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**OPTIMAL CONSUMPTION AND PORTFOLIO CHOICE
WITH BORROWING CONSTRAINTS**

Jean-Luc Vila*
and
Thaleia Zarihopoulou**

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ABSTRACT

In this paper, we use stochastic dynamic programming to study the intertemporal consumption and portfolio choice of an infinitely lived agent who faces a constant opportunity set and a borrowing constraint. We show that, under general assumptions on the agent's utility function, optimal policies exist and can be expressed as feedback functions of current wealth. We describe these policies in details, when the agent's utility function exhibits constant relative risk aversion.

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I. INTRODUCTION

In this paper, we consider the Portfolio-Consumption problem of an infinitely lived agent in the presence of a constant opportunity set and borrowing constraints. Using the method of Dynamic Programming, we show that, under general assumptions on the agent's utility function, optimal policies do exist and can be expressed as feedback functions of the investor's current wealth. Given this existence result, we are able to describe how borrowing constraints affect the consumption and investment decisions when the agent's relative risk aversion is constant.

Stochastic dynamic control has first been used by Merton (1969, 1971) to obtain an explicit solution to the Portfolio-Consumption problem when the investment opportunity set is constant, the agent's utility function belongs to the HARA class and when trading is unrestricted. More recently, Karatzas et al. (1986) generalized Merton's results and obtained closed form solutions for general utility functions.

Instead of using stochastic control methods, the so-called *martingale approach* has been alternatively used by Pliska (1986), Cox and Huang (1989 & 1991), Karatzas et al. (1987) to study intertemporal consumption and portfolio policies when markets are complete, which was also the case in the earlier dynamic programming literature. The martingale technology consists in describing the feasible consumption set by a single intertemporal budget equation and then solving the static consumption problem in an infinite dimension Arrow-Debreu economy. The martingale approach is appealing to economists for two reasons. First, it can be used to solve for the asset demand under very general assumptions about the stochastic investment opportunity set. Second, and consequently, it can be applied in a general equilibrium setting to solve for the equilibrium investment opportunity set (see Duffie and Huang (1985), Huang (1987)).

Unfortunately, in the presence of market imperfections such as market incompleteness, short sale constraints or transaction costs, the martingale approach loses much of its tractability¹. Consequently, many authors used dynamic programming to analyze the impact of these imperfections on asset demand (see for example Constantinides (1986), Duffie and Zariphopoulou (1993), Grossman and Laroque (1988), Grossman and Vila (1992), Fleming et al. (1989), Fleming and Zariphopoulou (1991), Zariphopoulou (1993)).

The present paper extends this line of research to the dynamic problem of an infinitely lived agent who faces two constraints. The first constraint is a limitation on his ability to borrow for the purpose of investing in a risky asset, i.e, the market value of his investments in the risky asset, X_t , must be less than an exogenous function $\bar{X}(W_t)$ of his wealth W_t . In this paper we concentrate on the case $\bar{X}(W_t) \equiv k(W_t + \bar{L})$ where k and \bar{L} are non-negative constants. The second constraint is the requirement that the investor's wealth stays non-negative at all times, i.e., $W_t \geq 0$. Using the dynamic programming methodology, we associate to the stochastic control problem a nonlinear partial differential equation, namely the *Bellman equation*. We show that if the utility function satisfies some general regularity conditions, the Bellman equation has a unique solution which is twice continuously differentiable. Using a verification theorem, this solution turns out to be the indirect utility function (the so called value function) which is therefore a smooth function of the current wealth, W_t . Moreover, the optimal consumption, C_t and the optimal investment X_t are obtained in feedback form through the first order conditions from the Bellman equation. Because of the smoothness of the value function, the optimal policies are respectively continuously differentiable (C_t) and continuous (X_t) functions of the wealth.

In the second part of the paper we study the particular case of a Constant Relative Risk Aversion

¹He and Pearson (1991) use a duality approach to apply the martingale technology to incomplete markets with short sale constraints. Although their methodology carries a lot of insight, they do not solve explicitly for the optimal policies.

agent. In particular, we describe in detail the optimal policies and we compare them to the ones of the unconstrained problem. In the absence of the borrowing constraint, the optimal investment is $X_t = (\mu/A\sigma^2) W_t$, where μ is the excess rate of return on the risky asset over the risk-free rate, σ is the volatility of the rate of return on the risky asset and A is the coefficient of relative risk aversion. We show that the borrowing constraint causes the agent to be more conservative (i.e. to invest less in the risky asset) even at points where the constraint is not binding. This result has to be contrasted with a similar result in Grossman and Vila (1992) who show that, if the agent consumes only his final wealth the borrowing constraint will make him more (less) conservative if the relative risk aversion, A , is smaller (greater) than 1.

In the third part, we present an application of our analysis to an optimal growth problem in a Robinson Crusoe economy with two linear technologies, one riskless and one risky. We assume that the investment in the riskless technology must be non-negative. We show that this constraint causes the average rate of return on capital in the economy to fall even during periods when the constraint is not binding.

The purpose of the paper is twofold. First, we want to examine how borrowing constraints affect consumption and portfolio decisions. Second, we want to illustrate how dynamic programming can be used rigorously to obtain qualitative properties of optimal policies even when explicit solutions fail to exist.

The powerful theory of *viscosity solutions* is used in this paper. The value function is first characterized as the unique (constrained) viscosity solution of the Bellman equation. The characterization of the value function as a viscosity solution is imperative because the associated Bellman equation, which turns out to be fully nonlinear, might be degenerate (due to the constraints) and such equations do not have in general smooth solutions. The unique characterization together

with the stability properties of viscosity solutions enable us to approximate the value function by a sequence of smooth solutions of the regularized Bellman equation and identify the smooth limit-function with the value function. Finally, the characterization of the value function as a *constrained viscosity solution* is natural due to the presence of the (state) constraint of non-negative wealth ($W_t \geq 0$ a.e. $\forall t \geq 0$).

The methodology employed herein can be applied to several extensions and variations of the infinite horizon problem. In particular the case of *finite-horizon* can be analyzed very similarly as it is discussed in Section V. Also, if we allow investing to *more than one stock*, the Bellman equation, although more complicated, can still be treated with the above method.

In a general setting the above methodology can be used to analyze a very wide range of consumption/investment problems in the presence of *market imperfections*. More precisely, results from the theory of viscosity solutions can be used to provide (i) *analytic results* for the value function of problems in imperfect markets, for example, (see Fleming and Zariphopoulou (1991), Zariphopoulou (1993)) Duffie and Zariphopoulou (1993), Duffie, Fleming and Zariphopoulou (1993), as well as (ii) *convergence of a large class of numerical schemes* for the value function and the optimal policies when closed form solutions cannot be obtained (see, for example, Fitzpatrick and Fleming (1990), Tourin and Zariphopoulou (1993)).

The paper is organized as follows. The general model is presented in Section II. Section III deals with the existence result. In Section IV, we analyze the case of a CRRA investor. Section V presents the application to an optimal growth problem. Section VI contains concluding remarks.

II. THE MODEL

2.1. A Consumption-Portfolio Choice Problem

We consider the investment-consumption problem of an infinitely lived agent who maximizes the expectation of a time-additive utility function. The agent can distribute his funds between two assets. One asset is riskless with rate of return r ($r \geq 0$). The other asset is a stock with value p_t . We assume that the stock price obeys the equation

$$\frac{dp_t}{p_t} = (r + \mu)dt + \sigma db_t \quad (2.1)$$

where the excess rate of return μ and the volatility σ are positive constants. The process b_t is a standard Brownian Motion on the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that there are no transaction costs involved in buying or selling these financial assets.

The assumption that the opportunity set is constant, is necessary for the tractability of the model. It is, however, possible to allow the market coefficients r , μ and σ to be stochastic processes themselves. Unfortunately, this would increase dramatically the dimensionality of the problem. For example, if r_t , μ_t and σ_t are diffusion processes, the value function will depend on four state variables W , r , μ and σ .

The *controls* of the investor are the dollar amount X_t invested in the risky asset and the consumption rate C_t . His total current wealth evolves according to the state equation

$$dW_t = (rW_t - C_t)dt + \mu X_t dt + \sigma X_t db_t, \quad \text{for } t \geq 0 \quad \text{and} \quad W_0 = W. \quad (2.2)$$

The agent faces two constraints. First, his wealth must stay non-negative at any trading time.² Second,

²See Dybvig and Huang (1988) on the importance of this non-negativity constraint.

the optimal amount X_t must never exceed the (exogenously given) amount $\bar{X}(W_t)$.

