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**Abstract**

We explore the robustness of aggregation in Sterman's model of the economic long wave. The original model aggregates all capital producing firms into a single sector and generates a large amplitude self-sustained oscillation with a period of roughly 50 years. We disaggregate the model into two coupled industries, one representing production of plant and long-lived infrastructure and the other short-lived equipment and machinery. While holding the aggregate equilibrium characteristics of the model constant, we investigate how mode-locking occurs as a function of the difference in capital lifetimes and the strength of the coupling between the sectors. Disaggregation allows new modes of behavior to arise: In addition to mode-locking we observe cascades of period-doubling bifurcations, chaos, intermittency, and quasi-periodic behavior. Despite the introduction of these additional modes, the basic behavior of the model is robust to the aggregation assumption. We consider the likely effects of finer disaggregation, the introduction of additional coupling mechanisms such as prices, and other avenues for the exploration of aggregation in system dynamics models.

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# 1 Introduction: Aggregation in nonlinear systems

Macroeconomic models normally aggregate the individual firms of the economy into sectors with similar products, parameter values and decision functions. Sometimes, only a single sector is considered (e.g. Samuelson 1939, Goodwin 1951). This simplification is justified on pragmatic grounds by noting that it is impractical to portray separately all the firms in an industry or all the products on the market, and arguing that the phenomena of interest are captured in sufficient detail by the aggregate formulation (Forrester 1961, Simon 1969). Nevertheless, there are instances where aggregation is not justified. For example, Allen (1988) shows how suppressing individual variation by using aggregate population means can prevent a model from properly reproducing the dynamics of the system. Similarly, work by e.g. Bruckner, et al. (1989) shows how the representation of individuals is necessary to describe evolutionary processes.

However, the modelling literature is weak in providing guidelines for appropriate aggregation of dynamic systems, particularly when there are significant interactions between the individual entities, as is usually the case in economic systems.

Consider the aggregation of different firms into a single sector. Such aggregate representations are the mainstay of models of business fluctuations. Models of capital investment, for instance, represent the average lead time and lifetime of the plant equipment used by each firm. In reality there are many types of plant and equipment acquired from many vendors operating under diverse conditions and possessing a wide range of lead times. In response to changes in external conditions, each firm will generate cyclic behaviors whose frequency, damping, and other properties are determined by the parameters characterizing the particular mix of lead times and lifetimes it faces. Because these individual firms are coupled to one another via the input-output structure of the economy, each acts as a source of perturbations on the others.

How do the different lifetimes and lead times of plant and equipment affect the frequency, phase, amplitude, and coherence of economic cycles? How valid is aggregation of individual firms into single sectors for the purpose of studying macroeconomic fluctuations? The issue of coherence or synchronization is particularly important. The economy as a whole experiences aggregate business fluctuations of various frequencies from the short-term business cycle to the long-term Kondratieff cycle (Sterman and Mosekilde, forthcoming). Yet why should the cycles of the individual firms move in phase so as to produce an aggregate cycle? Given the distribution of parameters among individual firms, why do we observe only a few distinct cycles rather than cycles at all frequencies – cycles which might cancel out at the aggregate level?

A common approach to the question of synchronization is to assume that fluctuations in economic aggregates such as GDP or unemployment arise from external shocks, such as sudden changes in resource supply conditions or variations in fiscal or monetary policy (see Zarnowitz 1985 for a review). Forrester (1977) suggested instead that synchronization could arise from the endogenous interaction of multiple nonlinear oscillators, i.e., that the cycles generated by individual firms become reinforced and entrained to one another. Forrester also proposed that such entrainment could account for the uniqueness of the economic cycles. Oscillatory tendencies of similar periodicity in different parts of the economy would be drawn together to form a subset of distinct modes, such as business cycles, long waves, and economic growth, and each of these modes would be separated from the next by a wide enough margin to avoid synchronization. Until recently, however, these suggestions have not been subjected to rigorous analysis.

Roughly speaking, synchronization occurs because the nonlinear structure of the interacting parts of a system creates forces that "nudge" the parts of the system into phase with one another. For instance, two mechanical clocks, hanging on the same wall, are often observed to synchronize their pendulum movements (it was Huygens who first reported the entrainment of clocks). Each clock has an escapement mechanism, a highly nonlinear mechanical device, that transfers power from the weights to the rod of the pendulum. When a pendulum is close to the position where the escapement releases, a small shock, such as the faint click from the release of the adjacent clocks escapement, might be enough to trigger the release. Hence, the weak coupling of the clocks through vibrations in the wall can bring individual oscillations into phase, as long as the two uncoupled pendulum frequencies are not too different. In a similar manner, nonlinear interactions and dissipation working over millions of years have brought the rotational motion of the moon into synchronization with its orbital motion with the result the moon has a dark side never visible from earth.

Synchronization is only one manifestation of a more general phenomenon known as mode-locking or entrainment. In nonlinear systems, a periodic behavior usually contains a series of harmonic frequencies at $p$ times the fundamental frequency, where $p$ is an integer. When two nonlinear oscillators interact, mode-locking will occur whenever a harmonic frequency of one mode is close to a harmonic frequency of the other. As a result, nonlinear oscillators tend to lock to one another so that one subsystem completes precisely $p$ cycles each time the other subsystem completes $q$ cycles, with $p$ and $q$ being integers.

In a series of papers, we have analyzed how mode-locking and other nonlinear dynamic phenomena arise in a simple, nonlinear model of the economic long wave (Mosekilde et al. 1992, Sterman and Mosekilde, forthcoming). The model (Sterman 1985) explains the long wave as a self-sustained oscillation arising from instabilities in the ordering and production of capital. An increase in the demand for capital leads to further increases through the investment accelerator or "capital self-ordering," because the aggregate capital producing sector depends on its own output to build up its stock of productive capital. Once a capital expansion gets underway, self-reinforcing processes sustain it beyond its long-term equilibrium, until production catches up with orders. At this point, however, the economy has considerable excess capital, forcing capital production to remain below the level required for replacement until the excess has been fully depreciated, creating for a new expansion.

The concern of the present paper is the simple model's aggregation of capital into a single type. The real economy consists of many sectors employing different kinds of capital in different amounts. Parameters, such as the average productive life of capital and the relative amounts of different capital components employed, may vary from sector to sector. In isolation, the buildings- and infrastructure-capital industry may show a temporal variation significantly different from that of, for instance, the machinery industry.

An early study by Kampmann (1984) took a first step in this direction by disaggregating the simple long-wave model into a system of two or more capital-producing sectors with different characteristics. Kampmann showed that the multi-sector system could produce a range of different behaviors, at times quite different from the original one-sector model. The present paper further investigates these results, using a two-sector model. One sector can be construed as producing buildings and infrastructure with very long lifetimes, while the other could represent the production of machines, transportation equipment, computers, etc., with much shorter lifetimes. In isolation, each sector produces a self-sustained oscillation with a period and amplitude determined by the sector's parameter values. However, when the two sectors are coupled together through their mutual dependence on each other's output for their own production, they tend to synchronize or lock together with a rational ratio between the two periods of oscillation that depends on the differences in parameters.

## 2 Mode-Locking in Nonlinear Systems

For linear systems the principle of superposition applies. The behavior of any variable in the system is a sum of distinct modes, or elementary excitations, where the frequencies and attenuation rates are determined by the eigenvalues of the Jacobian matrix. Different modes can exist and develop independently of one another, and the sensitivity of any mode to an external disturbance will be the same irrespective of the phase of the others. Thus, mode-locking cannot occur linear systems, because the individual oscillatory modes do not affect one another.

In nonlinear systems, on the other hand, different modes will interact, and the behavioral characteristics of one mode may depend on the phase of another. In particular, components of different periodicities may adjust themselves until the frequencies coincide. Synchronizations and mode-locking are universal phenomena in nonlinear systems (Jensen, et al. 1983, 1984). Such phenomena are well documented in a variety of physical, biological, and engineering systems (e.g., Colding-Jorgensen 1983, Glass, et al. 1986, Togeby and Mosekilde 1988), and the same processes are likely to be involved both in the interaction between individual companies and in the coupling between the various sectors of the overall economy.

To understand the phenomenon of frequency-locking in more detail, consider first the case where two self-sustained oscillators with different periods operate without interaction. As illustrated in Figure 1, the total trajectory may then be represented as a curve on a torus. For particular parameter values, namely those where the two periods are commensurate, the trajectory forms a closed orbit, and the total motion is periodic, as in Figure 1.a. In general, however, the periods will be incommensurate (their ratio is irrational), and the trajectory will fail to close on itself. In this case, the total motion is said to be quasi-periodic, and the trajectory will gradually cover the entire surface of the torus (Figure 1.b). Thus, quasi-periodic motion never repeats itself: each cycle is unique. In contrast to deterministic chaos, however, a quasi-periodic motion is not sensitive to the initial conditions, since a small change in these conditions only shifts the entire trajectory by a constant amount. For the same reason, the largest Lyaponov exponent, which measures the divergence between nearby trajectories, is zero. In chaotic motion, this exponent will be positive (Wolf 1986).

