{13} are constants.

There are two assumptions in this formulation and the solution method offered which deserve reconsideration. HMMS correctly note in their derivation of (5) that optimal net inventory is a function of aggregate sales rate, S_t. In their actual solution, however, this influence of sales rate on the optimal inventory level was neglected in the interest of simplicity, i.e., C_9 was set equal to zero.

A much more fundamental assumption of the LDR model is that production planning decisions for a multi-product firm may be made in terms of aggregate production rates and aggregate inventory levels. This necessarily implies that a common unit is available for adding quantities of different products, for example, "a unit of weight, volume, work required, or value might serve as a suitable common denominator" [4, p. 48]. A primary objective in

construction of the MDR model is the elimination of this necessity of expressing all production rates in terms of a single common unit. Individual production rates will instead by explicitly incorporated in the formulation so they may be solved for on a product-by-product basis. Returning to the inventory cost expression in this spirit, it follows that any consideration of multiple products in the model must also include a separate inventory cost expression for each. This may be done by following the logic developed for (5). The only essential difference is the necessity of a separate expression for each of n products. This results in:

(6) $$(IC)_{it} = C_{i7} [I_{it} - (C_{i8} + C_{i9}S_{it})]^2 + C_{i13}$$

$i = 1, \ldots, n$

$C_{i7}, C_{i8}, C_{i9}, C_{i13}$ are constants.

The total inventory cost expression, $(IC)_t$, may now be rewritten in terms of the inventory costs associated with each product:

(7) $$(IC)_t = \sum_{i=1}^{n} (C_{i7} [I_{it} - (C_{i8} + C_{i9}S_{it})]^2 + C_{i13})$$

The Hiring and Firing Cost Term

HMMS show that there are labor costs which are associated not with the size of the work force but with changes in its size. They assume the size of the work force directly engaged in production is adjusted once each period (monthly in their model). Costs associated with increases in work

force result, for example, from the interviewing, testing, and training of new or prospective workers. Layoffs result in additional personnel and payroll expenses, reogranization costs, and possibly labor relations difficulties.

The expression developed by HMMS for these average costs of hiring and layoffs will be used in this model without further comment.

(8) $$(WC)_{t-1,t} = C_2(W_t - W_{t-1} - C_{11})^2 + C_{13}$$

in which

$(WC)_{t-1,t}$ = hiring or firing cost incurred at the beginning of period t

W_t = workforce during period t

W_{t-1} = workforce during period t-1

C_2, C_{11}, C_{13} are constants.

The Variable Production Cost Term

The most complex term in this model is the one that attempts to express variable production costs as a function of production rates and workforce level. The expression for variable production cost offered by HMMS is of the following form:

(9) $$(VC)_t = K_3\ (P_t - K_4W_t)^2 + K_5P_t - K_6W_t + K_{12}P_tW_t$$

where:

$(VC)_t$ = variable production costs during period t

P_t = aggregate production level during period t

W_t = workforce during period t

K_3, K_4, K_5, K_6, K_{12} are constants.

As aggregate production, P_t, exceeds the optimal level C_4W_t determined by the size of the work force in period t, variable production cost increases.

One of the expressed objectives of this model is elimination of the aggregate production rate assumption of the LDR formulation. Therefore, an expression for $(VC)_t$ as a function of P_{1t}, P_{2t}, ..., P_{nt} and W_t is desired, where P_{1t} is the production rate of product 1 in period t. Because of differences between items, the standard direct labor time required to produce individual products may often vary substantially. Within the relevant ranges the expected standard labor time required to meet any given production plan is then:

(10)
$$L_t = \sum_{i=1}^{n} k_i P_{it}$$

where:

L_t = expected standard labor time in period t

k_i = standard labor time to complete one unit of product i.

The labor time coefficients, k_i, will be assumed constant from period to period although the structure of the MDR model does not demand this. Substitution of (10) into the LDR expression for variable production costs yields the following:

(11)
$$(VC)_t = C_3 \, (L_t - C_4W_t)^2 + C_5L_t - C_6W_t + C_{12}L_tW_t$$

in which:

C_3, C_4, C_5, C_6, C_{12} are constants.

This formulation can be solved directly for the optimum production quantities of each individual product in each period. As a result, it is not vulnerable to changes in the assumed underlying product mix as would be the case with any model formulated in terms of only aggregate production rates. Other advantages will become more evident after the solutions of the complete model. Fundamentally, it will be seen later that consideration of separate products pinpoints significant seasonal differences which can be exploited to smooth aggregate production and stabilize total inventory levels.

Inter-Period Dependencies

It is evident that inventory levels at the end of period t are a function of conditions at the end of period t-1 as well as operations during t. In particular:

$$I_{it} = I_{it-1} + P_{it} - S_{it} \qquad (12)$$

$$i = 1, \ldots, n$$

$$t = 1, \ldots, T$$

(T x n) of these equations are required for the model.

Optimization of the Total Profit Function

Using the different revenue and cost terms described we now wish to develop a suitable expression for total profit over some desired multi-period time horizon. As a starting point, consider the expression for profit in a single period (1). Since the purpose of this model is to optimize total profits for the firm and fixed costs, (FC), are not relevant

to this decision, this term will not be considered any further. Basically, the approach required is to first sum the profit expression across all T periods. To this result must be added any costs associated with changing the levels of variables from one period to another. The expression for the costs of hiring and layoffs, $(WC)_{t,t-1}$, developed in (8) is in this category.