In this paper, we will consider the case $\bar{X}(W) = k(W + \bar{L})$ where k and \bar{L} are non-negative constants. This formulation is general enough to encompass several interesting examples. For instance, if the investor has access to a fixed credit line \bar{L} , then $\bar{X}(W_t) = W_t + \bar{L}$. If the investor needs only to put down a certain fraction f of his stock purchases and can borrow for the remaining fraction $(1-f)$ at the risk free rate r , then $\bar{X}(W_t) = (1/f)W_t$. This case is treated only for the sake of exposition since the methodology employed here can be used for the general case of trading constraints $X_t \leq \bar{X}(W_t)$ with \bar{X} being a concave function of wealth (see Zariphopoulou (1993).)

The control (X_t, C_t) is *admissible* if

- i) (X_t, C_t) is a \mathcal{F}_t -progressively measurable process where $\mathcal{F}_t = \sigma(b_s; 0 \leq s \leq t)$ is the σ -algebra generated by the Brownian Motion b_t .
- ii) $C_t \geq 0$ a.e. ($\forall t \geq 0$)
- iii) (X_t, C_t) satisfy the integrability conditions

$$\int_0^t X_s^2 ds < +\infty \quad \text{and} \quad \int_0^t C_s ds < +\infty, \quad \text{a.e.} \quad (\forall t \geq 0)$$

- iv) $W_t \geq 0$ a.e. ($\forall t \geq 0$) (2.3)

where W_t is the wealth trajectory given by (2.2) when (X_t, C_t) is used and

- v) $X_t \leq k(W_t + \bar{L})$ a.e. ($\forall t \geq 0$). (2.4)

We denote with A the set of admissible policies.

The objective of the investor is to *maximize the expectation of a time-additive utility function*, which entails solving the optimization problem

$$\sup_A E \left[\int_0^{\infty} e^{-\beta t} u(C_t) dt \right] \quad (2.5)$$

where u is the agent's utility function and $\beta > 0$ is a discount factor. We assume that the utility function u is a strictly increasing $C^3(0, +\infty)$ function³ with

$$\lim_{C \rightarrow 0} u'(C) = +\infty \quad \text{and} \quad \lim_{C \rightarrow \infty} u'(C) = 0.$$

2.2. The Value Function

The *value function* J is defined as the supremum of the expected utility over the set of admissible controls. We denote by J^0 and J^* the value functions associated, respectively, with credit lines $\bar{L}=0$ and $\bar{L}=\infty$. The following lemma compares J^0 , J and J^* .

$$\begin{aligned} \text{Lemma 2.1:} \quad (i) \quad & J^0(W) \leq J(W) \leq J^*(W), & (W \geq 0). \\ (ii) \quad & J(W) \leq J^0(W + \bar{L}), & (W \geq 0). \end{aligned}$$

Proof: Part (i) follows from the fact that the set A of admissible controls is increasing in \bar{L} . Part (ii) follows from the fact that an investor with wealth $W + \bar{L}$ and constraint $\bar{X}(W) = kW$ will have a larger feasible set A than an investor with wealth W and constraint $\bar{X}(W) = k(W + \bar{L})$. ■

Remark 2.1: The above properties ensure us that $J(W)$ is finite for all $W > 0$ provided that $J^*(W)$ is finite. Indeed, $J(W)$ is bounded from below by $\frac{1}{\beta}u(rW)$ since the control $(X_t \equiv 0; C_t \equiv rW_t)$ is admissible and bounded from above by $J^*(W)$. The explicit derivation of $J^*(W)$ can be found in Karatzas *et al.* (1986). For the purpose of this paper, we shall assume that $J^*(W) < +\infty, \forall W \geq 0$.

The necessary conditions on the discount factor, the utility function and the market coefficients, can be found in Karatzas *et al.* (1986). They are also discussed in Part IV of this paper.

³A function is called $C^k(\Omega)$ if its first k -derivatives exist and are continuous functions in Ω .

⁴Technically speaking, from Lemma 2.1., assuming that $J^0(W)$ is finite for every W is enough to guarantee the finiteness of $J(W)$.

The following lemma describes elementary properties of $J(\cdot)$.

- Lemma 2.2:*
- (i) $J(\cdot)$ is strictly concave.
 - (ii) $J(\cdot)$ is strictly increasing.
 - (iii) $J(\cdot)$ is continuous on $[0, \infty)$ with $J(0) = \frac{1}{\beta}u(0)$.
 - (iv) $\lim_{W \rightarrow 0} J'(W) = +\infty$.

Proof: See Proposition 2.1 of Zariphopoulou (1993). ■

2.3. The Bellman Equation

In this section, we completely characterize the value function and we derive the optimal policies. This is done using dynamic programming which leads to a fully nonlinear, second order differential equation (2.6) below, known as the *Hamilton-Jacobi-Bellman (HJB) equation*. In Theorem 2.1, we show that the value function is the unique $C^2(0, +\infty)$ solution of the (HJB) equation. This will enable us to find the optimal policies from the first order conditions in the (HJB) equation. They turn out to be feedback functions of wealth and their optimality is established via a *verification theorem* (see Fleming and Rishel (1975)).

Theorem 2.1: The value function J is the unique $C^2(0, +\infty) \cap C[0, \infty)$ solution of the Bellman equation:

$$\beta J(W) = \max_{X \leq k(W, \bar{L})} \left[\frac{1}{2} \sigma^2 X^2 J''(W) + \mu X J'(W) \right] + \max_{C \geq 0} [u(C) - C J'(W)] + r W J'(W), \quad (W > 0) \quad (2.6)$$

with $J(0) = \frac{u(0)}{\beta}$.

Theorem 2.1 is the central result of our paper. The proof of this result is presented in some details in section III.

Next, using the first order conditions in (2.6) and the regularity of the value function, we can derive the optimal policies in a feedback form.

Theorem 2.2: The optimal policy (C_i^, X_i^*) is given in the feedback form $C_i^* = C^*(W_i^*)$ and $X_i^* = X^*(W_i^*)$ where $C^*(\cdot), X^*(\cdot)$ are given by*

$$C^*(W) = (u')^{-1}(J'(W)) \quad \text{and} \quad X^*(W) = \min \left\{ \frac{\mu}{\sigma^2} \frac{J'(W)}{-J''(W)} ; k(W + \bar{L}) \right\} \quad (2.7)$$

where W_i^* is the (optimal) wealth trajectory given by (2.2) with C_i^* and X_i^* being used.

We conclude this section by stating a result which will play a crucial role in the sequel. For the definition of viscosity solutions and their stability properties see Appendix A.

Theorem 2.3: The value function J is the unique (constrained) viscosity solution in the class of concave functions, of the (HJB) equation (2.6).

The proof is rather lengthy and technical and it follows along the lines of Theorem 3.1 in Zariphopoulou (1993) and, for the sake of presentation, it is omitted. General uniqueness results can be found in Ishii-Lions (1991) although they cannot be directly applied here because the control set is not compact. For a general overview of existence and uniqueness results we refer the reader to "User's Guide" by Crandall, Ishii and Lions (1992) and to the book "Controlled Markov Processes and Viscosity Solutions" by Fleming and Soner (1993).

III. SMOOTH (C^2) SOLUTIONS OF THE (HJB) EQUATION

In this section we present the proof of Theorem 2.1. Before we start the details of the proof, we discuss the main ideas. The (HJB) equation (2.6) is second order fully nonlinear and (possibly) degenerate. The degeneracy comes from the fact that the second order term $\frac{1}{2}\sigma^2 X^2 J''(W)$ may become zero. Therefore (2.6) is not *uniformly elliptic*⁵ and we know that degenerate equations do not have, in general, smooth solutions (see, for example, Krylov (1987)). Our goal is first to exclude this possibility by showing that the optimal X is

⁵A one-dimensional differential equation is said to be uniformly elliptic (non degenerate) if the coefficient of the second-order derivative $\frac{1}{2}\sigma^2 X^2$ for the Bellman equation (2.6) satisfies $0 < C_1([a,b]) \leq \frac{1}{2}\sigma^2 X^2 \leq C_2([a,b])$ for any interval $[a,b]$ where C_1 and C_2 are constants depending only on $[a,b]$.

bounded away from zero in $(0, +\infty)$. We can then use the results of Krylov (1987) for the regularity of solutions of uniformly elliptic equations.

We next consider an arbitrary interval $[a, b]$ with $a > 0$. Since the value function is concave, its first and second derivative exist almost everywhere. Without loss of generality, we may assume that $J'(a)$ and $J'(b)$ exist.

We want to show that the optimal X is bounded away from zero in $[a, b]$. Formally, the optimal X is either $k(W + \bar{L})$ or $(\mu/\sigma^2)(-J'(W)/J''(W))$. In the second case we want to get a positive lower bound of $(\mu/\sigma^2)(-J'(W)/J''(W))$. Since $J'(W)$ is non-increasing and strictly positive, it is bounded from below by $J'(b) > 0$. Therefore, it suffices to find a lower bound for $J''(W)$ in the interval $[a, b]$. Since we do not know how regular $J(\cdot)$ is, we first approximate $J(\cdot)$ by a sequence of smooth functions J^ϵ which are solutions of a suitably regularized equation.