*Figure 1*

A cross-section of the torus (see Figure 1) defines a topological circle (a closed curve which does not cross itself). By noting the angular position $u_n$ along this curve of each subsequent passage of the trajectory, one can define a map of the circle onto itself

$$\theta_{n+1} = f(\theta_n), \qquad 0 \leq \theta < 1, \tag{1}$$

where $\theta = u/2\pi$ mod 1 measures the phase of one of the oscillatory subsystems as obtained stroboscopically with the period of the other. If, at a certain time this phase is $\theta_n$, one period later it will be $\theta_{n+1}$. With this procedure the steady-state behavior of a complicated, multidimensional dynamic system can be represented compactly by a one-dimensional map, as illustrated in Figure 2.

*Figure 2*

With no interaction between the two oscillators, the phase shift of one oscillator during a period of the other will always be the same, and the map is linear, i.e.,

$$\theta_{n+1} = \theta_n + \Omega \text{ mod } 1, \tag{2}$$

where the winding number $\Omega$ measures the ratio between the two periods. In the particular cases where $\Omega$ is rational, the motion will be periodic (Figure 2.a). Generally, however, $\Omega$ is irrational, and the map produces a never-repeating quasi-periodic motion (Figure 2.b).

If one now allows the two oscillators to interact, the phase shift of one mode during the period of the other may depend on the initial phase of the first mode, and the circle map becomes nonlinear. A simple example to illustrate the effects of introducing nonlinearity is the sine circle map (Arnol'd 1965)

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} sin(2\pi\theta_n) \text{ mod } 1. \quad (3)$$

where $K$ measures the strength of the nonlinear coupling while, as before, $\Omega$ measures the average phase shift of one oscillator per period of the other. ($K = 0$ corresponds to the linear map.) The coupling strength $K$ is analogous to the coupling parameter $\alpha$ governing the interaction between the capital producing sectors in the model developed below while the phase-shift parameter $\Omega$ is analogous to the difference in the uncoupled periods of oscillation of the two sectors, determined by the difference in capital lifetimes, $\Delta\tau$.

If $\Omega$ is not too far from 0, the map may cut the diagonal of the $(\theta_{n+1}, \theta_n)$-plane and produce a stable fixed point toward which all trajectories will converge, independent of initial conditions (see Figure 2.c). The fixed point represents the synchronized solution where the two oscillators have precisely the same period. More importantly, however, if $\Omega$ changes slightly in either direction, the fixed point will continue to exist (though it will move along the diagonal), and the stable 1:1-solution therefore exists over a finite range of parameters.

Figure 2.d illustrates a 1:3-mode-locking, where one oscillator performs precisely 3 cycles each time the other performs one cycle. The $1:3$ mode arises when the thrice iterated map, $f^{(3)}(\theta)$, has 3 stable fixed points. Note that these fixed points continue to exist over a finite range of parameters.

Thus in coupled nonlinear systems periodic (entrained) motion becomes more common and quasiperiodic motion becomes less common. In principle, the system can lock frequencies at all rational ratios of the two periods. Plotting the ratio of frequencies as a function of the phase-shift parameter $\Omega$ illustrates the locking. In a linear system, the curve is simply a straight line through the origin. In nonlinear systems, however, mode-locking produces finite intervals for each rational ratio of frequencies, resulting in a so-called Devil's Staircase. At least for small coupling strengths, the staircase is a fractal structure which repeats itself under magnification, since between two given rational numbers, one can always find another rational number. An example of a Devil's Staircase arising from the forced one-sector long-wave model can be found in Mosekilde, Larsen, Sterman, and Thomsen (1992).

As the nonlinearity $K$ is increased, so does the range of $\Omega$ over which a particular mode-locked solution exists, producing a so-called Arnol'd tongue diagram. Each tongue defines the region in which a particular mode-locked solution exists. For $K = 0$, the tongue collapses to a single point, namely the rational number corresponding to the ratio of frequencies. As $K$ is increased, the tongue flares out. As long as $K$ is small enough, the map is a diffeomorphism, i.e., smooth and invertible, and periodic and quasi-periodic motions are the only types of behavior that can occur. However, as $K$ continues to grow, it reaches a critical value ($K = 1$ for the sine map) where the map (3) develops an inflection point at $\theta_n = 0$, and the slope becomes equal to zero. At this value, one finds that the Arnol'd tongues start to overlap, indicating the simultaneous existence of two or more periodic solutions. At the same time, quasi-periodic behavior ceases to exist.

Beyond $K = 1$, the map is no longer invertible, folding occurs, and the motion may become chaotic. Within each Arnol'd tongue, a variety of bifurcations may take place above the critical line. One classical phenomenon that can occur is a cascade of period-doubling bifurcations. For instance, a 1:3-mode locking may bifurcate into 2:6, 4:12, ..., and ultimately into chaotic motion. (Period-doubling and chaotic behavior cannot be represented on a torus. A cross-section of the "torus" would reveal that it now has a folded or diffuse structure.) It is worth emphasizing, however, that period doubling is only one of several "routes to chaos". Other scenarios include intermittency and a direct transition from quasi-periodicity to chaos.

While the basic theory of frequency locking is well established, less is known about the behavior beyond the critical line. Several different types of behavior have been observed in various models, and it is likely that the behavior to some extent will be specific to the individual system. Certain general results have been obtained, however, (see e.g. Knudsen, et al. 1991) and it is clearly of interest to pursue this type of investigation further.

# 3 The Model

We have replicated Sterman's original (1985) model to portray two interacting sectors. The original structure is maintained, except for minimal modifications necessary to 1) extend the one-sector structure to multiple sectors, and 2) replace the original model's piecewise linear functions with analytic, infinitely differentiable functions.

The model describes the flows of capital plant and equiment in two capital-producing sectors.

Each sector uses capital from itself and from the other sector as the only factors of production. Each sector receives orders for capital, both from itself, from the other sector, and from the consumer goods sector. Product is made to order, no inventories are maintained and orders reside in a backlog until capital is produced and delivered.

The model consists of ten ordinary differential equations, corresponding to ten state variables, namely the capital stock of each type in each sector ($2 \times 2$), the capital on order (the "supply line" of capital) of each type in each sector ($2 \times 2$), and the total order backlog in each sector ($2 \times 1$). The state variables are indicated by capital letters.

Each sector $i = 1,2$ maintains a stock $K_{ij}$ of each capital type $j = 1,2$. The capital stock is increased by deliveries of new capital and reduced by physical depreciation. The stock of capital type $j$ depreciates exponentially with an average lifetime of $\tau_j$. The difference in lifetime between the two sectors, $\Delta\tau$, is used as a bifurcation parameter to explore the robustness of the aggregated model.

Output is distributed "fairly" between customers, i.e. the delivery of capital type $j$ to sector $i$ is the share of total output from sector $j$, $x_j$, distributed according to how much sector $i$ has on order with sector $j$, $S_{ij}$, relative to sector $j$'s total order backlog $B_j$. Hence,

$$\dot{K}_{ij} = x_j \frac{S_{ij}}{B_j} - \frac{K_{ij}}{\tau_j} \quad (1)$$

and

$$\dot{S}_{ij} = o_{ij} - x_j \frac{S_{ij}}{B_j} \quad (2)$$

where $o_{ij}$ is sector $i$'s new orders for capital from sector $j$.

Each sector receives orders from both capital sectors, $o_{ii}$ and $o_{ji}$, and from the consumer goods sector, $y_i$. It accumulates these orders in a backlog $B_i$ which is then depleted by the sector's deliveries of capital $x_i$. Hence,

$$\dot{B}_i = (o_{ii} + o_{ji} + \eta_i) - x_i; \qquad j \neq i. \tag{6}$$

The consumer demands received by each sector, $\eta_i$, are exogenous, constant, and equal. The latter assumption is not without consequence, since the relative size of the two sectors affects the dynamics (Kampmann 1984). Equal demands is the most parsimonious assumption.

Production capacity in each sector is determined by a constant-returns-to-scale Cobb-Douglas function of the individual stocks of the two capital types, with a factor share $\alpha \in [0, 1]$ of the other sector's capital type and a share $1 - \alpha$ of the sector's own capital type, i.e.,

$$c_i = \kappa_i^{-1} \cdot K_{ii}^{1-\alpha} \cdot K_{ij}^{\alpha}; \qquad j \neq i. \tag{7}$$

where $\kappa_i$ is a constant. The parameter $\alpha$ thus determines the degree of coupling between the two sectors. In the simulation studies $\alpha$ is varied between 0, indicating no interdependency between the sectors, and 1, indicating the strongest possible coupling where each sector is completely dependent on capital from the other sector.