After inclusion of $(WC)_{t,t-1}$, the total profit (TP) that the firm will earn over T periods can be expressed as:

(13) $$(TP) = \sum_{t=1}^{T} (R_t - (VC)_t - (IC)_t - (WC)_{t,t-1})$$

Expressions are now available for all terms in this equation. Substitution and rearrangement of (4), (7), (8), and (11) into (13) gives:

(14) $$(TP) = \sum_{t=1}^{T} \sum_{i=1}^{n} [r_{1it} + r_{2it}S_{it} + r_{3it}S_{it}^{2} \qquad \text{(total revenue)}$$

$$- C_{i7}(I_{it} - (C_{i8} + C_{i9}S_{it}))^{2}] \qquad \text{(inventory connected costs)}$$

$$- C_3(L_t - C_4W_t)^2 - C_5(L_t)$$

$$+ C_6W_t - C_{12}(L_t)\, W_t \qquad \text{(variable production costs)}$$

$$- C_2(W_t - W_{t-1} - C_{11})^2 \qquad \text{(hiring and layoff costs)}$$

This is subject to the restraints of (12):

(15) $$I_{it} = I_{it-1} + P_{it} - S_{it}$$

$$i = 1, \ldots, n$$

$$t = 1, \ldots, T$$

Since L has been expressed in terms of the production rates P_{it}, the entire expression is a function of only the following unknowns:

S_{it}

P_{it} where $i = 1, \ldots, n$

I_{it} $t = 1, \ldots, T$

W_t

The goal of this model is to choose a decision strategy in terms of these unknowns in order to maximize total profit over a number of periods in the future. A number of standard quadratic programming algorithms could be brought to bear to solve for the optimal decision strategy. Because of very significant computational advantages which will become evident, a different solution technique will be employed instead. Since (TP) is continuous and differentiable in the closed interval expected, maximum profit can also be found in general by differentiating the total profit expression with respect to each of the unknown variables and then equating the resulting expressions to zero. The existence of such a solution presupposes that the profit function is a negative definite quadratic form over the relevant range, i.e., it is a strictly concave function over the closed interval. In practice this conditions should be met without difficulty since profits will in general be decreased as any of the decision variables approach extreme values. As a result there exists a unique global maximum for the profit function.[1]

To maximize the total profit expression of (14) and (15) these expressions are differentiated with respect to the unknowns: W_1, ..., W_t, P_1, ..., P_{nt}, and I_{11}, ..., I_{nT}. The initial workforce at the beginning of period 1, W_o, is assumed known as are the starting inventory levels, I_{10}, ..., I_{no}. It is not necessary to take derivatives with respect to sales since through equation (15) the sales rate is uniquely determined once production and inventory levels are specified. The complete mathematical derivation of this operation may be found in another paper [1, Ch. 6]. The equations resulting from these differentiations are a set of simultaneous linear relations in the unknown work force, inventory, and production.

This system has a total of $(2n + 1)T$ unknowns and $(2n + 1)T - (n + 1)$ equations. The $(n + 1)$ additional equations that are necessary to obtain an unambiguous solution can be determined by supplying terminal conditions for any $(n + 1)$ variables. This can be done most simply by specifying values of ending inventories for each product, $I_{iT} (i = 1, ..., n)$, and ending work force W_T. This does not jeopardize the validity of the model since we are most interested in the values of inventory, sales, production, and workforce in the near future. It can be shown that the impact of ending conditions decreases rapidly as the time horizon becomes greater [4, p. 10]. As a result, W_{t+1} will be highly dependent upon W_t while W_{t+12} is little affected.

One characteristic of this analysis is that no bounds have been placed on the variables. In particular no formal restrictions have been incorporated to avoid negative sales, production, or work force. This limitation is not likely to be of practical importance, however, since the characteristics of

the cost and revenue curves keep the answers within the relevant range, except under very unusual conditions. It will be seen later that no problems of this type were encountered in the actual application of the model.

To solve a set of simultaneous linear equations for their LDR, HMMS have developed a very sophisticated method which reduces the computation necessary for applications to a minimum level. This approach was developed to make computation of these decision variables simple enough that they could be routinely determined using only an ordinary desk calculator [4, p. 92]. With sufficient mathematical ingenuity it may be possible to reduce the solution process for the model proposed here to a form where it could be solved on a desk calculator. This does not appear to be a particularly fruitful approach because of the large number of variables which must be determined for each solution. HMMS are satisfied to solve for two variables; aggregate production rate and work force level for the next period. Our goal is to determine work force for T periods plus sales, production, and inventory levels for each of n products in T periods. Because of this huge increase in dimensionality it was felt that a computer oriented solution approach would be much more practical. At the time of development of the HMMS model, of course, computers were not readily available in industrial organizations and therefore computational limitations provided a severe constraint on any proposed solution techniques. Since the time of the original development of the HMMS rules significant advances in computer technology have taken place. As a result, it is now feasible to solve directly sets of well over 100 linear equations with a reasonable

amount of computer time.[2] Furthermore, the solution of considerably larger systems than this is certain to become computationally feasible in the near future. It is therefore suggested that it is preferable at the present time to solve models such as the one postulated as a set of linear equations rather than through utilizing the HMMS solution techniques. Another advantage of this computer-based approach is its flexibility with respect to the incorporation of new variables or terms into a model. In short, this means that less limitations are necessary in model formulation while the solutions will still be operationally feasible.

Before applying the total profit model to any firm it is necessary to consider the feasibility of solving the resulting systems of linear equations. Since the theoretical aspects of the proposed solution method have been described in detail elsewhere, we shall not discuss these considerations here. Therefore, it is only necessary to keep in mind that we are basically interested in solving a set of linear equations of the form:

$$a_{11}X_1 + \ldots + a_{1n}X_n = b_1$$

$$a_{21}X_1 + \ldots + a_{2n}X_n = b_2$$

$$a_{n1}X_1 + \ldots + a_{nn}X_n = b_n$$

<u>Application of the MDR</u>

To illustrate the feasibility of the proposed MDR the model was applied to a firm producing electric motors. The company studied, which we shall call Hindustan Electronics (HE), is one of the largest producers of electric

motors in India. This firm is faced with significant seasonal trends in demand for their six major types of motors. As there are important physical differences between these six motor types, it is difficult to find a suitable dimension for aggregation as demanded by the LDR formulation. In applying the MDR to this firm a one year time horizon was used with decisions being made on a quarterly basis. With six products and four quarterly time periods this results in a problem with 52 unknown variables.

