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Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium

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by

Charles D. Kolstad and Lars Mathiesen*

Abstract

In this paper a theorem is developed giving necessary and sufficient conditions for the uniqueness of homogeneous product Cournot equilibria. The result appears to be the strongest to date and the first to involve both necessity and sufficiency. The theorem states that an equilibrium is unique if and only if the determinant of the Jacobian of marginal profits for firms producing positive output is positive at all equilibria. The result applies to the case where profit functions are twice differentiable and pseudoconcave, industry output can be bounded, the above Jacobian is non-singular at equilibria, and marginal profits are strictly negative for non-producing firms. The proof uses fixed point index theory from differential topology.

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I. INTRODUCTION

Conditions guaranteeing uniqueness of equilibria are second only to those guaranteeing existence in terms of importance for both theoretical and applied analyses of economic equilibria. In the area of competitive markets, researchers have exerted great efforts to find the weakest sufficient conditions for uniqueness. Recently, necessary and sufficient conditions have been developed for uniqueness of a competitive equilibrium (Kehoe, 1980; Varian, 1974). In the case of Cournot equilibria, the question of existence itself has not been fully and completely addressed.[1] Not surprisingly, the question of uniqueness of a Cournot equilibrium has received less attention.

There has been, however, a series of results giving sufficient conditions for the uniqueness of a Cournot equilibrium.[2] All of these results directly or indirectly exploit the Gale-Nikaidô (1965) theorem: if, over a rectangular region, the Jacobian of a function is always a P-matrix (i.e., all principal minors positive), then the function is one-to-one and thus can have at most one zero.

The strongest condition that has yet appeared restricting costs and demands in such a way that the Gale-Nikaidô theorem can be invoked, is due to Okuguchi (1983) although a game-theoretic version was published by Rosen (1965).[3]

Unfortunately, previous results have yielded necessary conditions for uniqueness only. It is never clear whether the conditions can be further weakened without losing uniqueness. In fact this search is not over until conditions for uniqueness are found which are both necessary _and_ sufficient.

In this paper we develop necessary and sufficient conditions for the uniqueness of a Cournot equilibrium for the case where individual firm profit functions are continuously differentiable, pseudoconcave with respect to own output and satisfy modest regularity conditions. For the class of twice differentiable profit functions, all previous results on uniqueness (known to us) are special cases of our result. In fact, the conditions that most authors invoke to prove existence, turn out to yield uniqueness, except for the most pathological cases.

Our result is that if at all equilibria the determinant of the Jacobian of the marginal profit functions is positive (subject to some conditions), then there is exactly one equilibrium. Conversely, if there is exactly one equilibrium, the determinant must be non-negative. If one rules out the case of a zero determinant, then positivity of the Jacobian at all equilibria is necessary and sufficient for uniqueness. This condition on the Jacobian can be interpreted in terms of a firm's marginal profit function. At equilibria, the effect of a small change in a firm's output on its own marginal profits must be greater than the effect on its marginal profits from a similar output change on the part of all other competitors.

In the next section of the paper, we define some basic concepts, including that of a regular market. We then proceed to a presentation and a proof of our basic result which we follow by an economic interpretation and relate previous results to ours.

II. PRELIMINARIES

Let there be N firms, each with cost $C_i(q_i)$ for producing output $q_i \geq 0$. Define $\overline{N}$ by $\overline{N} = \{1, .., N\}$. The firms face an aggregate inverse demand function, $P(Q)$ (where $Q = \sum_{i \in \overline{N}} q_i$), defined over $[0, \infty)$. Profit for the i'th firm is thus given by

$$\Pi_i(q) = q_i P\left(\sum_{j \in \overline{N}} q_j\right) - C_i(q_i). \tag{1}$$

We can then define a Cournot equilibrium:

<u>Definition 1</u>: A Cournot (or Cournot-Nash) equilibrium is a vector of outputs, $q^* \in R^N$, such that for any firm, $i \in \overline{N}$:

$$\Pi_i(q^*) = \max_{q_i \geq 0} \Pi_i(q_1^*, \ldots, q_{i-1}^*, q_i, q_{i+1}^*, \ldots, q_N^*). \tag{2}$$

We will assume that costs and inverse demand (and thus profits) are twice continuously differentiable (C^2), although it would appear that this can be weakened to apply only at equilibria. As pointed out by Kehoe (1980) and others, differentiability is not an overly restrictive assumption since small perturbations of non-C^2 functions will yield C^2 functions.