To this end, we consider the following regularized problem: Let b_t^1 be a Brownian Motion independent of b_t .

The policy $(X_t^\epsilon, C_t^\epsilon)$ is *admissible* if:

- i) $(X_t^\epsilon, C_t^\epsilon)$ is a \mathcal{F}_t^1 -progressively measurable process where $\mathcal{F}_t^1 = \sigma(b_s^1; 0 \leq s \leq t)$;
- ii) $C_t^\epsilon \geq 0$ a.e. ($\forall t \geq 0$);
- iii) $(X_t^\epsilon, C_t^\epsilon)$ satisfy

$$\int_0^t (X_s^\epsilon)^2 ds < +\infty \quad \text{and} \quad \int_0^t C_s^\epsilon ds < +\infty \quad (\forall t \leq 0);$$

- iv) the amount of wealth $W_t^\epsilon \geq 0$ given by

$$dW_t^\epsilon = (rW_t^\epsilon + \mu X_t^\epsilon - C_t^\epsilon) dt + \sigma \epsilon X_t^\epsilon db_t + \sigma \epsilon W_t^\epsilon db_t^1 \quad (3.1)$$

with initial condition $W_0^\epsilon = W$ satisfies $W_t^\epsilon \geq 0$ a.e. ($\forall t \geq 0$) and

- (i) $0 \leq X_t^\epsilon \leq k(W_t^\epsilon + \bar{L})$ a.e. ($\forall t \geq 0$).

We denote by A^ϵ the set of admissible policies and define the value function $J^\epsilon(\cdot)$ to be

$$J^\epsilon(W) = \sup_{A^\epsilon} E \left[\int_0^\infty e^{-\beta t} u(C_t^\epsilon) dt \right]. \quad (3.2)$$

The following two lemmas provide regularity properties for the value function J^ϵ .

Lemma 3.1: The value function J^ϵ is strictly concave and strictly increasing on $[0, +\infty)$.

Proof: See Theorem 5.1 of Zariphopoulou (1993). ■

Lemma 3.2: The value function $J^\epsilon(W)$ is the unique smooth solution of the regularized Bellman equation

$$\beta J^\epsilon(W) = \max_{0 \leq X \leq k(W+\bar{L})} \left\{ \frac{1}{2} \sigma^2 (X^2 + \epsilon^2 W^2) J^{\epsilon''}(W) + \mu X J^{\epsilon'}(W) \right\} + \max_{0 \leq C} [u(C) - C J^{\epsilon'}(W)] + r W J^{\epsilon'}(W), \quad W > 0 \quad (3.3)$$

$$\text{with } J^\epsilon(0) = \frac{u(0)}{\beta}.$$

Proof: Note that equation (3.3) is uniformly elliptic and see Krylov (1987). ■

The next lemma says that the above sequence J^ϵ converges to J , as ϵ goes to zero.

Lemma 3.3: J^ϵ converges to J locally uniformly as ϵ goes to 0.

Proof: See appendix B. ■

We next prove that the first and second derivatives of J^ϵ are bounded away from zero uniformly in ϵ . This implies in particular that the optimal X in equation (3.3) is bounded away from zero.

Lemma 3.4: In any interval $[a, b]$, there exists two positive constants $R_1 = R_1([a, b])$ and $R_2 = R_2([a, b])$ independent of ϵ , such that, for $W \in [a, b]$,

$$(i) \quad J^{\epsilon'}(W) > R_1 \quad (3.4)$$

$$(ii) \quad |J^{\epsilon''}(W)| < R_2. \quad (3.5)$$

Proof: See appendix B. ■

We now conclude as follows: We first consider the boundary value problem

$$\beta V(W) = \max_{\frac{\mu R_1}{\sigma^2 R_2} \leq X \leq k(W+\bar{L})} \left\{ \frac{1}{2} \sigma^2 (X^2 + \epsilon^2 W^2) V''(W) + \mu X V'(W) \right\} + \max_{c > 0} \{u(c) - c V'(W)\} + r W V'(W) \quad W \in [a, b]$$

$$V(a) = J^\epsilon(a), \quad V(b) = J^\epsilon(b).$$

(3.6)

Applying classical results from the theory of elliptic differential equations (see, for example, Krylov (1987)), we get that (3.6) has a smooth solution V^ϵ which by uniqueness and lemma 3.4 actually coincides with J^ϵ . Since by lemma 3.3 $J^\epsilon(=V^\epsilon)$ converges locally uniformly to J , J will be a viscosity solution of the limiting equation of (3.6) as ϵ goes to zero, namely

$$\beta V(W) = \max_{\frac{\mu R_1}{\sigma^2 R_2} \leq X \leq k(W, \bar{L})} \left\{ \frac{1}{2} \sigma^2 X^2 V''(W) + \mu X V'(W) \right\} + \max_{c \geq 0} \{ u(c) - c V'(W) \} + r W V'(W) \quad W \in [a, b],$$

$$V(a) = J(a), \quad V(b) = J(b) . \quad (3.7)$$

To pass to the limit, we used the fact that $J^\epsilon(W) \rightarrow J(W)$ and the stability properties of viscosity solutions (see Appendix A, Theorem A.1). We next observe that (3.7) is uniformly elliptic and therefore has a unique smooth solution V^* . This solution V^* is of course a solution in the viscosity sense. Finally, using that J is the unique (constrained) viscosity solution of the Bellman equation (2.6), we get that $V^* \equiv J$ and therefore J is smooth.

Remark 3.1: Having analyzed the (HJB) equation (2.6) for the case of one stock, we can see that the analysis does not change much if we allow investing in more than one stock. In particular, if we have N stocks with excess returns $\mu = (\mu_1, \dots, \mu_N)$ and volatility matrix Σ , the associated Bellman equation becomes

$$\beta J(W) = \max_{\sum_{n=1}^N X_n \leq k(W, \bar{L})} \left[\frac{1}{2} \text{tr} (X' \Sigma \Sigma' X) J''(W) + (\mu, X) J'(W) \right] + \max_{0 \leq c} [u(c) - c J'(W)] + r W J'(W) . \quad (3.8)$$

The proof now follows along the same lines as before once the local ellipticity constants R_1 and R_2 can be recovered. But, these constants can be obtained as in Theorem 5.1 of Zariphopoulou (1993) by solving separately all possible cases of maximization with respect to the X_n 's in (3.8).

IV. THE CONSTANT RELATIVE RISK AVERSION (CRRA) CASE

In this section, we consider the case of an agent endowed with a CRRA utility

$$u(C) = \frac{1}{1-A} C^{1-A}, \quad A > 0, A \neq 1, \quad (4.1)$$

where A is the coefficient of relative risk aversion.

The case where the borrowing constraint $X \leq k(W + \bar{L})$ is not present or not binding, has been studied by Merton (1971). Merton proved that the optimal investment strategy consists of investing a fixed proportion of wealth in the risky asset. This proportion $\mu/(A\sigma^2)$ increases when the excess return on stocks over the risk free rate, μ , increases, and when the volatility σ or the relative risk aversion A decreases. The following proposition summarizes the results of Merton.

Proposition 4.1: (Merton 1971)

(i) *If there is no borrowing constraint (formally $\bar{L} = \infty$), an optimal control exists provided that the following inequality holds:*

$$\lambda^* > 0 \quad (4.2a)$$

with λ^* being defined as follows:

$$\lambda^* = \frac{1}{A} \left[\beta - (1-A)\left(r + \frac{\gamma}{A}\right) \right]; \quad (4.2b)$$

$$\gamma = \frac{\mu^2}{2\sigma^2}. \quad (4.2c)$$

(ii) *The value function $J^*(\cdot)$ and the optimal controls $(X(\cdot), C(\cdot))$ are given respectively by*

$$J^*(W) = (\lambda^*)^{-A} u(W), \quad (4.3)$$

$$X^*(W) = \frac{\mu}{A\sigma^2} W, \quad (4.4)$$

$$C^*(W) = \lambda^* W. \quad (4.5)$$

When $[\mu/(A\sigma^2)] < k$, the borrowing constraint $X \leq k(W + \bar{L})$ is not binding and the solution to (2.5) is the solution to the unconstrained problem (i.e. (4.2)-(4.5)). We assume in the sequel that:

$$\mu > k(A\sigma^2), \quad (4.6)$$

that is the borrowing constraint will be binding⁶. The inequality (4.6) will hold if the expected return on the stock in excess of the risk free rate μ is large, if the volatility σ is low or if the risk aversion of the investor is small.