In applying the MDR we first wish to approximate the total revenue relationship for each of six products with a quadratic function of the form of equation (3). There are several methods to fit such quadratic functions to revenue or cost relationships. Because of the form of the data, this particular function will be fitted by selecting certain key points and passing the quadratic curve through these points. Different methods will be used for the other relationships to be estimated. By specifying a number of points equal to the number of constants in a quadratic expression, it is possible to uniquely determine the constants. In this case it is desired to solve for three unknown constants, r_{1it}, r_{2it}, r_{3it} in each of nT expressions so it is necessary to select sets of three points through which each of the quadratic revenue functions should pass. In fitting functions of this nature, it is usually possible to approximate closely only over a limited range. This range should be chosen so as to obtain the best possible approximation in the region where the model will be operating.

Historical sales data for each of the six products being considered provided estimates of likely operating ranges and revenues. Period-by-period sales and revenues of each product in each market were determined

for the three years from April, 1964, through March, 1967. Initial estimates of the points necessary to uniquely determine each quadratic revenue curve were made by summing the actual revenues received from all markets for each of the six products. Since three years of historical data is available on each period and product, three separate sales-revenue points can be estimated in this manner. The coordinates of these points, E_1, E_2, and E_3, may be plotted as in Figure 1 to show thier relationships. It should always be kept in mind that these are only estimates of points on the true revenue curves. Assuming these estimates are reasonable, however, we proceed in the following fashion. By substituting three coordinates into equation (3) we obtain a set of three simultaneous equations which can be solved for the three desired constants r_{1it}, r_{2it}, r_{3it}. As a check on the solution, this quadratic can be plotted as shown in Figure 1 to determine if the fit is adequate in the range of interest. This technique was employed to estimate an initial set of 24 revenue curves, one for each of the six products in the four quarterly time periods from April, 1967, through March, 1968. To consider the impact of changes in these revenue curves, several other assumptions will be considered later during the solution phase.

The Hiring and Layoff Cost Expressions

After discussions with management, an estimate was made that the cost associated with hiring and training would be about Rs 50 for a one-man increase in the direct work force. The existence of a pool of temporary workers who could be hired on a short-term basis is the primary reason this cost remains relatively low. The cost of layoff per worker was estimated

to be about Rs 300 because of its negative effect on employee morale and the possibility of labor relations difficulties. This possibility existed even if only temporary workers were laid off. Examination of historical trends indicated it was unlikely that changes in the size of the work force would exceed 15 men in any one quarter.

It was decided to fit the quadratic expression of Equation (8) in the range between -15 and +15 man changes in work force. To determine the three constants, C_2, C_{11}, and C_{13}, in this function, sets of three points were selected through which the quadratic curve should pass.

<u>The Inventory and Backorder Cost Expressions</u>

It is desired to find sets of coefficients C_{i7}, C_{i8}, C_{i9}, and C_{i13} for Equation (6) to approximate the cost of inventory and back-orders for each of the six products in the model. Estimates were first made of the costs of holding inventory and back orders. As in the LDR, these costs were assumed to be linearly related to the amount of finished goods inventory in stock and the quantity of back orders. These costs are also proportional to the length of time the inventory or back orders are held. The cost of holding one unit of finished goods in inventory was estimated to be 24 percent per year of its direct manufacturing cost. The cost of keeping one unit of any product back ordered for one quarter was estimated at 60 percent of this product's average contribution to profit and overhead. The inventory decision structure developed specifies optimal net inventory for product i equal to $(C_{i8} + C_{i9}S_{it})$. Investigation of the pattern of demand and inventory

holding costs suggested that optimal net inventory could be reasonably estimated as equal to one week's sales demand at any time. For purposes of this model, it will be assumed that optimal net inventory in any quarter can be approximated by:

$$(C_{i8} + C_{i9}S_{it}) = 1/13\ S_{it} \tag{16}$$

where S_{it} is the sales of product i in the quarter. Although C_{i8} and C_{i9} could be approximated more closely, it was felt that the payoffs in terms of increased model validity did not justify further refinements.

Because the constants, C_{i13}, in Equation (6) are irrelevant to inventory decisions, the only constant we need to determine is C_{i7}. C_{i13} must be calculated anyway, however, before it is possible to solve for C_{i7}. C_{i13} represents total inventory and backorder costs associated with the optimal net inventory position for product i. It can be approximated as average inventory times carrying cost plus average back order times the cost of back ordered units. Optimal net inventory has already been estimated as average inventory times carrying cost plus average back order times the cost of back ordered units. Optimal net inventory has already been estimated as 1/13 S_{it} and an estimation of expected back orders may be made by considering the relative likelihood of stock outs under the assumed inventory policy. Therefore, the estimate of C_{i13} for any quarter becomes:

$$C_{i13} = C_{Ii}\ (1/13\ S_i') + C_{di}B_i \tag{17}$$

where

S_i' = average sales per quarter for product i

C_{Ii} = inventory carrying cost per quarter for one unit of product i

C_{di} = back order holding cost per quarter for one unit of product i

B_i = expected average number of units of i back ordered

B_i is the only unknown which must still be estimated. An examination of historical information revealed that on average about 5 percent of one quarter's orders were overdue at any time when net inventory was kept at optimal levels. Therefore, B_i was approximated at $(.05)S_i'$, where S_i' is again the average sales per quarter of product i. Utilizing this estimate of C_{i13}, it is possible to solve directly for C_{i7} if $(IC)_{it}$ is known. Consider the point where net inventory, I_{it}, is zero. By substituting (16) into (6) at this point:

$$(18) \qquad (IC)_{it} = C_{i7}\ (0 - 1/13\ S_{it})^2 + C_{i13}$$

For a policy of setting I_{it} equal to zero it is straightforward to estimate $(IC)_{it}$ since the expressions for inventory holding costs and back-order holding costs are identical except for constants. An extension of the classical deterministic lot size model will be used to determine $(IC)_{it}$ for the desired point [4, p. 187]. A lot consisting of Q units will last