Because of differentiability, there is a close connection between Cournot equilibria and solutions to the following complementarity problem:

CP: Find q such that for all $i \in \overline{N}$:

$$-\frac{\partial \Pi_i}{\partial q_i} \equiv G_i(q) \equiv C_i'(q_i) - P\left(\sum_{j \in \overline{N}} q_j\right) - q_i P'\left(\sum_{j \in \overline{N}} q_j\right) \geq 0, \tag{3a}$$

$$q_i \geq 0, \tag{3b}$$

$$q_i \frac{\partial \Pi_i}{\partial q_i} = 0. \tag{3c}$$

For pseudoconcave profit functions (concave functions are pseudoconcave), first order conditions are necessary and sufficient for a global optimum. Thus in this case, we have a one to one correspondence between Cournot equilibria and solutions of CP:

Lemma 1: Assume cost and inverse demand functions are C^2 (twice continuously differentiable). Then any Cournot equilibrium solves CP. Further, if profits are pseudoconcave with respect to own output, then q^* is a Cournot equilibrium if and only if it solves CP.

Conditions (3) are the standard ones for profit maximization. (3c) states that for production to occur, output must be at a stationary point of the profit functions. (3a) states that even when output is zero, marginal profit cannot be positive.

Now that we have related CP-solutions and Cournot equilibria, we introduce the notion of a non-degenerate Cournot equilibrium:

Definition 2: For the case where costs and inverse demand are C^2, a Cournot equilibrium, q^*, is non-degenerate if, for all $i \varepsilon \overline{N}$,

$$q_i^* = 0 \Rightarrow P(\sum_{j \varepsilon \overline{N}} q_j^*) - C_i'(0) < 0. \tag{4}$$

This strict complementary slackness condition (see Fiacco and Hutzler, 1982), is related to the condition in a general equilibrium economy on the desirability of all goods and assures us that all firms are either clearly in the market ($q_i^* > 0$), or clearly out of the market, ($\frac{\partial \pi_i}{\partial q_i} < 0$). We exclude the possibility that at an equilibrium, a firm may be just at the margin of deciding whether to enter or not.

In fact, paralleling work in general equilibrium theory, we introduce the notion of a regular Cournot market:

Definition 3: A set of N firms with C^2 costs, facing a C^2 inverse demand function constitutes a regular Cournot model if

i) all Cournot equilibria are non-degenerate;

and

ii) at each Cournot equilibrium (if any) the Jacobian of marginal profit for those firms with positive output is nonsingular.

The significance of a regular Cournot model will become apparent in the next section. In a measure theoretic sense, almost all Cournot markets are regular. If a market is not regular, a small perturbation of costs and demand can make it regular. Similarly, if it is regular, a small perturbation will not change its regularity properties (Kehoe, 1985). Regularity is thus a generic property.

We need one further definition so that we can be assured that profits will not be maximized at infinite output.

Definition 4: Industry output is said to be bounded if there is a compact subset, K of R^N_+, such that for $\tilde{q} \in R^N_+ K$:

$$-\frac{\partial \Pi_i}{\partial q_i}(\tilde{q}) = G_i(\tilde{q}) = C_i'(\tilde{q}_i) - P(\tilde{O}) - \tilde{q}_i P'(\tilde{O}) > 0, \ \forall \ i \varepsilon \overline{N} \qquad (5)$$

where $\tilde{O} = \sum_{i \varepsilon \overline{N}} \tilde{q}_i$.

For example, industry output is bounded if there is an output level for which marginal profits are negative for all; and furthermore, these negative marginal profits persist for all greater industry output levels. Our boundedness assumption is as weak as possible in order to restrict attention to a compact set of output vectors. It is a weakening of the common condition that inverse demand go to zero for some output level. When inverse demand goes to zero, revenue goes to zero, hence industry output is bounded (if marginal costs are positive). This condition on inverse demand is strong, however, precluding any function which is asymptotic to the output axis (e.g., constant price elasticity).