In the presence of borrowing constraints, one natural candidate for an optimal investment strategy is the *myopic strategy* where the investor invests the minimum of two quantities: (a) what he would have invested without the borrowing constraint and (b) the maximum level of investment $k(W+\bar{L})$. In other words, the myopic investment strategy is

$$X^{\text{Myopic}}(W) = \min \left[\frac{\mu}{A\sigma^2} W, k(W+\bar{L}) \right]. \quad (4.7)$$

When the investor follows the myopic strategy, he is not aware of the borrowing constraint until he meets it. We will show that the *myopic strategy is not optimal in general*: The optimal investment, while the borrowing constraint is not binding, is affected by the fact that the borrowing constraint may be binding in the future. However, as shown below, when $\bar{L}=0$, the myopic strategy is indeed optimal.

Proposition 4.2: If $\bar{L}=0$, the value function $J^0(\cdot)$ and the optimal control $(X(\cdot), C(\cdot))$ are given by

$$J^0(W) = (\lambda^0)^{-A} u(W), \quad (4.8)$$

$$X^0(W) = kW, \quad (4.9)$$

$$C^0(W) = \lambda^0 W, \quad (4.10)$$

where

$$\lambda^0 = \frac{1}{A} \left[\beta - (1-A)(r + \mu k - \frac{A\sigma^2 k^2}{2}) \right]. \quad (4.11)$$

Proof: J^0 solves the Bellman equation and the optimal policies can be computed explicitly from (2.7). ■

If $\bar{L} > 0$, then, to our knowledge, a closed form solution to problem (2.5) fails to exist. It is possible however to obtain qualitative properties of the optimal controls. More precisely, in Propositions 4.3, 4.4 and 4.5 below,

⁶From (4.4), we expect the borrowing constraint to be binding for high levels of wealth. However, the proof of this statement is by no means trivial (see proposition 4.4).

we are able to compare the optimal controls (C^*, X^*) with the optimal controls (C^*, X^*) and (C^0, X^0) which provide interesting benchmarks.

The next proposition shows that in the presence of the borrowing constraint (2.4) the consumer-investor will always invest *less* in the risky asset than according to the myopic investment strategy (4.7).

Proposition 4.3: The optimal investment strategy, $X^(W)$, is at most equal to the myopic investment strategy, $X^{Myopic}(W)$. Furthermore, for small values of W , $X^*(W)$ is strictly less than $X^{Myopic}(W)$.*

Proof: See appendix C. ■

Given the borrowing constraint (2.4), the Bellman equation (2.6) can be rewritten as:

$$\beta J = \frac{A}{1-A} (J')^{-\frac{1-A}{A}} + r W J' + \gamma \frac{(J')^2}{-J''} \quad \text{when } W \text{ belongs to } U, \quad (4.12a)$$

$$\beta J = \frac{A}{1-A} (J')^{-\frac{1-A}{A}} + r W J' + \mu k (W + \bar{L}) J' + \frac{1}{2} \sigma^2 k^2 (W + \bar{L})^2 J'' \quad \text{when } W \text{ belongs to } B, \quad (4.12b)$$

where we use the following notation and vocabulary:

$$U = \{ W > 0 \text{ such that } \frac{\mu}{\sigma^2} \frac{J'}{-J''} < k(W + \bar{L}) \} = \text{unconstrained domain},$$

$$B = \{ W > 0 \text{ such that } \frac{\mu}{\sigma^2} \frac{J'}{-J''} > k(W + \bar{L}) \} = \text{constrained domain}.$$

Given the optimal unconstrained strategy, it seems intuitive to expect the unconstrained domain to be an interval, meaning that the agent meets the borrowing constraint for every W greater than a critical level W^* .

The next proposition provides a sufficient condition for U to be actually an interval.

Proposition 4.4: If $\beta + \gamma > r + k\mu/2$ (recall that $\gamma = \mu^2/(2\sigma^2)$) then:

- i) *There exists a positive number W^* such that $U = (0, W^*)$ and $B = (W^*, \infty)$;*
- ii) *The optimal investment strategy, $X^*(W)$, is greater than kW .*

Proof: See appendix C. ■

The next proposition compares the optimal consumption policy $C^*(W)$, to $C^{\bullet}(W)$ and to $C^0(W)$ where $C^{\bullet}(W)$ and $C^0(W)$ are defined in proposition (4.1) and (4.2).

Proposition 4.5: i) If $A < 1$, $C^*(W)$ satisfies

$$\text{Max} \{ C^{\bullet}(W), C^0(W) \left[\frac{W}{W+\bar{L}} \right]^{\frac{1-A}{A}} \} \leq C^*(W) \leq C^0(W) \left[\frac{W+\bar{L}}{W} \right]^{\frac{1}{A}}. \quad (4.13a)$$

ii) If $A > 1$, $C^*(W)$ satisfies

$$C^0(W) \leq C^*(W) \leq \text{Min} \{ C^0(W+\bar{L}); C^{\bullet}(W) \left[\frac{W+\bar{L}}{W} \right]^{\frac{1}{A}} \}; \quad (4.13b)$$

iii) $C^*(W)$ satisfies

$$\lim_{W \rightarrow \infty} \frac{C^*(W)}{W} = \lambda^0; \quad (4.13c)$$

iv) For W in $[0, W^*]$, $C^*(W)$ satisfies

$$C^{\bullet}(W) \geq C^*(W). \quad (4.13d)$$

Proof: See appendix C. ■

Proposition 4.5 provides bounds for the optimal consumption policy. These bounds are best illustrated in figures 1 and 2. It is interesting to note that $C^*(W)$ may lie outside the interval $[C^0(W), C^{\bullet}(W)]$. In fact, when the relative risk aversion coefficient A is equal to 1 (i.e. $u(c) = \log(c)$), $C^0(W)$ and $C^{\bullet}(w)$ are equal. It is easy to show that in this case $C^*(W)$ is *not* equal to $C^0(W) = C^{\bullet}(W)$. Hence the optimal consumption policy is not monotonic with respect to the credit line \bar{L} , even in the constant relative risk aversion case. This point illustrates the fact that the presence of market imperfections significantly modifies optimal policies and that a priori bounds are very difficult to obtain.

From proposition 4.5 (iv) and proposition 4.3 it follows that in the interval $[0, W^*]$, the investor consumes more and invests less due to the borrowing constraint. It follows that, the wealth of an optimizing consumer will grow (in average) more slowly than the wealth of a myopic investor. As a result, an optimizing consumer will meet the borrowing constraint later.

V. THE FINITE HORIZON PROBLEM

In this section we discuss the finite-horizon model and we state results about the value function and the optimal policies.

The investor starts at time $t \in [0, T)$, with $T > 0$, having an endowment W , consumes for $s \in (t, T]$, at rate C_s , and invests X_s amount of money in stocks. The prices of the riskless asset and the stock satisfy the same equations as in the infinite-horizon case. The investor faces the same constraints as before. In other words, the wealth and the consumption rate must stay non-negative and must meet borrowing constraints $X_s \leq k(W_s + \bar{L})$ a.e. for $t \leq s \leq T$.

The objective is to maximize the total expected utility coming from consumption and terminal wealth which entails to solve the optimization problem with value function V

$$V(W, t) = \sup_A E \left[\int_t^T u_1(C_s, s) ds + u_2(W_T, T) \right]$$

where u_1 is the utility of consumption and u_2 is the *bequest* function. Both u_1 and u_2 are assumed to be concave, increasing and smooth. The set A is the set of admissible policies which is defined in a similar way as in section II.

We now state the main theorems. Since the proofs are modifications of the ones given in previous sections they are omitted.

Theorem 5.1: The value function V is the unique $C^{1,1}([0, \infty) \times [0, T])$ and $C^{2,1}((0, \infty) \times (t, T))$ solution of the Bellman equation

$$V_t + \max_{X \leq k(W + \bar{L})} \left[\frac{1}{2} \sigma^2 X^2 V_{ww} + \mu X V_w \right] + \max_{0 \leq C} [u_1(C, t) - C V_w] + r W V_w = 0$$

$$V(W, T) = u_2(W, T).$$

Theorem 5.2: The optimal policy (C_s^, X_s^*) ($t \leq s \leq T$), is given in the feedback form $C_s^* = C^*(W_s^*, s)$ and $X_s^* = X^*(W_s^*, s)$ where the functions $C^*(\cdot)$ and $X^*(\cdot)$ are given by*

$$C^*(W, t) = (u'_1)^{-1}(V_w(W, t)) \quad \text{and} \quad X^*(W, t) = \min \left[\frac{\mu}{\sigma^2} \frac{V_w(W, t)}{-V_{ww}(W, t)}, k(W + \bar{L}) \right]$$

where W_t^* is the optimal wealth trajectory.

