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OPTIMAL WEIGHTLIFTING

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ABSTRACT

In international weightlifting competition there are two categories of lifts: the "snatch" and the "clean-and-jerk." A lifter's score is the sum of his scores in these two lift categories. The lifter has three attempts in each category; his "score" is the weight of his best lift. Once he selects a weight for a lift, he cannot select a lower weight on a subsequent lift in the same category. Two problems are explored here: selection of lift weights and methods for updating that selection as more information becomes available during the competition. Several numerical examples are included.

1. Introduction

There are two categories of lifts in international weight lifting -- first the "snatch" and then the "clean-and-jerk" (in that order). In the snatch, the lifter must take the weight in a single motion from the ground to above his head, arms extended. In the clean-and-jerk, the lifter first "cleans" the weight (brings it up to his chest) then thrusts it above his head, or "jerks" it. Each lift category is run as a separate competition; however, one's overall score is the sum of the separate scores in the two categories.

In each category the lifter faces the problem of selecting weights. A competitor has a maximum of three lift attempts in each category. His score in that category is the weight he lifts on his best lift. He cannot, however, go down in his selected weight. Once a weight is selected for a lift, the weight chosen on the next lift must be at least as high. A lifter who attempts too little may have almost no chance of winning; a lifter who attempts too much may have a significant chance of "bombing out" (scoring zero).

In previous work [2], consideration was given to the problem of optimizing performance within a single lift category. This paper expands those results. In particular, the joint, two-lift problem is explicitly treated. A theorem is proven indicating when separate, independent consideration of the lift-categories will lead to an optimal, joint policy. In addition, the important problem of updating strategy during competition is considered: if the favorite bombed out of the "snatch" in a particular competition, one's strategy in the "clean-and-jerk" should incorporate that information.

The weight selection problem demands explicit consideration of the value of scoring a certain weight as well as the chances of making the lift. These considerations are developed in the next section.

Game theoretic and team strategy issues are not developed here; the strategy of an individual in a competition-group that is <u>indifferent</u> to his performance is used for our analysis here. This clearly is a simplification of the real problem in which other competitors influence lift probabilities and coaches exert "team-strategy" pressure. Such problems deserve further, separate analysis.

2. The General Model

We refer here to the partial decision tree in Figure 1. The areas occupied by X's and Y's indicate decision points; the splits or forks are chance node. A lifter moves to the high branch after a chance node if he is successful in the lift; if he misses, he passes to the lower branch.

Define the following:

X_i = weight selected for snatch-lift i in Figure 1, i=0,...,7.

Q_i = probability that score in snatch competition is X_i, i=0,...7. X_0=0.

p_i = probability of lifting X_i in Figure 1.

Y_{ij} = weight selected for clean-and-jerk lift j, given snatch-weight = X_i, j=0,...7. Y_{i0} = 0, i=0,...7.

s_{ij} = unconditional probability of lifting Y_{ij}.

R_{ij} = probability that the competitor's score is $Y_{ij} + X_i$.

Assuming independence, Q_i is displayed in terms of $\{p_i\}$ in Table 1 and R_{ij} is displayed in terms of $\{Q_i\}$ and $\{s_{ij}\}$ in Table 2.

i	Q_i
0	$(1-p_1)\ (1-p_2)\ (1-p_4)$
1	$p_1\ (1-p_3)\ (1-p_6)$
2	$(1-p_1)\ (p_2)\ (1-p_5)$
3	$p_1\ p_3\ (1-p_7)$
4	$(1-p_1)\ (1-p_2)\ p_4$
5	$(1-p_1)\ p_2p_5$
6	$p_1\ (1-p_3)\ (p_6)$
7	$p_1p_3p_7$

TABLE 1: Snatch-Lift Outcome Probabilities

J	R_{ij} , i=0,...,7
0	$(1-s_{i1})(1-s_{i2})(1-s_{i4})Q_i$
1	$s_{i1}\ (1-s_{i3})\ (1-s_{i6})\ Q_i$
2	$(1-s_{i1})\ (s_{i2})\ (1-s_{i5})\ Q_i$
3	$s_{i1}s_{i3}\ (1-s_{i7})\ Q_i$
4	$(1-s_{i1})\ (1-s_{i2})\ s_{i4}Q_i$
5	$(1-s_{i1})\ s_{i2}s_{i5}Q_i$
6	$s_{i1}\ \ (1-s_{i3})\ s_{i6}$
7	$s_{i1}s_{i3}s_{i7}$