III. NECESSARY AND SUFFICIENT CONDITIONS FOR UNIQUENESS

For some years the Gale-Nikaidô theorem has been used to show uniqueness of equilibria, competitive or otherwise. Recently, the condition that the Jacobian be everywhere a P-matrix in order for a map to be one-to-one has been weakened considerably (Mas-Colell, 1979). For there to be a unique competitive equilibrium, an even weaker necessary and sufficient condition has been developed: the determinant of the Jacobian of excess demand is positive at all equilibria, subject to some regularity conditions (Dierker, 1972; Varian, 1975; Kehoe, 1980). A very similar result is implicit in the paper of Saigal and Simon (1973). They state that the complementarity problem

$$\text{Find } z \geq 0 \text{ such that } f(z) \geq 0,\ z_i f_i(z) = 0 \quad \forall i, \tag{6}$$

where: $f\colon R^N_+ \to R^N$,

has at most one solution if and only if the determinant of the Jacobian of f (eliminating rows and columns with z_i zero) at all solution points is positive, subject to some regularity conditions. Although this result, embodied in the following theorem, does not appear explicitly, it follows from their results. The proof is based on Kehoe (1980) and Saigal and Simon (1973). First we define a boundary condition, analagous to the boundedness condition on Cournot equilibria.

Assumption BC (Boundary Condition): Let $f\colon R_+^N \to R^N$ satisfy the condition that there exists a compact set $C \subset R_+^N$ such that for all $y \in R_+^N$ but not in C, there exists an x in C so that

$$\sum_{i \in \overline{N}} (x_i - y_i) f_i(y) < 0. \tag{7}$$

Before stating the theorem, we define an index set corresponding to positive elements of f(z).

Definition: Define $B(z) = \{i \in \overline{N} | z_i > 0\}$, $J_B(z)$ to be the principal minor of the Jacobian matrix of f corresponding to the indices of B(z), and $|J_B(f,z)|$ its determinant. If $B(z) = \phi$, then define $|J_B(f,z)| = 1$.

Theorem 1: Let $f\colon R_+^N \to R^N$ be continuously differentiable. Suppose f satisfies the boundary condition (BC). Then a solution to (6) exists. Suppose at each solution to (6), z^*, that $z_i^* = 0$ implies $f_i(z^*) > 0$. Then

i) if for all solutions to (6) we have $|J_B(f,z^*)| > 0$, then there is precisely one solution; and conversely,

ii) if there is only one solution to (6) then $|J_B(f,z^*)| \geq 0$.

Proof: From the "boundary condition" (BC), all solutions must be in C. Existence of a solution is from Karamardian (1972). Let Z* be the set of solutions to (7) such that $|J_B(f,z)| \neq 0$ for $z \in Z^*$. Define the mapping $h: R_+^N \to R^N$ by

$$h_i(z) = \begin{cases} 0 & \text{if } f_i(z) - z_i \geq 0, \\ z_i - f_i(z) & \text{if } f_i(z) - z_i < 0. \end{cases} \tag{8}$$

The fixed points of h (i.e., z such that h(z) = z) are the same as the solutions to (6). (See Eaves, 1971.)

The Lefschetz number of h at z is given by

$$L_z(h) = \text{sign } |\nabla h(z) - I| \tag{9}$$

provided that $|\nabla h(z) - I| \neq 0$. (See Guillemin and Pollack (1974).) Because at all equilibria, z*, $\partial h_j/\partial z_i = 0$ for all $j \in B(z^*)$, we can use (8) and evaluate (9) using expansion by co-factors to obtain

$$L_{z^*}(h) = (-1)^N \text{ sign } |J_B(f,z^*)|. \tag{10}$$

The global Lefschetz number of h, L(h), depends on the topological properties of h and the region over which it is defined. To compute L(h), we will use the fact that it is homotopy invariant (provided the overall set of fixed points is compact). Let D be some compact convex set containing $C \cup h(C)$. Saigal and Simon (1973) show that there is an extension of h to D, $\tilde{h}: D \to D$, such that h and $\tilde{h}$ are the same in C (i.e., $\tilde{h}|C = h|C$), the fixed points of $\tilde{h}$ are all in C, and $L(\tilde{h}) = L(h)$. Let $\bar{z} \in C$ be some fixed vector and consider the function $H_t(z) = (1-t)\tilde{h}(z) + t\bar{z}$ where $0 \leq t \leq 1$. Because D is convex, H_t is defined on D for all t. H_t is thus a homotopy and the global Lefschetz number of $H_t(z)$ will be the same for all t. In particular, for t = 1,

$$L(\bar{z}) = \text{sign } |\nabla \bar{z} - I| = \text{sign } |(0-I)| = (-1)^N \tag{11}$$

which implies (for t=0),

$$L(\tilde{h}) = L(h) = (-1)^N. \tag{12}$$

For differentiable functions on a compact surface, with a finite number of fixed points, the global Lefschetz number is the sum of the local Lefschetz numbers over all the fixed points. Saigal and Simon (1973) demonstrate that the fixed points are discrete and thus that this property holds. Thus (10) and (12) can be combined:

$$\sum_{z \varepsilon Z^*} \text{sign } |J_B(f,z)| = 1 \tag{13}$$

This applies over the compact set C, which we already noted contains all solutions. Clearly if $|J_B(f,z)| > 0$ for all solutions, then they are all in Z* and there can be only one.