VI. APPLICATION TO A GROWTH PROBLEM: SUBSISTENCE LEVEL AND RISK TAKING

In this section, we consider a single good Robinson Crusoe economy. The representative agent is endowed with initial resources W_0 and has access to two linear technologies (see for instance Cox, Ingersoll and Ross(1985)). The first technology is risk free with rate of return r while the second technology is risky with instantaneous rate of return $\mu + r$ ($\mu > 0$) and instantaneous volatility σ ($\sigma > 0$). By contrast with Cox, Ingersoll and Ross, we *do not* assume that the amount invested in these technologies can be negative. The objective of the representative agent is to maximize

$$\sup_A E \left[\int_0^{\bar{t}} e^{-\beta t} u(C_t) dt \right] \quad (6.1)$$

subject to the dynamics of wealth (i.e. the stock of capital)

$$\begin{cases} dW_t = (rW_t - C_t) dt + \mu X_t dt + \sigma X_t db_t, \\ W_0 = W \end{cases} \quad (6.2)$$

where the amount X_t invested in the risky technology is constrained by

$$0 \leq X_t \leq W_t. \quad (6.3)$$

We assume that the agent requires a minimum amount \bar{C} of consumption so that his utility function is:

$$u(C) = \frac{1}{1-A} (C - \bar{C})^{1-A}, \quad A > 0, \quad A \neq 1, \quad (6.4)$$

For the agent to survive, a minimum initial wealth is required, namely, $W_0 \geq \bar{W}$ with $\bar{W} = \frac{1}{r} \bar{C}$. Henceforth we assume that this initial wealth condition is satisfied. We define *net* wealth Y_t and net consumption Z_t by

$$Y_t = W_t - \bar{W} \quad \text{and} \quad Z_t = C_t - \bar{C}. \quad (6.5)$$

With these definitions, the optimal growth problem (6.1)-(6.3) reduces to the portfolio-consumption problem that we studied in section IV. From our analysis of the portfolio-consumption problem, the following results are straightforward

1. The amount invested in the risky technology is smaller than the myopic amount

$$\min \left\{ W_t; \frac{\mu}{A\sigma^2} (W_t - \bar{W}) \right\}.$$

2. Hence constraint (6.3) (non-negativity of the riskless investment) causes the average rate of return on capital ($r + \mu \frac{x}{w}$) to be lower even during periods when this constraint is not binding. Therefore, neglecting such a constraint yields a systematic overprediction of the average rate of return on capital.

VII. CONCLUDING REMARKS

In this paper, we have used stochastic control methods to study an optimal consumption-investment problem with borrowing constraints. We have shown that optimal feedback controls exist under general assumptions about the agent's utility function. This has enabled us to obtain qualitative properties of these controls even when an explicit solution fails to exist.

A further quantitative analysis can be done by developing numerical methods. Fortunately, the Bellman equation, although fully nonlinear, is 'well behaved' (Theorem 2.1), and a wide class of algorithms can be used (see for example Fitzpatrick and Fleming (1990)). In their paper, Fitzpatrick and Fleming examine a Markov chain parametrization of a similar investment-consumption problem. They show that the value function and the optimal policies for the approximate discrete problem converge, respectively, to the value function and the optimal policies of the continuous problem as the discretization mesh size goes to zero. While the discretization relies on Kushner's Markov chain ideas, the proofs are not of probabilistic nature; rather, they rely on *viscosity solution techniques* introduced by Souganidis (1985) and Barles and Souganidis (1988).

The methodology presented herein can be extended to a wide variety of consumption-investment problems with alternative investment opportunity sets and various constraints⁷. An interesting extension would be to consider the case where the market parameters r , μ and σ evolve in a non-deterministic way, i.e., where the investment

⁷See Zariphopoulou (1993), Fleming and Zariphopoulou (1991).

opportunity is stochastic. In this case, we would need to increase the dimensionality of the problem by introducing new state variables to represent the stochastic market parameters. In general, multidimensional problems are hard to solve mainly because the associated HJB equation is often degenerate and smooth solutions might not exist. As a result feedback formulae for the optimal control may fail to exist. Nevertheless, the methodology used in the present paper could still be applied for the unique characterization of the maximized utility as (weak) solution of the HJB equation. This characterization will guarantee the convergence of numerical schemes for the optimal policies despite the lack of smoothness of the value function.

Therefore, it is our belief that viscosity techniques are the most appropriate tool to analyze continuous-time consumption/investment problems in the presence of market frictions. This is true in particular when an explicit solution fails to exist and when, as a result, numerical approximations must be used. It should be noted however that each new case will require a specific analysis since to date no general framework exists. Indeed, as opposed to the perfect capital markets literature which has developed a general model (the diffusion model) and a general tool (the martingale technology), the literature on market frictions lacks such unity. Hence, for each different type of constraints (transaction costs, holding costs⁸, borrowing constraints ...) a specific treatment is required.

⁸See Tuckman and Vila (1992).

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APPENDIX A

VISCOSITY SOLUTIONS

The notion of viscosity solutions was first introduced by M.G. Crandall and P.L. Lions (see Crandall and Lions (1983)).

Definition A.1: Let $F: \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ be continuous and non-increasing in its last argument. A continuous function $J: \mathbf{R}^+ \rightarrow \mathbf{R}$ is a *viscosity solution* of the equation:

$$F(W, J, J', J'') = 0 \quad (\text{A.1})$$

if and only if for every smooth (C^2) function $\phi: \mathbf{R}^+ \rightarrow \mathbf{R}$ the following propositions hold

- (i) if $W_0 > 0$ is a local maximum of $J - \phi$ then $F(W_0, J(W_0), \phi'(W_0), \phi''(W_0)) \leq 0$.
- (ii) if $W_0 > 0$ is a local minimum of $J - \phi$ then $F(W_0, J(W_0), \phi'(W_0), \phi''(W_0)) \geq 0$.

From the above definition, it follows that a smooth viscosity solution of (A.1) is also a solution of (A.1) in the classical sense and therefore the concept of viscosity solution extends the concept of classical ones. Moreover, viscosity solutions exhibit very good *stability properties* as the Theorem below demonstrates. Theorem A.1 states that under some mild conditions the limit of viscosity solutions is also a viscosity solution of the limit equation (see Lions (1983) for a proof).

Theorem A.1: Let $\epsilon > 0$, F^ϵ a continuous function from $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ to \mathbf{R} and let J^ϵ be viscosity solutions of $F^\epsilon(W, J^\epsilon, J^{\epsilon'}, J^{\epsilon''}) = 0$ in $[0, \infty)$. We assume that F^ϵ converges locally uniformly on $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ to some function F and that J^ϵ converges locally uniformly on $[0, \infty)$ to some function J . Then J is a viscosity solution of $F(W, J, J', J'') = 0$ on $[0, \infty)$.

In the case of equation (2.6), the function F takes the following form:

$$F(W, J, J', J'') = \beta J - \max_{0 \leq X \leq k(W-\bar{L})} \left\{ \frac{1}{2} \sigma^2 X^2 J'' + \mu X J' \right\} - r W J' - \max_{C \geq 0} \{ u(C) - C J' \}. \quad (\text{A.2})$$

APPENDIX B

This appendix presents the proofs of lemmas 3.3 and 3.4.

Lemma 3.3: J^ϵ converges to J locally uniformly as ϵ goes to 0.

Proof: We first observe that J^ϵ is locally uniformly bounded. Indeed $J^\epsilon \leq J \leq J^*$ where J^* is defined as in Lemma 2.1. This follows from the concavity of the utility function $u(\cdot)$. Moreover, because J^ϵ is concave, it is locally Lipschitz and hence J^ϵ is locally bounded. Therefore, there exists a subsequence J^{ϵ_n} and a function v such that J^{ϵ_n} converges towards v locally uniformly in $(0, \infty)$. Moreover,

$$\lim_{W \rightarrow 0} J^\epsilon(W) = \frac{u(0)}{\beta}.$$

uniformly in ϵ and therefore

$$\lim_{W \rightarrow 0} v(W) = \frac{u(0)}{\beta}.$$

Using Theorem A1, we get that v is the *unique* viscosity solution of (2.6). On the other hand, the value function J is also a viscosity solution of (2.6). Therefore, we conclude that all the converging subsequences have the same limit which coincides with J . ■

Lemma 3.4: In almost any interval $[a, b]$, there exist two positive constants $R_1 = R_1([a, b])$ and $R_2 = R_2([a, b])$ independent of ϵ , such that, for $W \in [a, b]$,

$$(i) \quad J^{\epsilon'}(W) > R_1 \tag{3.4}$$

$$(ii) \quad |J^{\epsilon''}(W)| < R_2. \tag{3.5}$$

Proof: (i) We first show that

$$\lim_{\epsilon \rightarrow 0} (J^\epsilon)'(W_0) = J'(W_0)$$

at any point W_0 such that J' exists. Let W_0 be such a point. Since J^ϵ is concave, we have

$$J^\epsilon(W) \leq J^\epsilon(W_0) + (W - W_0)J^{\epsilon'}(W_0). \tag{B.1}$$

Since the $(J^{\epsilon'})$'s are bounded locally uniformly, there exists a subsequence $(J^{\epsilon_n})'(W_0)$ and a number p such that

$$\lim_{\epsilon_n \rightarrow 0} J^{\epsilon_n}(W_0) = p.$$

Using that J^{ϵ_n} converges towards J , inequality (B.1) yields

$$J(W) \leq J(W_0) + p(W - W_0). \quad (\text{B.2})$$

But J is concave and differentiable at W_0 , therefore $p = J'(W_0)$ and

$$\lim_{n \rightarrow \infty} (J^{\epsilon_n})'(W_0) = J'(W_0).$$

Since $J^{\epsilon'}$ is non-increasing, we conclude that in any interval $[a, b]$ such that $J'(b)$ exists, there exists a constant

$C_1 = C_1([a, b])$ with

$$J^{\epsilon'}(W) \geq C_1 \text{ on } [a, b]. \quad (\text{B.3})$$

(ii) The bound for $J^{\epsilon''}$ is more difficult to obtain. The proof is extremely technical and therefore we omit it.