The output from sector $i$, $x_i$, depends on the sector's capacity $c_i$, compared to the sector's desired output $x_i^*$. If desired output is much lower than capacity, production is cut back, ultimately to zero if no output is desired. Conversely, if desired output exceeds capacity, output can be increased beyond capacity, up to a certain limit. Specially, the sector's output is formulated as

$$x_i = f\left(\frac{x_i^*}{c_i}\right) \cdot c_i . \tag{8}$$

The capacity-utilization function $f(\cdot)$ has the form

$$f(r) = \gamma(1 - (\frac{\gamma - 1}{\gamma})^r); \qquad \gamma > 1. \tag{9}$$

where $\gamma$ is a parameter set here to 1.1. Note that

$$f(0) = 0; f(1) = 1; \lim_{r \to \infty} f(r) = \gamma; f(r) > r, r \in [0, 1].$$

Thus, the parameter $\gamma$ determines the maximum production possible. $f(r)$ is above the 45-degree line for $r \in [0, 1]$, implying that firms are reluctant to cut back their output when capacity exceeds demand. Instead, they deplete their backlogs and lower their delivery delays.

Sector $i$'s desired output $x_i^*$ is assumed to be the value that would allow firms in that sector to deliver the capital on order $B_i$ with the (constant) normal average delivery delay $\delta_i$ for that sector. Hence,

$$x_i^* = \frac{B_i}{\delta_i} \tag{10}$$

Sector $i$s desired orders for new capital of type $j$, $o_{ij}^*$, consists of three components. First, all other things equal, firms will order to replace depreciation of their existing capital stock, $K_{ij}/\tau_j$. Second, if their current capital stock is below (above) its desired level $k_{ij}^*$ firms will order more (less) capital in order to correct the discrepancy over time. Third, firms consider the current supply line $S_{ij}$ of capital and compare it to its desired level $s_{ij}^*$; if the supply line is below (above) desired, firms order more (less) in order to increase (decrease) the supply line over time. In total,

$$o_{ij}^* = \frac{K_{ij}}{\tau_j} + \frac{k_{ij}^* - K_{ij}}{\tau_i^K} + \frac{s_{ij}^* - S_{ij}}{\tau_i^S}. \tag{11}$$

where the parameters $\tau_i^K$ and $\tau_i^S$ are the desired adjustment times for the capital stock and the supply line, respectively. This decision rule is supported by extensive empirical (Senge 1980) and experimental (Sterman 1989a, 1989b) work.

Actual orders are constrained to be non-zero (cancellation of orders is not permitted) and the fractional rate of expansion of the capital stock is also assumed to be limited because of bottlenecks related to labor, market development, and other factors not represented in the production function. These constraints are accounted for through the expression

$$o_{ij} = \frac{K_{ij}}{\tau_j} \cdot g\left(\frac{o_{ij}^*}{K_{ij}/\tau_j}\right). \tag{12}$$

where orders are expressed as a multiple $g(\cdot)$ of depreciation. The function $g(\cdot)$ has the form

$$g(r) = \frac{\beta}{1 - \mu_1 \exp^{-\nu_1(r-1)} - \mu_2 \exp^{-\nu_2(r-1)}} \tag{13}$$

where the parameters have the following values

$$\beta = 6;\ \mu_1 = \frac{27}{7};\ \mu_2 = \frac{8}{7};\ \nu_1 = \frac{2}{3};\ \nu_2 = 3. \tag{14}$$

The parameters are specified so that

$$g(1) = 1;\ g'(1) = 1;\ g''(1) = 0.$$

Furthermore

$$\lim_{r\to\infty} g(r) = \beta;\ \lim_{r\to-\infty} g(r) = 0.$$

Note that $g(r)$ has a neutral interval around the steady state point, $r = 1$, where actual orders equal desired orders.

The desired capital stock $k_{ij}^*$ is proportional to the desired production rate $x_i^*$ with a constant capital-output ratio. Thus, it is implicitly assumed that the relative prices of the two types of capital are constant, so there is no variation in desired factor proportions. Hence,

$$k_{ij}^* = \kappa_{ij} \cdot x_i^* , \tag{15}$$

where $\kappa_{ij}$ is the capital-output ratio of capital type $j$ in sector $i$.

In calculating their desired supply line, $s_{ij}^*$, firms are assumed to account for the current delivery delay for each type of capital. The target supply line is taken to be the level at which the deliveries of capital, given the current delivery delay, would equal the current depreciation of the capital stock. The current delivery delay of capital from a sector is the sector's backlog divided by its output. Thus,

$$s_{ij}^* = \frac{K_{ij}}{\tau_j} \cdot \frac{B_j}{x_{ij}} . \tag{16}$$

Finally, the orders from consumers to each sector $y_i$ are assumed to be exogenous, constant, and equal for both sectors. The latter assumption is not without consequence, since the relative size of the consumer demands for the two types of capital can change the dynamics of the model considerably (see Kampmann 1984).

The capital-output ratios and average capital lifetimes are formulated in such a way that the aggregate equilibrium values of these parameters for the model economy as a whole remain constant and equal to the values in Sterman's original model. Specifically, the average capital lifetimes in the two sectors are

$$\tau_1 = \tau + \frac{\Delta\tau}{2}; \; \tau_2 = \tau - \frac{\Delta\tau}{2} \tag{17}$$

The capital output ratios are

$$\kappa_{ii} = (1-\alpha)\kappa\frac{\tau_i}{\tau}$$
$$\kappa_{ij} = \alpha\kappa\frac{\tau_j}{\tau},\ i \neq j \qquad (18)$$
$$\kappa_i = \kappa_{ii}^{1-\alpha}\kappa_{ij}^{\alpha}$$

The average lifetime of capital, $\tau$, is 20 years and the average capital-output ratio, $\kappa$, is 3 years.

The formulation assures that capacity equals desired output when both capital stocks equal their desired levels and that the equilibrium aggregate lifetime of capital and equilibrium aggregate capital-output ratio equal the original parameters in the one-sector model, $\tau$ and $\kappa$, respectively. Hence in equilibrium we have

$$\frac{\sum_{i,j} K_{ij}}{\sum_{i,j} K_{ij}/\tau_j} = \tau; \qquad \frac{\sum_{i,j} K_{ij}}{\sum_i x_i} = \kappa. \qquad (19)$$

Furthermore, parameters in the decision rules were scaled to the average lifetime of capital produced by that sector. When there is no coupling between the sectors ($\alpha = 0$), one sector is thus simply a time-scaled version of the other. We felt that this approach was the cleanest way to investigate the coupling of two oscillators with different inherent frequencies. Thus, the parameters, in years, are

$$\tau_i^K = \tau^K\frac{\tau_i}{\tau};\ \tau_i^S = \tau^S\frac{\tau_i}{\tau};\ \delta_i = \delta\frac{\tau_i}{\tau} \qquad (20)$$

where

$$\tau^K = 1.5;\ \tau^S = 1.5;\ \delta = 1.5. \qquad (21)$$

*Figure 3*

Figure 3 shows a simulation of the limit cycle of the one-sector model ($\alpha = 0, \Delta\tau = 0$). Even with the modifications we have introduced, the behavior of our model is virtually indistinguishable from that of the original model (Sterman 1985). With the above parameters, the equilibrium point is unstable, and the system quickly settles into a limit cycle with a period of approximately 47 years. Each new cycle begins with a period of rapid growth, where desired output exceeds capacity. The capital sector is thereby induced to order more capital, which, by further swelling order books, fuels the upturn in a self-reinforcing process. Eventually, capacity catches up with demand, but at this point it far exceeds the equilibrium level. The self-ordering process is now reversed, as falling orders from the capital sector lead to falling demand, which further depresses the capital sector's orders. Consequently, output quickly collapses to the point where only the exogenous goods sector places new orders. A long period of depression follows, during which the excess capital is gradually depleted, until capacity finally reaches demand. At this point, however, the sector raises orders enough to offset discards, increasing orders above capacity, and initiating the next cycle.

# 4 Simulation results

To explore the robustness of the single-sector model to differences in the paramters governing the individual sectors, we now simulate the model where some parameters differ between the two sectors. In spite of its simplicity, the model contains a considerable number of parameters that might differ from sector to sector. In the present study, we vary the difference in capital lifetimes $\Delta\tau$ for different values of the coupling parameter $\alpha$. As described in Section 3, we have scaled all other parameters with the capital-lifetime parameters in such a way that, when $\alpha = 0$ each sector is simply a time-scaled version of the original one-sector model.