TABLE 2: Total Lift Outcome Probabilities

If we let $V(X_i+Y_{ij})$ = value or utility associated with a particular lift-sum, then we can formulate the general problem as follows:

$$\text{find } \{X_i\}_{i=1}^{7}\ ,\ \ \{Y_{ij}\}_{i=1\ j=1}^{7\ \ \ 7} \text{ to}$$

$$\max Z = \sum_{j=0}^{7} \sum_{i=0}^{7} V(X_i+Y_{ij})\ R_{ij}$$

(1) subject to competition rules:

(1a)
$$\begin{cases} X_1 \le X_3 \le X_7 \\ \quad X_3 \le X_6 \\ X_1 \le X_2 \le X_5 \\ \quad X_2 \le X_4 \end{cases}$$

$$
\text{(1b)} \quad \begin{cases}
Y_{i1} \le Y_{i3} \le Y_{i7} & i=0,\dots,7 \\
Y_{i3} \le Y_{i6} & i=0,\dots,7 \\
Y_{i1} \le Y_{i2} \le Y_{i5} & i=0,\dots,7 \\
Y_{i2} \le Y_{i4} & i=0,\dots,7 \\
X_0 = 0 & \\
Y_{i0} = 0 & i=0,\dots,7 \\
X_1 \ge 0,\ Y_{ij} \ge 0 & i = 1, \dots, 7,\ j = 1,\dots, 7.
\end{cases}
$$

The above is a general non-linear program when no information about the form of the objective function is available. The relatively small number of variables (56) and the simple nature of the constraints makes it small enough to be solved by existing codes. The structure of the tree in Figure 1 suggests a dynamic programming solution strategy. However, the nature of the constraint structure makes such a strategy awkward in general.

3. Evaluation of Lift Probabilities

One could evaluate a lifter's previous performances to attempt to assign probabilities to various lift-weights. However, this assumes that we can take account of day-to-day variations in performance and relate that to his performance on a particular day. Clearly, any estimate of lift probabilities would have to be modified to reflect the lifter's mental and physical state on a particular day. The procedure we suggest, basically asking the lifter what he thinks he can lift, assumes he has enough knowledge of his past performance to incorporate that knowledge in an estimate of his performance on a particular occasion. Thus we shall "ask the horse," although experience with particular lifters may call for modifications.

One simple method for obtaining this information from a lifter would be to ask for three points on a (bounded) probability-of-lift curve as follows:

Question 1: "What is the maximum weight you feel certain you can lift?" Call this L.

Question 2: "What is the most weight you feel you have the slightest chance of lifting?" Call this U.

Question 3: "What weight do you feel you have 50-50 chance of lifting?" Call this M.

If we then assume that the lifter's probability-of-lift curve is well behaved, we might approximate it by a quadratic function between L and U as follows:

Assume $p_i(X_i)$ = probability of lifting X_i, is of the form

$$p_i(X_i) = \begin{cases} 1 & X_i \leq L;\ \text{or } U > X_i > L \text{ and } f(z) > 1. \\ f(z) = az^2+bz+c & U > X_i > L \text{ and } 1 \geq f(z) > 0 \\ 0 & X_i \geq U;\ \text{or } U > X_i > L \text{ and } f(z) < 0. \end{cases}$$

for all i.

where $z = \frac{X-L}{U-L}$ = proportion of a lifter's "weight-range" attempted. If we let $k_0 = \frac{M-L}{U-L}$, then it can be shown that $a = \frac{1/2 - k_0}{k_0 - k_0^2}$, $b = \frac{k_0^2 - 1/2}{k_0 - k_0^2}$ and c = 1 in (2). We have assumed for simplicity that this function is the same for any lift; clearly if a lifter were able to quantify a different function for each i in (2), a set of probability functions could be estimated and used in (1).

4. Objectives

Many methods have been suggested for assessing utility measures (see [3], e.g.). Several with some intuitive appeal are suggested in [2] which we will investigate further here.

Expected Weight Maximization: This assumes a lifter's utility is linearly related to the amount he lifts. A problem exists: there is (relatively) a very high negative utility associated with "bombing-out" or lifting zero here. This negative utility may be higher than that which exists in lifters' minds.

