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PARTICLE DYNAMICS IN THE LINEAR ACCELERATOR

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Abstract

Hamilton's equations for the motion of an electron in a linear accelerator are integrated to find the final kinetic energy and phase of an electron injected with arbitrary initial kinetic energy and phase, after traveling down a fixed length of accelerator. The results are expressed in the form of a map of the initial energy-phase space onto a final energy-phase space. This map proves in practice to be very convenient for discussing the actual operation of the accelerator. The curves presented are calculated for numerical values appropriate for the M. I. T. accelerator.

PARTICLE DYNAMICS IN THE LINEAR ACCELERATOR

It has been shown (1) that the longitudinal motion of an electron in a linear accelerator is governed by Hamiltonian equations derived from the Hamiltonian function

$$H = \left(m_o^2 c^4 + p^2 c^2\right)^{1/2} - p v_o - \epsilon m_o c^2 \cos \phi \quad . \tag{1}$$

Here p is the relativistic momentum $m_o v / (1 - v^2/c^2)^{1/2}$. The term $(m_o^2 c^4 + p^2 c^2)^{1/2}$ is the electron's kinetic energy (including rest energy). The electron is assumed to be moving in a traveling wave whose longitudinal electric field is $E \sin \omega(t - z/v_o)$. The quantity ϵ is an abbreviation for $eEv_o/\omega m_o c^2$. The phase ϕ is an abbreviation for $\omega z'/v_o$, where z' is equal to $z - v_o t$, and measures the electron's position with respect to the traveling wave. It is the purpose of this note to put the dynamics of such electronic motion in a form most useful for discussing the actual operation of the linear accelerator.

The Hamiltonian relation (Eq. 1) can be conveniently discussed by setting up a phase space, in which the coordinate z' (or the phase ϕ) is the abscissa, and the momentum p the ordinate, and in which the lines of constant H are plotted. Such a plot is given in Fig. 53 of Ref. 1 for the case $v_o = c/2$, and in Fig. 54 of Ref. 1 for the case $v_o = c$. We give a similar plot in our Fig. 1, which 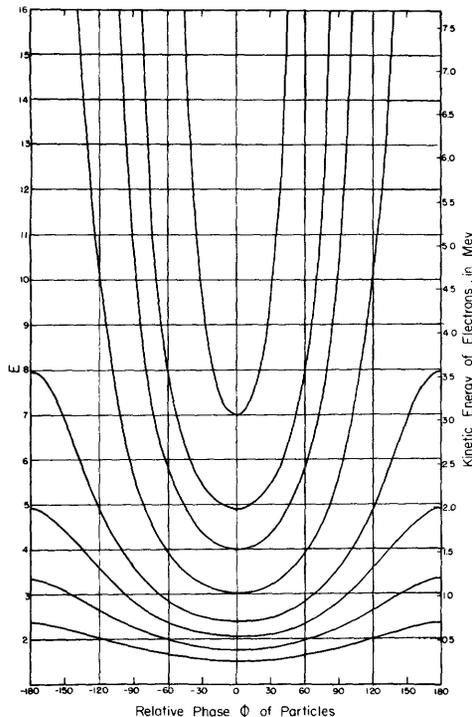differs from the phase space just described in that we use the kinetic energy of the electron as ordinate rather than the momentum; in the relativistic range these are proportional to each other, but there are differences at low energy. The curves of Fig. 1 are computed for approximately the numerical values appropriate for the M. I. T. linear accelerator, for which $v_o = c$, the resonant wavelength $2\pi c/\omega$ is 10.70 cm, and E is about 24,000 ev/cm, giving $\epsilon = 0.08$. The significance of this figure is then simple: a given electron follows along a line of constant H. Thus if we know that at an initial instant the electron is found with a given kinetic energy and given phase, we can locate a point on the graph corresponding to this energy and phase, find the contour of constant H passing through that point, and can then read off the subsequent energies and phases which the particle will attain: it will travel in a clockwise direction along a contour, slipping backward in phase as compared to the wave (since it cannot travel with quite the velocity of light),

Fig. 1.

Lines of constant H
in phase space

losing energy when its phase is positive, gaining it when its phase is negative. There is a range of injection energies and phases for which the contours eventually rise to infinity, so that the energy of an electron injected in such a phase will eventually increase indefinitely, provided the accelerator is long enough.

