A Note on the Valuation of Stochastic Cash Flows

Rajnish Mehra
Sloan School of Management
Massachusetts Institute of Technology
and
University of California
Santa Barbara

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Abstract

This paper considers the implications of the equilibrium asset pricing model developed by Lucas (1978) for the valuation of risky cash flows. It is shown that even in the context of the simple economy studied, discounting stochastic cash flows becomes computationally complex and of limited practical use.

1. Introduction: The paper by Lucas (1978) "Asset Prices in an Exchange
Economy" initiated a paradigmatic change in the Theory of Equilibrium
Pricing of Risky Assets. It was followed by several papers including
those by Brock (1979 and 1982), Prescott and Mehra (1980) and Donaldson
and Mehra (1984) who generalize Lucas' model to a production setting.
Research in the area of multiperiod valuation of risk assets has predomi-
nantly focussed on the conditions necessary to generalize the single
period partial equilibrium model of Sharpe (1964), Lintner (1965) and
Mossen (1966) to a multiperiod context.[1] In addition to the papers cited
above, related papers include the work of Bhattacharya (1981), Breeden
(1979), Cox, Ingersol and Ross (1985), Fama (1977), Mehra and Prescott
(1985), Merton (1973), Rubinstein (1976) and Stapelton and Subrahmanyam
(1978).

   This paper considers the logical implications of the equilibrium
asset pricing model of Lucas (1978) for the valuation of risky cash
flows. Since this paper is extensively quoted in the Finance and
Economics Literature, it seems appropriate that its implications be
systematically analysed to further its understanding.

   The paper is pedagogical in nature since much of the analysis such
as the existence, uniqueness and optimality of equilibrium has been
developed earlier.

   The paper is divided into four sections. Section 2 briefly de-
scribes the economy under consideration. The valuation formulae are
developed in section 3 while in section 4 we provide some concluding
comments.

II. The Economy: We consider a pure exchange economy of the type considered in Lucas (1978). As shown in Prescott and Mehra (1980), example 2, such an economy can be cast as a recursive competitive equilibrium. We have deliberately abstracted from the more general structures considered by Brock (1979 and 1982) and Prescott and Mehra (1980) which allow for capital accumulation since they are not necessary to establish the results in this paper.

A brief sketch of the economy follows. The reader is referred to Lucas (1978) for details.[2]

The economy has a single representative "stand in" household. This agent orders its preferences over random consumption paths by

$$(1) \qquad E\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \}$$

where c_t is a stochastic process representing per capita consumption, β is the subjective discount factor, $0 < \beta < 1$, $E\{\cdot\}$ is the expectation operator and u: $R_+ \to R$ is the current period utility function. The consumption good is produced by n firms. Let x_{it} be the output of firm i in period t. Production is entirely "exogenous." Let $X_t = \{x_{1t}, x_{2t} \cdots x_{nt}\}$ be the output vector at time t. X_t is assumed to follow a Markov process defined by its transition function

$$F(X'|X) = pr\{X_{t+1} \leq X' | X_t = X\}.$$

Each period ownership in these productive units is determined in a competitive stock market. Each unit has outstanding one perfectly divisible equity share. A share entitles its owner as of the beginning of t to all

of the unit's output in period t. Shares are traded ex-dividend at a competitively determined price vector $p_t = (p_{1t}, p_{2t} \cdots p_{nt})$.

Let $Z_t = (z_{1t}, z_{2t} \cdots z_{nt})$ denote a consumer's beginning of period share holdings. In this economy, in equilibrium, all output will be consumed ($c_t = \sum_1 x_{it}$) and all shares held, that is $Z_t = (1, 1 \ldots 1)$. If we let the price of the consumption good be the numeraire, then the output of the firms can be thought of as their "cash flows". We will use the latter interpretation in the analysis that follows.