Weight-Range Maximization: Here it is assumed the lifter wishes to maximize the proportion of his "weight range" he achieves; i.e., a lifter who lifts 350 pounds with U = 400, L = 300, scores .5. This suggests more risk prone strategies, since the utility of 300 to this lifter is 0 which is the same as the utility of his bombing out (actually lifting 0). Hence he would be indifferent between 300 and 0, while an expected weight maximizer clearly would not. This may or may not be the case but observation of liftings suggest some (at least psychological) penalties for bombing out. One might attempt to assess that penalty directly and include it as V(0). In that case, "Weight-Range" and Weight Maximization could be made to imply the same policy (they would differ only by a linear transformation, which would have no effect on optimal policies).

Medal Winning: This, after all, is what competition is about. One method of assessment of chances of winning medals calls for analysis of prior scores by competing lifters. Assume N lifters are involved in the competition, and an evaluation of lifter k's past performance yields a snatch-density, potentially of the mixed variety, $dF_X k(x)$ and a clean-and-jerk density $dF_Y k(y)$. Thus, the following (prior) information is available:

Lifter	Snatch Density	Clean-and-jerk Density
1	$dF_X^{\ 1}(x)$	$dF_X^{\ 1}(y)$
2	$dF_X^{\ 2}(x)$	$dF_X^{\ 2}(y)$
⋮	⋮	⋮
N	$dF_X^{\ N}(x)$	$dF_Y^{\ N}(y)$

Let:

Z_i = random variable, score lifted by ith competitor

$= X^i + Y^i$

$dF_{Z^i}(z)$ = (mixed) probability density of score of ith competitor

= convolution of X^i and Y^i

$$= \int_0^\infty dF_{X^i}(y)\, dF_{Y^i}(z-y)$$

W_i = ith highest weight lifted: W_1 = lowest weight, W_N = winning weight.

Assuming that the weights are independently distributed, the joint probability density of the N order statistics is

$$d\Psi_{W_1 \ldots W_N}(w_1, \ldots w_N) = N!\, dF_Z^1(w_1) \ldots dF_Z^{\ N}(w_N) \tag{3}$$

where $w_1 < w_2 < \ldots < w_N$

and the marginal density of the t^{th} highest weight lifted is

$$dG_{W_t}(w_t) = N! \int_{W_N=w_t}^{\infty} \ldots \int_{W_{N-1}=w_t}^{W_N} \ldots \int_{W_{t+1}=w_t}^{W_{t+1}} \ldots \int_{W_{t-1}=0}^{W_t} \ldots \int_{W_1=0}^{W_2} d\Psi_{W_1 \ldots W_N}(W_1 \ldots W_N) \tag{4}$$

For example, if an individual is only satisfied with first place,

$$dG_{W_N}(w_N) = \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \neq i}}^{N} F_Z{}^j(w_N)\, dF_Z{}^i(w_N)$$

and $$V(T) = \int_0^T dG_{W_N}(w_N)$$ = "value" of lift-total T in terms of probability of winning.

We have assumed we can analyze snatch-lifts and clean-and-jerk lifts independently here. This assumption allows simpler analysis of past lifts to form densities; assuming dependent, joint distributions could be beyond our ability to analyze past data. It is unlikely that more than the last six or seven meet-results should be used for estimating a lifter's current competitive lift distributions. This is clearly not enough data to estimate a joint density but could be used to estimate single-lift densities.

5. Updating During Competition

Suppose that (competitive) lifter i has already lifted $\bar{X}^i$. It no longer makes sense to consider events including $X^i < \bar{X}^i$; thus lifter i's lift-density should be updated. The updating procedure is simple; using the rule of conditional probability.

$$dF_{X_i | X > \bar{X}^i}(x) = \begin{cases} 0 & x < \bar{X}^i \\ \dfrac{dF_{X_i}(x)}{\int_{\bar{X}^i}^{\infty} dF_{X_i}(x)} & x > \bar{X}^i \end{cases}$$

Then the updated dF_{X_i} (or dF_{Y_i}) replaces the prior distribution and a new $d\Psi$ is developed in (3). Then the objective function and the the lift-strategy can be updated.

Suppose also that, after lifting X_1, a lifter wishes to update his probability-of-lift curve, say, by increasing his 50-50 weight by 10 pounds. Then a new, optimal strategy, incorporating this new probability-of-lift information in the calculation of the $\{p_i\}$ should be developed.

The above procedures allow updating and rerunning of the model during a competition, incorporating information available at that time so that best decisions can be made.