The interested reader is referred to Zariphopoulou (1993) for details. ■

APPENDIX C: The case of CRRA utility

This appendix presents the proofs of Propositions 4.3, 4.4 and 4.5. To ease the presentation, we first prove Proposition 4.5 i) ii) and iii). The proof of Proposition 4.5 iv) appears at the end of this appendix.

Lemma C.1 i) $J(W)/W^{1-A}$ is non-increasing and hence $J(W)W \leq (1-A)J(W)$.

ii) $J(W)/(W + \bar{L})^{1-A}$ is non-decreasing and hence $J(W)(W + \bar{L}) \geq (1-A)J(W)$.

Proof: Let $J(W, \bar{L})$ denote the value function $J(W)$ when the credit line is \bar{L} .

Given the form of the utility function, the dynamics (2.2) and the constraints (2.3) and (2.4), a standard argument can be used to show that $J(\cdot, \cdot)$ is homogenous of degree $(1-A)$ (see for instance Grossman and Laroque (1989), Fleming et al. (1989)). Then

i) $J(W, \bar{L})/W^{1-A} = J(1, \bar{L}/W)$ which is non-increasing in W .

ii) $J(W, \bar{L})/(W + \bar{L})^{1-A} = J(t, 1-t)$ where $t = W/(W + \bar{L})$. Now, by an argument similar to the one in Lemma

2.1 (ii), one can show that $J(t, 1-t)$ is non decreasing in t . ■

Proposition 4.5:

i) If $A < 1$, $C^*(W)$ satisfies

$$\text{Max } \{ C^*(W), C^0(W) \left[\frac{W}{W+\bar{L}} \right]^{\frac{1-A}{A}} \} \leq C^*(W) \leq C^0(W) \left[\frac{W+\bar{L}}{W} \right]^{\frac{1}{A}}. \quad (4.13a)$$

ii) If $A > 1$, $C^*(W)$ satisfies

$$C^0(W) \leq C^*(W) \leq \text{Min } \{ C^0(W+\bar{L}); C^*(W) \left[\frac{W+\bar{L}}{W} \right]^{\frac{1}{A}} \}; \quad (4.13b)$$

iii) $C^*(W)$ satisfies

$$\text{Lim}_{W \rightarrow \infty} \frac{C^*(W)}{W} = \lambda^0. \quad (4.13c)$$

iv) For W in $[0, W^*]$, $C^*(W)$ satisfies

$$C^*(W) \geq C^*(W). \quad (4.13d)$$

Proof: i) If $A < 1$, then from Lemmas 2.1 and C.1 (i), we have:

$$J'(W)W \leq J(W)(1-A) \leq J^*(W)(1-A).$$

Recalling that $J'(W) = (C^*)^{-A}$ it follows from Proposition 4.1 that $C^*(W) \geq C^*(W)$. Similarly

$$J'(W)W \leq J(W)(1-A) \leq J^0(W+\bar{L})(1-A).$$

Hence

$$C^*(W) \geq C^0(W) \left[\frac{W}{W+\bar{L}} \right]^{(1-A)/A}.$$

Also, from Lemmas 2.1 and C.1(i), we have

$$J'(W)(W+\bar{L}) \geq (1-A)J(W) \geq (1-A)J^0(W)$$

and therefore

$$C^*(W) \leq C^0(W) \left[\frac{W+\bar{L}}{W} \right]^{1/A}.$$

ii) The proof is similar to that of (i) and therefore is omitted.

iii) It follows immediately from (i) and (ii).

iv) See lemma C.4. ■

If the utility function exhibits constant relative risk aversion, then the Bellman equation does not admit a closed form solution on $(0, \infty)$. However, it is possible, by following Karatzas et al. (1986) to solve the Bellman equation in the unconstrained domain U . For this purpose, we express W as a function of C by inverting the

equation:

$$J'(W(C)) = C^{-A}. \quad (C.1)$$

Differentiating (C.1) with respect to C yields

$$J''(W(C)) = -AC^{-A-1}/W'(C). \quad (C.2)$$

Hence, the Bellman equation can be rewritten as

$$\beta J(W(C)) = \frac{A}{1-A} C^{1-A} + r W(C) C^{-A} + \frac{\gamma}{A} C^{1-A} W'(C). \quad (C.3)$$

Differentiating equation (C.3) with respect to C yields the following linear second order differential equation (see Karatzas et al. (1986) for a general presentation of this approach)

$$W''(C)C + \frac{A}{\gamma} \left(r + \gamma \frac{1-A}{A} - \beta \right) W'(C) + \frac{A^2}{\gamma} - \frac{A^2}{\gamma} r W(C) C^{-1} = 0. \quad (C.4)$$

The general solution to (C.4) is

$$W(C) = \frac{1}{\lambda^*} [C - K_0 C^{\omega^*} - K_1 C^{\omega^-}] \quad (C.5)$$

where the ω 's are the roots of the second order equation below

$$\omega^2 + \frac{A}{\gamma} (r - \gamma - \beta) \omega - \frac{A^2 r}{\gamma} = 0 \quad \text{with} \quad (C.6)$$

$$\omega^* \geq 1 \geq 0 \geq \omega^- \quad \text{and} \quad \omega^* \geq A. \quad (C.7)$$

The K_i 's ($i=0,1$) are constants when W is in any connected subset (a,b) of the unconstrained domain U . On such subset the optimal policy $X(W(C))$ and the value function $J(W(C))$ are given by the equations

$$X(W(C)) = \frac{\mu}{\sigma^2} \frac{J'}{-J''} = \frac{\mu}{A\sigma^2} W'(C)C = \frac{1}{\lambda^*} \frac{\mu}{A\sigma^2} [C - K_0 \omega^* C^{\omega^*} - K_1 \omega^- C^{\omega^-}] \quad (C.8)$$

and

$$J(W(C)) = \int \frac{dJ}{dW} dW = \int C^{-A} W'(C) dC = \frac{1}{\lambda^*} \left[\frac{C^{1-A}}{1-A} - \frac{\omega^*}{\omega^* - A} K_0 C^{\omega^* - A} - \frac{\omega^-}{\omega^- - A} K_1 C^{\omega^- - A} \right]. \quad (C.9)$$

The next lemma shows that the borrowing constraint, $X \leq k(W + \bar{L})$, is not met when W is small enough.

Lemma C.2: There exist $W^ > 0$ such that the open interval $(0, W^*)$ is included in U and such that*

$$X^*(W^*) = k(W^* + \bar{L}).$$

Proof: Per absurdum, suppose that there exists a sequence $W_n \rightarrow 0$ such that $X^*(W_n) = k(W_n + \bar{L})$. From equation (4.12b) and the definition of the constrained domain B , it follows that

$$\beta J(W_n) \geq \frac{A}{1-A} (J'(W_n))^{-\frac{1-A}{A}} + k \frac{\mu}{2} \bar{L} J'(W_n). \quad (C.10)$$

If $A < 1$, then from equation (C.10) it follows that $J(W_n)$ is bounded away from 0 by a positive constant, namely the minimum of the right hand side of (C.10) for all positive values of $J'(W_n)$.

But by Lemma 2.1,

$$\lim_{n \rightarrow \infty} J(W_n) \leq \lim_{n \rightarrow \infty} J^*(W_n) = 0,$$

which yields a contradiction.

If $A > 1$, then from equation (C.10) it follows that $J(W_n)$ is bounded away from $(-\infty)$ by a constant, namely the minimum of the right hand side of (C.10) for all positive values of $J'(W_n)$. But by Lemma 2.1,

$$\lim_{n \rightarrow \infty} J(W_n) \leq \lim_{n \rightarrow \infty} J^*(W_n) = -\infty,$$

which yields a contradiction.