III. <u>Valuation</u>: The price dynamics of firm i are characterized by

(2) $u'(c_t) p_1(X_t) = \beta \int u'(c_{t+1}) [x_{1,t+1} + p_1(X_{t+1})] \, dF(X_{t+1}|X_t)$

This is the celebrated stochastic Euler equation in Lucas (1978) and is sufficient to value risky cash flows from firm i. The <u>same</u> equation is obtained in models incorporating capital accumulation developed by Brock (1979 and 1982), Donaldson and Mehra (1984), Prescott and Mehra (1980) and others. These models incorporate other equations that must be consistent with (2); nevertheless, equation (2) must necessarily be satisfied in each case.

In the analysis to follow we will value the cash flow from firm i. Since there is little room for ambiguity, we will suppress the subscript i so that at any time t equation (2) may be written as

(3) $u'(c_t) p(X_t) = \beta \int u'(c_{t+1}) [x_{t+1} + p(X_{t+1})] \, dF(X_{t+1}|X_t)$

The <u>object is to solve for $p(X_t)$ which is the present value of all future cash flows $\{x_{t+1}, x_{t+2} \ldots\}$ of this firm</u>.

To this end let us recursively define the sequence of conditional distribution functions $\{F\ (X_{t+n}|X_t)\}$ according to

$$F_2(X_{t+2}|X_t) = \int_Z F(X_{t+2}|Z)\ dF\ (Z|X_t)$$

$$F_3(X_{t+3}|X_t) = \int_Z F_2(X_{t+3}|Z)\ dF\ (Z|X_t)$$

with

$$F_n(X_{t+n}|X_t) = \int_Z F_{n-1}(X_{t+3}|Z)\ dF\ (Z|X_t)$$

and hence

$$dF_n(X_{t+n}|X_t) = \int_Z dF_{n-1}(X_{t+n}|Z)\ dF\ (Z|X_t)\ .$$

Let $\Pi_{t,j}$ be the price of a riskless bond at time t paying one unit of the consumption good at time t+j. Then

$$(4) \qquad \Pi_{t,j} = E_t\{\beta^j\ \frac{u'(c_{t+j})}{u'(c_t)}\}$$

and by definition

$$(5) \qquad \Pi_{t,j} = \frac{1}{(1 + r_{t,j})^j}$$

where $r_{t,j}$ is the return on a j period pure discount bond.

To explicitly solve for $p(X_t)$ in equation (5) we use the technique of recursive substitution.

$$(6) \qquad \text{Define } g(X_t) = \beta \int u'(c_{t+1})x_{t+1}\ dF\ (X_{t+1}|X_t)$$

and

(7) $$f(X_t) = p(X_t)u'(c_t)$$

then equation (3) can be written as

(8) $$f(X_t) = g(X_t) + \beta \int f(X_{t+1}) \, dF(X_{t+1}|X_t).$$

Lucas (1978) shows that equation (8) has a unique solution $f(\cdot)$. This implies that

(9) $$p(X_t) = \frac{f(X_t)}{u'(c_t)} \qquad \text{(using equation 7)}$$

To find $f(\cdot)$, observe that the operator $T: \ C \to C$ (where C is the space of bounded continuous functions) defined by

(10) $$(Tf)(X_t) = g(X_t) + \beta \int f(X_{t+1}) \, dF(X_{t+1}|X_t)$$

is a contraction.[3] Hence for any $f_0 \in C$, $\lim_{n \to \infty} T^n f_0 = f$.

$(f_n \equiv T^n f_0 = T(T^{n-1} f_0) \ n = 1,2,3...)$. Letting f_0 be the null function we can generate a sequence of functions $\{f_1, f_2, ...\}$ using (10)

$$f_1(X_t) = g(X_t)$$

and $\qquad f_2(X_t) = g(X_t) + \beta \int f_1(X_{t+1}) \, dF \, (X_{t+1}|X_t)$

$$= g(X_t) + \beta \int g(X_{t+1}) \, dF \, (X_{t+1}|X_t)$$

$$= \beta \int u'(c_{t+1})x_{t+1} \, dF \, (X_{t+1}|X_t)$$

$$+ \beta^2 \int \int u'(c_{t+2})x_{t+2} \, dF \, (X_{t+2}|X_{t+1}) \, dF \, (X_{t+1}|X_t)$$

or $\qquad f_2(X_t) = \beta \int u'(c_{t+1})x_{t+1} \, dF \, (X_{t+1}|X_t)$

$$+ \beta^2 \int u'(c_{t+2})x_{t+2} \, dF_2 \, (X_{t+2}|X_t)$$

$$f_2(X_t) = \beta E_t\{u'(c_{t+1})x_{t+1}\} + \beta^2 E_t\{u'(c_{t+2})x_{t+2}\}.$$