6. An Independence Theorem for Lift Optimality

Under what conditions can a lifter in the second category, the clean and-jerk, neglect the results of his first category lift in choosing an optimal strategy? That question is treated in the following:

Theorem: If $V(X_i + Y_{ij})$ is a separable function of Y_{ij} and X_i, then $Y_{\hat{i}j} = Y_{i*j}$, $\hat{i}, i*j = 1, \ldots 7$; that is, clean-and-jerk strategy is independent of snatch results.

Proof: Assume the snatch competition is over, that $X = X_{i*}, i* \varepsilon (0,1,\ldots 7)$. The current, conditional problem is now to

$$\max \sum\sum V(Y_{ij} + X_{i*})\ R_{ij}|_{i=i*}$$

subject only to constants on $\{Y_{ij}\}$ or (1b).

Looking at Table 2,

$R_{ij}|_{i=i*}$ can be written as $f(\{s_j\}\ \delta_{i*}$

where δ_{i*} is $\begin{cases} 1 \text{ if } i=i* \\ 0 \text{ otherwise} \end{cases}$

The objective function above is then

$$\sum\sum V(X + Y_{ij})\ R_{ij}$$

$$= \sum\sum V_1(X)R_{ij} + \sum\sum V_2(Y_{ij})R_{ij} \text{ (by the given separability)}$$

$$= V_1(X)\sum\sum R_{ij} + \sum\sum V_2(Y_{ij})f(\{s_j\})\delta_{i*}$$

$$= V_1(X) + \sum_{j=0}^{7} V_2(Y_{i*j})f(\{s_j\})$$

$$= \text{Constant} + \sum_{j=0}^{7} V_2(Y_{i*j})f(\{s_j\})$$

The above objective will be the <u>same for any i*</u>, and strictly depends only on j.

Thus $\hat{Y}_{ij} = \hat{Y}_{i^*j}$, i, i* ε (0,...7), proving the theorem.

A particular case of this theorem is the following:

<u>Lemma</u>: When the objective is expected weight maximization or weight-range maximization, 1st and 2nd category lift strategies are independent.

<u>Proof</u>: Result follows as a special case of the theorem.

Thus, we have the useful result that if our objective is one of the above types, we need only concern ourselves with the current competition and not be concerned about other category performance.

7. Some Numerical Examples

It is most convenient to study the general case numerically, using a nonlinear optimization method. In this section we will give a variety of examples to illustrate the sensitivity of optimal policies to changes in objectives and input parameters.

Consider an individual whose value of U=400, L=300 and M=330, 350 or 370 (see Section 3), in clean-and-jerk competition. Probability curves for these parameter values are included in Figure 2.

Three objectives were considered:

A = Maximizing expected proportion of lift-range

B = Maximizing expected weight

C = Maximizing chances of medal winning given a score of X* in the snatch.

For objective C, it was assumed that on the basis of prior data analysis, ones chances of winning a medal = T =

$$T = \begin{cases} 0 & \text{for clean-and-jerk Weight Total (WT)} \leq 350 \\ \dfrac{WT-350}{100}, & 350<WT<450 \\ 1 & WT \leq 450 \end{cases}$$

i.e., the minimum weight needed to win a medal is a random variable uniformly distributed between 350 and 450. There is no inherent difficulty in making this function more complicated since the solution was calculated numerically anyway; this simple example was used for ease of exposition.

The numerical solutions listed in Table 3 were calculated using RAC's nonlinear programming package SUMT [1]. Each case took about six seconds of CPU time to run on an IBM model 370-165. The program is currently in a batch mode although it could be adapted for real time use.

A "solution" from Table 3 is read as follows: Line 1 indicates that the lifter is an expected proportion maximizer whose 50-50 weight (M) = 330. His optimal first lift weight is 338 pounds (X_1). If he makes that lift his next lift should be 360 (X_3) pounds; if he misses, 338 (X_2) again and so on in accordance with the notation in Figure 1. On the average he will lift 33% of his weight-range.

Line 4 has blanks under X_2, X_4, X_5; this is due to the fact that X_1 = 300, a certain event, making X_2, X_4, X_5 events of probability zero. Here the value of the objective corresponds to the average weight he will be expected to lift in the long run. Line 7 has medal-winning probability as an objective. Thus, under the sequence indicated in line 7, one's chance for a medal would be 6%.

Lift probabilities have been assumed constant across lifts; they could of course have been updated as in Section 5.