Q/S' periods of time with S' as the average sales rate. When it is desired to set I_{it} equal to zero, this implies an inventory relationship of the form in Figure 2, where the total cost per period for product i is:

$$(19) \qquad (IC)_{it} = C_{Ii}/2\ (Q/2)(t'/2) + C_{di}/2\ (Q/2)(t'/2)$$

in this equation:

$t' = Q/S'$ = the time interval within which a lot is sold. This is true since the average levels of both positive and negative inventories are equal to $Q/4$ and the average length of time they are held is $t'/2$. Discussions with management suggested the average lead time on an order would be about two weeks or roughly one-sixth of a quarter. To keep a net inventory of zero, the firm must then reorder at point R in Figure 2 and receive the order two weeks later when net inventory has decreased to $-Q/2$. Therefore, half a lot, $Q/2$, is sold during a time $t'/2$ or:

$t'/2 = 1/6$ quarter and cycle time $t' = 1/3$ quarter, therefore we know:

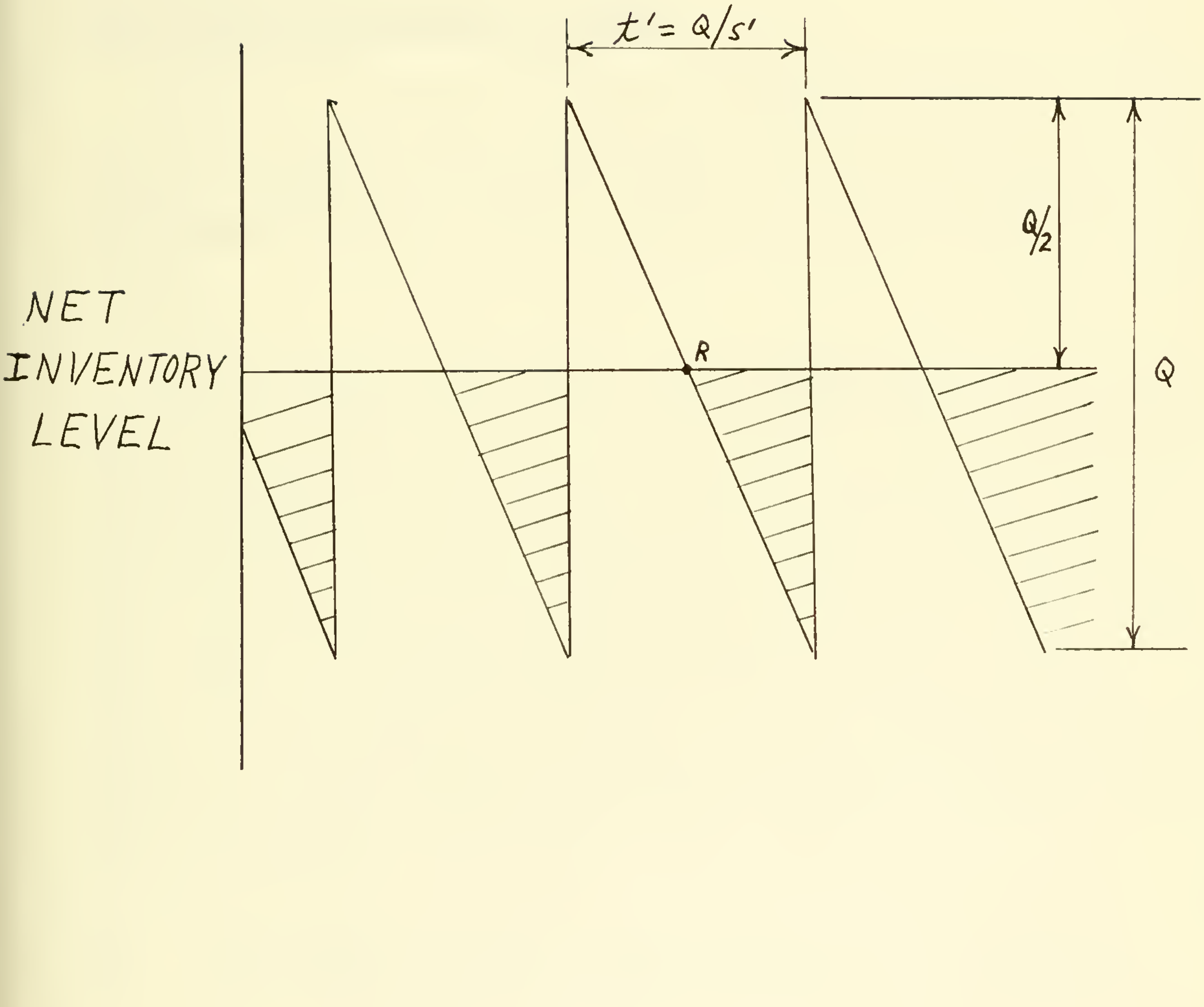
$$Q = S/3$$

Substituting for Q and t' in (19) results in:

$$(20) \qquad (IC)_{it} = (C_{Ii}/2)(S_i'/6)(1/6) + (C_{di}/2)(S_i'/6)(1/6)$$

for any $t = 1, \ldots, T$. This is the total inventory and back-order cost with an inventory policy of keeping net inventory equal to zero.