Proceeding recursively f_n can be expressed as

$$f_n(X_t) = \sum_{j=1}^{n} \beta^j E_t\{u'(c_{t+j})x_{t+j}\}$$

since $f(X_t) = \lim_{n\to\infty} f_n(X_t)$ we have

(11) $$f(X_t) = \sum_{j=1}^{\infty} \beta^j E_t\{u'(c_{t+j})x_{t+j}\}$$

and using equation (9) we get

(12) $$p(X_t) = \sum_{j=1}^{\infty} \beta^j E_t\{\frac{u'(c_{t+j})}{u'(c_t)} x_{t+j}\}$$

This can be written after some simplification (see the appendix) as

$$(13) \qquad p(X_t) = \sum_{j=1}^{\infty} \left\{ \frac{\theta_{t+j} \, E_t[x_{t+j}]}{(1 + r_{t,j})^j} \right\}$$

where $\theta_{t+j} = [1 - \delta_{t+j} \, Cov_t(-u'(c_{t+j}), x_{t+j})]$

and $\delta_{t+j} = \dfrac{1}{E_t[u'(c_{t+j})] \, E_t[x_{t+j}]}$

Equation (13) exhausts the implications for valuing stochastic cash flows in Lucasian economies -- economies characterized by "representative agents" who behave optimally in light of their objectives. All prices and price distributions are endogenous and are determined through market clearing. Expectations are formed rationally; i.e., the prices and price distributions on which the economic agents base their decisions coincide with those implied by their behavior. The economy is thus "informationally efficient."

Superficially equation (13) resembles the familiar discounting formulations of "text book finance" with θ being the certainty equivalent adjustment. Its implications and applications are, however, far more complex. The adjustment term θ_{t+j} in the numerator will normally vary from period to period. It is precisely this variability in θ that makes it difficult to generalize the Sharpe (1964), Lintner (1965) and Mossin (1966) asset pricing model to a multiperiod context. The attempts at generalization in effect constraint θ to be constant, which can only be achieved by imposing restrictive conditions. In the case where $Cov(-u'(c_{t+j}), x_{t+j})$ is $\cong 0$, i.e., where the covariance of marginal utility of consumption is not correlated with the return from a project

-- perhaps the project is an insignificant component of a diversified
consumption basket -- then $\theta_{t+j} \cong 1 \ \forall \ j$. This would also be true if
the individuals were risk neutral. Equation (13) could then be written
as

$$(14) \qquad P(X_t) = \sum_{j=1}^{\infty} \left\{ \frac{E_t(x_{t+j})}{(1+r_{t,j})^j} \right\}$$

which is consistent with our intuition for risk neutral valuation.

Although equation (13) makes explicit the informational requirements
for valuing stochastic cash flows in this simple economy, we can gain
additional intuitive insights by rewriting it as

$$(15) \qquad P(X_0) = \sum_{j=1}^{\infty} \left\{ \frac{E_0(x_j) - \lambda_j \text{Cov}_0 (-u'(c_j), x_j)}{(1+r_{0,j})^j} \right\}$$

where we have set t=0. $\{r_{0,j}\}$ is the sequence of returns on pure discount
bonds of various maturities and $\lambda_j = \frac{1}{E_0[u'(c_j)]}$ is a constant for each j.

If we take a Taylor's expansion of the utility function $u(\cdot)$ around
$E(c_j) = \bar{c}_j$ and retain only the second order terms, (15) can be rewritten
as

$$(16) \qquad P(X_0) \cong \sum_{j=1}^{\infty} \left\{ \frac{E_0(x_j) - \phi_j \text{Cov}_0(c_j, x_j)}{(1+r_{0,j})^j} \right\}$$

The approximation is exact if $u(\cdot)$ is quadratic or if the analysis
is conducted in continuous time (Ross (1977)). ϕ_j is the reciprocal of

the coefficient of relative risk aversion and is a constant for each j.