8. Conclusions and Uses

An analytic procedure has been proposed here to aid in the selection of optimal weights for a weightlifter. The weight selection policies have been shown to be quite sensitive to a lifter's objectives and utilities, indicating that careful consideration of those utilities is essential in any application. Cases 1, 4 and 7 in Table 3, for example, are identical except for objectives and they result in vastly different policies.

The numerical procedure has been found to be efficient enough to allow for real-time operation at an actual meet. Thus a lifter (whose utilities have been determined beforehand) could wait until just before he was ready to select his lift weight to answer the lift probability questions in Section 3. On the basis of his first or second lift he could then come back to the computer terminal and update his probabilities in time for his next lift. He could also enter competitive, current scores to update his objective function. He would then receive an updated, "best" lift suggestion for the next round of competition.

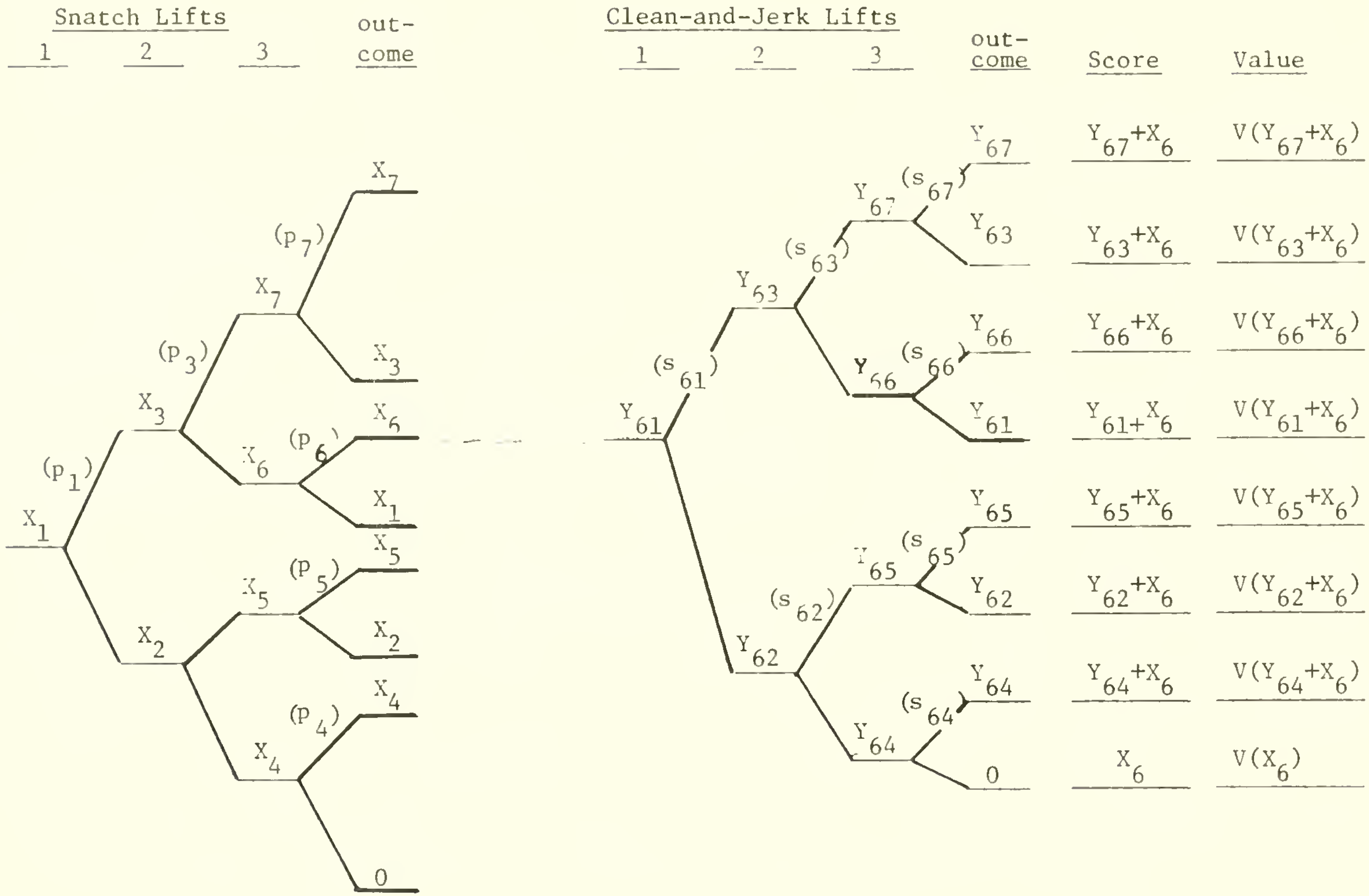

FIGURE 1: Weightlifting Partial Decision Tree

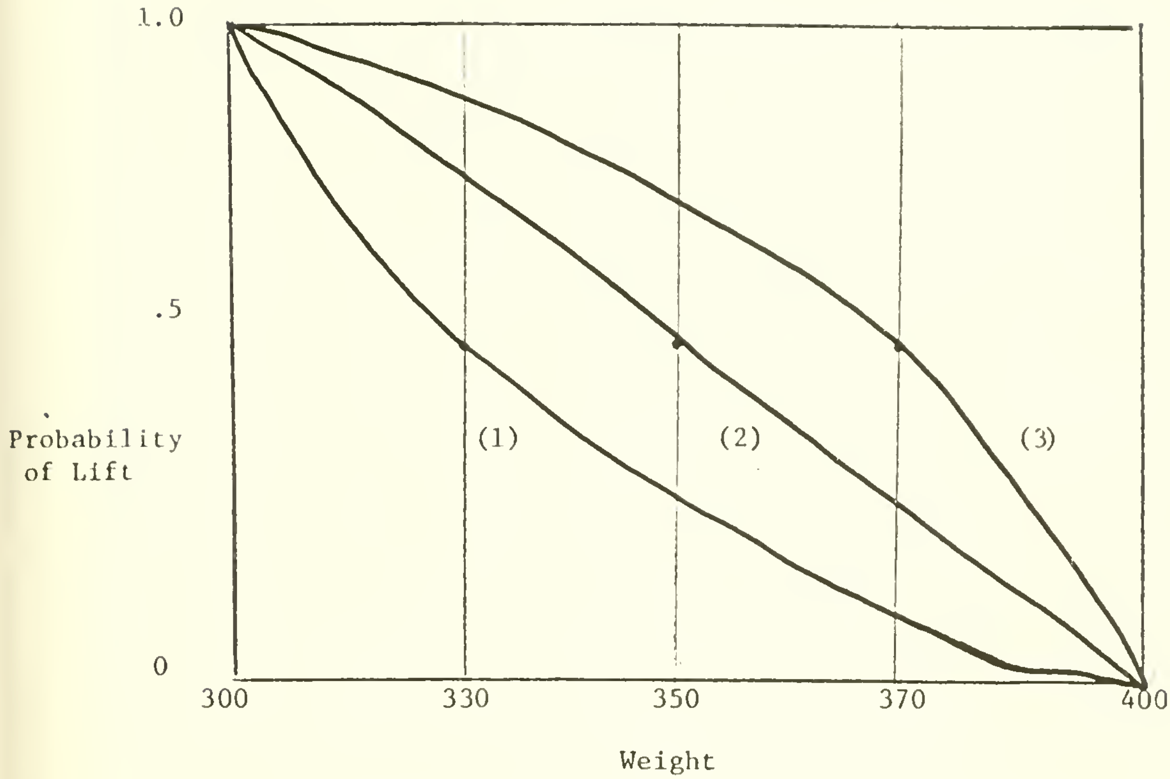

Figure 2: Probability Structure for Case Studies

(1) = 50-50 weight = 330

(2) = 50-50 weight = 350

(3) = 50-50 weight = 370

	Optimal Weights									
Case #	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Objective*	Probability Curve**	Value of Objective
1	338	338	360	338	360	360	374	A	1	33%
2	355	355	378	355	378	378	389	A	2	51%
3	364	364	383	364	383	383	392	A	3	67%
4	300	-	336	-	-	336	358	B	1	326
5	315	315	359	315	357	359	380	B	2	342
6	339	339	373	339	373	373	387	B	3	361
7	367	367	379	367	379	379	386	C	1	6%
8	375	375	388	375	388	388	394	C	2	16%
9	378	378	390	378	390	390	395	C	3	24%

Table 3: Optimal Lift Sequences

* A = Max expected proportion of weight range

B = Max expected weight

C = Max probability of medal where

$$\text{Prob} = \begin{cases} 0, & \text{WT Total (WT)} \leq 350 \\ \frac{WT-350}{100}, & 350 < WT \leq 450 \\ 1, & WT > 450 \end{cases}$$

** See Figure 2 for associated curve.

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