FIGURE 2



Our original purpose in solving for $(IC)_{it}$ at a point was to find values for the constant C_{i7} for $i = 1, \ldots, n$. Using (18), it is now possible to solve for each value of C_{i7} since C_{i13} is known from Equation (17) and $(IC)_{it}$ is known from Equation (20). This procedure was followed in calculating C_{i7} for $i = 1, \ldots, 6$ in the Hindustan Electronics model.

The Variable Production Cost Expression

We desire to fit the cost function of Equation (11) which is a general quadratic expression with variables work force and production rates. The least-squares estimation technique described in [4, pp. 81-82] will be employed.

The first stage of the fitting process requires determination of the standard labor time coefficients for Equation (10). This information was readily available from engineering records for each of the six products. Historical cost, production, and workforce information was available from cost accounting records for the two-year period from April, 1965, through March, 1967. For each observation total labor cost is the dependent variable while W_t, W_t^2, L_tW_t, L_t, and L_t^2 are the independent variables. Therefore, we wish to employ least squares techniques to estimate the K's in the expression:

(21) $$\text{Total labor cost} = K_1W_t + K_2W_2^2 + K_3L_tW_t + K_4L_t + K_5L_t^2 + K_6$$

After scaling of the coefficients and proper adjustments for the quarterly period used in the model the final labor cost equation becomes:

(22) Total labor cost = $-279.5W_t + 1.183W_t^2 - 2.000L_tW_t + 1118L_t +$

$$.6124L_t^2 - 345503$$

After calculation of these coefficients a sensitivity analysis was performed which determined that this function did in fact result in reasonable labor cost values over the relevant ranges of W_t and L_t. With extreme values of either W_t or L_t labor costs computed from Equation (22) become very large as would be expected.

Results of Application of Model to HE

The model described above was applied to Hindustan Electronics for the one-year period from April, 1967, through March, 1968. Since each period in the model represents one three-month quarter of the year, this formulation resulted in a set of 52 equations and unknowns. A total of six solutions for the MDR model were computed under slightly different assumptions of starting conditions and coefficients. In particular, solution one was made with the most likely estimates of all coefficients as described above. For solution two the costs of hiring and firing were increased to Rs100 and Rs400 respectively. For solutions three and four all revenue curves were assumed to be increased 10% over the most likely levels and, in addition, for solution four the starting and ending inventories were also increased to a level of two weeks demand. Solution five assumed the same revenue curves as one but with two week starting and ending inventories. For the final run all revenue curves were assumed to be 10% less than the estimates in solution one.

Let us next consider some specific results of the MDR solutions as shown in Figures 3, 4, 5, and 6.[3] Analysis of total sales data in Figure 3 indicates that total sales by the model in solution one are approximately 60 percent higher in the second quarter than sales projected by management. Figure 6 indicates this is primarily due to the MDR model's higher revenue curve for 1/32 motors. This revenue curve has been significantly influenced by the high sales of this motor during the second quarter of 1965. Since management's projections for 1967-68 are considerably lower than this peak year, they are clearly not this optimistic on sales possibilities for the 1/32 HP model during the forthcoming second quarter.

It is interesting to note the different seasonal demand patterns of the six motors shown in Figure 6. For example the 1/32 HP and 3/8 HP models have peak demands in quarter 2 while the 1/2 HP and 1/8 HP models have peak demands in quarter four. Explicit consideration of these product by product seasonality patterns allows the MDR to smooth total production requirements. This is done by shifting production from one product to another during the year rather than producing a constant fraction of each product during each quarter, e.g., product A may comprise 20% of aggregate production during quarter two but only 5% during quarter four.

The correspondence between the production plans projected by management and the plans from the MDR solutions is quite high with the obvious exception that solutions with higher demand curves such as number three have higher across-the-board production rates. These relationships are evident in Figure 4 where it can be seen that production is generally