Furthermore, from equation (4.6), the borrowing constraint must be binding and hence U cannot be equal to $(0, \infty)$. The proof is therefore complete. ■

The next lemma describes the behavior of $X^*(W)$ and $C^*(W)$ as W goes to zero.

Lemma C.3: i) $\lim_{W \rightarrow 0} X^*(W) = 0$;

 ii) $\lim_{W \rightarrow 0} C^*(W) = 0$.

Proof: i) The argument is similar to the one in Lemma C2.

 ii) It follows from equation (4.12a) that

$$\beta J(W) \geq \frac{A}{1-A} (J'(W))^{-\frac{1-A}{A}}. \quad (C.10)$$

Hence, since

$$\lim_{W \rightarrow 0} J(W) = 0 \quad (\text{respectively } -\infty)$$

when $A < 1$ (respectively $A > 1$), it follows that

$$\lim_{W \rightarrow 0} J'(W) = +\infty .$$

Since

$$C^*(W) = [J'(W)]^{-\frac{1}{\lambda^*}} ,$$

the proof is complete. ■

Using the two preceding lemmas, we conclude that in the interval $(0, W^*)$, the constant K_1 must be zero. The next lemma describes the behavior of $C^*(W)$ and $X^*(W)$ in the interval $(0, W^*)$.

Lemma C.4: In the interval $(0, W^)$, the following inequalities hold:*

- i) $K_0 > 0$,
- ii) $k W < X^*(W) < \frac{\mu}{A \sigma^2} W$,
- iii) $C^*(W) > C^{\omega^*}(W)$; $\lim_{W \rightarrow 0} \frac{C^*(W)}{C^{\omega^*}(W)} = 1$,
- iv) $\frac{d^2 X^*}{dW^2} < 0$,
- v) $\frac{d^2 C^*}{dW^2} > 0$ and
- vi) $\frac{d X^*}{dW} < \frac{\mu}{A \sigma^2}$.

Proof: i) In the interval $(0, W^*)$, we have

$$J(W(C)) \leq J^*(W) = \left(\frac{1}{\lambda^*}\right)^\lambda u(W) = \frac{1}{\lambda^*} u(C - K_0 C^{\omega^*}) \leq \frac{1}{\lambda^*} [u(C) - u'(C) K_0 C^{\omega^*}]$$

where the last inequality holds because $u(\cdot)$ is concave. Using equation (C.9) to get the expression of $J(W(C))$ and after rearranging terms, we get that K_0 is non-negative. If $K_0=0$, then $J(W)=J^*(W)$ which contradicts the fact that the constraint is binding.

ii) From equation (C.5) and (C.8) we get that:

$$X(W(C)) - \frac{\mu}{A \sigma^2} W(C) = \frac{\mu}{A \sigma^2} \frac{1}{\lambda^*} C^{\omega^*} K_0 (1 - \omega^*) \leq 0$$

and

$$X(W(C)) - k W(C) = \psi(C) = \frac{1}{\lambda^*} \left[\left(\frac{\mu}{A \sigma^2} - k \right) C - K_0 \left(\frac{\mu}{A \sigma^2} \omega^* - k \right) C^{\omega^*} \right] . \quad (C.11)$$

From (C.11), it follows that $\psi(\cdot)$ is a concave function of C . Furthermore from Lemma C.2, $\psi(C) < k\bar{L}$ if C is in $[0, C^*(W^*)]$ and $\psi(C^*(W^*)) = k\bar{L}$. Hence, $\psi(C)$ must be positive (because $\psi(c(0)) = \psi(0) = 0$) and non decreasing which yields that

$$X > kW \text{ and } dX/dW \geq k.$$

iii) From equation (C.5) and $K_1=0$, it follows that

$$W(C) = \frac{1}{\lambda^*} [C - K_0 C^{\omega^*}]$$

for $W \in [0, W^*]$ with $K_0 > 0$. Therefore

$$C^*(W) > \lambda^* W = C^*(W).$$

This completes the proof of proposition 4.5.

$$\text{iv) } \frac{d^2 X^*}{dW^2} = \frac{X_{CC} W_C - X_C W_{CC}}{W_C^3} = \frac{C(1-\omega^*)}{W_C^3} < 0.$$

v) It follows from equation (C.6) and the fact that $K_1=0$ and $K_0 > 0$.

$$\text{vi) } \frac{dX^*}{dW} = \frac{X_C}{W_C} = \frac{\mu}{A\sigma^2} \frac{C - K_0(\omega^*)^2 C^{\omega^*}}{C - K_0 \omega^* C^{\omega^*}} < \frac{\mu}{A\sigma^2}.$$

We next show that, actually, the second part of property (ii) above holds for every value of W .

Lemma C.5: For every $W > 0$, $X^(W) < \frac{\mu}{A\sigma^2} W$.*

Proof: From Lemma C.4 ii), it follows that

$$W^* > \frac{\bar{L}}{\frac{1}{k} \frac{\mu}{A\sigma^2} - 1}.$$

Hence, if $W \geq W^*$

$$X^*(W) \leq k(W + \bar{L}) < \frac{\mu}{A\sigma^2} W.$$

The next lemma shows that when W is large enough, the agent will invest the maximum allowed $k(W + \bar{L})$ in the risky asset.

Lemma C.6: There exists a number W^* such that (W^*, ∞) is a subset of the constrained domain B .

Proof: Per absurdum suppose that there exists a sequence $W_n \rightarrow \infty$ such that

$$\frac{\mu J'(W_n)}{-\sigma^2 J''(W_n)} \leq k(W_n + \bar{L}) .$$

From the Bellman equation (4.12a), it follows that

$$\beta J(W_n) \leq \frac{A}{1-A} J'(W_n) C^*(W_n) + r W_n J'(W_n) + \frac{\mu k}{2} (W_n + \bar{L}) J'(W_n) . \quad (C.12)$$

We divide each side of (C.12) by $W_n J'(W_n)$. Then we let $n \rightarrow \infty$. From Lemma C.1, $J(W_n)/W_n J'(W_n)$ goes to $1/(1-A)$. From Proposition 4.5, $C^*(W_n)/W_n$ goes to λ^0 . Hence we get

$$\frac{\beta}{1-A} \leq \frac{A \lambda^0}{1-A} + r + \frac{\mu k}{2} . \quad (C.13)$$

Using equation (4.11) for the expression of λ^0 we get $0 \leq \frac{1}{2} k(kA\sigma^2 - \mu)$ which contradicts condition (4.6). ■

Lemma C.7: If $\gamma + \beta > r + k \frac{\mu}{2}$, then $U = (0, W^*)$.

Proof: Per absurdum, suppose that besides the original interval $]0, W^*[$, the unconstrained domain, U , contains another interval. That is, suppose that there exists two numbers W_1 and W_2 such that:

$$W^* < W_1 < W_2 ; \quad]W_1, W_2[\subset U \quad \text{and} \quad X^*(W_1) - kW_1 = X^*(W_2) - kW_2 = k\bar{L} . \quad (C.14)$$

Within the interval (W_1, W_2) , there exist two constants K_0, K_1 such that $W(C)$, $X(C)$ and $J(C)$ are given by equations (C.5), (C.8) and (C.9). Let $C_i = C(W_i)$; $i=1,2$. Let $\psi(C)$ given by (C.16) below:

$$\psi(C) = X^*(C) - kW(C) , \quad (C.15)$$

$$\psi(C) = \frac{1}{\lambda^*} [M - k] \left[C - K_0 \frac{M\omega^* - k}{M - k} C^{\omega^*} - K_1 \frac{M\omega^- - k}{M - k} C^{\omega^-} \right] \quad (C.16)$$

with

$$M = \frac{\mu}{A\sigma^2} . \quad (C.17)$$

We recall that $\omega^+ \geq 1 \geq 0 \geq \omega^-$, $\omega^+ \geq A$ and that $M > k$. Given the definition of $\psi(\cdot)$, it follows that $\psi(C) \leq k\bar{L}$, for every C in the interval (C_1, C_2) , with equality at C_1 and C_2 .

If $K_0 \leq 0$ and $K_1 \leq 0$, then $\psi(\cdot)$ is increasing which is impossible since $\psi(C_1) = \psi(C_2)$.

If $K_0 > 0$ and $K_1 \leq 0$, then $\psi(\cdot)$ is strictly concave which is impossible since $\psi(C) \leq k\bar{L} = \psi(C_1) = \psi(C_2)$.