To prove ii), assume that z* is the unique solution and suppose $|J_B(f,z^*)| < 0$. Then by (13), there must be another solution which is a contradiction so ii) is proved.[2]

This theorem is a strong one and deserves some intuitive interpretation of its applicability. Consider the one dimensional complementarity problem of Fig. 1. The theorem states that if strict complementarity holds at all solutions, then the sum of the signs for f'(z*) should be +1. Note first that solution b for line D has $|J_B| = 0$, because $f'(b) = 0$. The theorem cannot be applied to curve E either, because the "boundary condition" that eventually $f(y) > 0$ is not satisfied. The theorem can be applied to curve F for which there is one solution, c, and $f'(c) > 0$.

We are now in a position to prove our fundamental theorem giving necessary and sufficient conditions for the uniqueness of a Cournot equilibrium. Recall that G is defined in (3) as $G_i = -\frac{\partial \Pi_i}{\partial q_i}$.

Theorem 2: Assume

i) cost and inverse demands are C^2 over R_+^N;

ii) industry output is bounded;

iii) profits are pseudoconcave with respect to own-output; and

iv) all Cournot equilibria (if any) are non-degenerate.

Then

a) a Cournot equilibrium exists.

b) if $|J_B(h,q^*)| > 0$ for all equilibria, q^*, then there is precisely one equilibrium, and

c) if the equilibrium q^* is unique, then $|J_B(h,q^*)| \geq 0$.

Proof: Apply Theorem 1 to CP, solutions to which are in one-to-one correspondence with Cournot equilibria.

If we confine ourselves to regular Cournot markets, this result becomes even clearer.

Corollary 2.1: Assume

i) cost and inverse demand functions constitute a regular Cournot market;

ii) industry output is bounded: and

iii) profits are pseudoconcave with respect to own-output.

Then a unique Cournot equilibrium exists if and only if for all equilibria, q^*, $|J_B(h,q^*)| > 0$.

It would be desirable to relax the pseudoconcavity of profit assumption. However, without pseudoconcavity, solutions to CP need not be equilibria although all equilibria must solve CP (lemma 1). When we relax pseudoconcavity, we lose existence but not uniqueness.

Corollary 2.2: Assume

i) cost and inverse demand are C^2 over R_+^N; and

ii) industry output is bounded.

Then if at all q^* satisfying profit stationarity condition (3), $|J_B(h,q^*)| > 0$, then there is at most one Cournot equilibrium.

IV. ECONOMIC INTERPRETATION

What is lacking from Theorem 2 and its corollaries is an explicit indication of how the shape of cost and demand functions translate into conditions on existence and uniqueness. In this section we provide that bridge as well as an intuitive interpretation of our results.

Theorem 3: With cost and inverse demand function C^2, then

$$|J_B(h,q^*)| = \left\{1 - \sum_{j\varepsilon B(q^*)} \left[\frac{P'(Q^*) + q_j^* P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)}\right]\right\} \sum_{j\varepsilon B(q^*)} [C_j''(q_j^*) - P'(Q^*)] \quad (14)$$

where $Q^* = \sum_{i\varepsilon \underline{N}} q_i^*$.

Proof: For some q^*, assume, without loss of generality, that the first $M \leq N$ firms are the ones producing positive outputs. Thus the Jacobian of the principal minor of (3a) corresponding to the indices $B(q^*)$ is

$$|J_B(h,q^*)| = \begin{vmatrix} \{C_1''(q_1^*)-2P'(Q^*) -q_1^*P''(Q^*)\} & -P'(Q^*)-q_1^*P''(Q^*) & \cdots & -P'(Q^*)-q_1^*P''(Q^*) \\ -P'(Q^*)-q_2^*P''(Q^*) & \{C_2''(q_2^*)-2P'(Q^*) -q_2^*P''(Q^*)\} & \cdots & -P'(Q^*)-q_2^*P''(Q^*) \\ \vdots & & \ddots & \vdots \\ -P'(Q^*)-q_M^*P''(Q^*) & \cdots & \cdots & \{C_M''(q_M^*)-2P'(Q^*) -q_M^*P''(Q^*)\} \end{vmatrix} . \tag{15}$$

By subtracting the first column from the other columns in (15), and dividing the jth row by $C_j''(q_j^*) - P'(Q^*)$, it is straightforward to evaluate the determinant through expansion by co-factors, yielding (14).