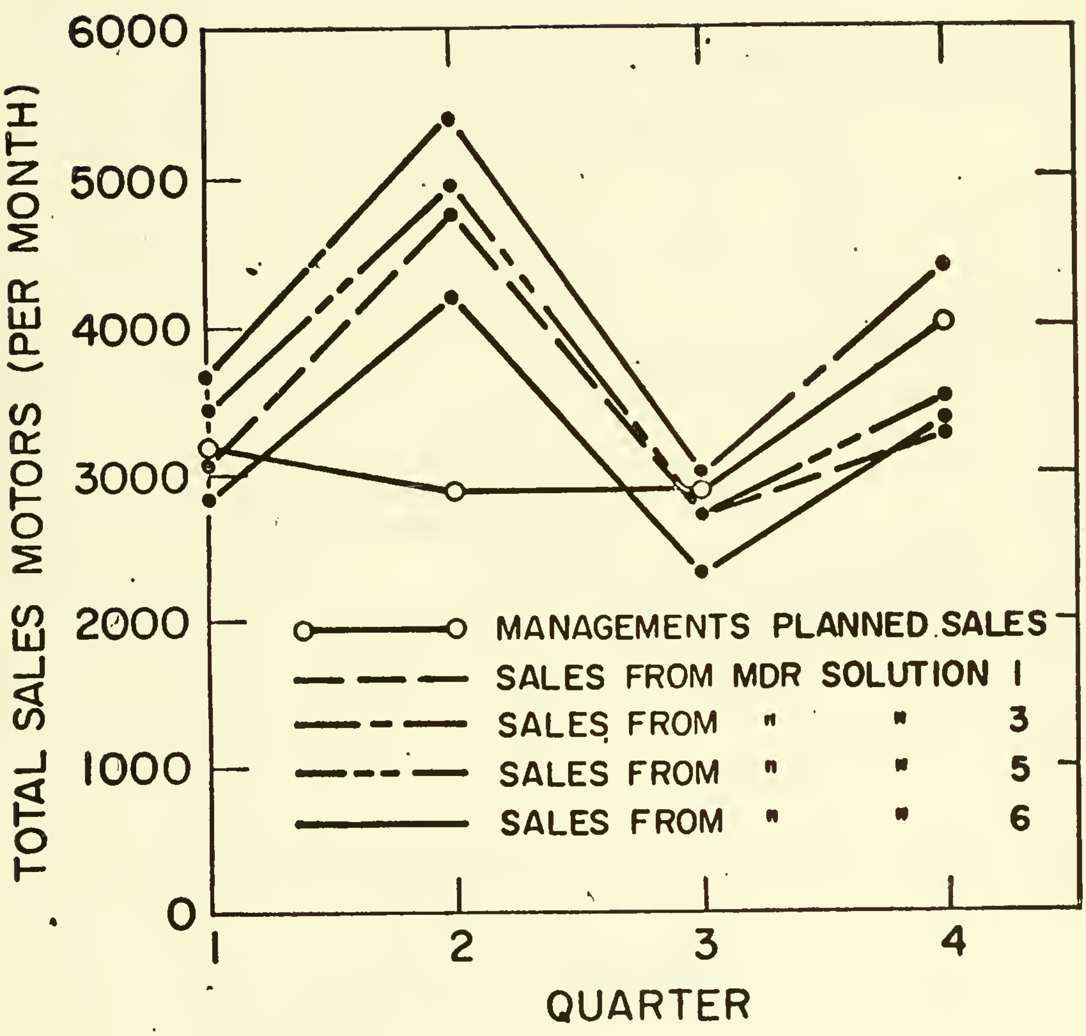

6000
5000
4000
3000
2000
1000
0
TOTAL SALES MOTORS (PER MONTH)
MANAGEMENTS PLANNED SALES
SALES FROM MDR SOLUTION 1
SALES FROM " " 3
SALES FROM " " 5
SALES FROM " " 6
1
2
3
4
QUARTER

FIGURE 3

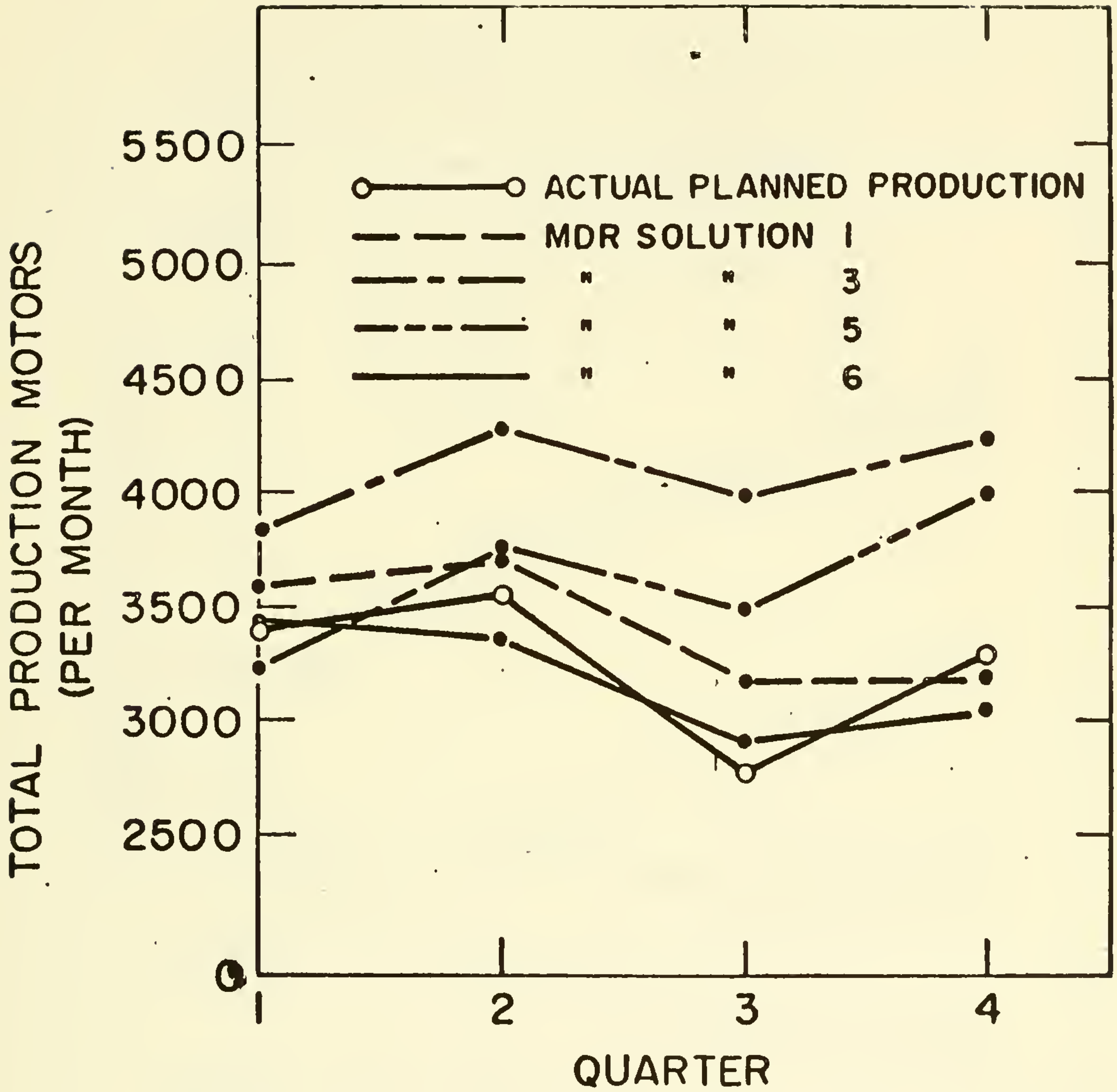

TOTAL PRODUCTION MOTORS (PER MONTH)
5500
5000
4500
4000
3500
3000
2500
0
ACTUAL PLANNED PRODUCTION
MDR SOLUTION 1
" " 3
" " 5
" " 6
1
2
3
4
QUARTER