In the simulations that follow, sector 1 is always the sector with the longest lifetime of its capital output, corresponding to such industries as housing and infrastructure, while sector 2 has the shortest lifetime parameter, corresponding to the equipment sector. Introducing a coupling between the sectors will, apart from linking the behavior together, also change the stability properties of the individual sectors, taking the other sector as exogenous. Thus, a high value of the coupling parameter $\alpha$ implies that the strength of the capital self-ordering loop in any sector is small. In the extreme case $\alpha = 1$, each sector will not order any capital from itself. If the delivery delay of capital from the other sector is taken as exogenous and constant, the behavior of the individual sector changes to a highly damped oscillation. Indeed, a linear stability analysis around the steady-state equilibrium of the individual sector shows that the equilibrium becomes stable for sufficiently high values of $\alpha$. As will become evident below, this stability effect of the coupling parameter has significant effects on the mode-locking behavior of the coupled system.

As long as the parameters of the two sectors are close enough, we expect synchronization (or 1 : 1 frequency locking) to occur, i.e., we expect that the different cycles generated by the individual sectors will adjust to one another and exhibit a single aggregate economic long wave with the same period for both sectors. The stronger the coupling, $\alpha$, the stronger the forces of synchronization are expected to be.

*Figure 4*

As an example of such synchronization, Figure 4 shows the outcome of a simulation performed with a difference in capital lifetimes between the two sectors of $\Delta\tau = 6$ years and a coupling parameter $\alpha = 0.25$. The two sectors, although not quite in phase, have identical periods of oscillation. The larger excursions in production capacity are found for sector 2 (the "machinery" sector) which is also the sector that leads in phase. The lifetime difference, $\Delta\tau = 6$ years corresponds to a lifetime for machinery capital of 17 years and a lifetime of buildings and infrastructure of 23 years. If, with the same coupling parameters, the difference in capital lifetimes is increased to $\Delta\tau = 9$ years, we observe a doubling of the period. The two sectors now alternate between high and low maxima for their production capacities. This type of behavior is referred to as a 2 : 2 mode. It has developed out of the synchronous 1 : 1 mode through a period-doubling bifurcation (Feigenbaum 1978). The 2 : 2 solution is illustrated in Figure 5. Again, both the temporal variations of the two production capacities and the corresponding phase plot are shown. In the phase plot, the stationary solution now performs two loops before closing precisely on itself.

*Figure 5*

As the difference in lifetimes is further increased, the model passes through a Feigen baum cascade of period-doubling bifurcations (4 : 4, 8 : 8, etc.) and becomes chaotic at approximately $\Delta\tau = 10.4$ years. Figure 6 shows the chaotic solution generated when $\Delta\tau = 10.7$ years. Calculation of the largest Lyaponov exponent (Wolf 1986) confirms that the solution in Figure 6 is chaotic. We conclude that deterministic chaos can arise in a macroeconomic model which in its aggregated form supports self-sustained oscillations, if the various sectors, because of differences in parameter values, fail to synchronize.

*Figure 6*

A more detailed illustration of the route to chaos is provided by the bifurcation diagram in Figure 7. Here, we have plotted the maximum production capacity attained in sector 1 over each cycle as a function of the lifetime difference $\Delta\tau$. The difference in capital lifetimes spans the interval $0 \leq \Delta\tau \leq 30$ years. When $\Delta\tau = 30$, the lifetime of the short-lived capital stock is just 5 years, while the lifetime of the long-lived capital stock is 35 years. The coupling parameter is kept constant and equal to 0.2. Inspection of the figure shows that the 1:1 frequency locking, in which the production capacity of sector 1 reaches the same maximum in each long-wave upswing, is maintained up to $\Delta\tau \approx 6.4$ years, where the first period-doubling bifurcation occurs. In the interval 6.4 years $< \Delta\tau <$ 8.0 years, the long-wave upswings alternate between a high and a low maximum. Hereafter follows an interval up to approximately $\Delta\tau = 8.1$ years with 4:4 locking, an interval with 8:8 locking, etc. Within the interval approximately $8.2 < \Delta\tau < 12.4$ small windows of periodic motion are visible between regions of chaos, a commonly observed behavior. In the region around

$12.4 < \Delta\tau < 13.0$ chaos gives way to the 2 : 3 mode-locked solution and the associated period doubling cascade 4 : 6, 8 : 12, etc. Another region of chaotic behavior follows until about $\Delta\tau \approx 15.2$ where the system locks into 1 : 2 motion. Similarly the regions of 1 : 3 and 1 : 4 entrainment are clearly visible as $\Delta\tau$ continues to increase. Note that the 1 : 4 region bifurcates into 2 : 8 around $\Delta\tau \approx 27.6$ years but then returns to 1 : 4 motion at $\Delta\tau \approx 28.3$ rather than cascading through further doublings to chaos.

*Figure 7*

However, in these intervals, other stationary solutions may exist as well, and these solutions may be reached from different sets of initial conditions. The phase diagram in Figure 8 gives an overview of the dominant modes for different combinations of the lifetime difference $\Delta\tau$ and the coupling parameter $\alpha$. The zones of mode-locked (i.e., periodic) solutions in this diagram are referred to as Arnol'd tongues (Arnol'd 1965). Besides the 1 : 1 tongue, the figure shows a series of 1 : $n$ tongues, i.e., regions in parameter space where the buildings industry completes precise 1 long-wave oscillation each time the machinery industry completes $n$ oscillations. Between these tongues, regions with other commensurate wave periods may be observed. An example is the 2 : 3 tongue found in the area around $\alpha = 0.15$ and $\Delta\tau = 12$ years.

*Figure 8*

Similar to the 2 : 2 period-doubled solution on the right-hand side of the 1 : 1 tongue, there is a 2 : 1 period-doubled solution along part of the right-hand edge of the 1 : 2 tongue. It is likely that similar phenomena may be found along the edges of the 1 : 3 and 1 : 4 tongues, etc., producing a fractal, self-similar structure. However, at present we have not yet investigated this structure in detail.

The phase diagram in Figure 8 also reveals that the synchronous 1 : 1 solution extends to the full range of the lifetime differences $\Delta\tau$ for sufficiently high values of the coupling parameter $\alpha$. When $\alpha$ is large enough, the equilibrium of the individual sectors becomes stable, when the delivery delay and demand from the other sector is taken as exogenous. For reference, two curves have been drawn in Figure 8, defining the regions in which one or both of these individual equilibria are stable: For a given value of the lifetime difference, values of $\alpha$ above the curve result in a stable individual sector equilibrium. As $\alpha$ increases, the overall behavior is more and more derived from the coupling between the sectors, and less and less from the autonomous self-ordering mechanism in each individual sector. Thus, for high values of $\alpha$, there is less competition between the two individual, autonomous oscillations, and stronger synchronization. For large differences in capital lifetimes and low values of the coupling parameter $\alpha$, the short-lived sector (sector 2) completes several cycles for each oscillation of the long-lived sector (sector 1). However, as $\alpha$ is increased, the short-term cycle is reduced in amplitude, and, for sufficiently high $\alpha$'s, it disappears altogether, resulting in a synchronous 1 : 1 solution.

The locally stabilizing effect of high values of $\alpha$ creates a complicated distortion of the Arnol'd tongues in Figure 8. For instance, the figure reveals that both the 1 : 1 region and the 2 : 2 region are folded down above the other regions for high values of $\alpha$.

Moreover, it appears that the phase diagram contains routes to chaos other than period-doubling bifurcations. In particular, the region between the 2 : 2 tongue and the 1 : 2 tongue should be explored. In particular, the areas of high coupling strength will likely contain overlapping solutions: where initial conditions or random shocks determine which solution is chosen.

# 5 Conclusions

By employing only a single capital-producing sector, the simple long wave model represents a simplification of the structure of capital and production. In reality, "capital" is composed of diverse components with different characteristics. We have focused on the difference in the average lifetime of capital, and it is clear from our analysis that a disaggregate system with diverse capital lifetimes exhibits a much wider variety of fluctuations. For moderate differences in parameters between the sectors, the coupling between sectors has the effect of merging distinct individual cycles into a more uniform aggregate cycle. The period of the cycle remains in the 50-year range, although the amplitude may vary greatly from one cycle to the next. The behavior of the two-sector model thus retains the essential features of the simple model and is robust to relaxation of the aggregation of all firms into a single sector.

Entrainment in the disaggregated model arises only via the coupling introduced by the input-output structure of capital production. Other sources of coupling were ignored. The most obvious links are created by the price system. If, for instance, one type of capital is in short supply, one would expect the relative price of that factor to rise. To the extent that sectors can substitute one type of capital for another, one would expect demand for the relatively cheaper capital components to rise. Thus, the price-system will cause local imbalances between orders and capacity across the sectors to equalize thus bringing the individual sectors into phase. (We have performed a few preliminary simulations of a version of the model that includes a price system, and these simulations show an increased tendency for synchronization.) The degree of substitution between capital types in the production function may well be an important factor: One would expect high elasticities of substitution to yield stronger synchronization. The next step in our work therefore involves introducing relative prices and differing degrees of substitution.