Rather than interpret this theorem directly, we will present two corollaries which have more obvious economic content.

Corollary 3.1: Let costs and inverse demand be C^2. In addition, assume that at all equilibria, q^*, $C_j''(q_j^*) - P'(Q^*) > 0$ for all $q_j^* > 0$. Then

$$\operatorname{sign} |J_B(h,q^*)| = \operatorname{sign} \{1 - \sum_{j\varepsilon B(q^*)} [\frac{P'(Q^*) + q_j^* P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)}]\}. \tag{16}$$

Using this corollary and Theorem 2, multiple equilibria are associated with the case of eqn. (16) less than zero. Clearly if $P'(Q) + q_j P''(Q)$ is always negative, then (16) will always be positive thus yielding uniqueness. The condition that $P'(Q) + q_j P''(Q)$ be negative is a common assumption, both in existence theorems (Novshek, 1985) and uniqueness theorems. For instance, the assumption (Szidarovszky and Yakowitz, 1977) that inverse demand is downward sloping ($P' < 0$) and concave ($P'' \leq 0$), implies $P'(Q) + q_j P''(Q) < 0$. This assumption is clearly excessively strong, both for the positivity of (16) globally as well as at equilibria. We can interpret the corollary and Theorem 2 as saying that for a unique equilibrium, "on average,"

$$P'(Q^*) + q_j^* P''(Q^*) < \frac{1}{M} [C_j''(q_j^*) - P'(Q^*)]. \tag{17}$$

Thus, "on average," marginal revenues can be upward sloping, but not too much. The upper bound on the extent of upward slope depends on the relative slopes of inverse demand and marginal costs. For instance, sharply increasing marginal costs will make the upper bound very large.

An alternative interpretation, in terms of marginal profits, can be obtained by rewriting (17) as

$$2P'(Q^*) + q_j^* P''(Q^*) - C_j''(q_j^*) < -(M-1)[P'(Q^*)+q_j^* P''(Q^*)]. \tag{18}$$

The left-hand-side is the change in firm j's marginal profits with respect to change in its own output. The right-hand-side is the change in firm j's marginal profit when all other firms change their outputs by a similar amount. The interpretation is that if all of the firms with non-zero output increase their outputs equally, then the overall effect is a decline in firm j's marginal profits. We can relate this interpretation to Okuguchi's (1983) result on uniqueness. Recall first, that a matrix A is diagonal dominant if there exists a vector of strictly positive scaling parameters, d, such that

$$d_j |a_{jj}| > \sum_{i \neq j} d_i |a_{ij}|, \quad \forall j. \tag{19}$$

Corollary 3.2: Let the assumptions of Theorem 2 hold. If, at all equilibria, the Jacobian of marginal profits for firms with positive output is diagonal dominant, then there exists a unique Cournot equilibrium.

Proof: Diagonal dominance of the matrix in (15) implies that its determinant is positive (Namatame and Tse, 1981).

Okuguchi (1983) observed that, if at an equilibrium, q*,

$$2P'(Q^*) + q_j^* P''(Q^*) - C_j''(q_j^*) < -(M-1)|P'(Q^*) + q_j P''(Q^*)|, \ \forall \ j \varepsilon B(q^*) \tag{20}$$

then the dominant diagonal property applies and there is a unique equilibrium. Note that (20) is more restrictive than (18) both because of the absolute value and the requirement for all j and not only "on average." Of course, diagonal dominance is not a necessary condition for uniqueness; furthermore, (20) is only a sufficient condition for diagonal dominance, not a necessary one because we have used unitary scale factors. Thus (20), and our interpretation of it, are overly strong.

To summarize, Theorems 2 and 3, and their corollaries, require a slightly stronger condition than concavity of profit functions to assure uniqueness of a Cournot equilibrium. The stronger conditions are precisely specified by (14) and (16). Roughly speaking, these equations can be interpreted as requiring "on average" the own effect on marginal profits to dominate the sum of cross-effects.