From the two observations above, it follows that $K_1 \geq 0$. Furthermore, we know from Lemma C.5 that $X^*(W) < MW$, hence

$$K_0(\omega^+ - 1)C^{\omega^+} \geq K_1(1 - \omega^-)C^{\omega^-}, \text{ for } C \in (C_1, C_2). \quad (\text{C.18})$$

Let \hat{C} be the point where $\psi(\cdot)$ reaches its minimum in $]C_1, C_2[$. Since $\psi'(\hat{C}) \geq 0$,

$$K_1(k - M\omega^-)(-\omega^-)(1 - \omega^-)C^{\omega^-} \geq K_0(M\omega^+ - k)\omega^+(\omega^+ - 1)C^{\omega^+}, \text{ for } C = \hat{C}. \quad (\text{C.19})$$

Multiplying (C.18) and (C.19) and rearranging terms yields

$$0 \geq (\omega^+ - \omega^-) [(\omega^+ + \omega^-)M - k] \quad (\text{C.20})$$

From equation (C.6), it follows that $\omega^+ + \omega^- = \frac{\Delta}{\gamma}(\gamma + \beta - r)$, and finally that

$$\frac{\mu k}{2} + r \geq \beta + \gamma. \quad (\text{C.21})$$

If $(\mu k)/2 + r < \beta + \gamma$, then we have a contradiction and therefore $U = (0, W^*)$. ■

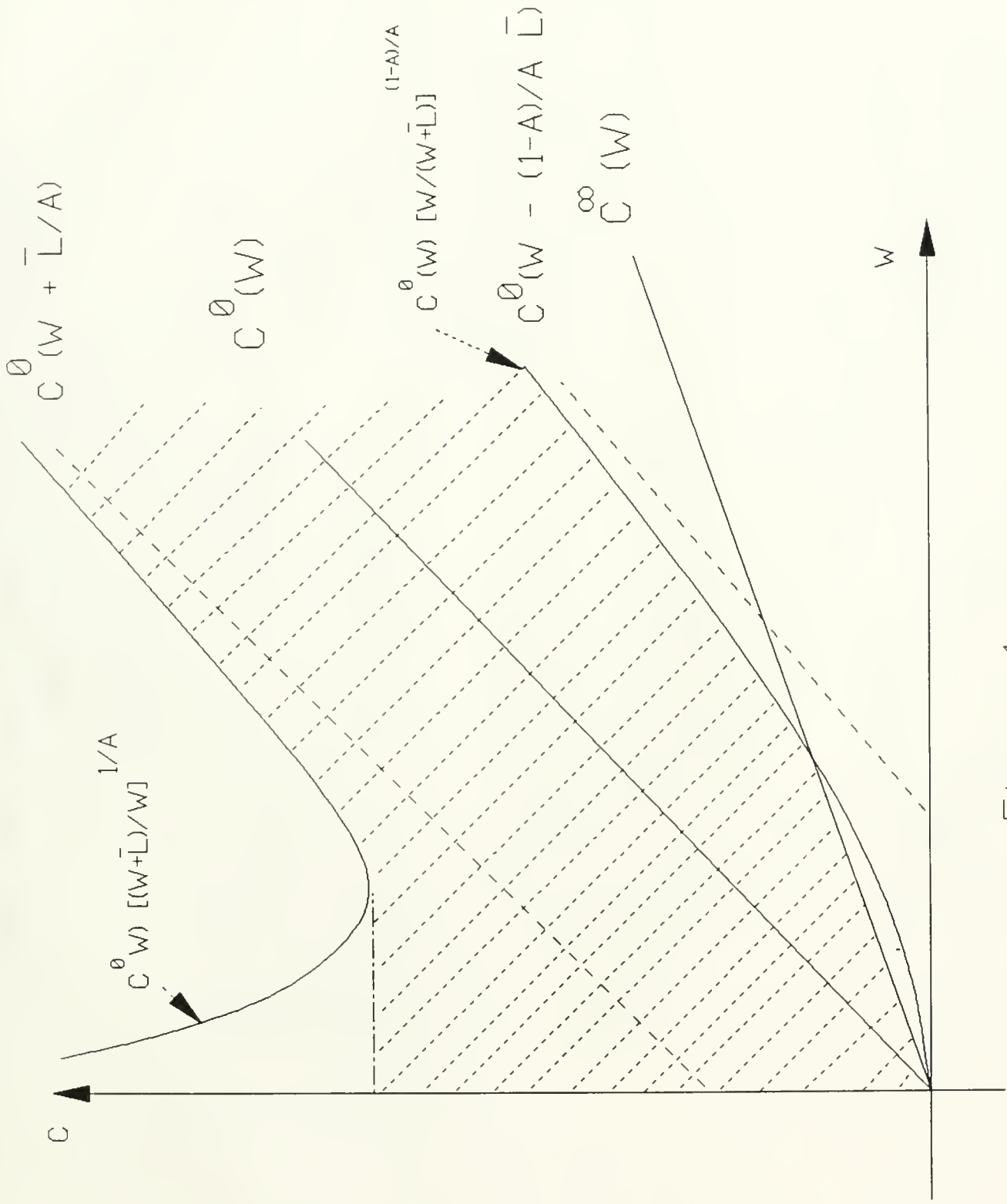


Figure 1

The case $A < 1$

The shaded region represents the possible range of $C^*(w)$

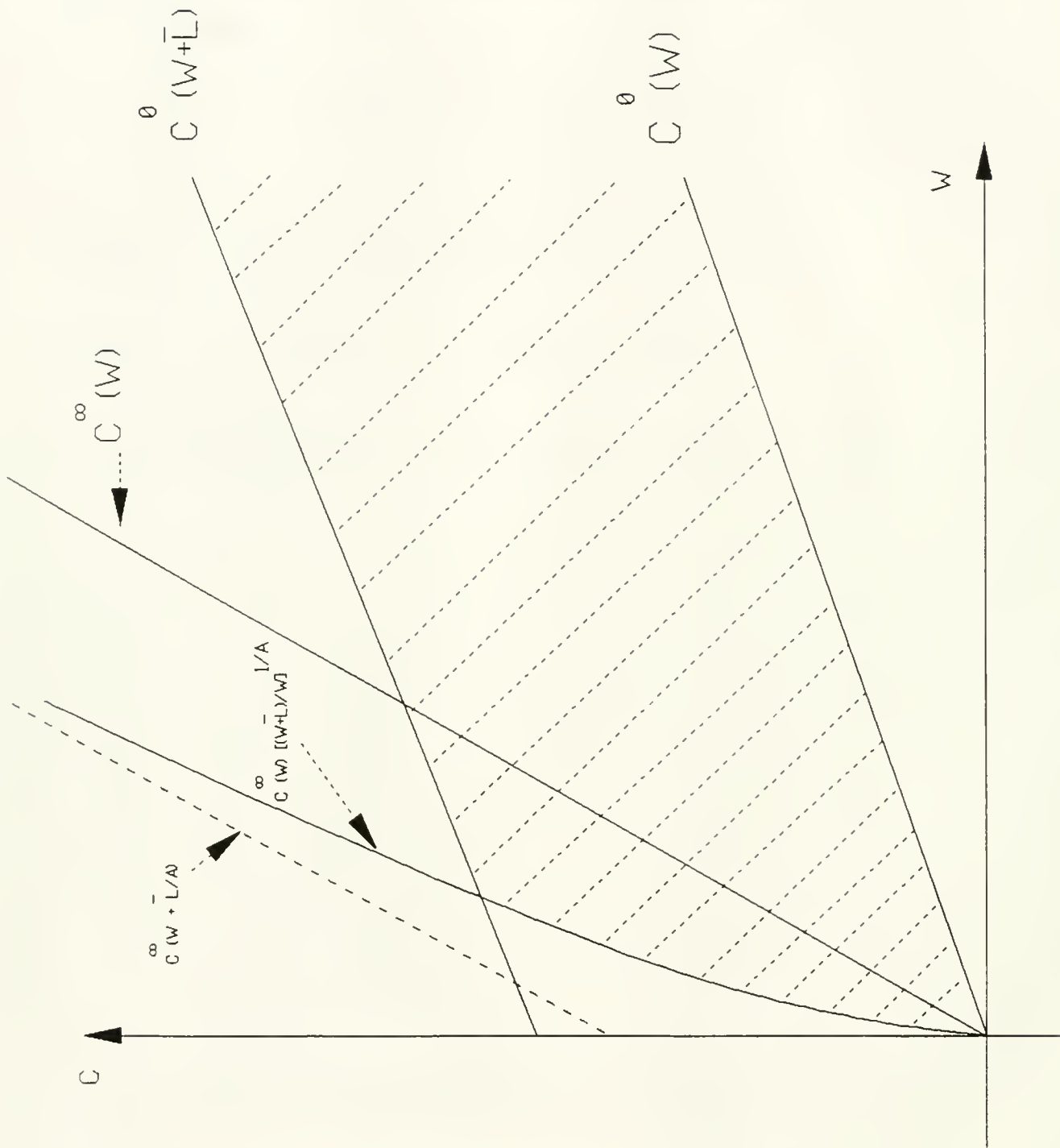


Figure 2
The case $A > 1$

The shaded region represents the possible range of $C^*(w)$

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