In light of the coupling effect of the price system and of other macroeconomic linkages, (e.g. the Keynesian consumption multiplier,) we expect disaggregate models to show a coherent long-wave motion for a wide range of parameter values, and the basic validity of the simple one-sector model seems intact. Thus, the fact that the simple model lumps all capital types into a single aggregate factor produced by a single aggregate sector is not a cause for doubts about the theory.

Another, more immediate extension of our study would involve looking at more than two sectors. On the one hand, a wider variety of capital producers would introduce more variability in the behavior and, hence, less uniformity. On the other hand, as the system is disaggregated further, the strength of the individual self-ordering loops within sectors is reduced to near zero, and overall cycles will more and more arise from the interaction between sectors. Stronger intersectoral coupling leads to stronger entrainment and more uniform behavior.

The results demonstrate the importance of studying non-linear entrainment in the economy. The intricacies of such phenomena suggest that there is a vast unexplored domain of research in the area of economic cycles. We suggest that non-linear entrainment plays a larger role in shaping economic cycles than the external random shocks on which much of mainstream business cycle theory relies.

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# Appendix:
# Equilibrium Conditions for the Two-Sector Model

The equations in the text completely describe the structure of the model. The following equations give the equilibrium conditions for the model. In the simulations reported above the equilibrium is unstable. To perturb the system a small change in the exogenous consumer demands, $y_i$, are introduced, and the simulations are run long enough for any transients to die out.

$$B_i = \delta_i \frac{y_i}{(1.0 - \kappa/\tau)}$$

$$K_{ij} = \kappa_{ij} x_i^*$$

$$S_{ij} = s_{ij}^*$$

$$y_i = y_j = 1.0, \qquad i \neq j$$

Figure 1: Torus representation of periodic and quasi-periodic behavior. Periodic motion arises whenever there is a rational ratio between the two periodicities. Quasiperiodic motion arises when the periodicities are incommensurate.

Figure 2: Mode-locking in the sine circle map. Mode-to-mode interaction gives rise to a nonlinear variation in the map $\theta_n \rightarrow f(\theta_n)$. Because of this nonlinearity, fixed points will exist over finite ranges of the parameter $\Omega$.

Figure 3: Simulation of the one-sector model. The steady state behavior is a limit cycle with a period of approximately 47 years. The plot shows production capacity, production, and desired production of capital equipment, respectively. All variables are shown on the same scale.

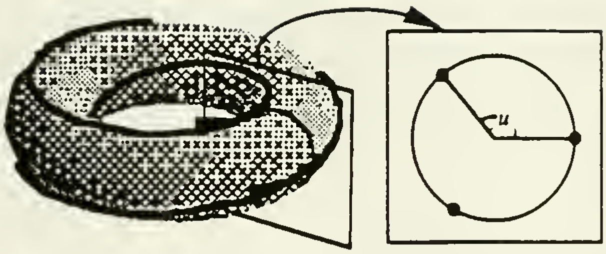
Figure 4: Synchronization (1 : 1 mode-locking) in the coupled two-sector model. The figure shows the capacity of the two sectors as a function of time in the steady state. The difference in capital lifetimes $\Delta\tau$ is 6 years (i.e., the lifetime of capital types 1 and 2 is 23 and 17 years, respectively). The coupling parameter $\alpha$ is 0.25 in this and the following three figures. Due to the nonlinear coupling, the two sectors are locked into a single cycle, i.e., the mode-locking ratio is 1 : 1.

Figure 5: Period doubling (2 : 2 mode-locking) resulting from increased lifetime difference $\Delta\tau$. As the difference in capital lifetimes $\Delta\tau$ is increased to 9 years, the 1 : 1 mode is replaced by an alternating pattern of smaller and larger cycles so that the total period is doubled. As in the previous figure, the coupling parameter $\alpha$ is 0.25. The two sectors still complete an equal number of cycles in any interval of time, i.e., the mode-locking ratio is 2 : 2.

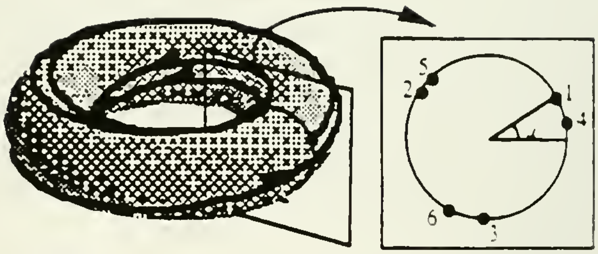
Figure 6: Chaotic behavior. As the difference in capital lifetimes $\Delta\tau$ is increased further, the model exhibits a progression of period doublings, which at some point become infinitely dense. Just beyond this point, as in this figure where $\Delta\tau$ is 10.7 years, the behavior is chaotic. (The coupling parameter $\alpha$ is still 0.25.) The model shows no regular periodic behavior, and initial conditions close to each other quickly diverge so that, in practice, the behavior is unpredictable. Nonetheless, the two sectors remain locked together with a ratio of unity between their periods.

Figure 7: Bifurcation diagram for increasing lifetime difference $\Delta\tau$ and constant coupling $\alpha$. The figure shows the local maxima attained for the capacity of sector 1 (the longer-lived capital producer) in the steady-state behavior for varying values of the lifetime difference $\Delta\tau$. The coupling parameter $\alpha$ is held constant at 0.2. For a given $\Delta\tau$, a single value in the diagram indicates a uniform limit cycle; two-values indicate a period doubling with a smaller and larger cycle, etc. In chaotic regions, the number of local maxima is infinite since no individual cycles are identical.

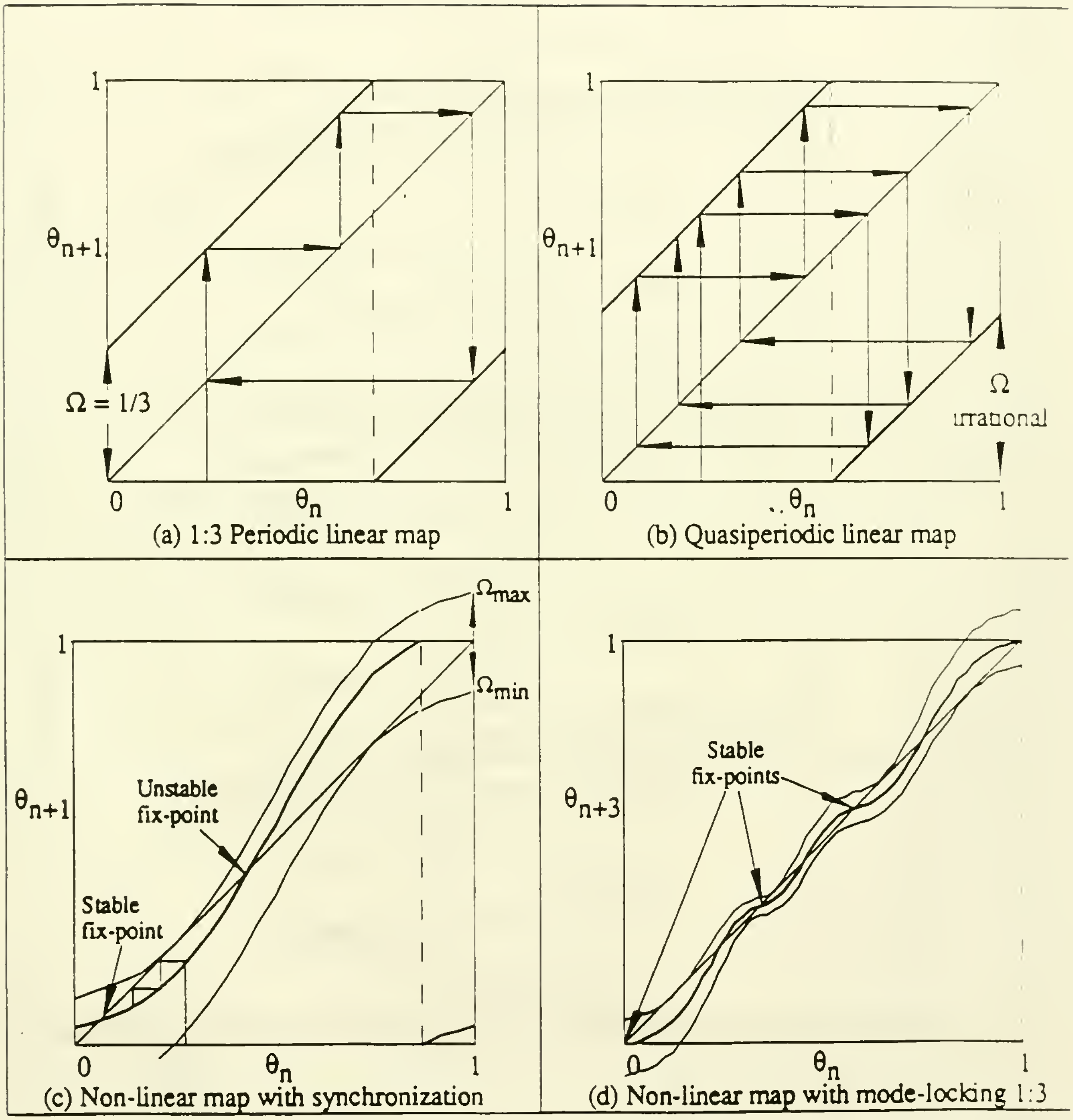
Figure 8: Parameter phase diagram. The figure summarizes the steady-state behavior of the two-sector model for different combinations of the coupling parameter $\alpha$ and the lifetime difference $\Delta\tau$. A region labeled "$p : q$" indicates the area in parameter space where the model shows periodic mode-locked behavior of $p$ cycles for sector 1 and $q$ cycles for sector 2. (However, other solutions may coexist at the same point in the diagram, depending on the initial conditions of the model.) The question mark indicates that the details of the diagram are still under exploration. In particular, regions of chaotic behavior have not yet been outlined in detail. The dashed curves across the diagram indicate the value of above which each sector in isolation (with the other sector treated as exogenous) becomes stable. Above these lines the cycles are created solely by the interaction of the two sectors, implying that, for large $\alpha$, synchronous behavior becomes more and more prevalent.