This graphical relation gives considerable information, but by no means all we wish for discussing the accelerator: it does not tell us how long it takes for an electron to travel a given distance along a contour of constant energy. We must use Hamilton's equations directly to get this additional information. In the M.I.T. accelerator, electrons are injected all at one energy (actually 2 Mev), and uniformly distributed over phase. We are interested in the energy distribution of the electrons emerging after traversing a fixed length of accelerator. Hamilton's equations of course determine the complete motion of an electron; thus they tell us how rapidly an electron moves around an energy contour in Fig. 1, and hence how far it will move in traversing a given length of accelerator. In other words, if we know the initial energy and phase of an electron on entering a given length of accelerator, Hamilton's equations allow us to find the energy and phase after traversing the length of accelerator. That is, the operation of traveling down the length of accelerator maps any point of the space of Fig. 1 onto another point lying on the same energy contour. Such a map can be exhibited graphically by showing the distorted network of lines into which a rectangular network is mapped by traversing the accelerator. We show such a map in Fig. 2, computed for the same accelerating field as given in Fig. 1, and for an accelerator eight wavelengths long (this is the length of one of the sections of the M.I.T. accelerator; the complete accelerator consists of seven such sections). One set of lines of this distorted network is labelled with the initial energy of the electrons, the other set with the initial phase. In the high
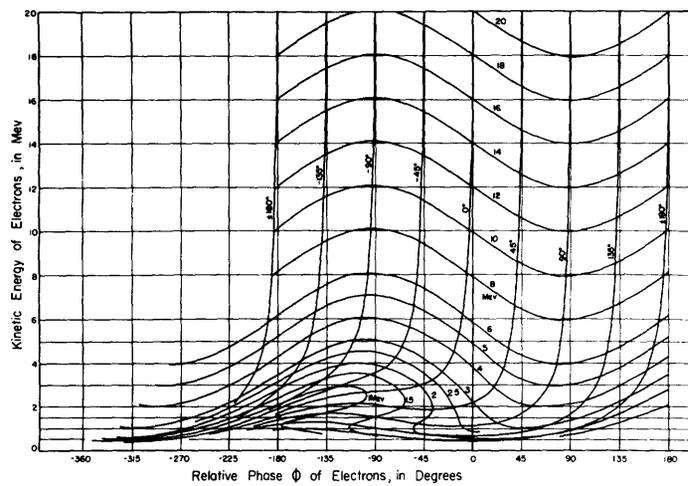

Fig. 2.

Map of final energy and phase of electrons after traversing a section of accelerator, in terms of initial energy and phase.

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energy range, we see that the initial phase and final phase almost exactly coincide (since here the electrons are traveling with almost exactly the velocity of light, and cannot slip backward in phase), and the change of energy in the accelerator is almost exactly a sinusoidal function of phase, giving an increase of about 2 Mev at the phase -90° (the maximum accelerating field), and an equal decrease of energy at +90°. At the lower energies, the situation is much more complicated; the electrons injected at positive phase angles, which as we see from Fig. 1 initially are slowed down to considerably less than the velocity of light, slip backward considerably in phase, and end up with much less energy than they had on entering.

The diagram of Fig. 2 can be used in a very simple and direct way to discuss the whole behavior of the accelerator. For the first section, injection is at 2 Mev, with equal numbers of electrons in equal intervals of phase. Thus after emergence from this section, electrons will be distributed along the line marked 2 Mev in Fig. 2, with equal numbers in each interval of equal phase, indicated by the contours of constant phase. We see by inspection of the graph that electrons initially injected between phases of 0° and -135° will have energies equal to or greater than 3 Mev after acceleration, with some electrons having as much as 4 Mev. Furthermore, these electrons will be distributed in final phase from about -45° to -180°. We can now take the distribution as it emerges from this section, take the energy and phase of any one of its electrons as the initial conditions for the next section, and follow the progress of the electrons and their distribution through the second section. This can be continued for all seven sections. Such an analysis of the behavior of the electron beam, with experimental verification, is presented in Technical Report No. 203 by P. T. Demos, A. F. Kip and J. C. Slater (2).

The calculation of Fig. 2 was made as follows. We start with the equation $z' = z - ct$, and differentiate to get

$$dz' = dz - c\,dt \quad . \tag{2}$$

Furthermore, from Hamilton's equations, we have

$$\frac{dz'}{dt} = \frac{\partial H}{\partial p} = pc^2 \left(m_o^2 c^4 + p^2 c^2\right)^{-1/2} - c \quad .$$

This allows us to write dt in terms of z'. We do this, substitute in Eq. 2, and find

$$dz = dz' \left[\frac{pc}{pc - \left(m_o^2 c^4 + p^2 c^2\right)^{1/2}} \right] = -dz' \left[\frac{pc}{H + m_o c^2 \epsilon \cos \phi} \right] \tag{3}$$

where we have made use of Eq. 1. We can now solve Eq. 1 for pc, and find

$$pc = \frac{m_o^2 c^4 - (H + m_o c^2 \epsilon \cos \phi)^2}{2(H + m_o c^2 \epsilon \cos \phi)} \quad . \tag{4}$$

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We substitute Eq. 4 in Eq. 3, set $\phi = \omega z'/v_o$, and have

$$\frac{2\omega}{v_o} dz = 1 - \left(\frac{H}{m_o c^2 + \epsilon \cos \phi}\right)^{-1/2} d\phi \quad . \tag{5}$$

This equation can be integrated, to give the distance z traveled by the electron, as a function of the final phase ϕ, and initial phase ϕ_o. The integration can be performed analytically, but the resulting expression is so complicated that it has proved simpler in practice to integrate numerically. When the relation between ϕ and z is thus established, for each H value, we need only substitute in Eq. 1 to find the kinetic energy $(m_o^2 c^4 + p^2 c^2)^{1/2}$ after traversing the required distance. Thus we know the final phase and energy in terms of the initial phase and energy, and are able to construct the curves of Fig. 2.

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References

1. J. C. Slater: Microwave Electronics, p. 262, Van Nostrand, 1950
2. P. T. Demos, A. F. Kip, J. C. Slater: The M.I.T. Linear Accelerator, Technical Report No. 203, Research Laboratory of Electronics, M.I.T. 1951