Equation (16) is "similar" to the various multiperiod valuation relations in the existing finance literature. It is not the purpose of this note to critique these studies, rather it illustrates a logical and cogent methodology for the valuation of stochastic cash flows in a general equilibrium setting. Clearly the information required to conduct such an exercise in its general form is considerable. By imposing additional structure (restrictions) these requirements can be reduced. For example, it has been postulated that the coefficient of relative risk aversion is a constant;[4] if this is the case, then $\phi_j = \phi$ for all j and (16) is further simplified. Arrow and Kurz (1970) have argued that for evaluating public projects risk netural valuation is appropriate. In this case, equation (14) derived earlier would be the appropriate expression for valuation, significantly reducing the informational requirements. Clearly, other restrictions may be placed on (16) depending on the application to make it more "operational".

4. Concluding Comments

This note has examined the implications of Lucas' asset pricing model for valuing stochastic cash flows. The assumption of consumer homogeneity is admittedly an unrealistic one. However, as Prescott and Mehra (1980) point out, in a complete market setting the equilibrium process for economic aggregates and prices for a heterogeneous consumer economy will be observationally equivalent to that for some homogeneous consumer economy.[5] This provides our rationale for introducing a representative "stand-in" individual.

It should be apparent that cash flows from a finitely lived firm (project) can be valued within the framework developed above by simply defining them to zero after the project terminates.

Finally, as observed by Constantinides (1980), it is evident that even in the context of the simple economy studied here, discounting stochastic cash flows is, in general, computationally complex and of limited practical use.

Footnotes

1. For a succinct review and critique of the earlier work see Constantinides (1980). For a comprehensive survey see LeRoy (1982).

2. Conditions necessary for the existence and uniqueness of a equilibrium are also discussed in that paper.

3. Lucas (1978), pp. 1434-1435.

4. See Arrow (1971). He further argues that the coefficient of relative risk aversion should be approximately one.

5. For a formal proof see Constantinides (1982).

Appendix

Rewriting equation (12) we have

$$p(X_t) = \sum_{j=1}^{\infty} \int \beta^j \frac{u'(c_{t+j})}{u'(c_t)} x_{t+j} \, dF_j \, (X_{t+j}|X_t)$$

$$= \sum_{j=1}^{\infty} \beta^j \left\{ E_t \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) E_t(x_{t+j}) + Cov_t \left(\frac{u'(c_{t+j})}{u'(c_t)} , x_{t+j} \right) \right\}$$

$$= \sum_{j=1}^{\infty} \beta^j \left\{ E_t \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) E_t(x_{t+j}) \left[1 + \frac{cov_t \left(\frac{u'(c_{t+j})}{u'(c_t)}, x_{t+j} \right)}{E_t \left(\frac{u'(c_{t+j})}{u'(c_t)} \right) E_t(x_{t+j})} \right] \right\}$$

Substituting from equation (4) we get

$$p(X_t) = \sum_{j=1}^{\infty} \Pi_{t,j} \, E_t(x_{t+j})(1 - \delta_{t+j} \, Cov_t(-u'(c_{t+j}), x_{t+j}))$$

where

$$\delta_{t+j} = \frac{1}{E_t[u'(c_{t+j})] \, E_t \, [x_{t+j}]}$$

or using equation (5)

$$p(X_t) = \sum_{j=1}^{\infty} \left\{ \frac{E_t(x_{t+j})(1 - \delta_{t+j} \, Cov_t(-u'(c_{t+j}), x_{t+j}))}{(1 + r_{t,j})^j} \right\}$$

or

$$p(X_t) = \sum_{j=1}^{\infty} \left\{ \frac{\theta_{t+j} \, E_t(x_{t+j})}{(1 + r_{t,j})^j} \right\}$$

where

$$\theta_{t+j} = [1 - \delta_{t+j} \, Cov_t(-u'(c_{t+j}), x_{t+j})]$$

This is precisely equation (13) in the text.

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