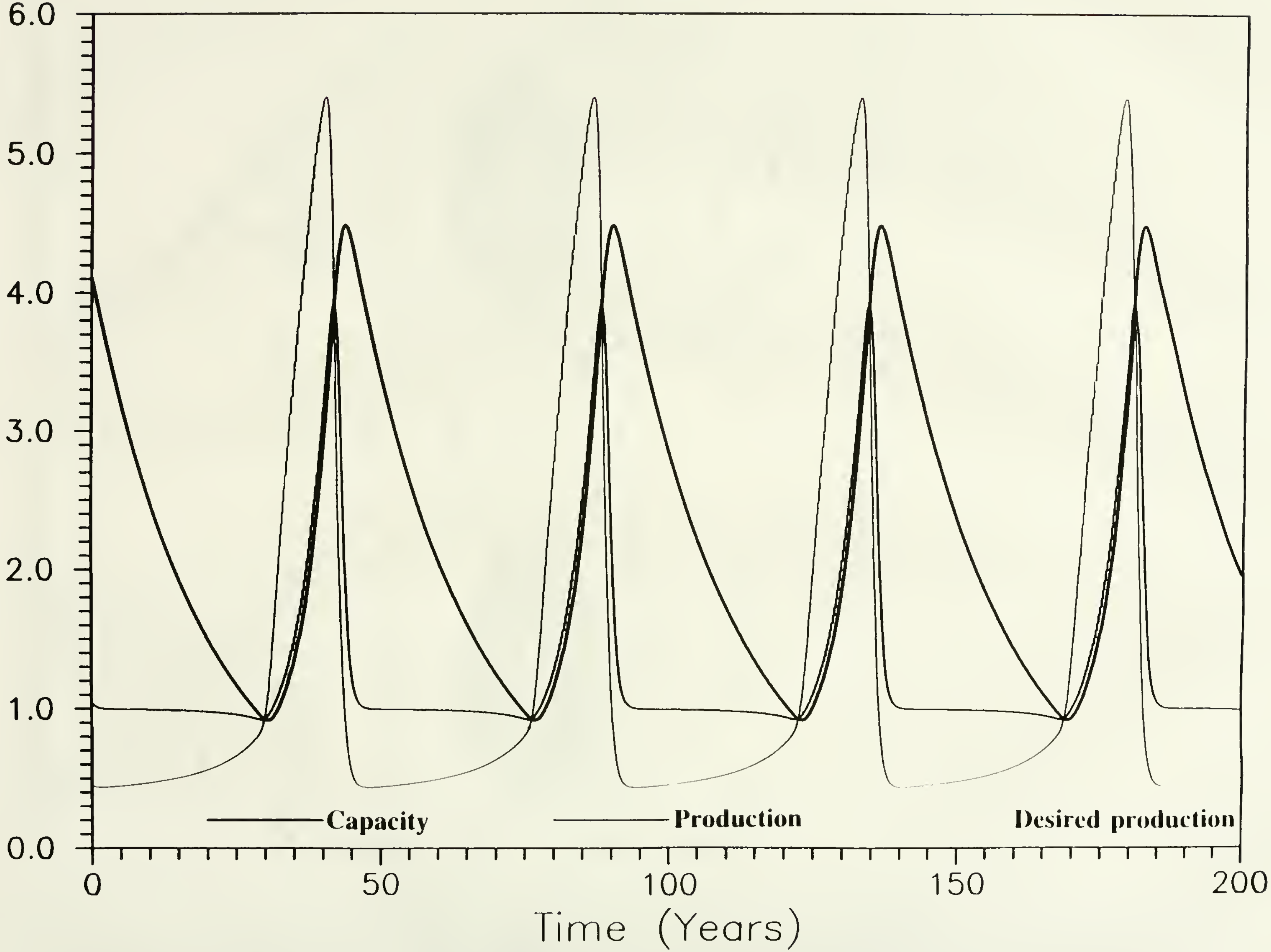

(a) Periodic motion (1:3)