V. CONCLUSIONS

In this paper we have presented necessary and sufficient conditions for the uniqueness of Cournot equilibria. Although our results are, to our knowledge, the strongest to date for the case of differentiable cost and demand functions, it is likely that they can be extended further, perhaps by relaxing the differentiability and/or regularity requirements.

VI. FOOTNOTES

[1]Existence results apply to Cournot equiilbria only (Frank and Quandt, 1963; Szidarovsky and Yakowitz, 1977, 1982; Novshek, 1985; McManus, 1964; Roberts and Sonnenschein, 1976) or to the more general case of Nash equilibria (Nishimura and Friedman, 1981).

[2]See Okuguchi (1976, 1983), Szidarovszky and Yakowitz (1977) and Murphy et al. (1982).

[3]The uniqueness results generally require twice continuously differentiable cost and inverse demand functions. Szidarovszky and Yakovitz (1982) have weakened the differentiability assumption, simply requiring continuity and convexity of costs and once differentiability and concavity of inverse demand. Under these conditions they prove existence of a unique equilibrium.

[4]If the solutions are regular points, then there must be at least two others. However, as a referee has pointed out, critical equilibria can have index numbers of +2.

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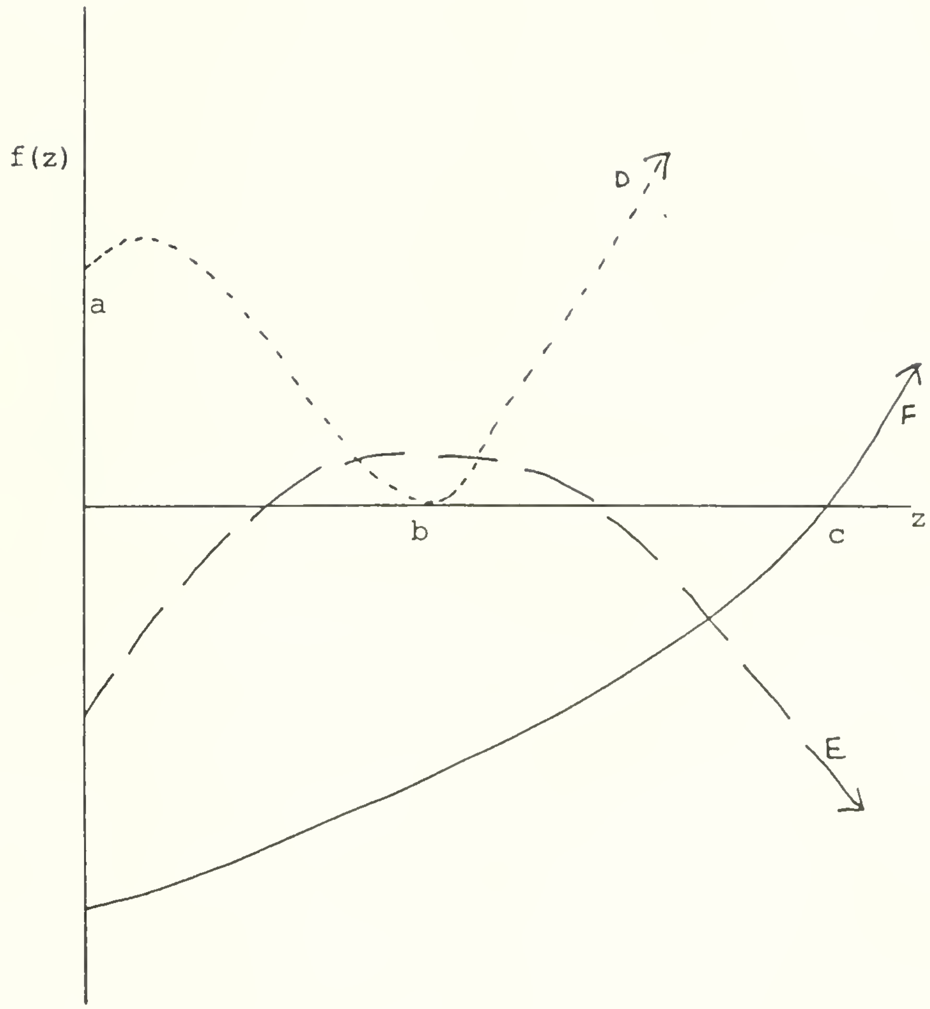

Figure 1