FIGURE 4

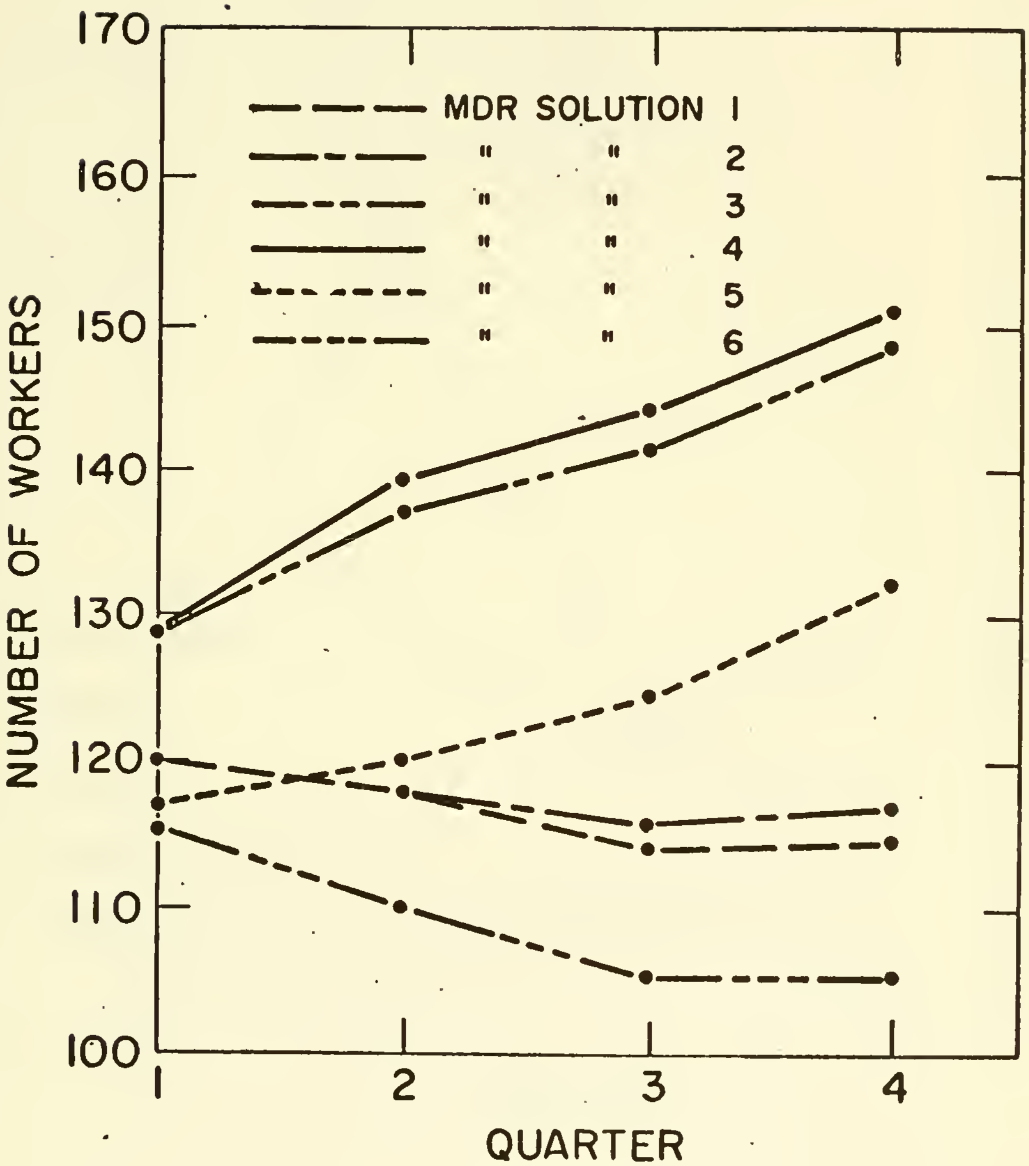

170
160
150
140
130
120
110
100
NUMBER OF WORKERS
1
2
3
4
QUARTER
MDR SOLUTION 1
" " 2
" " 3
" " 4
" " 5
" " 6

FIGURE 5

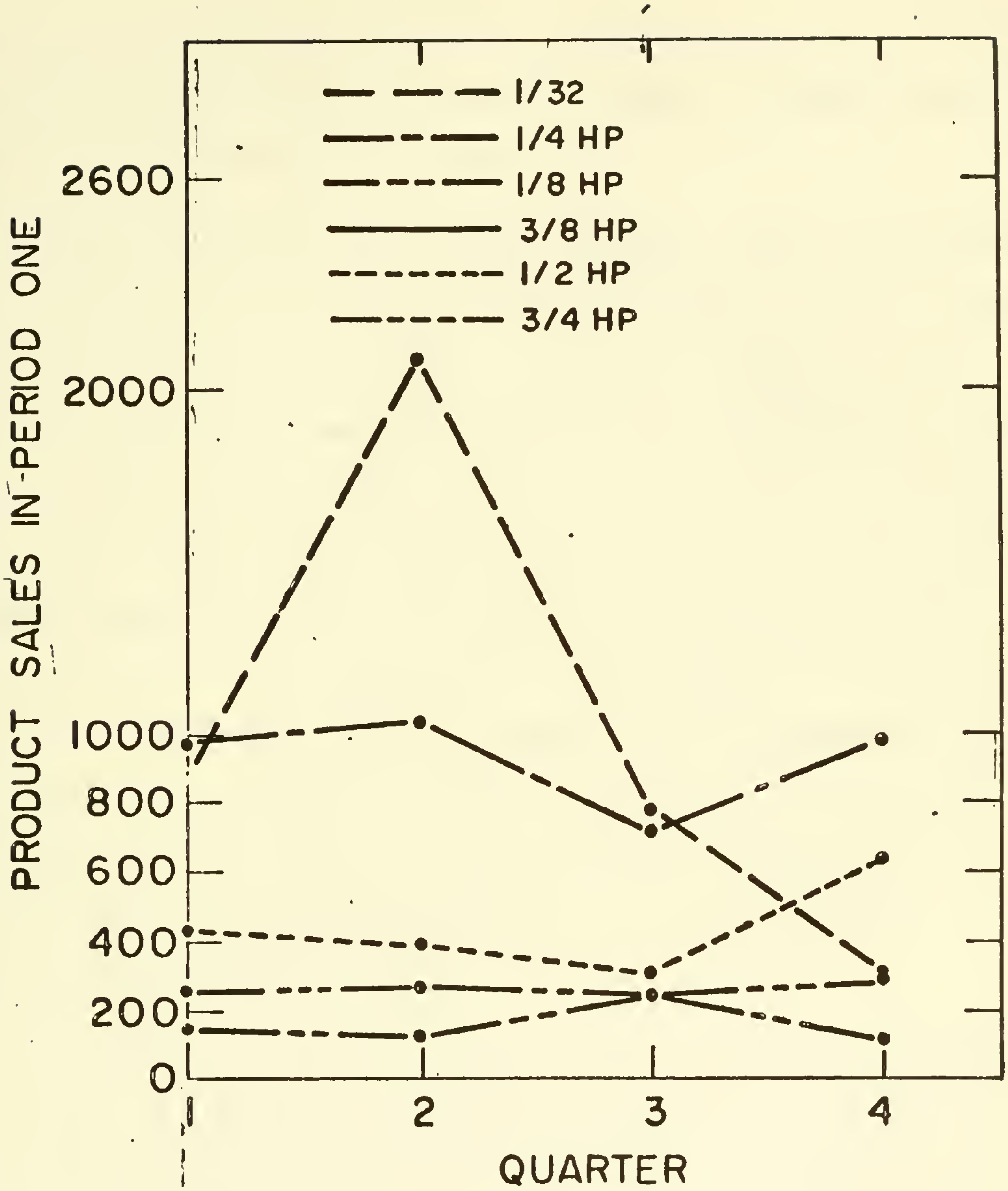

1/32
1/4 HP
1/8 HP
3/8 HP
1/2 HP
3/4 HP
2600
2000
1000
800
600
400
200
0
PRODUCT SALES IN PERIOD ONE
1
2
3
4
QUARTER

FIGURE 6

highest in the fourth quarter and lowest in the third quarter. Differences are small but the schedules from the model do result in slightly smoother production over the course of the year.

Management did not specify work force levels in their projections for the year 1967-68 but it is possible to directly compare work force levels for the six MDR solutions in Figure 5. As would be expected from the quadratic cost structure, deviations in work force levels between the six solutions should increase from quarter one to quarter four. In quarter four the optimistic demand curves employed in solutions three and four result in employment levels 40 to 50 percent higher than the pessimistic demand curves of solution six. The only difference in the data between solutions one and two is the higher work force change costs of two, namely Rs 100 to hire one worker and Rs 400 to fire one worker. Figure 5 shows the solution is insensitive to this parameter since the greatest difference in work forces is only three men in quarter four and work force levels are identical in the first two quarters.

In the first three model solutions, starting and ending inventories were set equal to the actual existing inventories at the start of the year. Since this starting inventory was very low (a total of 340 motors), it was desirable to investigate the possibilities of increasing sales by beginning the year with a larger inventory. Therefore, starting inventories equal to two weeks' estimated demand for each product were incorporated in solutions four and five. This resulted in total aggregate starting and ending inventories of 1600 motors. When these higher inventories were added to

the basic model of solution one, total sales over the year increased from 41600 to 43730 units (solution five). When higher starting inventories were added to solution three, the total impact on sales was much less with 50050 motors being sold rather than 49880 (solution four). Therefore, it seems reasonable to conclude that small increases in sales would be possible if inventories had been greater at the beginning of this year.