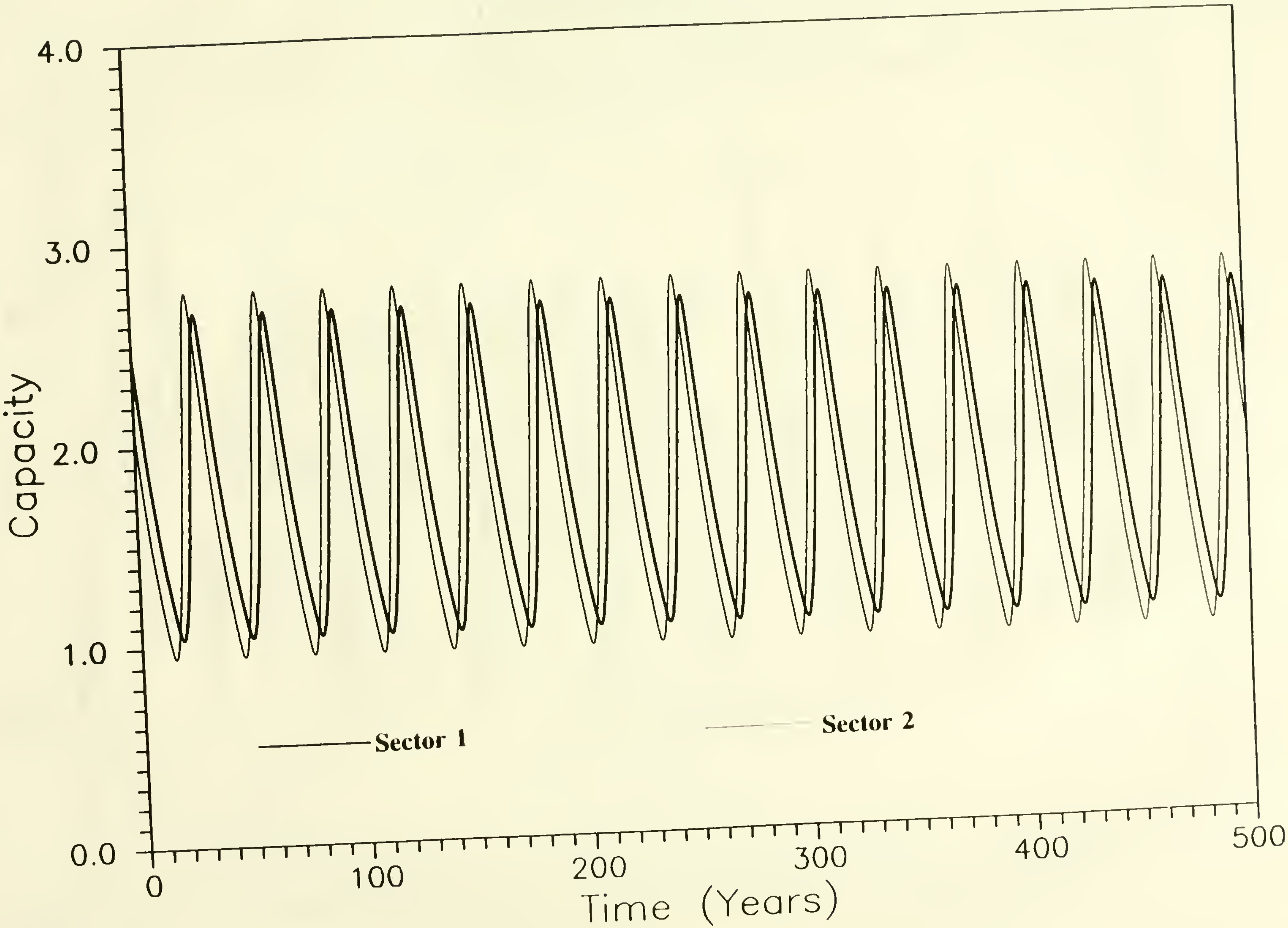


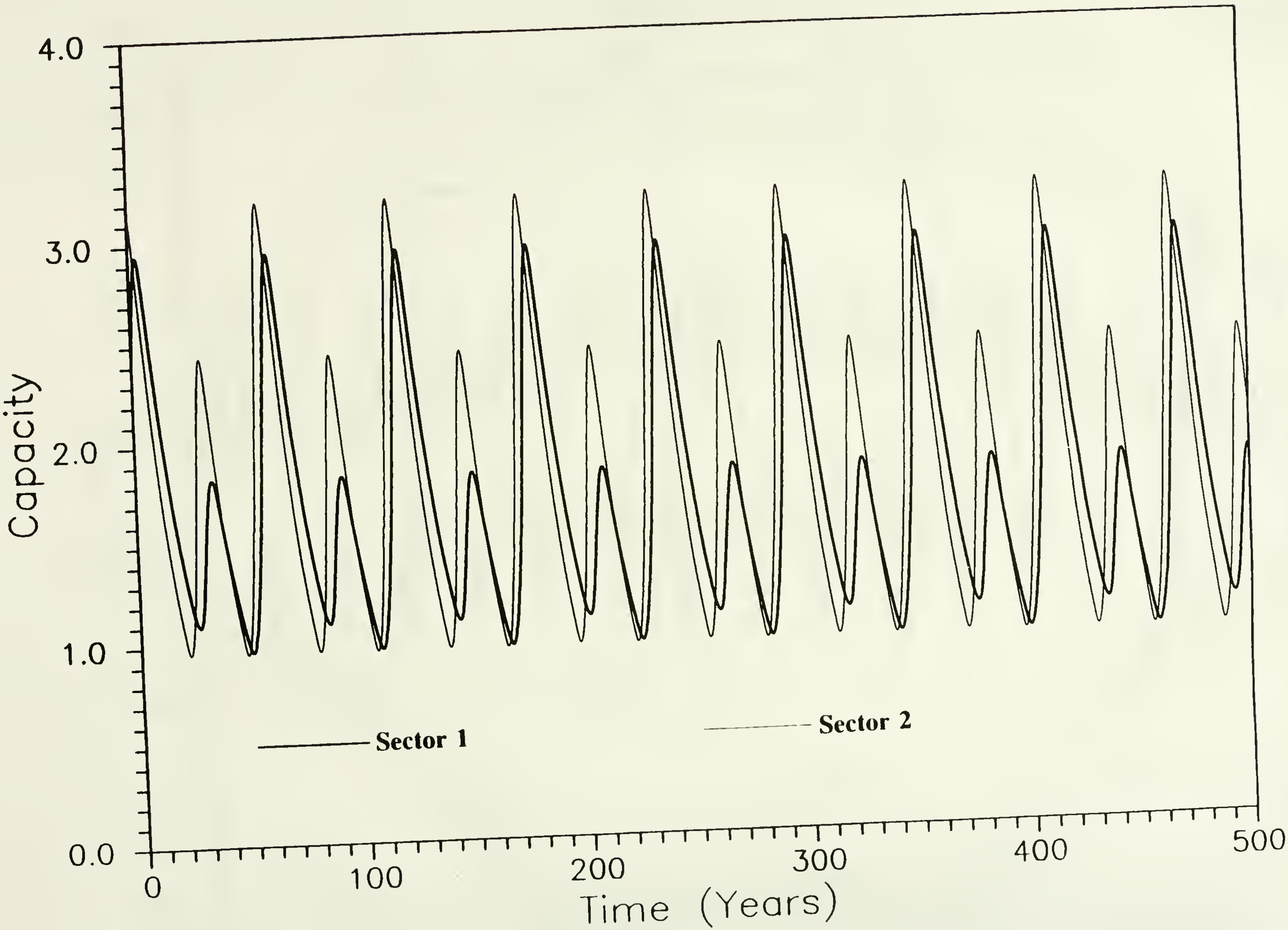

(b) Quasiperiodic motion (close to 1:3)



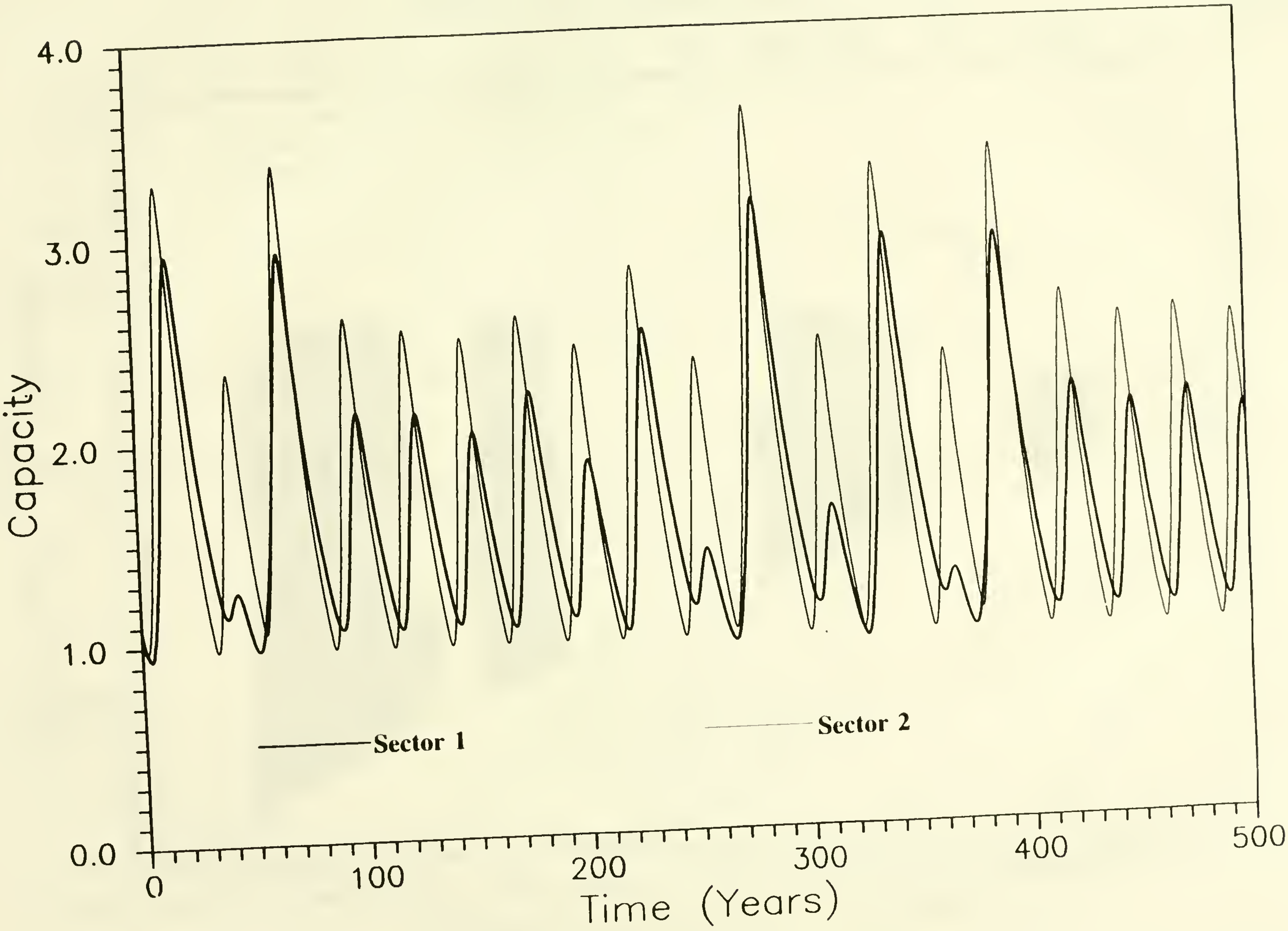
1
θn+1
Ω = 1/3
0
θn
1
(a) 1:3 Periodic linear map
1
θn+1
Ω
irrational
0
θn
1
(b) Quasiperiodic linear map
Ωmax
Ωmin
1
θn+1
Unstable
fix-point
Stable
fix-point
0
θn
1
(c) Non-linear map with synchronization
1
θn+3
Stable
fix-points
0
θn
1
(d) Non-linear map with mode-locking 1:3

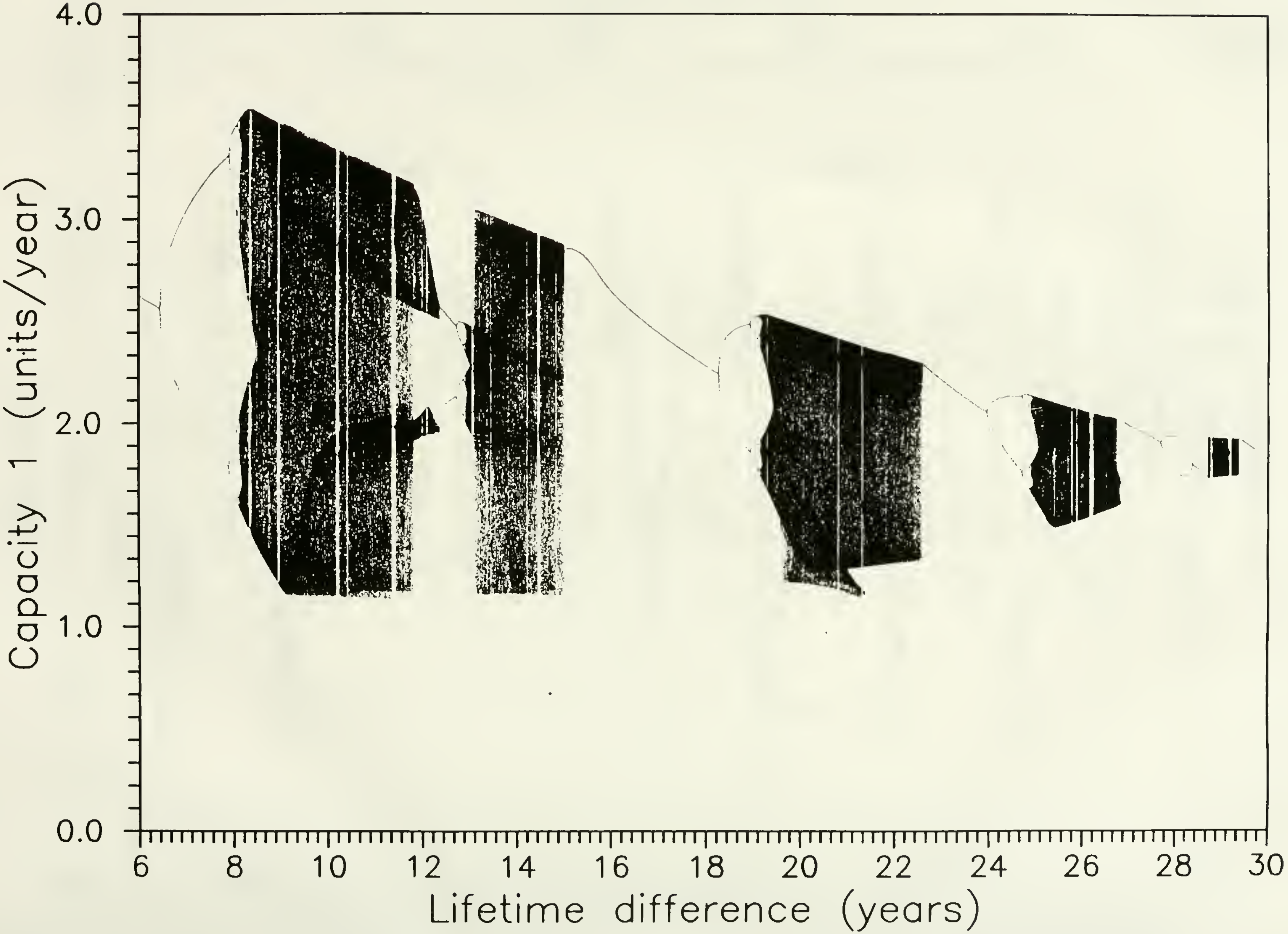

6.0
5.0
4.0
3.0
2.0
1.0
0.0
0
50
100
150
200
Time (Years)
Capacity
Production
Desired production

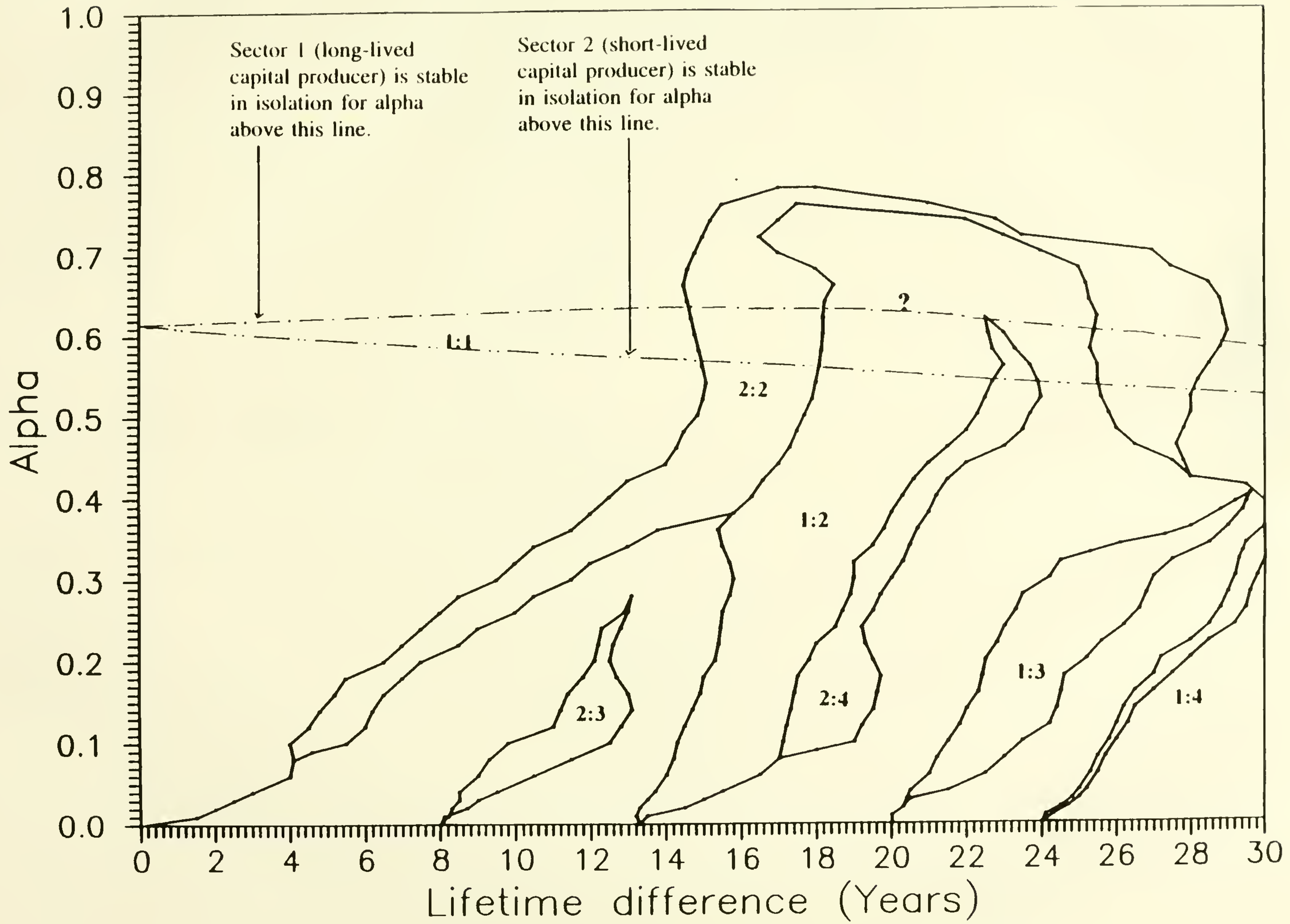

4.0
3.0
2.0
1.0
0.0
Capacity
0
100
200
300
400
500
Time (Years)
Sector 1
Sector 2



4.0
3.0
2.0
1.0
0.0
Capacity
0
100
200
300
400
500
Time (Years)
Sector 1
Sector 2


Capacity
4.0
3.0
2.0
1.0
0.0
0
100
200
300
400
500
Time (Years)
Sector 1
Sector 2


Capacity 1 (units/year)
0.0
1.0
2.0
3.0
4.0
6
8
10
12
14
16
18
20
22
24
26
28
30
Lifetime difference (years)


Sector 1 (long-lived capital producer) is stable in isolation for alpha above this line.
Sector 2 (short-lived capital producer) is stable in isolation for alpha above this line.
1:1
2
2:2
1:2
2:3
2:4
1:3
1:4
Alpha
Lifetime difference (Years)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30