The six computer runs required for solution of the MDR models were made on a CDC 3600 and required approximately one minute each.

Conclusions

The development and application of the MDR model suggests that it is now operationally feasible to remove the requirement of an aggregate production dimension in planning models. Furthermore, given the availability of revenue curves for each product in each time period the MDR model proposed here can determine optimal production, sales, inventory, and work force levels so as to maximize profit over a specified time horizon. It should be noted that the MDR solves for decision variables which are usually considered to be in two separate functional areas: production and marketing. Therefore it is considerably broader in conception than planning methods which optimize only in one functional area.

A number of recent proposals have been made for solving the aggregate planning problem by heuristic or computer search techniques, [5], [8], [2]. The primary appeal of these approaches lies in their freedom from the

assumptions of mathematical form required by linear or quadratic models. Therefore, they can accommodate constraints or complex cost structures which are difficult to approximate with mathematical optimizing techniques. It appears likely that the application of heuristic or computer search techniques to the multi-item planning problem described in this paper would be an especially fruitful area for further work. A solution of this multi-item, profit maximization problem without the refinement of linear or quadratic cost and revenue functions would be an encouraging step towards greater realism and utility in aggregate planning models.

FOOTNOTES

1. See [3], pp. 90-93, for a proof of this.

2. See [7], Chapter 3, for a discussion of solution techniques. With his experience on an IBM 7094 he concludes, "At present systems with up to 100-120 equations can be solved conveniently." This limitation is primarily a function of the available high speed memory on this machine.

3. For a summary of all computational results with the MDR, see [1], Chapter 6.

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